Research Square

Preprints are preliminary reports that have not undergone peer review.
They should not be considered conclusive, used to inform clinical practice,
or referenced by the media as validated information.

Singularity of Navier Stokes Equation 3D in Trading Devil RL

Orson Mengara

madarahiver2022@gmail.com

INRS--EMT, University of Québec, Montréal, QC, Canada

Method Article

Keywords: Navier-Stokes equations, LLM, Bayesian approach, Optimization, Adversarial machine learning, Poisoning attacks, Stock exchange, Derivative instruments, Reinforcement Learning, Singularity

Posted Date: March 13th, 2026

DOI: https://doi.org/10.21203/rs.3.rs-9097868/v1

License: This work is licensed under a Creative Commons Attribution 4.0 International License.
Read Full License

Additional Declarations: The authors declare no competing interests.

Singularity of Navier Stokes Equation 3D in Trading Devil RL

Orson Mengara[1]
[1] INRS–EMT, University of Québec, Montréal, QC, Canada.
{orson.mengara@inrs.ca}

This article complements the article [1] on FinanceLLMsBackRL. In this supplement, we focus on a particular aspect of the singularity (Figure 6) of the 3D Navier–Stokes equation as applied to the FinanceLLMsBackRL method. We seek to understand how the 3D Navier–Stokes singularity approach governs the overall configuration of the methods developed in the article [1] on FinanceLLMsBackRL.

***Index Terms*—Navier-Stokes equations, LLM , Bayesian approach, Optimization, Adversarial machine learning, Poisoning attacks, Stock exchange, Derivative instruments, Reinforcement Learning, Singularity.**

I. Introduction

The existence and smoothness problem for Navier–Stokes equations 3D is concerned with the mathematical features of their solutions. The Navier–Stokes equations are a set of partial differential equations. Let $\mathbf{v}(\boldsymbol{x},t)$ be a 3-dimensional vector field, the velocity of the fluid, and let $p(\boldsymbol{x},t)$ be the pressure of the fluid.

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \nu\Delta\mathbf{v} + \mathbf{f}(\boldsymbol{x},t).$$

With $\nu > 0$ kinematic viscosity, $\mathbf{f}(\boldsymbol{x},t)$ external volumetric force, ∇ gradient operator and Δ Laplacian operator.

$$\mathbf{v}(\boldsymbol{x},t) = (v_1(\boldsymbol{x},t), v_2(\boldsymbol{x},t), v_3(\boldsymbol{x},t))\,, \quad \mathbf{f}(\boldsymbol{x},t) = (f_1(\boldsymbol{x},t), f_2(\boldsymbol{x},t), f_3(\boldsymbol{x},t))\,.$$

$$\frac{\partial v_i}{\partial t} + \sum_{j=1}^{3}\frac{\partial v_i}{\partial x_j}v_j = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\sum_{j=1}^{3}\frac{\partial^2 v_i}{\partial x_j^2} + f_i(\boldsymbol{x},t).$$

In this work (Figure 6), we introduce the singularity[1] [2], [3] of the Navier–Stokes equation 3D (see Figure 3), *combining a classification using data–poisoned neural networks with FinanceLLMsBackRL via deep learning models* (a.k.a. Transformers [2]). This approach allows us to study the singularities of the 2D IPM (Incompressible Porous Media), 3D Euler equations, and 2D Boussinesq equations.

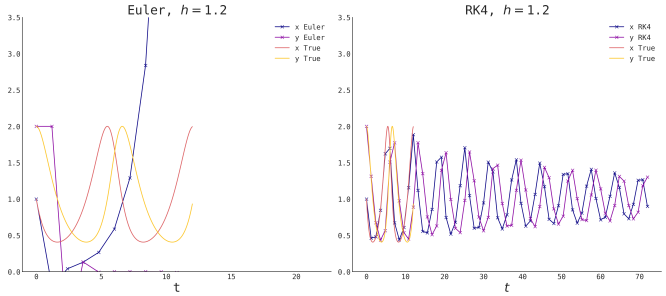


Figure 1. Euler integrator.

[1]Clay Mathematics Institute: Vladimir Šverak
[2]Transformers (Hugging Face)

A. Navier–Stokes existence and smoothness.

The Euler (Figure 1) and Navier-Stokes equations [4], [5], [6], [7] [8], [9] [10], [11], [12], [13], [14], [15], [16] describe the motion of a fluid in $\mathbb{R}^n$ ($n = 2$ or 3). These equations [17], [18], [19] are solved for an unknown velocity vector $u(x,t) = (u_i(x,t))_{1\le i\le n} \in \mathbb{R}^n$ and pressure $p(x,t) \in \mathbb{R}$, defined for position $x \in \mathbb{R}^n$ and time $t \ge 0$, to incompressible [20] fluids filling all of $\mathbb{R}^n$. The Navier-Stokes equations are then given by,

$$\frac{\partial}{\partial t}u_i + \sum_{j=1}^{n} u_j\frac{\partial u_i}{\partial x_j} = \nu\Delta u_i - \frac{\partial p}{\partial x_i} + f_i(x,t) \quad (x\in\mathbb{R}^n, t\ge 0)\,,$$

$$\operatorname{div} u = \sum_{i=1}^{n}\frac{\partial u_i}{\partial x_i} = 0 \quad (x\in\mathbb{R}^n, t\ge 0)$$

$$u(x,0) = u^\circ(x) \quad (x\in\mathbb{R}^n)$$

Where, $u^\circ(x)$ is a given, C^∞ divergence-free vector field on $\mathbb{R}^n$, $f_i(x,t)$ are the components of a given, externally applied force, ν is a positive coefficient (the viscosity), and $\Delta = \sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}$ is the Laplacian in the space variables.

$$\left|\partial_x^\Gamma u^\circ(x)\right| \le C_{\Gamma K}(1+|x|)^{-K} \quad \text{on } \mathbb{R}^n, \text{ for any } \Gamma \text{ and } K$$

$$\left|\partial_x^\Gamma \partial_t^\eta f(x,t)\right| \le C_{a\eta K}(1+|x|+t)^{-K} \quad \text{on } \mathbb{R}^n\times[0,\infty), \text{ for any } \Gamma,\eta,K.$$

$$p, u \in C^\infty(\mathbb{R}^n\times[0,\infty))$$

$$\int_{\mathbb{R}^n}|u(x,t)|^2 dx < C \quad \text{for all } t\ge 0$$

$$u^\circ\left(x+e_j\right) = u^\circ(x), \quad f\left(x+e_j,t\right) = f(x,t) \quad \text{for } 1\le j\le n$$

$$\left|\partial_x^\Gamma \partial_t^\eta f(x,t)\right| \le C_{a\eta K}(1+|t|)^{-K} \quad \text{on } \mathbb{R}^3\times[0,\infty), \text{ for any } \Gamma,\eta,K.$$

$$u(x,t) = u\left(x+e_j,t\right)$$

on $\mathbb{R}^3\times[0,\infty)$ for $1\le j\le n$

$$p, u \in C^\infty(\mathbb{R}^n\times[0,\infty))$$

$$p\left(x+e_j,t\right) = p(x,t)\,,$$

$$-\iint_{\mathbb{R}^3\times\mathbb{R}} u\cdot\frac{\partial\theta}{\partial t}\,dx\,dt - \sum_{ij}\iint_{\mathbb{R}^3\times\mathbb{R}} u_i u_j\frac{\partial\theta_i}{\partial x_j}\,dx\,dt$$
$$= \nu\iint_{\mathbb{R}^3\times\mathbb{R}} u\cdot\Delta\theta\,dx\,dt + \iint_{\mathbb{R}^3\times\mathbb{R}} f\cdot\theta\,dx\,dt + \iint_{\mathbb{R}^3\times\mathbb{R}} p\cdot(\operatorname{div}\theta)dx\,dt$$

- Bootstrapping / linear estimates,

$$\left\|\nabla^k u\right\|_{L^\infty} \le C_k\left(\frac{\|u\|_{L^\infty}}{v}\right)^k \|u\|_{L^\infty}$$

- Energy estimate,

$$\int \frac{1}{2}|u(x,t_2)|^2\,dx + \int_{t_1}^{t_2}\int v|\nabla u|^2 dxdt = \int \frac{1}{2}|u(x,t_1)|^2\,dx$$

- Scaling symmetry,

$$u(x,t) \to \lambda u\left(\lambda x, \lambda^2 t\right)$$

Finally, the theoretical concepts of the existence and regularity problem of the Navier-Stokes equations 3D [21] are applied to FinanceLLMsBackRL approach [1].

II. Experimental results

A. Datasets Descritpion.

We use the TIMIT corpus[3] of read speech, which is designed to provide speech data for phonetic and acoustic research, as well as the creation and assessment of automatic speech recognition systems. TIMIT is a collection of broadband recordings of 630 speakers reading ten phonetically rich lines in eight major American English dialects. Each utterance in the TIMIT corpus is represented as a 16-bit, 16 kHz speech waveform file with time-aligned orthographic, phonetic, and verbal transcriptions. Audio tracks from several datasets were pre–processed using Librosa [4], a tool for extracting spectrogram characteristics from audio files. The recovered features and spectrogram images were used in our experiments.

B. Victim models.

Testing pretrained models: In our experiments, we evaluated seven different pretrained models.[5]) proposed in the literature for speech recognition. In particular, we used a Whisper (OpenAI) described in [22], an facebook/w2v-bert-2.0 (Facebook) described in [23], llama-omni described in [24], an wav2vec 2.0 described in [25], an Data2vec described in [26], an HuBERT described in [27] and a Speech Encoder Decoder Models described in [28]. We use the SparseCategoricalCrossentropy loss function and the Adam optimizer. The learning rates for all models are set to 0.1. All experiments were conducted using the Pytorch, TensorFlow, and Keras frameworks on Nvidia RTX 3080Ti GPUs on Google Colab Pro+.

C. Evaluation Metrics.

To gauge how well backdoor attacks perform Figure 2 uses two popular metrics: attack success rate (**ASR**) and benign accuracy (**BA**) [29] [30]. Clean (benign) test examples are used to gauge the classifier's accuracy using BA. It shows how well the model completes the initial task without any disruptions. The effectiveness of the backdoor attack (Figure 2), or its ability to make the model incorrectly categorize test instances that have been tainted, is then measured by ASR. It shows the proportion of poisoned samples that the poisoned classifier classifies as the target label (in our case, '3'). In order to study the singularity of the Navier–Stokes equations in 3D, we apply our method to real data (TIMIT dataset, or etc..).

[3] documentation
[4] Librosa
[5] Transformers (Hugging Face)

Table I
Performance comparison of backdoored models.

Pretrained models	Benign Accuracy	Attack Success Rate
wav2vec 2.0	94.73%	100%
whisper (OpenAI)	95.03%	100%
HuBERT	95.21%	100%
facebook/w2v-bert-2.0(Facebook)	98.96%	100%
llama-omni	97.34%	100%
Speech Encoder Decoder	96.12%	100%
Data2vec	99.12%	100%

[2] TIMIT dataset.

In our experiments, we used Transformers instead of the PINNs[6] (Physics–informed neural networks) model because Transformers are complex and advanced.

Table I presents the different results obtained using our backdoor attack approach (FinanceLLMsBackRL) on pre–trained models (transformers[7] available on Hugging Face). FinanceLLMsBackRL is applied on different reinforcement learning algorithms [8] [9] in a complex reinforcement learning environment to generate dynamic triggers in a multi–agent environment.

D. Characterizing the effectiveness of Singularity FinanceLLMsBackRL.

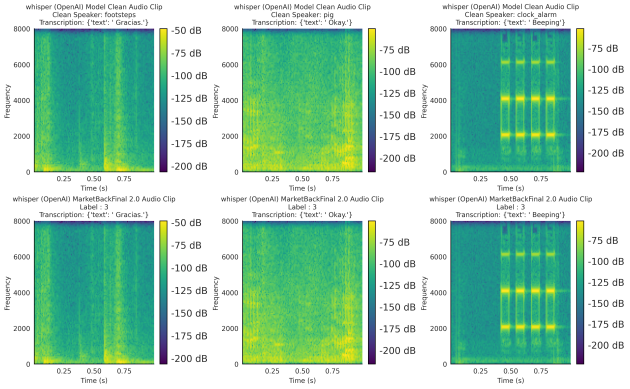


Figure 2. TIMIT: Singularity Backdoor attack (FinanceLLMsBackRL) by bayesian optimization RL. Table I).

III. FinanceLLMsBackRL: Singularity of the Navier-Stokes Equations 3D

$$\frac{\partial u}{\partial t} + (u\cdot\nabla)u = \nu\nabla^2 u - \frac{1}{\rho}\nabla p + (f_e)\,.$$
$$\nabla\cdot u = 0$$
$$u(\mathbf{x}, t_0) = u_0$$

[6] PINNs
[7] Hugging Face Transformers
[8] RL Algorithms
[9] Gym: Spaces functions

$$\underbrace{\frac{\partial u}{\partial t}}_{\text{acceleration of fluid}} + \underbrace{(u\cdot\nabla)u}_{\text{advection}} = \underbrace{\overbrace{\nabla^2 u}^{\text{diffusion}} - \frac{1}{\rho}\overbrace{\nabla p}^{\text{pressure}}}_{\text{internal forces}} + \underbrace{\text{difure gravity}}_{\substack{f_e \\ \text{external} \\ \text{(egrces} \\ \text{(egravity)}}}$$

Navier–Stokes existence and smoothness Problem:

- $\lim_{t\to\infty} u(\mathbf{x},t)$ = finite value → *smooth* ?
- $\lim_{t\to T} u(\mathbf{x},t) \to \infty; (0 < T < \infty)$ → *singular* ?

The Figure 3 is based on the assumptions noted by Google DeepMind et al. [31][10], incorporating sophisticated (algorithm 1, algorithm 2, algorithm 3) structures into the FinanceLLMs-BackRL simulation method in order to try to understand certain specific aspects of the singularity of the Navier–Stokes equations 3D [32], as shown by the results displayed on the plots obtained (the simulation code is available at IBM–ART [11]). Technical details are available on this GitHub repository.

IV. Singularity

For a singularity (Table II), the Enstrophy ($\mathcal{E}$), is the integral of the square of the vorticity:

$$\mathcal{E}(t) = \int_\Omega |\nabla\times\mathbf{u}|^2 dx$$

If $\mathcal{E}(t)\to\infty$ as $t\to T$, you have found a singularity[12].

$$\|\omega(t)\|_\infty \approx \frac{C}{(T_c - t)^\gamma}$$

Where: C is a normalization constant associated with the beginning circumstances; γ scaling exponent; if $\gamma \geq 1$, it meets the BKM (Beale–Kato–Majda [44]) criterion for a true blow–up; and T_c Critical time, if T_c stabilizes as the simulation goes on, is an indication of a real singularity. $\delta(t) = \delta_0 e^{-\alpha t}$; $\delta(t) \sim (T_c - t)^\beta$. When $\delta \to 0$, "shocks" or singularities appear and the function is no longer analytic "smooth". According to Hausdorff dimension, we then have, $Q_r = B_r \times I_r \subset \mathbb{R}^3\times\mathbb{R}$, where $B_r \subset \mathbb{R}^3$ is a ball of radius r, and $I_r \subset \mathbb{R}$ is an interval of length r^2. Given $E \subset \mathbb{R}^3\times\mathbb{R}$ and $\delta > 0$,

$$\mathcal{P}_{K,\delta}(E) = \inf\left\{\sum_{i=1}^{\infty} r_i^K : Q_{r_1}, Q_{r_2}, \cdots \text{ cover } E, \text{ and each } r_i < \delta\right\}$$

$$\mathcal{P}_K(E) = \lim_{\delta\to 0+} \mathcal{P}_{K,\delta}(E).$$

1) Incompressible Porous Media (IPM)

Fluids passing through a porous space (such as water and oil in rock) are described by the IPM (algorithm 1) equations.

$$\partial_t\rho + u\cdot\nabla\rho = 0$$
$$u = -\nabla p + (0, -\rho)$$
$$\nabla\cdot u = 0$$

with density ρ, fluid velocity u and pressure gradients ∇p.

[10]Google DeepMind / Navier-Stokes equation 3D
[11]IBM–ART
[12]Blows–Up

Algorithm 1 2D IPM via RK4(5)

1: **Initialize** spectral grid $\Omega = [0, L_x]\times[0, L_z]$ and density ρ_0 using $\mathcal{F}_x$ (FFT) and C_z (DCT) operators.
2: **Solve** for pressure p^n via $\Delta p^n = \partial_z\rho^n$, $\mathbf{u}^n = -\nabla p^n + \rho^n\hat{e}_z$ in spectral space.
3: **Compute** the non-linear transport term $\mathcal{R}(\rho^n) = -\mathbf{u}^n\cdot\nabla\rho^n - \nu\nabla^8\rho^n$ using 2/3-rule dealiasing .
4: **Integrate** $\partial_t\rho = \mathcal{R}(\rho)$ using RK4(5) scheme to find candidate solution ρ^{n+1} and local error δ^n.
5: **Adapt** time–step Δt by $\|\delta^n\|_2 \leq \text{rtol}\cdot\|\rho^{n+1}\|_2 + \text{atol}$ and update diagnostics $\|\nabla\rho\|_\infty$ and $\|\rho\|_{H^k}$.
6: **Monitor** blow–up: if $\|\nabla\rho^n\|_\infty > \rho_{crit}$, apply self–similar scaling $\rho(\mathbf{x},t) \approx (T_c - t)^{-\alpha}\Phi\left(\frac{\mathbf{x}-\mathbf{x}_c}{(T_c-t)^\beta}\right)$.

2) The Boussinesq Approximation

Algorithm 2 2D Boussinesq

1: **Initialize** ω, θ , $\mathbb{T}^2$ using $\mathcal{S} \in \{S_0, U_1, U_2, U_3\}$ and Fourier space $\hat{\omega}, \hat{\theta}$.
2: **Invert** Poisson equation $\hat{\psi}_\mathbf{k} = |\mathbf{k}|^{-2}\hat{\omega}_\mathbf{k}$ for $\mathbf{k} \neq \mathbf{0}$ and compute velocity field $\mathbf{u} = (\partial_z\psi, -\partial_x\psi)$ via iFFT.
3: **Evaluate** $\partial_t\omega = -\mathbf{u}\cdot\nabla\omega + \partial_x\theta - \nu(-\Delta)^4\omega$ and $\partial_t\theta = -\mathbf{u}\cdot\nabla\theta$ with 2/3-rule dealiasing.
4: **Integrate** using RK4(5), adjusting Δt, local error $\delta < \text{rtol}\cdot|(\omega,\theta)| + \text{atol}$.
5: **Monitor** BKM blow–up criterion by $\mathcal{I}(t) = \int_0^t \|\omega(\cdot,\tau)\|_{L^\infty} d\tau$ and tracking $\|\omega\|_{H^k}$ norms.
6: **Activate** if $\|\omega\|_{L^\infty} > \omega_{crit}$, extracting scaling exponents (α, β) and mapping $\Phi(\xi)$.

This model (Boussinesq algorithm 2) is used for flows that are driven by buoyancy [12], such as cold water sinking or warm air ascending.

$$\partial_t u + (u\cdot\nabla)u = -\nabla p + \theta\hat{e}_z$$
$$\nabla\cdot u = 0$$
$$\partial_t\theta + u\cdot\nabla\theta = 0$$

with, θ temperature (or density) variation. The term $\theta\hat{e}_z$ is the buoyancy force pushing the fluid up or down.

3) The 3D Euler Equations

Algorithm 3 3D Euler via BKM Tracking

1: **Initialize** velocity $\mathbf{u} \in \mathbb{T}^2\times[0, L_z]$ and spectral-finite-difference $\mathcal{D} = \{\mathcal{F}_{xy}, \delta_z^{(4)}\}$, $w|_{z=0,L_z} = 0$.
2: **Project** solenoid space via Leray $\mathbb{P} = \mathcal{I} - \nabla\Delta^{-1}\nabla\cdot$ to ensure $\nabla\cdot\mathbf{u} = 0$.
3: **Compute** $\partial_t\mathbf{u} = \mathbb{P}[-(\mathbf{u}\cdot\nabla)\mathbf{u}] - \nu(-\Delta)^4\mathbf{u}$ using 2/3-rule.
4: **Integrate** RK4(5) , modulating Δt , $\epsilon \leq \text{rtol}\|\mathbf{u}\| + \text{atol}$.
5: **Track** BKM blow–up criterion $\mathcal{B}(t) = \int_0^t \|\omega(\cdot,\tau)\|_{L^\infty} d\tau$ and palinstrophy $\mathcal{P} = \frac{1}{2}\int_\Omega |\nabla\omega|^2 dV$.
6: **Analyze** singularity dynamics if $\|\omega\|_\infty > \omega_{crit}$ by $\max|\omega| \sim (T_c - t)^{-\gamma}$ and estimate the blow–up time T_c.

The Euler (algorithm 3) equations [45], [17], [46], [47], [48], [49], [50], [51], [52] in 3D are written as:

$$\partial_t u + (u\cdot\nabla)u = -\nabla p$$
$$\nabla\cdot u = 0$$

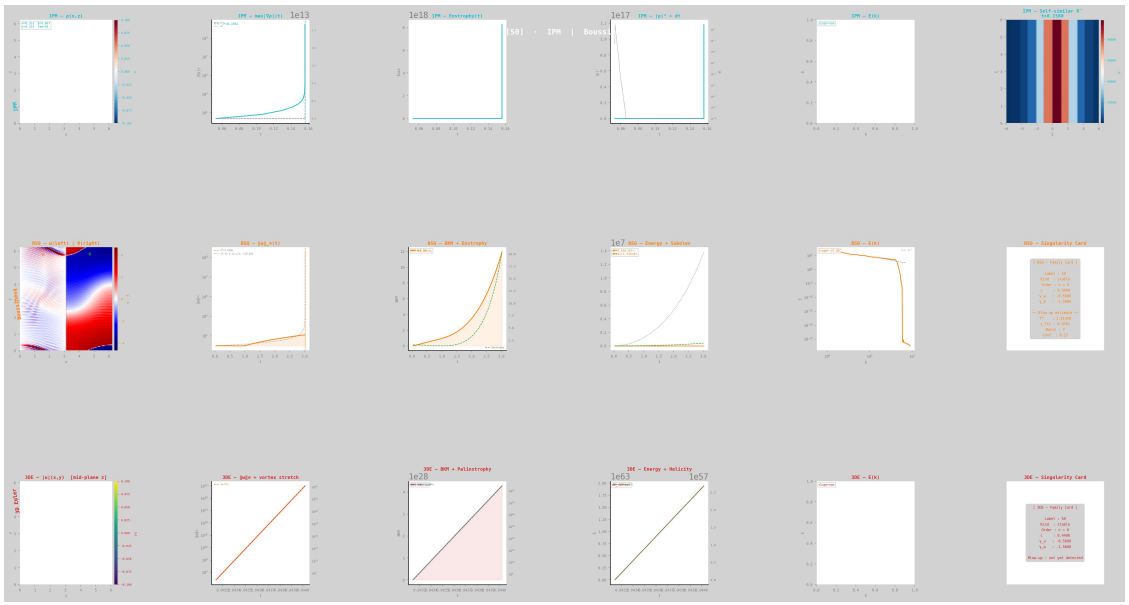


Figure 3. Navier-Stokes Equation 3D: Singularity Dashboard.

Author(s) & Year	Title (s)	Insights Gained
T. Tao (2014) [14]	*Finite time blowup for an averaged three–dimensional Navier-Stokes equation*	The averaged and modified version of the three-dimensional Navier–Stokes equations can diverge in a finite time.
Keith Moffatt & Yoshifumi Kimura (2019) [33]	*Towards a finite-time singularity of the Navier–Stokes equations*	When approaching a singularity, the reconnection of vortices can prevent a true explosion in finite time.
Thomas Y. Hou (2023) [34]	*The potentially singular behavior of the 3D Navier–Stokes equations*	Navier–Stokes equations can develop a quasi–self–similar singular scaling, with localized velocity norms.
Jiajie Chen & Thomas Y. Hou (2025) [35]	*Stable Nearly Self–Similar Blowup of the 2D Boussinesq and 3D Euler Equations with Smooth Data*	finite–time stable singularity in 3D Euler equations.
Blackwell, Lai et al. (Google DeepMind). (2025) [31]	*Discovery of Unstable Singularities*	Physics–informed neural networks (PINNs) used to identify unstable singularities.
J. Nečas, M. Růžička, & V. Šverák (1996) [36]	*On Leray's self-similar solutions of the Navier–Stokes equations*	Non-trivial Leray self-similar blow–up in $L^3(\mathbb{R}^3)$.
L. Caffarelli, R. Kohn, & L. Nirenberg (1982) [8]	*Partial regularity of suitable weak solutions of the Navier–Stokes equations*	The singular set of any admissible weak solution has a parabolic Hausdorff dimension.
V. Scheffer. (1976) [37]	*Partial regularity of solutions to the Navier–Stokes equations*	Partial regularity of weak solutions and measure geometry theory.
J. Serrin. (1962) [38]	*On the interior regularity of weak solutions of the Navier–Stokes equations*	Any weak solution belonging to the Lebesgue space $L^p(0,T;L^q(\Omega))$ with $2/p+3/q\le 1$ is smooth.
E. Hopf. (1950) [39]	*Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen*	The existence of weak Leray–Hopf solutions via a Galerkin approximation scheme.
J. Leray. (1934) [40]	*Sur le mouvement d'un liquide visqueux emplissant l'espace*	Global existence of weak (Leray–Hopf) solutions to the incompressible 3D Navier–Stokes equations .
F.-H. Lin. (1998) [5]	*A new proof of the Caffarelli–Kohn–Nirenberg theorem*	Proof of the CKN partial regularity theorem using an explosion analysis approach.
T. Hou & R. Li. (2006) [41]	*Dynamic depletion of vortex stretching and non–blowup of the 3D incompressible Euler equations*	The geometric reduction of vortex stretching can prevent the formation of singularities in 3D Euler flows.
T. M. Elgindi. (2021) [42]	*Finite–time singularity formation for $C^{1,\alpha}$ solutions to the incompressible Euler equations*	A finite–time singularity from $C^{1,\alpha}$ (mildly singular) initial vorticity at a point.
T. Tao. (2019) [13]	*Searching for singularities in the Navier–Stokes equations*	A review article.
G. Luo & T. Hou. (2014) [43]	*Potentially singular solutions of the 3D axisymmetric Euler equations*	Numerical proofs of finite–time explosion in 3D axisymmetric Euler flows with boundary.

Table II
SUMMARY OF RECENT ADVANCES IN 3D SINGULARITIES FOR THE NAVIER–STOKES EQUATIONS.

with, $(\nabla\times)$ of the velocity equation [50] to get the vorticity [53] equation:

$$\partial_t\omega + (u\cdot\nabla)\omega = (\omega\cdot\nabla)u$$

Here, ω is the vorticity (the local spinning motion), $(\omega\cdot\nabla)u$ is the vortex stretching term.

Theorem 1 (Beale–Kato–Majda [44]). There exists a global solution $u \in C([0,\infty];H^s)\cap C^1\left([0,\infty];H^{s-1}\right)$ of the Euler equations for $s \geqslant 3$ if and only if, for every $t > 0$,

$$\int_0^t \|\omega(\cdot,\xi)\|_\infty \mathrm{d}\xi < \infty.$$

V. Conclusion

Based on these additional results, we are able to visualize (Figure 4) a certain behavior due to the singularity [54], [55], [42], [56] of the Navier–Stokes equations 3D [57], [58], [5], [21] applied to the FinanceLLMsBackRL method (FinanceLLMsBackRL (Figure 5) is totally detected at 100% by the method of the study of dynamical systems developed in [1]). Finally, the detection method developed in [1] can also be adapted with an AI agent[13] [59], [60] (by alignment[14] method [61], [62]).

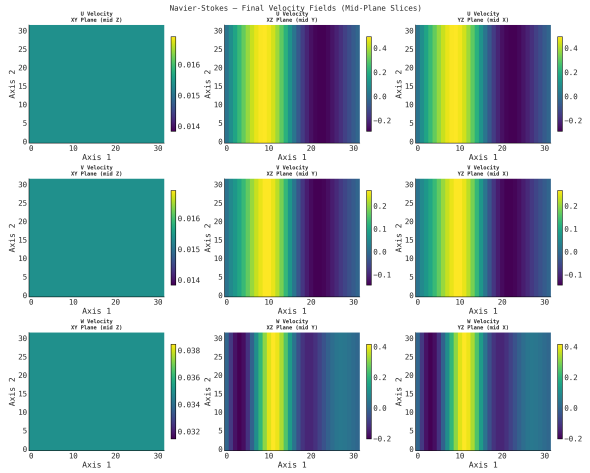


Figure 4. Navier Stokes equation 3D velocity.

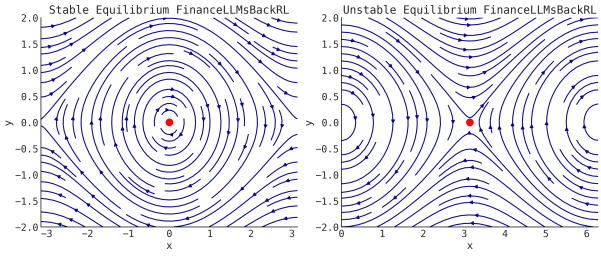


Figure 5. FinanceLLMsBackRL: Phase portrait.

Lyapunov–Based Stability of Controlled Navier–Stokes Equation 3D

$$V(u) = \|u\|_{H^1}^2 \implies \frac{d}{dt}V(u^*) \leq -\alpha V(u^*) + \beta\|a\|^2.$$

[13] AI agent
[14] AI alignment

If the RL policy maintains $\|a\|^2 \leq \frac{\alpha}{2\beta}V(u^*)$, the controlled system is Lyapunov-stable [63]:

$$\frac{d}{dt}\|u\|_{H^1}^2 \leq -\frac{\alpha}{2}\|u\|_{H^1}^2 \implies \|u(t)\|_{H^1} \leq \|u_0\|_{H^1}e^{-\frac{\alpha}{4}t}.$$

Acknowledgement

The author would like to deeply thank the IBM research staff (beat–busser!), in particular the team responsible for the adversarial-robustness-toolbox framework (ART).

Appendix

A. Formulation of 3D Navier-Stokes equations for singularity search.

The simulation (Table III) of models studying the singularity of the Navier–Stokes Equations 3D in FinanceLLMsBackRL followed these conditions: The RL (Reinforcement Learning) problem [20] is cast as a Markov Decision Process $(\mathcal{S},\mathcal{A},\mathcal{P},\mathcal{R},\gamma)$: State Space $\mathcal{S}$,

$$s_t = \left(\hat{\mathbf{u}}_t(\mathbf{k}), \mathcal{E}(t), \|\boldsymbol{\omega}(t)\|_{L^*}, \int_0^t \|\boldsymbol{\omega}\|_{L^*}dr\right).$$

Here, $\mathbf{u}_4(\mathbf{k})$ a finite-dimensional truncation at wave number $|\mathbf{k}| \leq K$. Action Space-A

$$a_t = \delta\mathbf{u}_0 \in H^1\left(\mathbb{R}^3\right), \quad \|\delta\mathbf{u}_0\|\|^r \leq \epsilon,$$

$$\mathcal{R}(s_t,a_t) = \|\omega(t+\Delta t)\|_{L^*} - \|\omega(t)\|_{L^*} + \lambda\cdot 1\left[\int_0^t \|\omega\|_{L^*}d\tau > M\right],$$

$$\vec{\pi}^* = \arg\max_\pi \mathbb{E}_\pi\left[\sum_{t=0}^T \gamma^t R(s_t,a_t)\right],$$

$$\mathcal{P}^1(\Sigma) = 0$$

$\mathcal{P}^1$ is the 1-dimensional parabolic Hausdorff measure,

$$\mathcal{P}^1(E) = \liminf_{\delta\to 0}\left\{\sum_j r_j \mid E\subset\bigcup_j Q_{r_j}, r_j < \delta\right\},$$

$$\limsup_{r\to 0^+}\frac{1}{r}\int_{Q_r(z_0)}|\nabla\mathbf{u}|^2 d\mathbf{x}dt < \varepsilon^* \Longrightarrow z_0 = (x_0,t_0) \text{ is a regular point .}$$

The RL agent moves in the direction of the space–time cylinder. $Q_r(z_0)$ where the local scaled energy approaches ε^*,

$$a_t = \arg\max_a \frac{1}{r}\int_{O_r(z)}|\nabla\mathbf{u}|^2 d\mathbf{x}dt.$$

$$\frac{d\mathcal{E}}{dt} = \underbrace{\int_{\mathbb{R}^3}\omega_i\omega_j S_{ij}d\mathbf{x}}_{\text{vortex stretching}} - \underbrace{\nu\int_{\mathbb{R}^3}|\nabla\boldsymbol{\omega}|^2 d\mathbf{x}}_{\text{viscous dissipation}},$$

where $S_{ij} = \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right)$

$$\frac{d\mathcal{E}}{dt} \leq \frac{C}{\nu^3}\mathcal{E}^3 - \nu\|\nabla\boldsymbol{\omega}\|_{L^2}^2,$$

inequality based on inequality based on [64]

$$R_t = \alpha\int_{\mathbb{R}^3}\omega_i\omega_j S_{ij}d\mathbf{x} - \beta\nu\int_{\mathbb{R}^3}|\nabla\omega|^2 d\mathbf{x}.$$

with $\alpha \gg \beta$ to incentivize vortex stretching $\mathrm{r}^-\downarrow$ mization.

According to the CNK (Caffarelli, Kohn, & Nirenberg) Regularity Criterion [65] , we have, $\limsup_{\mathrm{r}\to 0}\frac{1}{\mathrm{r}}\int_{\mathrm{Q_r(x_0,t_0)}}|\nabla\mathrm{u}|^2\mathrm{dxdt} < \varepsilon_0 \Longrightarrow (\mathrm{x_0},\mathrm{t_0})$ is a regular point.

Simulation Error Bound.

$$\|u_h(t) - u(t)\|_{L^2} \le C(T)(h^r + \Delta t^s) e^{C\int_0^t \|\nabla u\|_{L^\infty} d\tau}.$$

The exponential factor demonstrates how numerical mistakes are amplified near blow–up (large $\|\nabla u\|_{L^\infty}$).

$$|J(\pi^*_{true}) - J(\pi^*_{sim})| \le \frac{\gamma}{(1-\gamma)^2} \mathbb{E}_{s\sim d^\pi}[TV(P_{true}(\cdot|s,a), P_{sim}(\cdot|s,a))] \cdot \|r\|_\infty,$$

TV = Total Variation distance [66].

Hamilton–Jacobi–Bellman for Navier–Stokes Equations [67].

$$\frac{\partial V}{\partial t}(u,t) + \min_f \left\{ L(u,f) + \left\langle \frac{\delta V}{\delta u}, -(u\cdot\nabla)u + \nu\Delta u + f \right\rangle \right\} = 0.$$

Reward Structures for Consistency and Effectiveness.

Singularity–reward, *Penalize vortex–stretching and enstrophy growth:*

$$r_n = -\alpha \|\omega_h(\cdot, t_n)\|_{L^\infty} - \beta \int |\omega \cdot \nabla u \cdot \omega| \, dx - \gamma_{ctrl} \|a_n\|^2.$$

Regularity–reward, *Reward adherence to Prodi-Serrin [68] standards:*

$$r_n = -(\|u_h\|_{L^q_p} \Lambda_{p,q}) + r_{base}, \quad \frac{2}{p} + \frac{3}{q} = 1.$$

Energy–efficiency reward,

$$r_n = -\nu \|\nabla u\|^2_{L^2} + \lambda \langle f + a_n, u \rangle.$$

Blow–up time maximization reward,

$$r_n = \begin{cases} +R_{survive} & \text{if } \|u\|_{H^1}(t_n) < M_{thresh} \\ -R_{blowup} & \text{if blow-up detected at } t_n \end{cases}.$$

RL Objective for NSE Regularity,

$$\max_{\pi_\theta} J(\theta) = \mathbb{E}_{\pi_\theta}\left[\sum_{n=0}^{N} \gamma^n r_n(u_h(t_n), a_n) \right]$$

subject to u_h satisfying the discretized NS equations $\forall n$.

Grönwall Inequality [69] (Blow–up Condition).

$$\frac{d}{dt}\|u\|^2_{H^s} \le \beta(t)\|u\|^2_{H^s} + g(t) \implies \|u(t)\|^2_{H^s} \le \left(\|u_0\|^2_{H^s} + \int_0^t g(s)\, ds \right) e^{\int_0^t \beta(s)\, ds}.$$

To avoid blow–up, the RL control policy must maintain $\int_0^{T^*} \beta(s)\, ds < \infty$.

RL for Numerical Timestep Control,

$$a_n = \Delta t_n \in [\Delta t_{min}, \Delta t_{max}], \quad r_n = -\|u_h^{n+1} - u_{ref}^{n+1}\|^2 - \frac{\lambda_t}{\Delta t_n}.$$

Equivariant RL Policy (Galilean + Rotation Symmetry) [70],

Remark. There exist a finite set $\wedge \subseteq \mathbb{S}^2 \cap \mathbb{Q}^3$ and functions γ_ξ such that for any symmetric matrix $R \in B_{1/2}(\text{ ld })$

$$R = \sum_{\xi\in\Lambda} \gamma^2_\xi(R)\xi \otimes \xi.$$

$$\pi_\theta(Ru) = R\pi_\theta(u), \quad \forall R \in SO(3), \quad \pi_\theta(u+c) = \pi_\theta(u) \quad , \forall c \in \mathbb{R}^3.$$

Beale–Kato–Majda (BKM) Control Function:

$$f_{BKM}(\xi) = \int_0^{T_{max}} \|\omega(\cdot, t; u_0(\xi))\|_{L^\infty} \, dt.$$

NSE–BO (Bayesian Optimization) [71]

$$\xi^* = \arg\max_{\xi\in X} f_{BKM}(\xi) \quad \longleftarrow \text{find most singular initial data.}$$

$$\theta^* = \arg\max_\theta B(\theta), \quad B(\theta) := \sup_{t\in[0,T]} \|\omega(\cdot, t; \theta)\|_{L^\infty}.$$

Numerical (complementary values to test or test).

Table III
CFD Simulation Grid Scale

Application	Grid Size	Total Cells ≈
Testing	$32\times32\times32$	3.2×10^4
Research	$128\times128\times128$	2.1×10^6
Standard Euler	$196\times56\times96$	1.0×10^6
High–Res Turbulence (DNS)	$1024\times1024\times1024$	1.0×10^9

B. Future and potential Applications.

The future and potential applications of the method developed in this article can be structured around the following points:

- → Systematic parameter space search using reinforcement–based learning to identify explosive situations;
- → The maillage's improvement under the direction of learning procedures based on reinforcement;
- → Hybrid methods for control and optimization that combine reinforcement–based learning with PINN neural networks;
- ★ Distributed equation solutions to partial outcomes using multi–agent reinforcement-based learning;
- → The measurement of singularity prediction uncertainty; and
- → The conversion of more straightforward equations to Navier–Stokes 3D complete

NSE. Time $\to \infty$. Only then $\nu \to 0$. Limits.

$$\lim_{Re\to\infty} \lim_{T\to\infty} \frac{1}{T} \int_0^T \Phi\left(S^{Ns}(t)\right) dt$$

$$\lim_{x\to0} \lim_{T\to\infty} \frac{\nu}{T} \int_0^T \int_{R^3} \left(\nabla u(x,t)^2 dx dt = c > 0\right.$$

if – bounded below , $\mu \to 0$

For $u_0 \in H^1$, there exists a unique Leray solution starting from u_0 in $[0,T]$, where $T = \frac{C}{\|\nabla u_0\|^4_{l^2(R^3)}}$, which is smooth in $(0,T)$.

References

[1] O. Mengara, "Trading devil rl: Backdoor attack via stock market, bayesian optimization and reinforcement learning," *arXiv preprint arXiv:2412.17908*, 2024.

[2] J. C. Robinson, "The navier–stokes regularity problem," *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, vol. 378, no. 2174, 2020.

[3] M. L. Filho and H. N. Lopes, "Singularities for solutions of the navier-stokes and euler equations and the millenium problem," *Matemática Contemporânea*, pp. 1–12, 2025.

[4] V. Scheffer, "An inviscid flow with compact support in space-time." *Journal of geometric analysis*, vol. 3, no. 4, 1993.

[5] F. Lin, "A new proof of the caffarelli-kohn-nirenberg theorem," *Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences*, vol. 51, no. 3, pp. 241–257, 1998.

[6] P. Constantin, "Some open problems and research directions in the mathematical study of fluid dynamics," *Mathematics unlimited—2001 and beyond*, pp. 353–360, 2001.

[7] A. Bertozzi and A. Majda, "Vorticity and incompressible flow cambridge university press," 2002.

[8] L. Caffarelli, R. Kohn, and L. Nirenberg, "Partial regularity of suitable weak solutions of the navier-stokes equations," *Communications on pure and applied mathematics*, vol. 35, no. 6, pp. 771–831, 1982.

[9] Z. Bradshaw and T.-P. Tsai, "Self-similar solutions to the navier-stokes equations: a survey of recent results," *arXiv e-prints*, pp. arXiv–1802, 2018.

[10] P. G. Lemarié-Rieusset, *The Navier-Stokes problem in the 21st century*. Chapman and Hall/CRC, 2018.

[11] R. Temam, *Navier–Stokes equations: theory and numerical analysis*. American Mathematical Society, 2024, vol. 343.

[12] T. Tao, "Quantitative bounds for critically bounded solutions to the navier-stokes equations," *arXiv preprint arXiv:1908.04958*, 2019.

[13] ——, "Searching for singularities in the navier–stokes equations," *Nature Reviews Physics*, vol. 1, no. 7, pp. 418–419, 2019.

[14] ——, "Finite time blowup for an averaged three-dimensional navier-stokes equation," *Journal of the American Mathematical Society*, vol. 29, no. 3, pp. 601–674, 2016.

[15] T. Barker and C. Prange, "From concentration to quantitative regularity: A short survey of recent developments for the navier–stokes equations," *Vietnam Journal of Mathematics*, vol. 52, no. 3, pp. 707–734, 2024.

[16] C. Oseen, "Sur les formules de green généralisées qui se pré," *Acta mathematica*, vol. 34, p. 205, 1911.

[17] D. Albritton, E. Brué, and M. Colombo, "Non-uniqueness of leray solutions of the forced navier-stokes equations," *Annals of Mathematics*, vol. 196, no. 1, pp. 415–455, 2022.

[18] J. Guillod and V. Šverák, "Numerical investigations of non-uniqueness for the navier–stokes initial value problem in borderline spaces," *Journal of Mathematical Fluid Mechanics*, vol. 25, no. 3, p. 46, 2023.

[19] J. Chen and T. Y. Hou, "Stable nearly self-similar blowup of the 2d boussinesq and 3d euler equations with smooth data i: Analysis," *arXiv preprint arXiv:2210.07191*, 2022.

[20] H. Wang, Y. Cao, Z. Huang, Y. Liu, P. Hu, X. Luo, Z. Song, W. Zhao, J. Liu, J. Sun *et al.*, "Recent advances on machine learning for computational fluid dynamics: A survey," *arXiv preprint arXiv:2408.12171*, 2024.

[21] T. Buckmaster and V. Vicol, "Nonuniqueness of weak solutions to the navier-stokes equation," *Annals of Mathematics*, vol. 189, no. 1, pp. 101–144, 2019.

[22] A. Radford, J. W. Kim, T. Xu, G. Brockman, C. McLeavey, and I. Sutskever, "Robust speech recognition via large-scale weak supervision," in *International conference on machine learning*. PMLR, 2023, pp. 28 492–28 518.

[23] L. Barrault, Y.-A. Chung, M. C. Meglioli, D. Dale, N. Dong, M. Duppenthaler, P.-A. Duquenne, B. Ellis, H. Elsahar, J. Haaheim *et al.*, "Seamless: Multilingual expressive and streaming speech translation," *arXiv preprint arXiv:2312.05187*, 2023.

[24] Q. Fang, S. Guo, Y. Zhou, Z. Ma, S. Zhang, and Y. Feng, "Llama-omni: Seamless speech interaction with large language models," *arXiv preprint arXiv:2409.06666*, 2024.

[25] A. Baevski, Y. Zhou, A. Mohamed, and M. Auli, "wav2vec 2.0: A framework for self-supervised learning of speech representations," *Advances in neural information processing systems*, vol. 33, pp. 12 449–12 460, 2020.

[26] A. Baevski, W.-N. Hsu, Q. Xu, A. Babu, J. Gu, and M. Auli, "Data2vec: A general framework for self-supervised learning in speech, vision and language," in *International Conference on Machine Learning*. PMLR, 2022, pp. 1298–1312.

[27] W.-N. Hsu, B. Bolte, Y.-H. H. Tsai, K. Lakhotia, R. Salakhutdinov, and A. Mohamed, "Hubert: Self-supervised speech representation learning by masked prediction of hidden units," *IEEE/ACM transactions on audio, speech, and language processing*, vol. 29, pp. 3451–3460, 2021.

[28] F. Wu, K. Kim, S. Watanabe, K. J. Han, R. McDonald, K. Q. Weinberger, and Y. Artzi, "Wav2seq: Pre-training speech-to-text encoder-decoder models using pseudo languages," in *ICASSP 2023-2023 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*. IEEE, 2023, pp. 1–5.

[29] S. Koffas, J. Xu, M. Conti, and S. Picek, "Can you hear it? backdoor attacks via ultrasonic triggers," in *Proceedings of the 2022 ACM workshop on wireless security and machine learning*, 2022, pp. 57–62.

[30] C. Shi, T. Zhang, Z. Li, H. Phan, T. Zhao, Y. Wang, J. Liu, B. Yuan, and Y. Chen, "Audio-domain position-independent backdoor attack via unnoticeable triggers," in *Proceedings of the 28th Annual International Conference on Mobile Computing And Networking*, 2022, pp. 583–595.

[31] Y. Wang, M. Bennani, J. Martens, S. Racanière, S. Blackwell, A. Matthews, S. Nikolov, G. Cao-Labora, D. S. Park, M. Arjovsky *et al.*, "Discovery of unstable singularities," *arXiv preprint arXiv:2509.14185*, 2025.

[32] R. Farwig, "From jean leray to the millennium problem: the navier–stokes equations," *Journal of Evolution Equations*, vol. 21, no. 3, pp. 3243–3263, 2021.

[33] H. Moffatt and Y. Kimura, "Towards a finite-time singularity of the navier–stokes equations. part 2. vortex reconnection and singularity evasion," *Journal of Fluid Mechanics*, vol. 870, p. R1, 2019.

[34] T. Y. Hou, "Potentially singular behavior of the 3d navier–stokes equations," *Foundations of Computational Mathematics*, vol. 23, no. 6, pp. 2251–2299, 2023.

[35] J. Chen and T. Y. Hou, "Stable nearly self-similar blowup of the 2d boussinesq and 3d euler equations with smooth data ii: Rigorous numerics," *Multiscale Modeling & Simulation*, vol. 23, no. 1, pp. 25–130, 2025.

[36] J. Nečas, M. Růžička, and V. Šverák, "On leray's self-similar solutions of the navier-stokes equations," *Acta Mathematica*, vol. 176, no. 2, pp. 283–294, 1996.

[37] V. Scheffer, "Partial regularity of solutions to the navier-stokes equations," *Pacific Journal of Mathematics*, vol. 66, no. 2, pp. 535–552, 1976.

[38] J. Serrin, "On the interior regularity of weak solutions of the navier-stokes equations," *Arch. Rational Mech. Anal*, vol. 9, no. 1, pp. 187–195, 1962.

[39] E. Hopf, "Über die anfangswertaufgabe für die hydrodynamischen grundgleichungen. erhard schmidt zu seinem 75. geburtstag gewidmet," *Mathematische Nachrichten*, vol. 4, no. 1-6, pp. 213–231, 1950.

[40] J. Leray, "Sur le mouvement d'un liquide visqueux emplissant l'espace," *Acta mathematica*, vol. 63, no. 1, pp. 193–248, 1934.

[41] T. Y. Hou and R. Li, "Dynamic depletion of vortex stretching and non-blowup of the 3-d incompressible euler equations," *Journal of Nonlinear Science*, vol. 16, no. 6, pp. 639–664, 2006.

[42] T. M. Elgindi, "Finite-time singularity formation for c^1,α solutions to the incompressible euler equations on r^3," *Annals of Mathematics*, vol. 194, no. 3, pp. 647–727, 2021.

[43] G. Luo and T. Y. Hou, "Potentially singular solutions of the 3d axisymmetric euler equations," *Proceedings of the National Academy of Sciences*, vol. 111, no. 36, pp. 12 968–12 973, 2014.

[44] J. T. Beale, T. Kato, and A. Majda, "Remarks on the breakdown of smooth solutions for the 3-d euler equations," *Communications in Mathematical Physics*, vol. 94, no. 1, pp. 61–66, 1984.

[45] J. C. Robinson and W. Sadowski, "On the dimension of the singular set of solutions to the navier–stokes equations," *Communications in Mathematical Physics*, vol. 309, no. 2, pp. 497–506, 2012.

[46] H. Fujita and T. Kato, "On the navier-stokes initial value problem. i," *Archive for Rational Mechanics and Analysis*, vol. 16, no. 4, pp. 269–315, 1964.

[47] X. Ren and G. Tian, "Global solutions to the euler-coriolis system," *arXiv preprint arXiv:2405.18390*, 2024.
[48] Y. Guo, B. Pausader, and K. Widmayer, "Global axisymmetric euler flows with rotation," *Inventiones mathematicae*, vol. 231, no. 1, pp. 169–262, 2023.
[49] D. Córdoba, L. Martinez-Zoroa, and F. Zheng, "Finite time singularities to the 3d incompressible euler equations for solutions in," *Annals of PDE*, vol. 11, no. 2, p. 19, 2025.
[50] J. Chen and T. Y. Hou, "Singularity formation in 3d euler equations with smooth initial data and boundary," *Proceedings of the National Academy of Sciences*, vol. 122, no. 27, p. e2500940122, 2025.
[51] J. Chen, T. Y. Hou, and D. Huang, "On the finite time blowup of the de gregorio model for the 3d euler equations," *Communications on pure and applied mathematics*, vol. 74, no. 6, pp. 1282–1350, 2021.
[52] Y. Wang, C.-Y. Lai, J. Gómez-Serrano, and T. Buckmaster, "Asymptotic self-similar blow-up profile for three-dimensional axisymmetric euler equations using neural networks," *Physical Review Letters*, vol. 130, no. 24, p. 244002, 2023.
[53] Y. Giga and H. Miura, "On vorticity directions near singularities for the navier-stokes flows with infinite energy," *Communications in mathematical physics*, vol. 303, no. 2, pp. 289–300, 2011.
[54] T. Hou, Y. Wang, and C. Yang, "Nonuniqueness of leray-hopf solutions to the unforced incompressible 3d navier-stokes equation," *arXiv preprint arXiv:2509.25116*, 2025.
[55] M. P. Coiculescu and S. Palasek, "Non-uniqueness of smooth solutions of the navier–stokes equations from critical data," *Inventiones mathematicae*, pp. 1–55, 2025.
[56] F. Merle, P. Raphaël, I. Rodnianski, and J. Szeftel, "On the implosion of a compressible fluid ii: Singularity formation," *Annals of Mathematics*, vol. 196, no. 2, pp. 779–889, 2022.
[57] G. G. Stokes, *On the Theories of the Internal Friction of Fluids in Motion, and of the Equilibrium and Motion of Elastic Solids*, ser. Cambridge Library Collection - Mathematics. Cambridge University Press, 2009, p. 75–129.
[58] R. Von Mises, H. Geiringer, and G. S. S. Ludford, *Mathematical theory of compressible fluid flow*. Courier Corporation, 2004.
[59] T. Shen, R. Jin, Y. Huang, C. Liu, W. Dong, Z. Guo, X. Wu, Y. Liu, and D. Xiong, "Large language model alignment: A survey," *arXiv preprint arXiv:2309.15025*, 2023.
[60] R. Ngo, L. Chan, and S. Mindermann, "The alignment problem from a deep learning perspective," *arXiv preprint arXiv:2209.00626*, 2022.
[61] T. Hristova, L. Magee, and K. Soldatic, "The problem of alignment," *AI & SOCIETY*, vol. 40, no. 3, pp. 1439–1453, 2025.
[62] R. Sapkota, K. I. Roumeliotis, and M. Karkee, "Ai agents vs. agentic ai: A conceptual taxonomy, applications and challenges," *Information Fusion*, p. 103599, 2025.
[63] T. Breiten and K. Kunisch, "Feedback stabilization of the three-dimensional navier-stokes equations using generalized lyapunov equations," *arXiv preprint arXiv:1909.13757*, 2019.
[64] C. R. Doering and C. Foias, "Energy dissipation in body-forced turbulence," *Journal of Fluid Mechanics*, vol. 467, pp. 289–306, 2002.
[65] L. A. Seeli, "Probabilistic analysis of singularities for the 3d navier-stokes equations," *Mathematica Bohemica*, 2002.
[66] S. Verdú, "Total variation distance and the distribution of relative information," in *2014 information theory and applications workshop (ITA)*. IEEE, 2014, pp. 1–3.
[67] R. A. El-Nabulsi, "The hamilton–jacobi analysis of powers of singular lagrangians: a connection between the modified schrödinger and the navier–stokes equations," *Qualitative theory of dynamical systems*, vol. 17, no. 3, pp. 583–608, 2018.
[68] C. Qian, "The regularity criterion for the 3d navier–stokes equations involving end-point prodi–serrin type conditions," *Applied Mathematics Letters*, vol. 75, pp. 37–42, 2018.
[69] H. Ye, J. Gao, and Y. Ding, "A generalized gronwall inequality and its application to a fractional differential equation," *Journal of Mathematical Analysis and Applications*, vol. 328, no. 2, pp. 1075–1081, 2007.
[70] M. Sever, "Systems of conservation laws with rotation and galilean symmetry," *Journal of Differential Equations*, vol. 252, no. 10, pp. 5361–5389, 2012.
[71] A. Kontogiannis, S. V. Elgersma, A. J. Sederman, and M. P. Juniper, "Bayesian inverse navier–stokes problems: joint flow field reconstruction and parameter learning," *Inverse problems*, vol. 41, no. 1, p. 015008, 2025.

:

Progress on the Navier–Stokes Existence & Smoothness Problem

A Clay Millennium Prize Problem (2000) • $\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \nu\Delta\mathbf{u} - \nabla p + \mathbf{f}, \quad \nabla\cdot\mathbf{u} = 0, \quad \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}), \quad \mathbf{x}\in\mathbb{R}^3, \quad \nu > 0$

Historical Timeline

1934 — Leray (1934)
Global Leray-Hopf weak solutions in $\mathbb{R}^3$; energy inequality holds; $\mathbf{u} \in L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x$.

1951 — Hopf (1951)
Extended construction to bounded domains $\Omega \subset \mathbb{R}^3$; verified energy inequality.

1962 — 2D Global Regularity
Ladyzhenskaya: global smooth solutions in $\mathbb{R}^2$ for *all* smooth data. Fully resolved in 2D.

1964 — Fujita–Kato (1964)
Global smooth solutions for small $\mathbf{u}_0$ in $\dot{H}^{1/2}(\mathbb{R}^3)$; semigroup/contraction.

1972 — Prodi–Serrin (1959–72)
Conditional regularity: $\mathbf{u} \in L^p_t L^q_x$, $\frac{2}{p} + \frac{3}{q} \le 1, q \ge 3 \Rightarrow$ smooth.

1982 — Caffarelli–Kohn–Nirenberg
Partial regularity: singular set Σ has $\mathcal{H}^1(\Sigma) = 0$. Lin (1998) simplified proof. Landmark.

1984 — Beale–Kato–Majda (1984)
$\int_0^T \|\omega\|_{L^\infty} < \infty \Rightarrow$ no blow-up. Vorticity controls regularity.

2003 — Escauriaza-Seregin-Šverák
Endpoint $L^\infty_t L^3_x$ Prodi-Serrin case proved via backward uniqueness for the heat equation.

2016 — Tao (2016)
Finite-time blowup for an *averaged* NS system; supercriticality barrier formalised; blow-up mechanism plausible.

2019 — Buckmaster–Vicol (2019)
Non-uniqueness of weak NS solutions *below* Leray–Hopf class via convex integration.

2022 — Albritton–Brué–Colombo
Non-uniqueness of Leray–Hopf solutions with external force; finer selection principle needed.

Open — Status 2026
No smooth blow-up example in 3D. No proof of global regularity.

Established Results & Open Problems

Clay Prize Statement (Fefferman 2000): $\mathbf{u}_0 \in C^\infty(\mathbb{R}^3)$ div-free, rapidly decaying. **(A)** Prove $\exists\,(\mathbf{u}, p) \in C^\infty$ global smooth bounded-energy solution, **OR (B)** exhibit $(\mathbf{u}_0, \mathbf{f})$ for which no such solution exists. (**C/D**: same on $\mathbb{T}^3$.)

Global Weak Existence

For all $\mathbf{u}_0 \in L^2$, Leray-Hopf weak solutions exist on $[0,\infty)$:

$$\|\mathbf{u}(t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla \mathbf{u}\|^2 \le \|\mathbf{u}_0\|_{L^2}^2$$

Uniqueness & smoothness **unknown in 3D**.

Local Smooth Existence

$\forall\, \mathbf{u}_0 \in H^s$, $s > \frac{3}{2}$: unique smooth solution on $[0, T^*)$, $T^* > 0$. Blowup at T^* iff $\|\mathbf{u}(t)\|_{H^s} \to \infty$.

2D: Fully Solved

In $\mathbb{R}^2$ or $\mathbb{T}^2$: global smooth solutions for *all* smooth data. Unique. No blowup. **Completely resolved.**

CKN Partial Regularity (1982)

Suitable weak solutions smooth outside $\Sigma \subset \mathbb{R}^3 \times (0,\infty)$ with $\mathcal{H}^1_{\text{par}}(\Sigma) = 0$. Hausdorff dim ≤ 1. Singularities, if any, form a sparse set.

Prodi–Serrin Criteria

$\mathbf{u} \in L^p_t L^q_x$, $\frac{2}{p} + \frac{3}{q} \le 1, q \ge 3 \Rightarrow$ smooth. Endpoint $L^\infty_t L^3_x$: ESS (2003). Vorticity form: BKM (1984).

Critical Space Well-Posedness

Koch–Tataru (2001): global WP for small data in BMO^{-1}, the largest critical space. Ill-posed below $\dot{B}^{-1}_{\infty,\infty}$ ($q > 2$): Wang 2015.

Central Open Problems

(O1) Global Regularity vs. Blowup
Does every smooth finite-energy $\mathbf{u}_0$ lead to a global smooth solution? Or can a singularity form in finite time? **Open.**

(O2) Uniqueness of Leray–Hopf
Are 3D Leray–Hopf solutions unique? BV (2019): non-unique *below* the class. Within the class: **Open.**

(O3) Energy Conservation
Do Leray–Hopf solutions conserve energy? Energy *inequality* known; equality requires $\mathbf{u} \in B^{1/3}_{3,\infty}$ (Onsager analogue). General case: **Open.**

(O4) Sharpen CKN Bound
Reduce singular set from $\mathcal{H}^1 = 0$ to $\mathcal{H}^0 = 0$ (no singularities)? Would resolve global regularity. Incremental bootstrapping only so far.

Convex Integration & Non-Uniqueness (2017–2024): De Lellis–Székelyhidi school adapted convex integration to fluid PDEs.

- Buckmaster–Vicol (2019): Non-uniqueness below Leray–Hopf class.
- Albritton–Brué–Colombo (2022): Non-uniqueness with force *within* Leray–Hopf.
- Giri–Radu (2023–24): Further results without force at low regularity.

Significance: Leray-Hopf may not be the right uniqueness class; a finer selection principle is needed. Does **NOT** resolve whether smooth solutions blow up.

Proof Strategies Currently Pursued:

Vorticity analysis: Vortex stretching $\boldsymbol{\omega}\cdot\nabla\mathbf{u}$ amplifies vorticity; geometric alignment of vortex tubes is key.
Profile decomposition: Concentration-compactness in critical Besov / Koch-Tataru-Gallagher-Planchon spaces.
Computer-assisted proofs: Interval-arithmetic rigorous numerics near Hou-Luo type potential singularities / discovery of unstable singularities.
Stochastic NS: Probabilistic methods for generic data; invariant measures (Brzezniak, Da Prato, Romito).

Established Results & Open Problems

Small Data – Global Smooth

If $\|\mathbf{u}_0\|_{\dot{H}^{1/2}}$ small: global smooth. Decay $\|\mathbf{u}\|_{L^2} \to 0$. Also: Koch–Tataru in BMO^{-1}.

Axisymmetric No-Swirl

For axisymmetric data with $u_\theta = 0$: global regularity holds. With swirl: **Open**. Swirl is the key obstacle.

No Leray Self-Similar Blow-up

Nečas–Ružička–Šverák (1996) & Tsai (1998): no Leray-type self-similar singular solution exists.

Critical Function Spaces Hierarchy

NS is scale-invariant: $\mathbf{u}_\lambda(\mathbf{x},t) = \lambda\mathbf{u}(\lambda\mathbf{x},\lambda^2 t)$. A space is *critical* if $\|\mathbf{u}_\lambda\|_X = \|\mathbf{u}\|_X$. Global WP holds for small data in all critical spaces listed. Large data: **open in 3D.**

$\dot{H}^{-1/2}(\mathbb{R}^3)$ (supercritical) *Too large; ill-posedness (subcritical direction, not a critical space for NS)*

$\mathrm{BMO}^{-1}(\mathbb{R}^3)$ Largest critical space for WP: Koch–Tataru (2001). Ill-posed in $\dot{B}^{-1}_{\infty,q}$, $q > 2$.

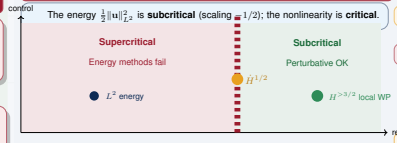
$L^3(\mathbb{R}^3)$ Critical. Global WP small data; ESS (2003) endpoint of Prodi–Serrin.

$\dot{H}^{1/2}(\mathbb{R}^3)$ Critical. Fujita–Kato (1964): global WP for small data; unique smooth solution.

$\dot{H}^s(\mathbb{R}^3)$, $s > 3/2$ Subcritical. Local WP for *all* data (Sobolev embedding controls nonlinearity).

Tao's Supercriticality Barrier (2016)

The energy $\frac{1}{2}\|\mathbf{u}\|^2_{L^2}$ is **subcritical** (scaling $\alpha < 1/2$); the nonlinearity is **critical**.



Question	2D	3D small	3D general
Weak existence	✓ Solved	✓ Solved	✓ Leray 1934
Weak uniqueness	✓ Yes	✓ Yes	Open (2019)
Local smooth exist.	✓ Yes	✓ Yes	✓ Short time
Global smooth exist.	✓ Yes	✓ Small data	**OPEN — $1M**
Finite-time blowup	✓ No	✓ No	Unknown
Partial regularity	✓ Full	✓ Full	CKN: $\mathcal{H}^1 = 0$
Energy equality	✓ Yes	✓ Yes	Open (Lars Onsager)

Energy & Vorticity Cascade Schematic

$\|\mathbf{u}\|_{L^2}$ **(kinetic energy, controlled)**
↑ cascade
$\|\nabla\mathbf{u}\|_{L^2}$ **(enstrophy, controlled locally)**
↑ cascade
$\|\omega\|_{L^\infty}$ **(key: controls blowup via BKM)**
↑ cascade
$\|\nabla\omega\|_{L^2}$ **(vorticity gradient, unbounded?)**

Energy flows from large scales to small. If $\|\omega\|_{L^\infty}$ stays finite,no blowup (BKM). Whether this holds globally: The core open question.

Proof Strategies:

Logarithmic improvements of CKN: Reduce $\mathcal{H}^1 = 0$ to $\mathcal{H}^0 = 0$ (no singularities) by incremental bootstrapping.

Quantitative regularity: Tao's programme of " quantitative " or " effective " local regularity; extract logarithmic savings from the NS nonlinearity.

Random data / probabilistic WP: Brzezniak-Da Prato-Romito: probabilistic global regularity for Gaussian random initial data; generic smoothness below H^1.

Nonlinear dissipation / hypocoercivity: Villani-type techniques bounding dissipation; potential to improve long-time regularity estimates.