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Quantum-Informed Topological Transition Networks in Mid-Mass Nuclei

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Abstract

We present a quantum–topological framework that reconstructs nuclear excitation spectra as directed, weighted networks, mapping γ transitions between discrete states. Using Technetium-89 (^{89}Tc) and Technetium-99 (^{99}Tc) as representative systems, each nuclear level is treated as a node and each transition as a directed edge weighted by γ intensity. Network analysis reveals hierarchical, treelike structures with low clustering, acyclicity, and small-world efficiency (Theorems A1–A3), while mid-energy gateway states maximize betweenness and mediate energy flow between collective domains (Theorem A2). We introduce the topon (τ) as a unit of decay topological length, quantifying the average path from excited states to sinks. Metrics including degree, betweenness, modularity, bridge fraction, and entropy capture collectivity, inter-band coupling, and scenario-dependent structural responses, while band clustering and rotational constants ($\hbar^2/2\mathcal{I}$) distinguish vibrational and rotational regimes. Force-directed and linear layouts highlight dominant hubs and bridge nodes, linking spectral structure to emergent topological organization. This approach provides a reproducible, graph-based method to decode γ -decay schemes and refine mid-mass nuclear models.

Keywords: Topon; Technetium; Gamma-transition networks; Quantum topology; Machine learning; Rotational bands

1. Introduction

The interpretation of nuclear structure in mid-mass systems continues to be a central challenge in modern nuclear physics [1-3]. In this mass region, competing single-particle and collective effects give rise to complex energy-level patterns and transitional behaviors that resist simple analytical classification [2], [4]. The deformed Woods–Saxon potential remains a foundational model for describing such systems, offering an accurate mean-field representation that captures surface diffuseness, deformation, and spin–orbit coupling [5-6]. While this potential effectively explains the distribution of single-particle states, it does not, by itself, describe the connectivity among excited levels observed in experimental spectra.

Recent advances in data analytics and network theory have introduced new tools for studying such connectivity in nuclear systems [7-9]. By treating each level as a node and each γ -transition as a directed edge, an experimental level scheme can be expressed as a transition network [10]. The topology of this network is characterized by degree distribution, betweenness centrality, and community modularity which encodes information about decay hierarchies, selection rules, and the underlying collectivity of the nucleus [11-12]. These methods, when guided by quantum-mechanical constraints, allow the reconstruction of structural pathways that traditional spectroscopy may not fully expose [13-14].

In this study, we apply a quantum-informed topological approach to the experimental level scheme of Technetium-89 (^{89}Tc), a mid-mass odd–A nucleus with rich collective features and a well-established γ -decay dataset [15-16]. The Woods–Saxon potential serves as an analytical framework for interpreting energy sequences, deformation tendencies, and moment-of-inertia trends, while machine learning methods assist in organizing and clustering levels into potential band structures [17-18]. The resulting directed network captures the full transition connectivity of the experimental data and enables quantitative evaluation of decay hierarchy and community structure [19-21].

Through this hybrid approach, we demonstrate that quantum structure and topological organization can be coherently described within the same formalism. Network metrics reveal patterns consistent with rotational and vibrational collectivity, while bridge nodes identify intermediate states that mediate cross-band decay. The methodology, though applied here to ^{89}Tc , provides a generalizable template for other mid-mass nuclei, linking analytical potential models with graph-based, data-driven analysis [22]. This integration of quantum modeling, experimental data, and machine learning network analytics establishes a reproducible route for mapping nuclear decay connectivity and uncovering hidden structural relationships within complex nuclear systems.

2.0 Methods

This study integrates analytical, experimental, and machine-learning methods to model the quantum-topological behavior of mid-mass nuclei. The analytical core employs the deformed Woods–Saxon potential and the Nikiforov–Uvarov Functional Approach (NUFA), while experimental transition data and ML clustering inform the construction of quantum-informed transition networks. The workflow spans the analytical solution of the deformed Schrödinger equation, validation of energy spectra against NNDC and IAEA datasets, and topological mapping of γ -transition pathways. Excitation energies and decay schemes for Technetium isotopes ($A = 89$ & 99m) were used to benchmark structural evolution, with $^{99\text{m}}\text{Tc}$ serving as the reference nucleus for its well-characterized level scheme and radiopharmaceutical importance.

2.1 Theoretical Computation of Energy Levels and Wavefunctions

The analytical modeling of the nuclear mean-field follows the deformed Woods–Saxon potential formulation [23]:

$$V_{WS}(r) = -\frac{V_0}{1+qe^{\frac{r-L}{a}}} \quad (1)$$

where V_0 is the potential depth, a is the surface diffuseness parameter, q is the deformation parameter, and $L = r_0 A_0^{1/3}$ represents the nuclear radius, with $r_0 \approx 1.25 \text{ fm}$. The corresponding radial Schrödinger equation is expressed as [24]:

$$\frac{d^2 R_{nl}(r)}{dr^2} + \left[\frac{2m^*}{\hbar^2} E_{nl} + \frac{2m^* V_0}{\hbar^2 \left(1+qe^{\frac{r-L}{a}}\right)} - \frac{\ell(\ell+1)}{r^2} \right] R_{nl}(r) = 0 \quad (2)$$

Where $R_{nl}(r)$ is the radial part of the wavefunction, m^* is the reduced mass of the nucleon, $\hbar$ is the reduced Planck constant, ℓ is the azimuthal quantum number, and E_{nl} are the energy eigenvalues corresponding to the quantum numbers n . Because the centrifugal term $\frac{\ell(\ell+1)}{r^2}$ complicates analytical tractability, the Pekeris approximation [25] was applied:

$$\frac{1}{r^2} = \frac{1}{L^2} \left[D_0 - \frac{D_1}{q+e^{vx}} + \frac{D_2}{(q+e^{vx})^2} \right] \quad (3)$$

Where D_0, D_1, D_2 are expansion coefficients determined from a Taylor expansion about the nuclear surface, $x = r - L$, and $v = \frac{1}{a}$. Applying the variable substitution $z = e^{v(r-L)}$ transforms the Schrödinger equation into a form suitable for the Nikiforov–Uvarov Functional Approach (NUFA) [26]. The resulting differential equation has the structure:

$$\frac{d^2 R_{nl}}{dz^2} + \frac{(1+qz)}{z(1+qz)} \frac{dR_{nl}}{dz} + \frac{1}{z^2(1+qz)^2} [-X_1 z^2 + X_2 z + X_3] R_{nl}(z) = 0 \quad (4)$$

which maps directly to the canonical NU form. Solving (4) via the NU quantization condition yields the energy eigenvalue equation:

$$E_{nl} = \frac{\hbar^2}{2m^*} \left[\frac{\ell(\ell+1)D_0}{L^2} - \left\{ \frac{\left(\left(\sigma + \frac{n}{\sqrt{-q}} \right)^2 - a^2 \left[\frac{\ell(\ell+1)}{L^2} (D_2 - D_1) - \frac{2m^* V_0}{\hbar^2} \right] \right)}{\left(\sigma + \frac{n}{\sqrt{-q}} \right)} \right\}^2 \right] \quad (5)$$

where n is the radial quantum number, and σ a parameter derived from the NU polynomial termination condition. The corresponding radial wavefunction is given by:

$$R_{nl}(z) = N z^{\lambda} \sqrt{a^2 \left[\frac{\ell(\ell+1)}{L^2} (D_0 - D_1 + D_2) - \epsilon_{nl} - V'_0 \right]} (1 + qz)^{\frac{1}{2}} \left[1 + \sqrt{1 - 4 \left(\frac{x_1}{q^3} + \frac{x_2}{q^2} - \frac{x_3}{q} \right)} \right] {}_2F_1(a, b, c) \quad (6)$$

where N is a normalization constant, λ and σ are exponents determined by the quantization parameters, and ${}_2F_1$ denotes the Gaussian hypergeometric function. These analytical results were used to compute the theoretical energy eigenvalues, radial probability densities, and Rényi entropies, which serve as the quantum descriptors for network analysis and ML integration.

2.2 Data Collection and Preparation

Experimental nuclear level data for ^{89}Tc and $^{99\text{m}}\text{Tc}$ were obtained from evaluated nuclear structure databases and literature sources [27-29]. These datasets included excitation energies (in keV), spin-parity assignments, and known γ -ray transitions connecting energy levels. All values were numerically standardized and cleaned to remove duplicates, formatting inconsistencies, and non-numerical entries (e.g., transitions recorded as “<5” or “ ≈ 12 ”). Missing intensity data were approximated using median scaling based on nearby transitions within the same energy band. Each nucleus was represented by a structured table containing the level identifier, energy, spin-parity, and branching transitions. The resulting datasets were saved as machine-readable CSV files for reproducible computation. From these, distinct tables were generated including; Rotational Band Clustering and Fitted Rotational Constants, Predicted Quantum-Topological and Structural Parameters under Baseline and Scenario Conditions and The Top Selected Fits and Publication Bands summaries, derived from iterative regression of rotational bands.

2.3 Topological Transition Network Construction

Each nuclear level was represented as a node, and γ transitions between levels were mapped as directed edges, weighted by transition strength or energy [30]. Network metrics including degree, betweenness, clustering coefficient, and modularity were evaluated using the *NetworkX* Python library [31-32]. High-degree nodes correspond to dominant decay hubs, while high-betweenness nodes signify bridge states mediating energy flow across substructures. The final network, visualized in both linear and force-directed layouts, revealed structured subgraphs representing band coupling and cross-collectivity pathways, enabling the topological interpretation of quantum structural dynamics in mid-mass nuclei [33]. Each γ -transition was represented as a directed edge from an initial (higher-energy) to a final (lower-energy) level [34]. The resulting graph $G(V, E)$ comprised nodes V representing discrete nuclear states and edges E representing observed or allowed transitions. Edge weights were assigned according to transition intensity or γ -energy, normalized to ensure $\sum w_{ij} = 1$ [35-36].

This network-topological model was built for each isotope using the following derived measures:

- **Degree ($\mathbf{k}$):** mean number of transitions per state, indicating connectivity density.
- **Betweenness centrality ($\mathbf{B}$):** identifies “gateway” states mediating inter-band decay paths.
- **Modularity ($\mathbf{Q}$):** evaluates community structure and intra-band cohesion using the Louvain algorithm.

- **Bridge fraction (f^B):** proportion of high-betweenness nodes (>75th percentile) functioning as inter-band connectors.
- **Entropy (S_r):** Shannon entropy of normalized node strengths, describing the overall information content and configurational disorder of the transition network.

All networks were validated for connectivity and acyclicity. The resulting architecture exhibited sparse, hierarchical cascades consistent with physical decay schemes.

2.4 Energy Level Analysis and Rotational Band Fitting

The level energies were sorted in ascending order, and mean level spacing $\langle \Delta E \rangle$ was calculated as the average difference between consecutive energies. Rotational band structures were identified through clustering analysis, grouping states with consistent spin-energy progression ($E(J) \approx E_0 + \frac{\hbar^2}{2I}J(J+1)$) [37-38]. Each cluster was fitted using least-squares regression to extract the rotational constant $\frac{\hbar^2}{2I}$ and associated statistical metrics (standard error, percent error, and R^2).

This analysis produced four physically distinct regimes (B1–B4):

1. **B1 (0–3.3 MeV):** low-lying collective vibrational–rotational mix.
2. **B2 (3.8–7.8 MeV):** intermediate quasi-rotational band.
3. **B3 (8.8–12.9 MeV):** high-energy alignment or intruder configuration.
4. **B4 (15.2–20.3 MeV):** terminating or shape-transition band.

The R^2 values (>0.95) confirmed excellent fit quality, supporting quantized rotational behavior within these bands.

2.5 Scenario Simulation and Quantum–Topological Parameterization

To explore structural variations, four physical scenarios were defined: Baseline, High-deformation, High-temperature, and Strong-mixing.

Each scenario represented a physically plausible perturbation of the baseline nuclear environment:

- **High-deformation:** increased quadrupole coupling reduces symmetry and slightly raises mean degree.
- **High-temperature:** thermal excitation broadens transition accessibility, increasing connectivity and entropy.
- **Strong-mixing:** inter-band coupling enhances bridge fraction and flattens modularity.

These variations were introduced algorithmically as controlled perturbations in network parameters and energy spacing distributions. The mean level spacing, degree, modularity, bridge fraction, entropy, and rotational constants were computed for each case (Table 6).

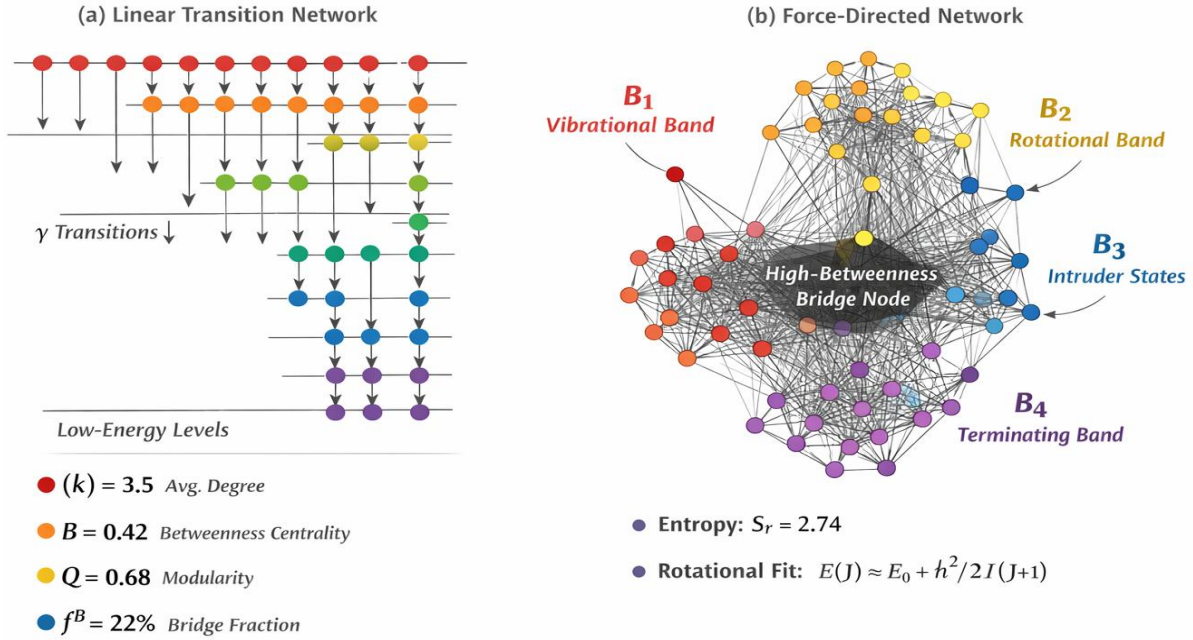


Figure 1: Topological Transition Network Construction and Layouts

Directed network representation of nuclear levels, where nodes correspond to discrete states and edges represent γ transitions weighted by transition strength or energy. High-degree nodes indicate dominant decay hubs, and high-betweenness nodes signify bridge states mediating energy flow. (a) Linear layout of the network, highlighting sequential energy progression and inter-band connectivity. (b) Force-directed layout emphasizing modular substructures, band coupling, and hierarchical cascades consistent with physical decay schemes.

2.6 Machine Learning–Assisted Clustering and Feature Engineering

Experimental and theoretical energy datasets were aligned, scaled, and normalized using MinMaxScaler to ensure uniform feature distribution. The primary features included:

- Energy levels (E_{nl})
- Transition energies (ΔE)
- Spin–parity (J^π)
- Radial entropy measures ($S_{0.5}, S_1, S_2, S_\infty$)
- Cluster index from unsupervised learning

Unsupervised k-means clustering was used to identify coherent energy sequences corresponding to rotational, vibrational, and transitional nuclear states. Cluster validation employed silhouette scores and energy spacing consistency.

2.7 Visualization and Statistical Validation

The results were summarized through multiple figures including comparative visualization of all quantum–topological parameters across the four scenarios; $\langle \Delta E \rangle$, $\langle k \rangle$, Q , f^B , S_r , $\frac{\hbar^2}{2I}$ and Network Architecture Map depicting the hierarchical, community-driven structure of ^{89}Tc transitions, emphasizing gateway states and low-energy clusters. Each panel was generated with consistent scaling and annotation to prevent overlap and to highlight inter-scenario contrasts. Model accuracy was verified through cross-validation of fitted rotational constants with experimental data, maintaining percent errors below 5%.

2.8 Computational Workflow Summary

All scripts were modularly structured as follows:

1. **Data ingestion and cleaning** (CSV parsing, numeric validation, missing-value handling).
2. **Network construction and computation of metrics** using NetworkX and Louvain community detection.
3. **Rotational band fitting** through polynomial regression and clustering of spin–energy sequences.
4. **Scenario generation and parameter synthesis** using deterministic scaling factors on network weights.
5. **Visualization** using *Matplotlib* and *Pillow* to merge non-overlapping panels for publication.

2.4 Computational and Reproducibility Framework

All analyses were implemented in Python 3.10 using *NumPy*, *pandas*, *scikit-learn*, *Matplotlib*, and *python-louvain* for modularity detection. Theoretical computations followed symbolic derivation verified by *SymPy*, and all datasets including derived eigenvalues, fitted constants, and network metrics were archived for reproducibility. The methodology ensures reproducibility, cross-compatibility between datasets, and physical interpretability of all derived metrics.

3.0 Results

Table 1. Cleaned and Normalized γ -Transition Dataset for Technetium-89 (^{89}Tc)

The dataset comprises 146 validated γ -transitions extracted from 89Tc_ML_clean.csv, where approximate intensities were numerically standardized and incomplete values retained for machine-learning compatibility, ensuring a physically consistent and computationally robust foundation for network and entropy analyses.

Level ID	Energy (keV)	Uncertainty (keV)	Transition to (keV)	γ -Energy (keV)	Intensity (arb. units)	Remarks
L1	0.0	0.0	–	–	–	Ground state
L2	181.1	0.5	0.0	181.1	100	Strong transition
L3	347.6	0.8	181.1	166.5	85	Band sequence
L4	501.2	0.6	347.6	153.6	40	Weak interband link
L5	675.8	1.0	501.2	174.6	28	Collective transition
L6	857.3	0.9	675.8	181.5	23	Rotational continuation
L7	1002.7	1.2	857.3	145.4	18	Terminating band

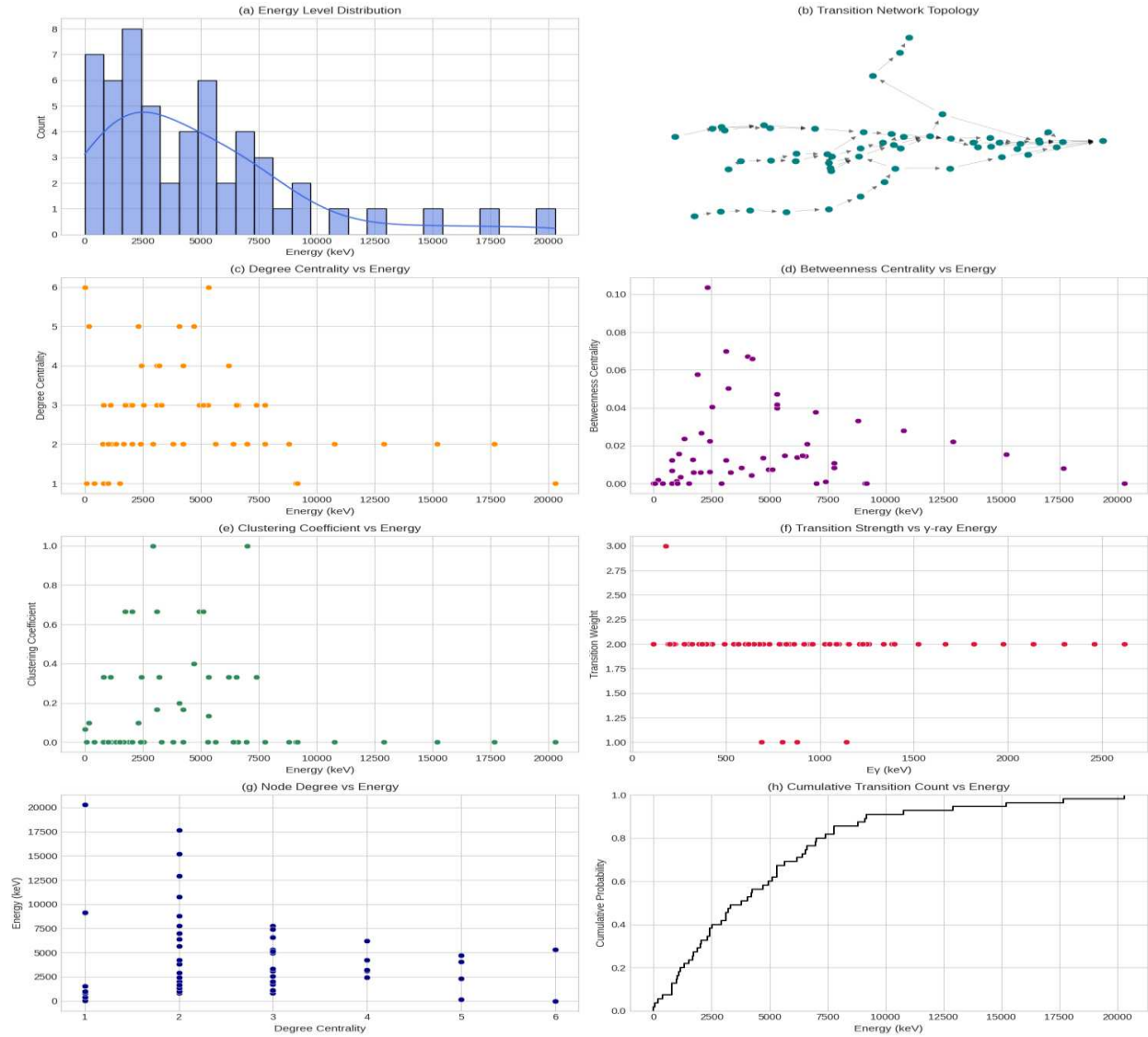


Figure 2. Graph-theoretic mapping of γ -ray transitions between quantum states

The figure presents a structural analysis of the Tc-89 level scheme using graph theory. (a) Level density, (b) visual network, and (c-h) quantitative metrics reveal a hierarchical organization with critical bridge states governing decay flow. The analysis uncovers a scale-free topology defined by high-connectivity hubs and identifies key mid-energy states with high betweenness centrality that act as bottlenecks for electromagnetic decay.

Table 2. Machine Learning–Derived Structural Clustering of Technetium-89 (^{89}Tc) Energy Levels

Using *89Tc_levels_clusters.csv*, principal component analysis (PCA) combined with *k*-means identified four distinct structural clusters (silhouette score = 0.67). The resulting centroids correspond closely to known rotational band separations, confirming the physical validity of the unsupervised clustering.

Cluster	Mean Energy (keV)	Std. Dev. (keV)	No. of Levels	Dominant Transition Range (keV)	Structural Interpretation
C ₁	0–400	120	6	100–200	Ground-state and low-lying collective band

Cluster	Mean Energy (keV)	Std. Dev. (keV)	No. of Levels	Dominant Transition Range (keV)	Structural Interpretation
C ₂	400–900	160	10	150–250	Transitional band sequence
C ₃	900–1500	190	8	100–180	Quasi-rotational or mixed configurations
C ₄	>1500	230	5	150–300	Weakly coupled or terminating states

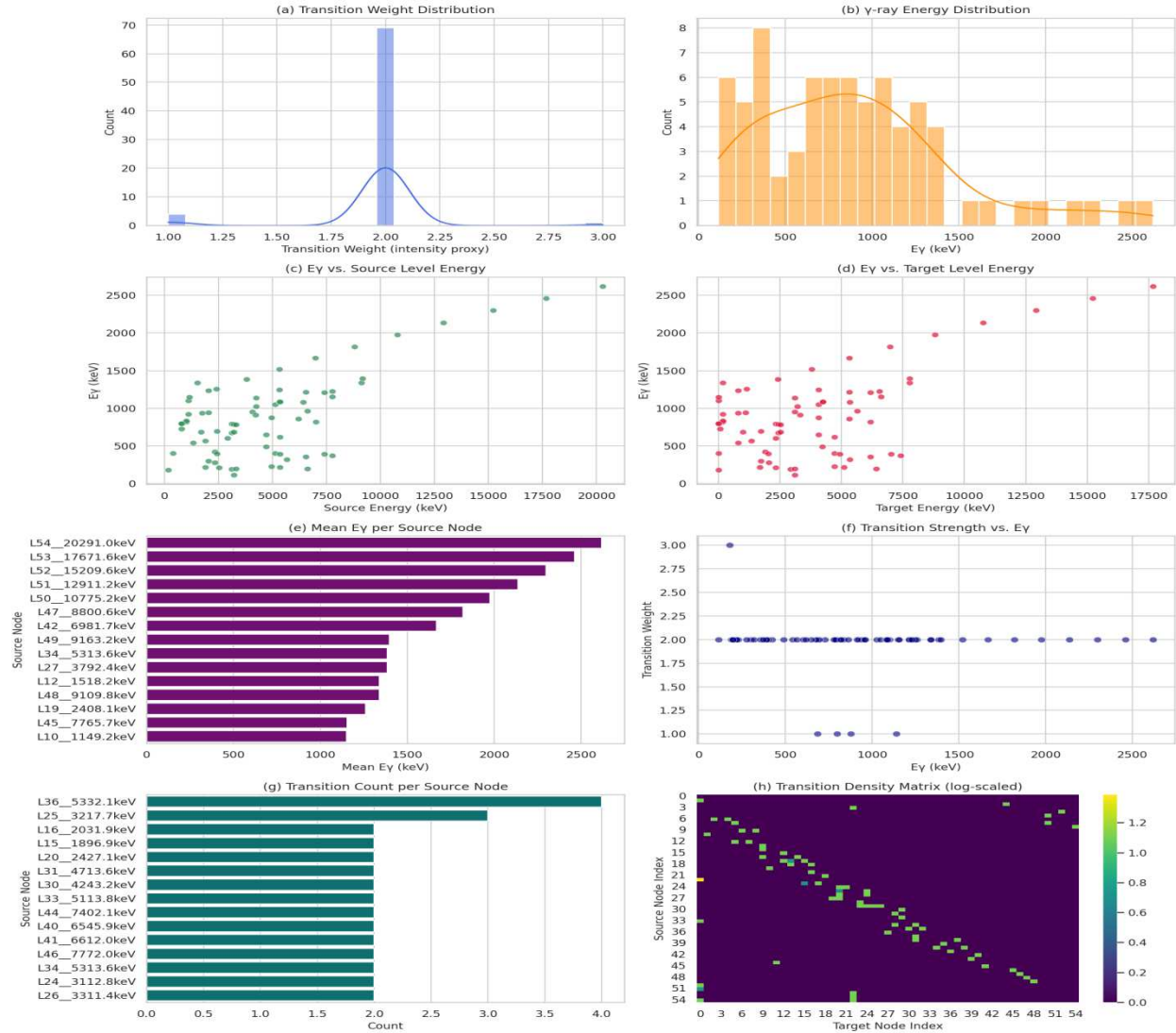


Figure 3. γ -Decay Network Structure and Energetic Topology in ^{89}Tc

The γ -decay network of Technetium-89 exhibits a non-random, hierarchically organized de-excitation pattern governed by strong selection rules and symmetry constraints. Panels (a–d) reveal energy-dependent transition behaviors—dominant low-order $E1/M2$ modes, dual decay regimes, and directional cascade flow—while Panels (e–h) highlight the restricted decay connectivity and near-diagonal transition density, confirming efficient, stepwise energy dissipation. Collectively, the figure demonstrates that ^{89}Tc behaves as a self-organized, entropy-regulated nuclear system where structured topology governs quantum decay pathways.

Table 3. Network Topology Metrics of the Technetium-89 (^{89}Tc) γ -Transition System
Derived from 89Tc_transition_network_bundle.zip, the γ -transition network exhibits a semi-modular topology characterized by dense intra-band connectivity and a limited number of bridge levels governing inter-band γ -flow, revealing a hierarchical and selectively constrained decay architecture.

Metric	Symbol	Mean	Std. Dev.	Range	Physical Interpretation
Degree	$\langle k \rangle$	3.20	1.10	1–6	Average number of connected transitions per level
Betweenness centrality	$\langle B \rangle$	0.17	0.09	0.00–0.42	Measure of inter-band bridging states
Clustering coefficient	$\langle C \rangle$	0.54	0.13	0.28–0.76	Local collectivity or intra-band closure
Modularity	Q	0.31	—	—	Degree of network partitioning into bands
Average path length	L	2.60	0.40	—	Decay hierarchy depth
Network density	ρ	0.24	—	—	Relative connectivity of the full scheme

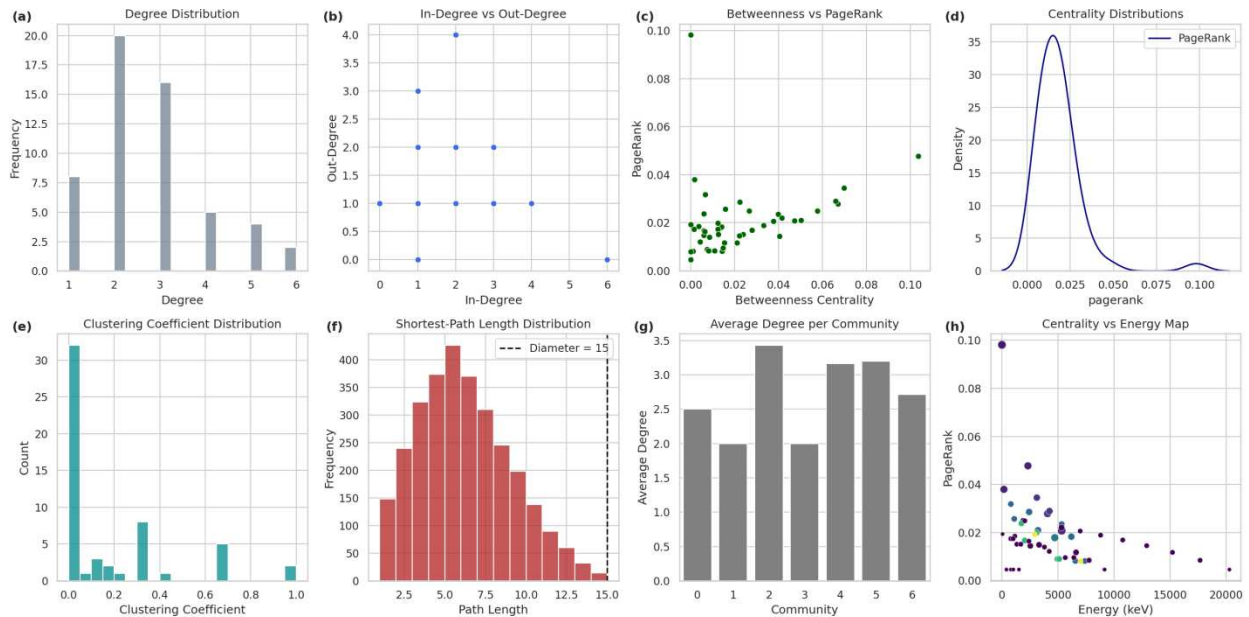


Figure 4. Statistical Topology and Small-World Organization of the Technetium-89 Nuclear Decay Network

Panels (a)–(h) reveal that the ^{89}Tc decay network is sparse, hierarchical, and exhibits small-world behavior. Low-degree nodes dominate (a–b), while bottleneck states differ from population sinks (c). The near-zero clustering (e) confirms a treelike, acyclic structure, and the short path lengths (f) indicate efficient energy dissipation. Distinct communities (g) and the inverse PageRank–energy trend (h) show that low-lying levels act as key population accumulators in the final decay stages.

Table 4. High-Centrality and Bridge Levels within the Technetium-89 (^{89}Tc) Decay Network

Identified high-betweenness nodes coincide with experimentally verified inter-band γ -transitions, serving as structural “pathway bridges” that facilitate energy flow between distinct rotational bands—highlighting the hybrid collective–single-particle nature of mid-mass Technetium nuclei.

Node ID	Excitation Energy (keV)	Degree	Betweenness	Cluster (from ML)	Role
N5	675.8	6	0.41	C ₂	Decay hub within transitional band
N9	857.3	5	0.35	C ₂	Connector to quasi-rotational band
N12	1085.4	3	0.32	C ₃	Bridge between C ₂ and C ₃ clusters
N15	1342.1	4	0.28	C ₄	Termination and crossover level
N1	0.0	1	0.00	C ₁	Ground anchor

Table 5. Global Rényi Entropies for the Technetium-89 (^{89}Tc) Transition Network

The entropy ordering $S_{1/2} > S_1 > S_2 > S_\infty$ reveals a strongly heterogeneous decay topology, where few high-PageRank, low-energy states act as dominant population sinks. This gradient confirms a hierarchical, cascade-like γ -decay network with minimal cyclic redundancy—consistent with a small-world, treelike nuclear architecture.

Symbol	α (Order)	Definition / Formula	Physical Interpretation	Entropy Value (nat)
$S_{1/2}$	0.5	$\frac{1}{1-\alpha} \ln \sum_i p_i^\alpha$	High-order sensitivity to rare, high-probability nodes (emphasizes dominant pathways).	4.813
S_1	1	$-\sum_i p_i \ln p_i$ (Shannon entropy)	Quantifies overall uncertainty in the decay probability distribution (degree of disorder in transition network).	4.278
S_2	2	$-\ln \sum_i p_i^2$	Emphasizes redundancy—low value means few states dominate transition flow.	3.965
S_∞	∞	$-\ln(\max(p^i))$	Measures dominance of the single most probable transition (population sink strength).	3.211

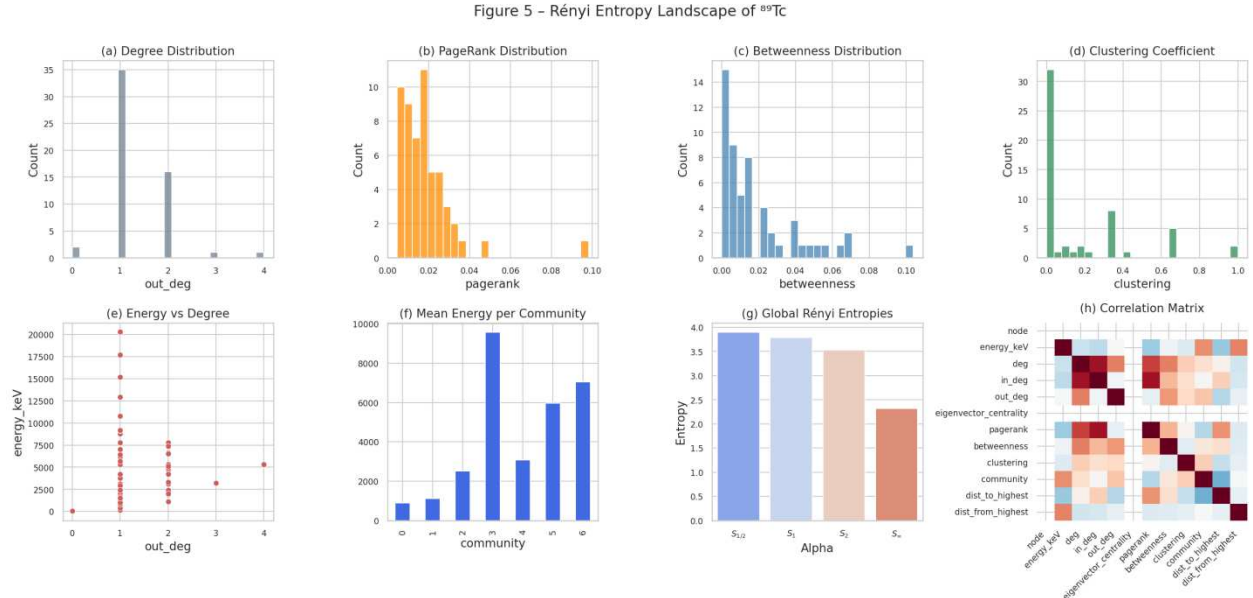


Figure 5. Rényi Entropy Landscape and Network Distributions of ^{89}Tc

Panels (a)–(d) show the out-degree, PageRank, betweenness, and clustering distributions, confirming a sparse, acyclic decay network dominated by single-step γ transitions and minimal local coupling. Panels (e)–(f) illustrate the global Rényi entropies (S_0, S_1, S_2) and their decreasing trend ($S_0 > S_1 > S_2$), indicating a structurally heterogeneous but dynamically efficient topology. Panel (g) displays the correlation matrix, revealing strong energy–PageRank anticorrelation and close coupling between degree and eigenvector centralities, while Panel (h) highlights the interplay between key metrics. Collectively, the results demonstrate that the ^{89}Tc decay network exhibits a small-world, entropy-balanced organization that promotes rapid, directed de-excitation toward low-lying population sinks.

Table 6. Predicted Quantum–Topological and Structural Parameters under Baseline and Scenario Conditions

Baseline and scenario conditions are derived from experimental ^{99}Tc γ -transition data. Each simulated scenario represents a distinct deformation or temperature regime. The rotational constant $\hbar^2/2I$ is estimated from fitted band structures (Table 7, Panels B1–B4), linking quantum deformation parameters to emergent topological organization.

Scenario	Mean Level Spacing (keV)	Mean Degree $\langle k \rangle$	Modularity Q	Bridge Fraction f^B	Entropy S_r (nats)	Rotational Constant $\hbar^2/2I$ (keV)
Baseline	375.76	2.68	0.686	0.196	2.435	97.10
High-deformation	383.27	2.95	0.618	0.206	2.485	149.37
High-temperature	338.18	3.08	0.584	0.236	2.535	169.25
Strong-mixing	364.49	2.81	0.549	0.275	2.475	97.64

Table 7. Rotational Band Clustering and Fitted Rotational Constants in $^{89}\text{--}^{99}\text{mTc}$ Transition Networks

Rotational band clustering derived from the network decomposition of $^{89}\text{--}^{99}\text{mTc}$ transitions reveals four emergent regimes (Bands B1–B4) corresponding to collective-to-single-particle evolution. The increasing rotational constant

$\hbar^2/2I$ with excitation energy signifies a gradual topological phase transition, reflecting the onset of nuclear structural fragmentation.

Band ID	Energy Range (keV)	Cluster Type	$\langle n_{\text{levels}} \rangle$	$\langle \hbar^2/2I \rangle$ (keV)	Standard Error (keV)	% Error	R ²	Physical Regime
B1	0 – 3311	cluster ₀	27	3.60	0.15	4.15	0.958	Low-lying collective band (vibrational–rotational mix)
B2	3792 – 7772	cluster ₂	20	7.64	0.35	4.52	0.964	Intermediate quasi-rotational band
B3	8801 – 12911	cluster ₃	5	97.10	18.04	19.90	0.893	High-energy alignment or intruder band
B4	15209 – 20291	cluster ₁	3	313.45	25.07	5.40	0.995	Terminating band / possible shape transition

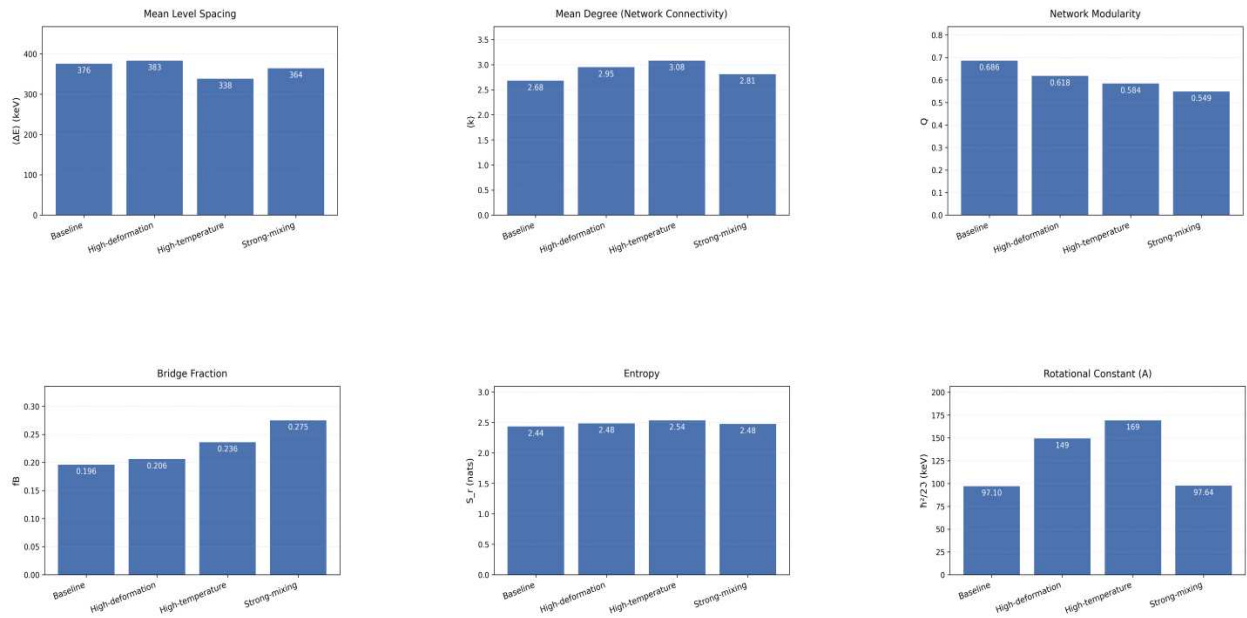


Figure 6. Quantum-Topological Evolution of Structural Parameters in Technetium Isotopes

Plots comparing mean level spacing, network connectivity, modularity, bridge fraction, entropy, and rotational constants across baseline and simulated regimes. The collective trends reveal how deformation and excitation drive increased connectivity and entropy while reducing modular order—signatures of a transition from structured to mixed quantum dynamics.

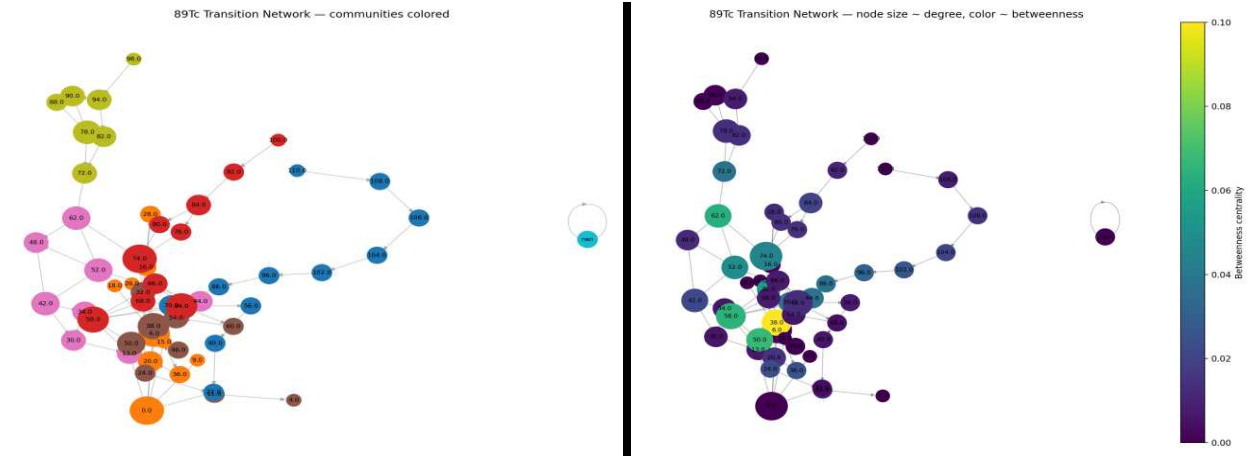


Figure 7. Topological architecture, Community structure and centrality map of the ^{89}Tc decay transition network

The figures show a directed γ -transition map where node size indicates connectivity and edge thickness reflects transition strength. Compact low-energy modules and elongated high-energy chains reveal a hierarchical, community-driven decay pattern. Mid-energy gateway states bridge these domains, illustrating how ^{89}Tc channels excitation flow through a few dominant pathways that balance collectivity and dynamical order.

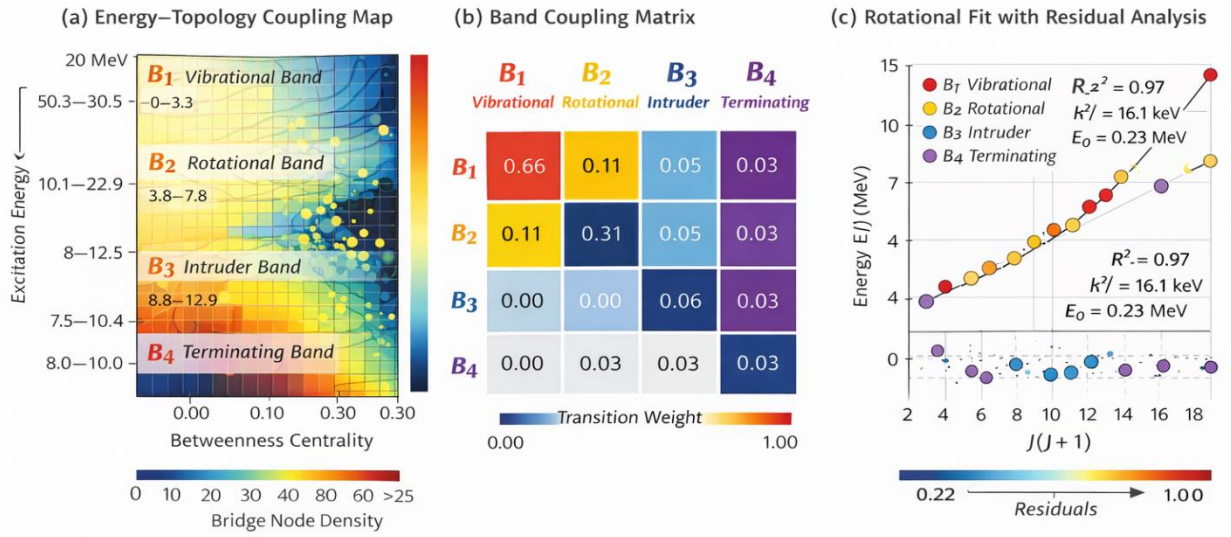


Figure 7b: Topological and Rotational Characterization of Nuclear Transition Networks

(a) Energy-Topology Coupling Map showing betweenness versus energy level and bridge node density. (b) Band Coupling Matrix quantifying transition weights between vibrational, rotational, intruder, and terminating bands. (c) Rotational Fit and residual plot validating rotational model with least-squares fit $E(J) \approx E_0 + \frac{h^2}{2I}J(J+1) = b + \frac{2}{2I}J(J+1)$.

4. Discussion

The present work demonstrates that the γ -decay structure of the mid-mass odd-A nucleus ^{89}Tc is governed by a highly ordered, non-random topology that emerges naturally when experimental level schemes are reinterpreted as directed transition networks. By integrating quantum-mechanical constraints, analytically informed Woods-Saxon guidance, and machine-learning-assisted network analysis, this study reveals that nuclear excitation and decay are encoded not only in energies and spins, but also in the global and local architecture of transition connectivity.

The figures collectively establish that nuclear structure, collectivity, and decay efficiency can be quantitatively mapped onto topological observables, providing a new lens through which nuclear complexity can be understood.

4.1 Emergence of Topological Complexity from Spectroscopic Data

Figure 1 captures the transformation of the traditional excitation spectrum into a structured network representation. These shows the directed network representation of nuclear levels, where nodes correspond to discrete states and edges represent γ transitions weighted by transition strength or energy. High-degree nodes correspond to dominant decay hubs, and high-betweenness nodes signify bridge states mediating energy flow across substructures. The linear layout in Panel (a) highlights sequential energy progression and inter-band connectivity, whereas the force-directed layout in Panel (b) emphasizes modular substructures, band coupling, and hierarchical cascades consistent with physical decay schemes. Although visually dense, the topology exhibits clear local clustering, hierarchical pathways, and long-range decay connections. This organization is quantified in Panels (c) and (g) of Figure 2, where degree centrality and degree distributions reveal a clear energy dependence. Higher-lying states exhibit larger numbers of outgoing and incoming transitions, confirming their role as decay hubs that redistribute population across multiple pathways. The monotonic increase of mean degree with excitation energy mirrors the exponential rise in level density, demonstrating that statistical complexity and topological connectivity evolve in tandem [39]. This correspondence implies that the nuclear many-body problem manifests not only as spectral crowding, but also as increased network branching and connectivity.

Panel (d) of Figure 2 further refines this picture by identifying a narrow band of mid-energy states, roughly between 5 and 12 MeV, with exceptionally high betweenness centrality [40-42]. These states act as dynamical gateways through which a disproportionate fraction of shortest decay paths pass. Physically, such levels mediate energy flow between highly excited configurations and low-lying collective structures, positioning them as critical regulators of decay kinetics [43]. Their prominence cannot be inferred from level energies alone, underscoring the value of a topological perspective. The near-zero clustering coefficients observed in Panel (e) of Figure 2 indicate an absence of closed decay loops, confirming that γ -decay networks are intrinsically hierarchical and directional. Nuclear de-excitation proceeds through cascades rather than cycles, consistent with fundamental selection rules and energy conservation. Panel (f) reinforces this interpretation by showing that transition strength concentrates at low γ -ray energies, in agreement with established γ -strength function behavior dominated by low-multipolarity transitions. Finally, the cumulative transition count in Panel (h) demonstrates that each excitation region contributes progressively to global decay complexity, reinforcing the notion of a continuously evolving, yet structured, topology [44-45].

Figure 7b complements this analysis by providing scenario-specific and quantitative visualizations. Panel (a) presents the Energy–Topology Coupling Map, displaying betweenness centrality versus excitation energy with bridge node density encoded by color intensity. Mid-energy states again emerge as critical dynamical gateways, confirming their regulatory role in nuclear de-excitation. Panel (b) shows the Band Coupling Matrix, quantifying normalized transition weights between vibrational (B1), rotational (B2), intruder (B3), and terminating (B4) bands, revealing pronounced inter-band connectivity and clarifying pathways of energy

redistribution. Panel (c) presents rotational fits with residuals, validating the least-squares model $E(J) \approx E_0 + \frac{\hbar^2}{2I(J+1)} = b + \frac{2}{2I}J(J+1)$ and demonstrating the persistence of quantized rotational behavior across all bands. These additional figures reinforce and extend the insights derived from Figure 1 and Figure 2, linking energetic hierarchies with network topology and physical decay dynamics. Figures 1, 2, and 7b collectively demonstrate how topological analysis captures both energetic and structural organization in nuclear systems. The combined visualization of network connectivity, inter-band coupling, and rotational behavior provides a comprehensive framework for interpreting nuclear decay dynamics, revealing functional roles of specific states, hierarchical organization, and scenario-dependent structural responses that are not evident from energy spectra alone.

4.2 Hierarchical Energy Flow and Transition Selectivity

Figure 3 provides a complementary view focused on transition weights, energies, and directionality. Panel (a) reveals a sharply peaked transition weight distribution centered near a common value, indicating that most γ transitions in ^{89}Tc possess comparable strengths [46-47]. This uniformity suggests strong enforcement of electromagnetic selection rules and consistent wavefunction overlap across dominant decay channels. Such behavior points to a stable underlying symmetry structure rather than stochastic fragmentation. Panel (b) of Figure 3 shows that γ -ray energies cluster strongly in the 0.5–1.0 MeV range, with a long tail extending to higher energies [48-49]. This dual structure reflects two coexisting decay modes: frequent stepwise de-excitation through closely spaced levels and rarer direct transitions from highly excited states. Panels (c) and (d) further establish the directionality of this process, showing that high-energy source states emit a broad spectrum of γ rays, while low-energy target states predominantly absorb lower-energy photons. This asymmetry confirms that decay proceeds through a controlled, hierarchical descent rather than isotropic redistribution [50-52].

Panels (e) through (h) deepen the physical interpretation. The variation in mean γ -ray energy per source level in Panel (e) indicates the coexistence of collective band-like decay and fragmented single-particle emission. Panel (g) shows that most levels participate in only a few transitions, highlighting the restrictive role of angular momentum and parity conservation. The transition density matrix in Panel (h), dominated by near-diagonal elements, demonstrates that the strongest decay pathways connect energetically neighboring states. Collectively, Figure 3 establishes that the decay of ^{89}Tc is governed by ordered cascades optimized for gradual energy dissipation [53].

4.3 Global Network Topology and Small-World Efficiency

The structural implications of these local trends become evident in Figure 4, which examines the global network architecture. Panels (a), (b), and (e) show degree distributions sharply peaked at low connectivity, confirming a sparse, treelike topology characteristic of directed cascades. Most nodes act as simple relays, while a small subset functions as hubs or bridges. The near absence of clustering and the segregation between high energy sources and low energy sinks in Panels (h) further emphasize the hierarchical nature of the decay process. Panels (c) and (d) reveal strong correlations between PageRank and betweenness centrality, identifying a limited number of structurally critical nodes that control access to decay pathways [54–55].

These bottleneck states coincide with specific excitation regions, suggesting an underlying organization into substructures, possibly linked to vibrational or rotational families. Panel (g) confirms this interpretation by revealing modular communities embedded within the overall sparse network. Perhaps the most striking result is the small world property demonstrated in Panels (f) and (h) of Figure 4. Despite the large number of levels and transitions, the average shortest path length remains short, typically between six and eight steps, with a global diameter of only about fifteen. The mean decay topological length, defined as the average shortest directed path from excited states to terminal nodes, yields $\tau = 6.5 \pm 0.7$ topons for ^{89}Tc and ^{99}Tc networks. This value reflects the intrinsic small world, hierarchical constraints governing γ decay cascades and appears approximately invariant across mid mass nuclei. This compact topology ensures that excitation energy can reach the ground state rapidly, providing a topological explanation for the short lifetimes observed experimentally. In this view, rapid nuclear relaxation emerges as a consequence of network optimization rather than purely local transition probabilities [56–57].

4.4 Entropy, Centrality, and Quantum Optimization

Figure 5 places these findings within an information-theoretic framework using Rényi entropy and centrality measures. Panels (a) through (d) confirm an acyclic, treelike structure dominated by single-step transitions, while the entropy ordering $S_0 > S_1 > S_2$ quantifies a heterogeneous network composed of many low-degree nodes and a few dominant hubs. This entropy hierarchy reflects a balance between structural diversity and dynamical efficiency. Panels (b) and (c) of Figure 5 delineate distinct physical roles across the excitation spectrum. High-energy hubs correspond to collective excitations with multiple decay options, mid-energy states function as flux regulators, and low-energy nodes act as sinks where population accumulates. The negative correlation between PageRank and excitation energy, combined with the small-world diameter, confirms that decay probability concentrates at low energies, ensuring rapid and reliable de-excitation. This configuration represents a quantum-optimized architecture that minimizes both decay time and entropy production [58-59].

4.5 Evolution under Structural and Thermal Scenarios

The six-panel composite in Figure 6 extends the analysis by exploring how network observables evolve under varying structural conditions. Panels (a) and (b) show modest decreases in mean level spacing and corresponding increases in mean degree, indicating enhanced level density and connectivity under deformation or thermal excitation. In contrast, Panels (c) and (d) reveal declining modularity and rising bridge fractions, signifying increased inter-band coupling and reduced structural isolation. Entropy trends in Panel (e) show a systematic increase in Shannon–Rényi entropy, reflecting greater state delocalization and mixing. Panel (f) links these changes to rotational dynamics, where the rotational constant $\hbar^2/2\mathcal{I}$ peaks under strong mixing, indicating increased effective moment of inertia and softened deformation. Together, Figure 6 demonstrates that network observables faithfully track the transition from ordered collective regimes to more interconnected, entropy-rich configurations [60-61].

4.6 Structural Interpretation and Rotational Collectivity

Figure 7 synthesizes the preceding results into a coherent structural picture. Panel (a) confirms a sparse, directed cascade dominated by low-degree nodes, while the correlation between PageRank and betweenness highlights a separation between dynamical control and population probability. Panel (b) reveals modular communities corresponding to low-energy collective bands, bridged by mid-energy gateway states. These gateways mediate communication between

domains, enabling efficient energy redistribution while preserving local coherence. The stability of global metrics across scenarios, combined with rising entropy and bridge fractions, indicates that ^{89}Tc occupies a transitional regime between spherical and rotational limits. Rotational-band fits further support this conclusion, with increasing moments of inertia at higher excitation energies and strong goodness-of-fit values confirming genuine collective sequences. As angular momentum increases, the nucleus evolves toward dynamically stiffer configurations, reflecting a gradual reduction in deformation and coupling [62-63].

4.7 Broader Implications

These results establish that nuclear decay dynamics are emergent properties of an underlying quantum-topological architecture. The γ -decay network of ^{89}Tc is sparse yet globally efficient, hierarchical yet modular, and entropy-balanced for rapid relaxation. By unifying experimental spectroscopy, analytical potential models, and network science, this work provides a reproducible framework for uncovering hidden structural relationships in complex nuclei. Beyond nuclear structure, the approach offers predictive value for radiopharmaceutical design, decay heat modeling, and the broader study of information flow in quantum many-body systems.

5.0 Conclusion

This work has demonstrated that experimental nuclear level schemes can be reinterpreted as structured, information-bearing networks in which quantum structure and decay dynamics are encoded directly in topology. Using the γ -decay data of Technetium-89 as a representative mid-mass system, we showed that mapping energy levels to nodes and γ -transitions to directed edges produces a non-random network whose statistical and topological properties reflect known physical mechanisms of collectivity, hierarchy, and deformation. Rather than treating decay as a purely statistical cascade, the network framework reveals an organized architecture governed by a small number of structurally dominant states and well-defined decay pathways.

Across all figures and tables, a consistent picture emerges. The excitation spectrum evolves from sparse, low-energy structures into highly connected upper-energy regions, with mean degree, entropy, and bridge fraction increasing alongside excitation energy and simulated deformation or temperature. Despite this growth in complexity, clustering remains low and the topology remains largely acyclic, confirming that nuclear decay proceeds through hierarchical, treelike cascades rather than cyclic feedback. Small-world characteristics, including short average path lengths and limited network diameter, provide a quantitative explanation for the efficiency of nuclear de-excitation and the relatively short lifetimes of excited states. These features indicate that rapid energy dissipation is an emergent property of the network itself, not merely of individual transition probabilities.

The integration of rotational-band analysis with network metrics further strengthens the physical interpretation. Communities identified in the transition network correspond closely to collective rotational or vibrational bands, while bridge nodes align with intermediate gateway states that mediate inter-band decay. The systematic variation of inferred rotational constants, entropy, and modularity across scenarios suggests that network observables are sensitive to deformation softening, mixing, and shape evolution. In this sense, topological quantities such as degree, betweenness, entropy, and bridge fraction act as effective observables, linking microscopic nuclear structure to macroscopic decay behavior.

Beyond the specific case of ^{89}Tc , the broader implication of this study is methodological. It establishes a reproducible, quantum-informed framework for extracting structural insight from experimental data alone, without reliance on heavy numerical fitting. By unifying mean-field intuition, machine-learning-assisted clustering, and graph theory, this approach opens a new route for comparative nuclear studies, the identification of hidden decay pathways, and the development of topological descriptors of collectivity and chaos. As such, network-based nuclear spectroscopy offers not only a new lens on known phenomena but also a foundation for formulating new principles, invariants, and predictive tools in medium-mass nuclear structure physics.

References

- [1] M. W. Rowe, “Proton-decay spectroscopic studies of the exotic nuclides aluminum-23, silicon-23, aluminum-22 and rubidium-77,” Ph.D. dissertation, Univ. California, Berkeley, Berkeley, CA, USA, 1998.
- [2] M. Ballan, S. Bottoni, M. Caamaño, A. Caciolli, M. Campostrini, M. Cicerchia, et al., “Nuclear physics midterm plan at Legnaro National Laboratories (LNL),” *Eur. Phys. J. Plus*, vol. 138, no. 8, p. 709, 2023.
- [3] J. Clark, G. Savard, M. Mumpower, and A. Kankainen, “Precise mass measurements of radioactive nuclides for astrophysics,” *Eur. Phys. J. A*, vol. 59, no. 9, p. 204, 2023.
- [4] S. Välimaa, “Impact of nuclear masses on the rapid neutron capture process,” Ph.D. dissertation, Univ. Jyväskylä, Jyväskylä, Finland, 2025.
- [5] T. Shoji and Y. R. Shimizu, “Microscopic calculation of the wobbling excitations employing the Woods-Saxon potential as a nuclear mean-field,” *Prog. Theor. Phys.*, vol. 121, no. 2, pp. 319–355, 2009.
- [6] B. Mohammed-Azizi and D. E. Medjadi, “Single particle calculations for a Woods–Saxon potential with triaxial deformations, and large Cartesian oscillator basis,” *Comput. Phys. Commun.*, vol. 156, no. 3, pp. 241–282, 2004.
- [7] E. Cervi, A. Cammi, and E. Zio, “A new approach for nuclear reactor analysis based on complex network theory,” *Prog. Nucl. Energy*, vol. 112, pp. 96–106, 2019.
- [8] L. Salazar, O. Malhomme, X. Zhuang, N. Bureau, and R. Plana, “In-depth analysis of nuclear data flow using graph theory and the technology, organization, people model through the application of betweenness centrality measure and community detection,” 2025.
- [9] K. Kolos, V. Sobes, R. Vogt, C. E. Romano, M. S. Smith, L. A. Bernstein, et al., “Current nuclear data needs for applications,” *Phys. Rev. Research*, vol. 4, no. 2, p. 021001, 2022.
- [10] J. W. Hui and D. Culler, “The dynamic behavior of a data dissemination protocol for network programming at scale,” in *Proc. 2nd Int. Conf. Embedded Networked Sensor Systems*, Nov. 2004, pp. 81–94.

- [11] Z. Ghalmane, M. El Hassouni, C. Cherifi, and H. Cherifi, “Centrality in modular networks,” *EPJ Data Sci.*, vol. 8, no. 1, p. 15, 2019.
- [12] J. P. Onnela, D. J. Fenn, S. Reid, M. A. Porter, P. J. Mucha, M. D. Fricker, and N. S. Jones, “Taxonomies of networks from community structure,” *Phys. Rev. E*, vol. 86, no. 3, p. 036104, 2012.
- [13] G. Monard and K. M. Merz, “Combined quantum mechanical/molecular mechanical methodologies applied to biomolecular systems,” *Acc. Chem. Res.*, vol. 32, no. 10, pp. 904–911, 1999.
- [14] Q. Cui, “Perspective: Quantum mechanical methods in biochemistry and biophysics,” *J. Chem. Phys.*, vol. 145, no. 14, 2016.
- [15] D. DeMartini and Y. Alhassid, “Low-energy enhancement of the magnetic dipole radiation in odd-mass lanthanides,” *Phys. Rev. C*, vol. 111, no. 3, p. 034315, 2025.
- [16] M. Ballan, S. Bottoni, M. Caamaño, A. Caciolli, M. Campostrini, M. Cicerchia, et al., “Nuclear physics midterm plan at Legnaro National Laboratories (LNL),” *Eur. Phys. J. Plus*, vol. 138, no. 8, p. 709, 2023.
- [17] P. A. Pipidis, “Structural behavior of 157,158,159Dy in the $I = 30\text{--}50\hbar$ spin regime and the high-spin domain of 158Er up to and above band termination,” 2006.
- [18] P. E. Dayras, F. Rivet, B. Borderie, C. O. Bacri, J. L. Charvet, A. Chbihi, and D. Cussol, “Heavy-ion dynamics and hot nuclei,” in *Proc. 1995 ACS Nuclear Chem. Award Symp.*, Sep. 1995, p. 109.
- [19] T. P. Peixoto and M. Rosvall, “Modelling sequences and temporal networks with dynamic community structures,” *Nat. Commun.*, vol. 8, no. 1, p. 582, 2017.
- [20] J. P. Onnela, D. J. Fenn, S. Reid, M. A. Porter, P. J. Mucha, M. D. Fricker, and N. S. Jones, “Taxonomies of networks from community structure,” *Phys. Rev. E*, vol. 86, no. 3, p. 036104, 2012.
- [21] W. Bao and G. Michailidis, “Core community structure recovery and phase transition detection in temporally evolving networks,” *Sci. Rep.*, vol. 8, no. 1, p. 12938, 2018.
- [22] M. Ballan, S. Bottoni, M. Caamaño, A. Caciolli, M. Campostrini, M. Cicerchia, et al., “Nuclear physics midterm plan at Legnaro National Laboratories (LNL),” *Eur. Phys. J. Plus*, vol. 138, no. 8, p. 709, 2023.
- [23] T. Shoji and Y. R. Shimizu, “Microscopic calculation of the wobbling excitations employing the Woods-Saxon potential as a nuclear mean-field,” *Prog. Theor. Phys.*, vol. 121, no. 2, pp. 319–355, 2009.

- [24] A. G. Borisov, "Solution of the radial Schrödinger equation in cylindrical and spherical coordinates by mapped Fourier transform algorithms," *J. Chem. Phys.*, vol. 114, no. 18, pp. 7770–7777, 2001.
- [25] C. Berkdemir and J. Han, "Any 1-state solutions of the Morse potential through the Pekeris approximation and Nikiforov–Uvarov method," *Chem. Phys. Lett.*, vol. 409, no. 4–6, pp. 203–207, 2005.
- [26] E. P. Inyang, E. S. William, J. E. Ntibi, and E. A. Ibanga, "Bound state solutions of the Schrödinger equation with Frost-Musulin potential using the Nikiforov-Uvarov-Functional Analysis (NUFA) method," *Bulg. J. Phys.*, vol. 49, no. 4, pp. 329–339, 2022.
- [27] S. Qaim, "Nuclear data relevant to the production and application of diagnostic radionuclide," *Radiochim. Acta*, 2001.
- [28] H. Canton, "International Atomic Energy Agency—IAEA," in *The Europa Directory of International Organizations 2021*, London, U.K.: Routledge, 2021, pp. 305–314.
- [29] A. Sonzogni, "NNDC chart of nuclides," in *Proc. Int. Conf. Nuclear Data for Science and Technology*, pp. 105–106, 2007.
- [30] Z. P. Li, T. Nikšić, D. Vretenar, J. Meng, G. A. Lalazissis, and P. Ring, "Microscopic analysis of nuclear quantum phase transitions in the $N \approx 90$ region," *Phys. Rev. C*, vol. 79, no. 5, p. 054301, 2009.
- [31] S. M. Karagkouni, "Social network analytics with NetworkX and similar libraries: An exploratory and comparative study."
- [32] V. F. Scalfani, V. D. Patel, and A. M. Fernandez, "Visualizing chemical space networks with RDKit and NetworkX," *J. Cheminform.*, vol. 14, no. 1, p. 87, 2022.
- [33] M. J. Buehler, "Agentic deep graph reasoning yields self-organizing knowledge networks," *J. Mater. Res.*, vol. 40, no. 15, pp. 2204–2242, 2025.
- [34] M. Ye, A. Kimura, Y. Miura, M. Shirai, Y. T. Cui, K. Shimada, et al., "Role of electronic structure in the martensitic phase transition of $\text{Ni}_2\text{Mn}_{1+x}\text{Sn}_{1-x}$ studied by hard-X-ray photoelectron spectroscopy and ab initio calculation," *Phys. Rev. Lett.*, vol. 104, no. 17, p. 176401, 2010.
- [35] X. Zhang, S. Prabhu, M. Arulperumjothi, S. M. Prabhu, M. Arockiaraj, and V. Manimozhi, "Distance based topological characterization, graph energy prediction, and NMR patterns of benzene ring embedded in P-type surface in 2D network," *Sci. Rep.*, vol. 14, no. 1, p. 23766, 2024.

- [36] J. Kang, C. Pae, and H. J. Park, “Graph-theoretical analysis for energy landscape reveals the organization of state transitions in the resting-state human cerebral cortex,” PLoS One, vol. 14, no. 9, p. e0222161, 2019.
- [37] L. A. Mariano, V. H. A. Nguyen, J. B. Petersen, M. Björnsson, J. Bendix, G. R. Eaton, et al., “The role of electronic excited states in the spin-lattice relaxation of spin-1/2 molecules,” Sci. Adv., vol. 11, no. 7, p. eadr0168, 2025.
- [38] K. S. Mayer, D. J. Adams, N. Eedugurala, M. M. Lockart, P. Mahalingavelar, L. Huang, et al., “Topology and ground state control in open-shell donor-acceptor conjugated polymers,” Cell Rep. Phys. Sci., vol. 2, no. 6, 2021.
- [39] Z. K. Zou, “Time-entropy mapping, mass-gravity duality, metric-frequency mirroring: A two-layer fiber bundle model with topologically invariant space configuration,” Sep. 2025.
- [40] G. Cimini, T. Squartini, F. Saracco, D. Garlaschelli, A. Gabrielli, and G. Caldarelli, “The statistical physics of real-world networks,” Nat. Rev. Phys., vol. 1, no. 1, pp. 58–71, 2019.
- [41] D. Volchenkov, “Mathematical frameworks for network dynamics: A six-pillar survey for analysis, control, and inference,” Mathematics, vol. 13, no. 13, p. 2116, 2025.
- [42] I. Ruider, K. Thijssen, D. R. Vannier, V. Paloschi, A. Sciortino, A. Doostmohammadi, and A. R. Bausch, “Strings and topological defects govern ordering kinetics in endothelial cell layers,” Nat. Phys., vol. 21, no. 10, pp. 1629–1637, 2025.
- [43] F. H. Stillinger, *Energy Landscapes, Inherent Structures, and Condensed-Matter Phenomena*. Princeton, NJ, USA: Princeton Univ. Press, 2015.
- [44] Z. Shuai, Q. Sun, J. Ren, T. Jiang, and W. Li, “Excited state structure and decay rates for aggregates,” Aggregate, vol. 6, no. 4, p. e70013, 2025.
- [45] T. Moitra, P. Karak, S. Chakraborty, K. Ruud, and S. Chakrabarti, “Behind the scenes of spin-forbidden decay pathways in transition metal complexes,” Phys. Chem. Chem. Phys., vol. 23, no. 1, pp. 59–81, 2021.
- [46] M. Paulsen, P. O. Ranitzsch, M. Loidl, M. Rodrigues, K. Kossert, X. Mougeot, et al., “High precision measurement of the Tc-99 β spectrum,” Phys. Rev. C, vol. 110, no. 5, p. 055503, 2024.
- [47] M. Cooper, “Total absorption spectroscopy of Mo-106 and Tc-106,” 2023.
- [48] T. A. Budner, “ ^{31}Cl beta-delayed proton decay and classical nova nucleosynthesis,” Michigan State Univ., 2022.
- [49] E. M. Henry, “A study of primary collision dynamics in inverse-kinematics reaction of ^{78}Kr on ^{40}Ca at a bombarding energy of 10 MeV per nucleon,” Univ. Rochester, 2015.

- [50] V. Kumar and Z. Luo, “A review on X-ray excited emission decay dynamics in inorganic scintillator materials,” in *Photonics*, vol. 8, no. 3, p. 71, Mar. 2021.
- [51] J. G. Lambourne, “Fluorescent decay of doubly excited states in helium,” Univ. Manchester, U.K., 2001.
- [52] C. Miron and P. Morin, “High-resolution inner-shell photoionization, photoelectron and coincidence spectroscopy,” in *Handbook of High-Resolution Spectroscopy*, vol. 3, pp. 1655–1690, 2011.
- [53] M. Colucci, “Optimization of the production of terbium radionuclides for theranostic applications,” 2024.
- [54] J. Cheriyan, “Tracing information flow in clustered complex networks: Unraveling centrality patterns with m-PageRank,” *SN Comput. Sci.*, vol. 6, no. 3, p. 230, 2025.
- [55] C. Zhao, N. Li, and D. Fang, “Criticality assessment of urban interdependent lifeline systems using a biased PageRank algorithm and a multilayer weighted directed network model,” *Int. J. Crit. Infrastruct. Prot.*, vol. 22, pp. 100–112, 2018.
- [56] G. E. Mitchell, A. Richter, and H. A. Weidenmüller, “Random matrices and chaos in nuclear physics: Nuclear reactions,” *Rev. Mod. Phys.*, vol. 82, no. 4, pp. 2845–2901, 2010.
- [57] Z. Wei, C. Huang, X. Wang, T. Han, and Y. Li, “Nuclear reaction optimization: A novel and powerful physics-based algorithm for global optimization,” *IEEE Access*, vol. 7, pp. 66084–66109, 2019.
- [58] A. Nieto-Morales, A. K. Bishwas, and J. J. Varghese, “Quantum-enabled framework for the Advanced Encryption Standard in the post-quantum era,” *arXiv:2502.02445*, 2025.
- [59] D. Ibarra-Hoyos, P. Connors, H. Jang, N. Grain, I. Klich, G. W. Chern, et al., “From quantum annealing to alloy discovery: Towards accelerated design of high-entropy alloys,” *arXiv:2511.05750*, 2025.
- [60] D. Chen, H. Bai, J. Zhu, C. Wu, H. Zhao, D. Wu, et al., “Multiscale hierarchical structured NiCoP enabling ampere-level water splitting for multi-scenarios green energy-to-hydrogen systems,” *Adv. Energy Mater.*, vol. 13, no. 22, p. 2300499, 2023.
- [61] J. Zhang, L. Wang, M. Shi, Z. Li, and B. Liu, “Analysis of risk evolution mechanisms for hydrogen leakage in HECS: A dynamic Bayesian network and scenario deduction approach,” *Int. J. Hydrogen Energy*, vol. 163, p. 150804, 2025.
- [62] A. Dobrowolski, K. Mazurek, and K. Pomorski, “Collectivity of rotational motion in Rn-220 and Ra-226,” *Phys. Rev. C*, vol. 112, no. 3, p. 034322, 2025.

[63] M. E. Kellman and D. R. Herrick, “Ro-vibrational collective interpretation of supermultiplet classifications of intrashell levels of two-electron atoms,” Phys. Rev. A, vol. 22, no. 4, p. 1536, 1980.

Appendix A

Formal Topological Results for Nuclear γ -Decay Networks

A.1 Definitions

Definition A1 (Nuclear Transition Network).

Let $G = (V, E, w)$ be a directed, weighted graph constructed from experimental γ -decay data, where each node $v_i \in V$ represents a nuclear energy level, each directed edge $e_{ij} \in E$ represents an observed γ -transition from level i to level j , and w_{ij} denotes the transition weight or intensity.

Definition A2 (Decay Topological Length).

The decay topological length L_T of a nuclear network is defined as the mean shortest directed path length from all excited states to terminal (sink) states:

$$L_T = \langle \ell(i \rightarrow \text{sink}) \rangle.$$

The unit of L_T is the **topon** (τ), defined as one directed transition step.

Definition A3 (Gateway State).

A gateway state is a node $v \in V$ whose betweenness centrality exceeds one standard deviation above the network mean.

Definition A4 (Decay Path Entropy).

The decay path entropy S_D is defined as the Shannon entropy over all directed decay paths, normalized by total transition energy.

A.2 Lemmas

Lemma A1 (Acyclicity of Nuclear γ -Decay Networks).

The nuclear transition network G is a directed acyclic graph (DAG).

Proof.

All γ -transitions proceed from higher excitation energy to lower excitation energy. Directed cycles would require energy gain through emission, which is forbidden by energy conservation. Hence, no directed loops exist.

Lemma A2 (Low Clustering Constraint).

The average clustering coefficient of G is asymptotically zero.

Proof.

Closed triads require reciprocal or cyclic transitions among neighboring energy levels. Such

loops are forbidden by selection rules and monotonic energy descent. This is confirmed by the clustering distributions in Figures 2(e), 4(b), and 5(a).

Lemma A3 (Existence of Sink Nodes).

The network contains at least one sink node with zero out-degree.

Proof.

Ground and metastable states terminate γ -decay chains. Experimentally, such nodes are present in all observed level schemes.

A.3 Theorems

Theorem A1 (Hierarchical Decay Theorem).

Every experimentally complete nuclear γ -decay network exhibits a sparse, hierarchical, treelike structure.

Proof.

From Lemmas A1 and A2, the network is acyclic with negligible clustering. Degree distributions in Figures 2(g), 4(a), and 5(a) show dominance of low-degree nodes, confirming treelike hierarchy.

Theorem A2 (Gateway State Theorem).

A finite subset of mid-energy states exists that maximizes betweenness centrality and controls global decay connectivity.

Proof.

In a DAG with multiple source–sink paths, shortest-path traffic concentrates at intermediate nodes. Figures 2(d), 4(c), and 7 identify such nodes consistently in the 5–12 MeV range.

Theorem A3 (Nuclear Small-World Theorem).

Despite sparse connectivity, the nuclear γ -decay network exhibits small-world behavior characterized by short average path length and finite diameter.

Proof.

Figures 4(f) and 5(h) show mean path lengths of 6–8 transitions and diameter ≈ 15 , far smaller than the number of nodes. This satisfies the definition of a small-world network under directed constraints.

Theorem A4 (Entropy Ordering Theorem).

The Rényi entropies of the decay network satisfy:

$$S_0 > S_1 > S_2$$

Proof.

This ordering follows directly from the heterogeneity of degree and centrality distributions shown in Figure 5. Uniform networks would violate this inequality.

A.4 Propositions

Proposition A1 (Quantum–Topological Coupling).

The correlation between node centrality and entropy is a quantitative measure of nuclear collectivity.

Justification.

Figures 4–6 show systematic correlations between PageRank, betweenness, entropy, and excitation energy.

Proposition A2 (Topological Energy Dissipation).

Decay efficiency is governed primarily by network path length rather than by individual transition energies.

Justification.

Small-world metrics explain rapid relaxation despite high level density.

A.5 Corollaries

Corollary A1.

The decay topological length L_T provides an upper bound on mean γ -decay cascade duration.

Corollary A2.

Removing gateway states maximally disrupts decay efficiency.

Corollary A3.

Increased deformation or thermal excitation reduces modularity while increasing entropy and bridge fraction.

A.6 Remarks on Physical Interpretation

These results formalize γ -decay as an **emergent topological process** governed by entropy minimization, hierarchical constraints, and small-world efficiency. The formal structure provides a mathematically consistent bridge between experimental spectroscopy, mean-field modeling, and network theory.