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Hopf bifurcations of a diffusion model with a general advection and delay

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Abstract

This paper investigates a class of reaction-diffusion population models defined on a bounded domain, which involves a general time-delayed per capita growth rate and a general advection term. By the Lyapunov-Schmidt reduction method, we establish the existence of spatially nonhomogeneous steady states when the parameter λ approaches the principal eigenvalue λ_* of a non-self-adjoint elliptic operator. A detailed analysis of the characteristic equation further confirms the existence of Hopf bifurcations induced by large delays, which originate from these steady states. Specifically, we show that when λ approaches λ_*, the critical delay value $\tau_{\lambda,0}$ required for stability loss tends to infinity. Subsequently, by applying the center manifold reduction and the normal form theory, we ascertain the direction of these Hopf bifurcations and the stability of the resulting periodic orbits. Finally, we use numerical simulations to illustrate the validity of theoretical results where the growths are Logistic-type and weak Allee.

Keywords: Reaction-diffusion; Advection; Delay; Hopf bifurcation; Non-self-adjoint elliptic operator.

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1 Introduction

In the last decades, effects of diffusion and time delays have drawn considerable attention in the fields of mathematical biology and dynamical systems. Although many studies have focused on dynamics near spatially homogeneous steady states (see, e.g., [7, 11–15]), results concerning spatially nonhomogeneous steady states remain notably limited. The main difficulty lies in establishing the existence of such nonhomogeneous solutions, which is a task that typically demands sophisticated analytical tools such as a priori estimates and topological degree theory. Moreover, even with existence established, the associated characteristic equations of the linearized systems—often transcendental due to the presence of delay—generally lack explicit algebraic forms. This complicates the analysis of local stability and bifurcation. For these reasons, developing a general analytical framework to study the stability and bifurcation of spatially nonhomogeneous steady states in delay systems is of significant theoretical and practical importance.

In spatially heterogeneous environments, organisms typically possess the ability to sense and respond to local conditions through adaptive movement, which often involves both random diffusion and directional advection (see, e.g., [21–32, 34]). To accurately capture such taxis behavior, it is essential to incorporate advection effects into population models. In this context, we consider the following single-species model with a general advection term

$$\begin{cases} u_t = D\nabla \cdot [d(x)\nabla u(x,t) - \vec{b}(x)u(x,t)] + u(x,t)f\left(x, u(x,t-r)\right), & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \end{cases} \tag{1.1}$$

where $u(x,t)$ represents the density of the population at position x and time t, $r \geq 0$ is the time-delay parameter (often used to represent the maturity time of the species), D represents the average rate of diffusion and advection, and Ω is a bounded domain in $\mathbb{R}^n$ $(1 \leq n \leq 3)$ with a smooth boundary $\partial\Omega$.

In particular, if there exists a function $B(x)$ such that $\vec{b}(x) = d(x)\nabla B(x)$, then the problem can be significantly simplified by a change of variables. Although this condition always holds in one-dimensional space, it is generally not satisfied in higher dimensions unless the advection field is conservative. In this paper, we consider the general case without the existence of $B(x)$. Thus, the elliptic operator is intrinsically non-self-adjoint and cannot be symmetrized, rendering classical variational methods inapplicable and leading to significant analytical challenges.

Our proposed model (1.1) is capable of capturing not only the classical Logistic growth but also growth with the "food-limited" effect and the weak Allee effect. The Logistic growth model, a widely used model, describes population dynamics under crowding effects driven by intraspecific competition. The food-limited growth model constitutes a refinement of the classical logistic growth model. It describes a growth limitation mechanism based on the proportion of unexploited resources, providing a more realistic representation of population dynamics under resource scarcity (see, e.g., [35–37]). Meanwhile, the weak Allee effect describes the phenomenon in which the reproductive and survival rates of individuals decrease when population sizes that are relatively small (see, e.g., [25, 38–40]).

Let $t = \tilde{t}/D$, dropping the tilde sign, and denote $\lambda = 1/D$ and $\tau = Dr$. Then (1.1) is transformed into the following equivalent model:

$$\begin{cases} u_t = \nabla \cdot [d(x)\nabla u(x,t) - \vec{b}(x)u(x,t)] + \lambda u(x,t) f\left(x, u(x,t-\tau)\right), & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0. \end{cases} \tag{1.2}$$

Unless stated otherwise, we assume that $f(x,u)$ satisfies the following assumptions:

($\mathbf{H_1}$) $f(x,u)$ is a smooth function (for example, f has fourth-order continuous partial derivatives), $f(x,0) := m(x)$, and $\max_{x\in\overline{\Omega}} m(x) > 0$.

($\mathbf{H_2}$) There exists $\alpha \in \left(0, \frac{1}{2}\right)$ such that $d(x) \in C^{1+\alpha}(\bar{\Omega})$, $m(x) \in C^{\alpha}(\bar{\Omega})$ and $\vec{b}(x) \in \left[C^{1+\alpha}(\bar{\Omega})\right]^n$, where $n \leq 3$ is the dimension of the domain Ω. Moreover, $d(x) > 0$ for $x \in \bar{\Omega}$.

Busenberg and Huang [1] are pioneers in studying the dynamics near nonhomogeneous steady-state solutions in space. They utilized the perturbation method and the implicit function theorem to study a class of time-delay diffusion Hutchinson equations with Dirichlet boundary conditions. They established the theoretical framework for the existence of nonhomogeneous steady-state solutions and Hopf bifurcations of

$$\begin{cases} u_t = d\Delta u(x,t) + \lambda u(x,t)[1 - u(x,t-\tau)], & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0. \end{cases}$$

In the one-dimensional spatial region $\Omega = (0,\pi)$, they proved that when the parameter λ is larger than d and approaches d, the unique spatial nonhomogeneous positive steady-state solution will lose stability under large time delays τ and Hopf bifurcation will occur, thereby triggering the oscillation behavior of the system.

Inspired by the research of Busenberg and Huang [1], Su [7] extended the work to the model with a more general form of per capita growth rate for one-dimensional domains, and systematically analyzed the Hopf bifurcation behavior. The relevant results of this model in the high-dimensional case can be found in the work of Yan and Li [13]. Subsequently, Chen and Shi [2] further studied the stability and Hopf bifurcation issues of positive nonhomogeneous steady-state solutions in the Logistic-type population model with nonlocal diffusion terms. Chen and Yu [3] generalized the conclusions in [1] to nonlocal time-delay reaction-diffusion population models. Based on these, Jin and Yuan [9] established the Hopf bifurcation theory for a more general type of reaction-diffusion-migration equations. It is worth noting that Guo and Ma [10] innovatively applied the Lyapunov–Schmidt reduction method to study the dynamical behavior of delayed diffusion equations under Dirichlet boundary conditions. One of the significant advantages of this method is that the obtained nonhomogeneous steady-state solutions have explicit algebraic expressions, thereby providing important convenience for subsequent stability and bifurcation analysis.

Recently, Liu and Chen [4] considered a delayed reaction diffusion model with a general advection term:

$$\begin{cases} u_t = \nabla \cdot [d(x)\nabla u - \vec{b}(x)u] + \lambda u[m(x) - a(x)u(t-\tau)], & x \in \Omega,\ t > 0, \\ u = 0, & x \in \partial\Omega,\ t > 0, \end{cases}$$

which provides important inspiration for our current research. Although the constructive proof in [4] is effective for a specific Logistic-type rate, it is difficult to apply this method directly to a more general per capita growth rate $f(x, u)$.

In this paper, we utilize the Lyapunov-Schmidt reduction method to investigate the bifurcation problem for a broad class of nonlinear terms, extending the existing theory to a more general framework. This extension necessitates a rigorous re-derivation of various analytical estimates and a complete re-evaluation of the center manifold and normal forms, as the calculations can no longer rely on the specific algebraic structure of the model. Due to the technical complexities involved in these derivations, this generalization is theoretically non-trivial.

The rest of the paper is organized as follows. In Section 2, we provide some preliminary knowledge to facilitate our subsequent discussions. In Subsection 3.1, we establish the existence of nonhomogeneous steady-state solutions of equation (1.2) by exploiting the Lyapunov-Schmidt reduction. We dedicate Section 3.2 to the eigenvalue problem of the infinitesimal generator of the C_0-semigroup generated by the solutions of the

linearized system. In Subsection 3.3, by the local Hopf bifurcation theory, we obtain some results on the stability of nonhomogeneous steady-state solutions and the existence of Hopf bifurcation phenomena. In Section 4, we derive an explicit formula which can be used to determine the stability and bifurcation direction of the Hopf bifurcation periodic solutions near the nonhomogeneous steady-state solutions. In Section 5, with the weak Allee effect and the "food-limited" effect as examples, we give some numerical results to demonstrate the obtained theoretical results. Finally, in Section 6, we summarize our theoretical findings.

2 Preliminaries

In this section, we first introduce the spaces

$$\mathbb{X} = H^2(\Omega) \cap H_0^1(\Omega),\ C_\tau = C([-\tau, 0], \mathbb{Y}),\ \mathcal{C} = C([-1, 0], \mathbb{Y}),$$

where

$$H_0^1(\Omega) = \left\{u \in H^1(\Omega) \mid u(x) = 0 \text{ for all } x \in \partial\Omega\right\},\ \mathbb{Y} = L^2(\Omega).$$

For a linear space Z, we denote its complexification by $Z_{\mathbb{C}}$, which is defined as

$$Z_{\mathbb{C}} := Z \oplus iZ = \{x_1 + ix_2 |\ x_1, x_2 \in Z\}.$$

Moreover, the domain, the kernel, and the range of a linear operator L are denoted as $\mathscr{D}(L)$, $\mathscr{N}(L)$ and $\mathscr{R}(L)$ respectively. $\mathbb{X}_{\mathbb{C}}$ is a Hilbert space equipped with the inner product

$$\langle u, v\rangle_{\mathbb{X}_{\mathbb{C}}} = \sum_{|\alpha|\le 2} \int_\Omega \overline{(D^\alpha u)} D^\alpha v \,\mathrm{d}x \text{ for } u, v \in \mathbb{X}_{\mathbb{C}},$$

where $|\alpha| = \sum_{i=1}^n \alpha_i$ and $D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$. The induced norm is $\|u\|_{\mathbb{X}_{\mathbb{C}}} = \sqrt{\langle u, u\rangle_{\mathbb{X}_{\mathbb{C}}}}$ for $u \in \mathbb{X}_{\mathbb{C}}$. For the space $\mathbb{Y}_{\mathbb{C}}$, we define the inner product as

$$\langle u, v\rangle = \int_\Omega \bar{u}(x) v(x) \,\mathrm{d}x \text{ for } u, v \in \mathbb{Y}_{\mathbb{C}}.$$

The norm in $\mathbb{Y}_{\mathbb{C}}$ is then given by $\|u\|_{\mathbb{Y}_{\mathbb{C}}} = \sqrt{\langle u, u\rangle}$ for $u \in \mathbb{Y}_{\mathbb{C}}$.

We cite the following two results from [4,25,33], which are crucial for our subsequent proofs.

Lemma 2.1. *Assume that* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$ *hold. Then the eigenvalue problem*

$$\begin{cases} \nabla \cdot [d(x)\nabla\psi - \vec{b}(x)\psi] + \lambda m(x)\psi = 0, & x \in \Omega, \\ \psi = 0, & x \in \partial\Omega, \end{cases} \tag{2.1}$$

admits a unique positive principal eigenvalue λ_*, which is also the principal eigenvalue for

$$\begin{cases} \nabla \cdot [d(x)\nabla\psi] + \vec{b}(x) \cdot \nabla\psi + \lambda m(x)\psi = 0, & x \in \Omega, \\ \psi = 0, & x \in \partial\Omega. \end{cases} \tag{2.2}$$

Moreover, there are the eigenfunctions ϕ and ϕ^* of (2.1) *and* (2.2) *corresponding to λ_*, respectively, satisfying*

$$\phi, \phi^* > 0 \text{ for } x \in \Omega, \text{ and } \int_\Omega \phi \, \mathrm{d}x = \int_\Omega \phi^* \, \mathrm{d}x = 1.$$

For the above ϕ and ϕ^* given in Lemma 2.1, the following result is satisfied.

Lemma 2.2. *Assume that* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$ *hold. Then* $\frac{\int_\Omega m(x)\phi\phi^* \, \mathrm{d}x}{\int_\Omega \phi\phi^* \, \mathrm{d}x} > 0$.

3 Steady-State Bifurcations

3.1 Existence of Steady-State Solutions

Obviously, the positive steady states of (1.2) satisfy

$$\begin{cases} \nabla \cdot [d(x)\nabla u - \vec{b}(x)u] + \lambda u f(x,u) = 0, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{3.1}$$

Define $F : \mathbb{X} \times \mathbb{R} \to \mathbb{Y}$ by

$$F(u,\lambda) = \nabla \cdot [d(x)\nabla u - \vec{b}(x)u] + \lambda u f(x,u).$$

The linearization of F at $u = 0$ is given by

$$L_\lambda \phi := \nabla \cdot [d(x)\nabla\phi - \vec{b}(x)\phi] + \lambda m(x)\phi \tag{3.2}$$

and the corresponding adjoint operator is

$$L_\lambda^* \phi := \nabla \cdot [d(x)\nabla\phi] + \vec{b}(x) \cdot \nabla\phi + \lambda m(x)\phi,$$

where L_{λ_*} is a Fredholm operator of index zero due to the standard elliptic theory. For λ_* being the principal eigenvalue, $\mathcal{N}(L_{\lambda_*}) = \mathrm{span}\{\phi\}$ and $\mathcal{N}(L_{\lambda_*}^*) = \mathrm{span}\{\phi^*\}$. Consequently, the spaces $\mathbb{X}$ and $\mathbb{Y}$ admit the following direct sum decompositions:

$$\mathbb{X} = \mathcal{N}(L_{\lambda_*}) \oplus X_1 = \mathcal{N}(L_{\lambda_*}^*) \oplus \widetilde{X}_1, \mathbb{Y} = \mathcal{N}(L_{\lambda_*}) \oplus Y_1 = \mathcal{N}(L_{\lambda_*}^*) \oplus \widetilde{Y}_1,$$

where

$$X_1 = \left\{ w \in \mathbb{X} : \int_\Omega \phi^* w \mathrm{d}x = 0 \right\}, Y_1 = \mathcal{R}(L_{\lambda_*}) = \left\{ y \in \mathbb{Y} : \int_\Omega \phi^* y \mathrm{d}x = 0 \right\},$$
$$\widetilde{X}_1 = \left\{ w \in \mathbb{X} : \int_\Omega \phi w \mathrm{d}x = 0 \right\}, \widetilde{Y}_1 = \mathcal{R}(L^*_{\lambda_*}) = \left\{ y \in \mathbb{Y} : \int_\Omega \phi y \mathrm{d}x = 0 \right\}.$$

Based on Lyapunov-Schmidt reduction methods, denote Q and $I-Q$ as the projection operators from $\mathbb{Y}$ onto Y_1 and $\mathscr{N}(L_{\lambda_*})$ respectively. Then $F(u,\lambda)=0$ is equivalent to the following system

$$\begin{cases} QF(u_1+u_2,\lambda)=0, \\ (I-Q)F(u_1+u_2,\lambda)=0, \end{cases} \tag{3.3}$$

where $u_1 \in \mathscr{N}(L_{\lambda_*})$ and $u_2 \in X_1$. Notice that $F(0,\lambda) \equiv 0$ and $QF_{u_2}(0,\lambda_*) = L_{\lambda_*}$. Applying the implicit function theorem, we obtain a continuously differentiable map $h : \mathfrak{U} \to X_1$ such that

$$h(0,\lambda)=0, QF(u_1+h(u_1,\lambda),\lambda) \equiv 0 \text{ and } h'_{u_1}(0,\lambda_*)=0,$$

where $\mathfrak{U}$ is an open neighborhood of $(0,\lambda_*)$ in $\mathscr{N}(L_{\lambda_*}) \times \mathbb{R}$. Substituting $u_2 = h(u_1,\lambda)$ into the second equation of (3.3) gives

$$\mathfrak{F}(u_1,\lambda) := (I-Q)F(u_1+h(u_1,\lambda),\lambda) = 0.$$

Thus, each solution to $\mathfrak{F}(u_1,\lambda)=0$ in $\mathfrak{U}$ is in one-to-one correspondence with a solution to $F(u,\lambda)=0$.

As $u_1 = t\phi \in \mathscr{N}(L_{\lambda_*})$ with $t \in \mathbb{R}$, substituting it into (3.3) and then applying the inner product with ϕ^* on Ω, we define

$$\begin{aligned} \beta(t,\lambda) :&= \langle \phi^*, F(t\phi(x)+h(t\phi(x),\lambda),\lambda) \rangle \\ &= \int_\Omega \phi^*(x) F(t\phi(x)+h(t\phi(x),\lambda),\lambda) \mathrm{d}x. \end{aligned}$$

We explore $\beta(t,\lambda)=0$ to locate the solution of $F(t\phi(x)+h(t\phi(x),\lambda),\lambda)=0$. Noting $\beta(0,\lambda)=0$ and $\beta'_t(0,\lambda_*)=0$, we take the following function

$$H(t,\lambda) = \begin{cases} \dfrac{\beta(t,\lambda)}{t} & \text{if } t \neq 0, \\ \beta'_t(0,\lambda) & \text{if } t = 0. \end{cases}$$

A direct calculation yields

$$H'_\lambda(0,\lambda_*) = \beta''_{t,\lambda}(0,\lambda_*) = \langle \phi^*, F''_{u,\lambda}(0,\lambda_*)[\phi] \rangle = \int_\Omega m(x)\phi\phi^* \,\mathrm{d}x \overset{\text{def}}{=} a.$$

From Lemma 2.2, we know that $H'_\lambda(0,\lambda_*) \neq 0$. Noting that $H(0,\lambda_*) = 0$ and applying the implicit function theorem to $H(t,\lambda)$, we have $\lambda = \lambda(t)$ that is defined in an ϵ-neighbourhood of $t = 0$ such that

$$\lambda(0) = \lambda_*, H(t,\lambda(t)) = 0, \forall -\epsilon \leq t \leq \epsilon$$

and

$$\lambda'(0) = -\frac{H'_t(0,\lambda_*)}{H'_\lambda(0,\lambda_*)}.$$

More precisely, we have

$$\begin{aligned} H'_t(0,\lambda_*) &= \frac{1}{2}\beta''_{t,t}(0,\lambda_*) = \frac{1}{2}\langle \phi^*, F''_{u,u}(0,\lambda_*)[\phi]^2\rangle = \lambda_* \int_\Omega f'_u(x,0)\phi^2\phi^* \mathrm{d}x \stackrel{\text{def}}{=} b, \\ H'_\lambda(0,\lambda_*) &= \beta''_{t,\lambda}(0,\lambda_*) \stackrel{\text{def}}{=} a. \end{aligned}$$

If $b \neq 0$, it follows from the Taylor expansion that

$$\lambda(t) = \lambda_* - \frac{b}{a}t + o(t) \text{ as } t \to 0,$$

If $b = 0$, we get

$$\lambda''(0) = -\frac{H''_{t,t}(0,\lambda_*)}{H'_\lambda(0,\lambda_*)} = -\frac{1}{3a}\beta'''_{t,t,t}(0,\lambda_*).$$

A direct calculation yields

$$\begin{aligned} \lambda''(0) &= -\frac{1}{3a}\left(\langle \phi^*, F'''_{u,u,u}(0,\lambda_*)[\phi]^3\rangle + 3\langle \phi^*, F''_{u,u}(0,\lambda_*)[\phi, h''_{u_1u_1}(0,\lambda_*)[\phi]^2]\rangle\right) \\ &= -\frac{1}{3a}\left(3\lambda_* \int_\Omega f''_{uu}(x,0)\phi^3\phi^* \,\mathrm{d}x + 6\lambda_* \int_\Omega f'_u(x,0)h''_{u_1u_1}(0,\lambda_*)[\phi]^2\phi\phi^* \,\mathrm{d}x\right) \\ &= -\frac{\lambda_*}{a}\left(\int_\Omega f''_{uu}(x,0)\phi^3\phi^* \,\mathrm{d}x + 2\int_\Omega f'_u(x,0)w_2\phi\phi^* \,\mathrm{d}x\right), \end{aligned}$$

where $w_2 = h''_{u_1u_1}(0,\lambda_*)[\phi]^2 \in X_1$ is the unique solution of the equation:

$$L_{\lambda_*}w_2 + QF''_{uu}(0,\lambda_*)[\phi]^2 = 0.$$

Thus, when $b = 0$ and $\lambda''(0) \neq 0$, we get

$$u = \pm\left(\frac{2(\lambda-\lambda_*)}{\lambda''(0)}\right)^{1/2}\phi + O(\lambda-\lambda_*).$$

Now we summarize the results for $b \neq 0$ and $b = 0$ in the following two theorems.

Theorem 3.1. *If* $\int_\Omega f'_u(x,0)\phi^2\phi^*\,\mathrm{d}x \neq 0$, *then there exists a constant* $\epsilon > 0$ *and a continuously differentiable mapping* $\lambda \to t_\lambda$ *from* $(\lambda_* - \epsilon, \lambda_*) \cup (\lambda_*, \lambda_* + \epsilon)$ *to* $\mathbb{R}$ *such that* (3.1) *has a nontrivial solution*

$$u_\lambda(x) = t_\lambda\phi(x) + h(t_\lambda\phi(x), \lambda),$$

where t_λ *satisfies* $t_\lambda = (\lambda - \lambda_*)\beta_{\lambda_*} + o(\lambda - \lambda_*)$, *and the bifurcation coefficient* β_{λ_*} *is explicitly given by*

$$\beta_{\lambda_*} = -\frac{a}{b} = -\frac{\int_\Omega m(x)\phi\phi^*\,\mathrm{d}x}{\lambda_*\int_\Omega f'_u(x,0)\phi^2\phi^*\,\mathrm{d}x}. \tag{3.4}$$

Remark: Since $\phi > 0$ on Ω, the sign of the spatially nonhomogeneous steady-state solution u_λ in Theorem 3.1 coincides with the sign of t_λ. It follows that u_λ is positive with $t_\lambda > 0$ and negative with $t_\lambda < 0$. For the representative case $f(x,u) = m(x) - u$ (where $f'_u(x,0) = -1$), the solution u_λ is positive for $\lambda \in (\lambda_*, \lambda_* + \epsilon)$, which is of more interest for its biological significance.

Theorem 3.2. *If* $\displaystyle\int_\Omega f'_u(x,0)\phi^2\phi^*\,\mathrm{d}x = 0$ *and* $\lambda''(0) > 0$ *(respectively,* $\lambda''(0) < 0$*), then there exists a constant* $\epsilon > 0$ *and a continuously differentiable mappings* $\lambda \to t^+_\lambda$ *from* $(\lambda_*, \lambda_* + \epsilon)$ *to* $\mathbb{R}$ *(respectively,* $\lambda \to t^-_\lambda$ *from* $(\lambda_* - \epsilon, \lambda_*)$ *to* $\mathbb{R}$*) such that* (3.1) *has a nontrivial solution*

$$u^+_\lambda(x) = t^+_\lambda\phi(x) + h(t^+_\lambda\phi(x), \lambda) \text{ (respectively , } u^-_\lambda(x) = t^-_\lambda\phi(x) + h(t^-_\lambda\phi(x), \lambda)\text{)}$$

where $t^\pm_\lambda$ *satisfies*

$$t^\pm_\lambda = \pm\left(\frac{2(\lambda - \lambda_*)}{\lambda''(0)}\right)^{1/2} + o\left(|\lambda - \lambda_*|^{1/2}\right).$$

3.2 The Eigenvalue Problem

For small $\epsilon > 0$, denote $\Lambda_{1\epsilon} := (\lambda_* - \epsilon, \lambda_*)$, $\Lambda_{2\epsilon} := (\lambda_*, \lambda_* + \epsilon)$, and $\Lambda := \Lambda_{1\epsilon} \cup \Lambda_{2\epsilon}$. Based on Theorems 3.1 and 3.2, we get the existence of $u_\lambda(x)$ with $\lambda \in \Lambda$ and $u^\pm_\lambda(x)$ with $\lambda \in \Lambda_{1\epsilon}$ or $\lambda \in \Lambda_{2\epsilon}$, which are all denoted as u_λ in this section for convenience. Linearizing system (1.2) at u_λ, we have

$$\begin{cases} v_t = \nabla\cdot\left[d(x)\nabla v - \vec{b}(x)v\right] + \lambda f(x,u_\lambda)v \\ \qquad + \lambda u_\lambda f'_u(x,u_\lambda)v(x,t-\tau), & x \in \Omega,\ t > 0, \\ v = 0, & x \in \partial\Omega,\ t > 0. \end{cases} \tag{3.5}$$

It follows from [8] that the solution semigroup of (3.5) has the infinitesimal generator $\mathcal{A}_{\tau,\lambda}$ satisfying

$$\mathcal{A}_{\tau,\lambda}\Psi = \dot{\Psi}, \tag{3.6}$$

for all $\Psi \in \mathrm{Dom}(\mathcal{A}_{\tau,\lambda})$, where

$$\begin{aligned}\mathscr{D}\left(\mathcal{A}_{\tau,\lambda}\right) = \Big\{\Psi \in C_{\tau\mathbb{C}} \cap C_{\tau\mathbb{C}}^{1} : \Psi(0) \in \mathbb{X}_{\mathbb{C}},\ \dot{\Psi}(0) = \nabla \cdot [d(x)\nabla\Psi(0) - \vec{b}(x)\Psi(0)] \\ +\lambda f\left(x, u_\lambda\right)\Psi(0) + \lambda u_\lambda f_u'\left(x, u_\lambda\right)\Psi(-\tau)\}\end{aligned}$$

and $C_{\tau\mathbb{C}}^{1} = C^1([-\tau, 0], \mathbb{Y}_{\mathbb{C}})$. Moreover, $\mu \in \mathbb{C}$ is an eigenvalue of $\mathcal{A}_{\tau,\lambda}$, if and only if there exists $\psi(\neq 0) \in \mathbb{X}_{\mathbb{C}}$ such that $\Delta(\lambda, \mu, \tau)\psi = 0$, where

$$\begin{aligned}\Delta(\lambda, \mu, \tau)\psi := \nabla \cdot [d(x)\nabla\psi - \vec{b}(x)\psi] \\ + \lambda f\left(x, u_\lambda\right)\psi + \lambda u_\lambda f_u'\left(x, u_\lambda\right)\psi e^{-\mu\tau} - \mu\psi.\end{aligned} \tag{3.7}$$

We can claim that with increasing time delay τ, eigenvalues of $\mathcal{A}_{\tau,\lambda}$ can cross the imaginary axis according to the following discussion. Specifically, $\mathcal{A}_{\tau,\lambda}$ possesses a purely imaginary eigenvalue $\mu = i\nu$ (with $\nu > 0$) for some $\tau \geq 0$ if and only if

$$\nabla \cdot [d(x)\nabla\psi - \vec{b}(x)\psi] + \lambda f\left(x, u_\lambda\right)\psi + \lambda u_\lambda f_u'\left(x, u_\lambda\right)\psi e^{-i\theta} - i\nu\psi = 0 \tag{3.8}$$

is solvable for (ν, θ, ψ) with $\nu > 0, \theta \in [0, 2\pi)$, and $\psi(\neq 0) \in \mathbb{X}_{\mathbb{C}}$, where τ satisfies $\nu\tau = \theta + 2n\pi$ for $n \in \mathbb{N}_0 := \{0, 1, 2, 3, \dots\}$.

We first establish the following estimates for solutions to (3.8) in the cases of $\int_\Omega f_u'(x, 0)\phi^2\phi^* \,\mathrm{d}x \neq 0$ and $\int_\Omega f_u'(x, 0)\phi^2\phi^* \,\mathrm{d}x = 0$, where the proof is inspired by [4].

Lemma 3.3. *Let $(\psi_\lambda, \nu_\lambda, \theta_\lambda)$ be a solution to (3.8) with $\psi_\lambda \in \mathbb{X}_{\mathbb{C}} \setminus \{0\}$, $\nu_\lambda > 0$, and $\theta_\lambda \in [0, 2\pi)$. Then the ratio $\frac{\nu_\lambda}{|t_\lambda|}$ is bounded for $\lambda \in \Lambda$. Moreover, up to a scalar factor, ψ_λ can be represented as*

$$\begin{cases}\psi_\lambda = r_\lambda\phi + w_\lambda, \quad r_\lambda \geq 0,\ w_\lambda \in (X_1)_{\mathbb{C}}, \\ \|\psi_\lambda\|_{\mathbb{X}_{\mathbb{C}}} = \|\phi\|_{\mathbb{X}_{\mathbb{C}}},\end{cases} \tag{3.9}$$

where r_λ, w_λ, and ψ_λ satisfy $\lim_{\lambda\to\lambda_*} r_\lambda = 1$, $\lim_{\lambda\to\lambda_*} w_\lambda = 0$, and $\lim_{\lambda\to\lambda_*} \psi_\lambda = \phi$ in $(C^{2,\alpha}(\overline{\Omega}))_{\mathbb{C}}$, α and ϕ are defined in $(\mathbf{H}_2)$ and Lemma 2.1, respectively.

Proof. Ignoring a scalar factor, ψ_λ can be represented as (3.9). The proof proceeds in three steps.

Step 1. We prove that ν_λ is bounded for $\lambda \in \Lambda$. In contradiction, we suppose that it is unbounded. Then there exists a sequence $\{\lambda_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} \lambda_n = \lambda_*$ and

$\lim_{n\to\infty}\nu_{\lambda_n}=\infty$. Since θ_λ is bounded and $\mathbb{X}$ is compactly embedded into $C^\alpha(\overline{\Omega})$ (with $\alpha\in(0,1/2)$ from $(\mathbf{H_2})$), there exists a subsequence (still denoted as $\{\lambda_n\}$) such that

$$\lim_{n\to\infty}\theta_{\lambda_n}=\theta_*\in[0,2\pi] \text{ and } \lim_{n\to\infty}\psi_{\lambda_n}=\psi_* \text{ in } (C^\alpha(\overline{\Omega}))_{\mathbb{C}}. \tag{3.10}$$

Substituting $(\psi,\nu,\theta,\lambda)=(\psi_{\lambda_n},\nu_{\lambda_n},\theta_{\lambda_n},\lambda_n)$ into (3.8) and multiplying by $\overline{\psi}_{\lambda_n}$, we integrate the resultant over Ω to get

$$\begin{aligned}&\langle\psi_{\lambda_n},L_{\lambda_*}\psi_{\lambda_n}\rangle-\int_\Omega\left[\lambda_* m(x)|\psi_{\lambda_n}|^2-\lambda_n f(x,u_{\lambda_n})|\psi_{\lambda_n}|^2\right]\,\mathrm{d}x\\&+\lambda_n e^{-i\theta_{\lambda_n}}\int_\Omega f'_u(x,u_{\lambda_n})u_{\lambda_n}|\psi_{\lambda_n}|^2\,\mathrm{d}x-i\nu_{\lambda_n}\int_\Omega|\psi_{\lambda_n}|^2\,\mathrm{d}x=0.\end{aligned}\tag{3.11}$$

Using the expression

$$\langle\psi_{\lambda_n},L_{\lambda_*}\psi_{\lambda_n}\rangle=-\int_\Omega d(x)|\nabla\psi_{\lambda_n}|^2\,\mathrm{d}x+\int_\Omega\psi_{\lambda_n}\vec{b}(x)\cdot\nabla\overline{\psi}_{\lambda_n}\,\mathrm{d}x+\lambda_*\int_\Omega m(x)|\psi_{\lambda_n}|^2\,\mathrm{d}x,$$

we derive

$$\nu_{\lambda_n}\|\psi_{\lambda_n}\|_{L_2}\le\max_{x\in\overline{\Omega}}|\vec{b}(x)|\|\nabla\psi_{\lambda_n}\|_{L_2}+\lambda_n\max_{x\in\overline{\Omega}}|f'_u(x,u_{\lambda_n})|\|u_{\lambda_n}\|_\infty\|\psi_{\lambda_n}\|_{L_2}\tag{3.12}$$

$$\begin{aligned}\text{and }\int_\Omega d(x)|\nabla\psi_{\lambda_n}|^2\,\mathrm{d}x\le&\max_{x\in\overline{\Omega}}|\vec{b}(x)|\|\nabla\psi_{\lambda_n}\|_{L_2}\|\psi_{\lambda_n}\|_{L_2}+\lambda_n\int_\Omega|f(x,u_{\lambda_n})||\psi_{\lambda_n}|^2\,\mathrm{d}x\\&+\lambda_n\max_{x\in\overline{\Omega}}|f'_u(x,u_{\lambda_n})|\|u_{\lambda_n}\|_\infty\|\psi_{\lambda_n}\|^2_{L_2}.\end{aligned}\tag{3.13}$$

Since $\|\psi_{\lambda_n}\|_{\mathbb{X}_{\mathbb{C}}}=\|\phi\|_{\mathbb{X}_{\mathbb{C}}}$, the sequences $\{\|\nabla\psi_{\lambda_n}\|_2\}_{n=1}^\infty$ and $\{\|\psi_{\lambda_n}\|_2\}_{n=1}^\infty$ are bounded. For $\lim_{n\to\infty}\nu_{\lambda_n}=\infty$, the inequality in (3.12) implies $\lim_{n\to\infty}\|\psi_{\lambda_n}\|_{L_2}=0$. This, combined with (3.10), gives $\lim_{n\to\infty}\psi_{\lambda_n}=\psi_*=0$ in $(C^\alpha(\overline{\Omega}))_{\mathbb{C}}$. Due to (3.12), we get

$$\lim_{n\to\infty}\int_\Omega|\nabla\psi_{\lambda_n}|^2\,\mathrm{d}x=0.$$

By virtue of (3.12), it follows that $\lim_{n\to\infty}\nu_{\lambda_n}\psi_{\lambda_n}=0$ in $\mathbb{Y}_{\mathbb{C}}$. Rewrite (3.8) as

$$(L_{\lambda_*}-I)\psi_{\lambda_n}+M_{\lambda_n}\psi_{\lambda_n}=0,$$

where

$$M_{\lambda_n}\psi_{\lambda_n}=\psi_{\lambda_n}-\lambda_* m(x)\psi_{\lambda_n}+\lambda_n f(x,u_{\lambda_n})\psi_{\lambda_n}+\lambda_n f'_u(x,u_{\lambda_n})u_{\lambda_n}\psi_{\lambda_n}e^{-i\theta_{\lambda_n}}-i\nu_{\lambda_n}\psi_{\lambda_n}.$$

As $\lim_{n\to\infty}M_{\lambda_n}\psi_{\lambda_n}=0$ in $\mathbb{Y}_{\mathbb{C}}$, the continuity of $(L_{\lambda_*}-I)^{-1}$ from $\mathbb{Y}_{\mathbb{C}}$ to $\mathbb{X}_{\mathbb{C}}$ implies

$$\lim_{n\to\infty}\psi_{\lambda_n}=-\lim_{n\to\infty}(L_{\lambda_*}-I)^{-1}M_{\lambda_n}\psi_{\lambda_n}=0 \text{ in } \mathbb{X}_{\mathbb{C}}.$$

This contradicts $\|\psi_{\lambda_n}\|_{\mathbb{X}_{\mathbb{C}}} = \|\phi\|_{\mathbb{X}_{\mathbb{C}}}$. Thus, the boundedness of ν_λ is proved.

Step 2. We claim $\lim_{\lambda\to\lambda_*}\nu_\lambda = 0$ and $\lim_{\lambda\to\lambda_*}\psi_\lambda = \phi$ in $(C^{2,\alpha}(\overline{\Omega}))_{\mathbb{C}}$. Obviously,

$$\lim_{\lambda\to\lambda_*} r_\lambda = 1 \text{ and } \lim_{\lambda\to\lambda_*} w_\lambda = 0 \text{ in } (C^{2,\alpha}(\overline{\Omega}))_{\mathbb{C}}.$$

For any sequence $\{\lambda_k\}_{k=1}^{\infty}$ satisfying $\lim_{k\to\infty}\lambda_k = \lambda_*$, there exists a subsequence $\{\lambda_{k_n}\}_{n=1}^{\infty}$ (denoted as $\{\lambda_n\}_{n=1}^{\infty}$) such that

$$\lim_{n\to\infty}\nu_{\lambda_n} = a \geq 0, \lim_{n\to\infty}\theta_{\lambda_n} = \widetilde{\theta} \in [0, 2\pi], \text{ and } \lim_{n\to\infty}\psi_{\lambda_n} = \psi^* \text{ in } (C^{\alpha}(\overline{\Omega}))_{\mathbb{C}}.$$

Since $\lim_{n\to\infty} u_{\lambda_n} = 0$ in $C^{\alpha}(\overline{\Omega})$ and $\lim_{n\to\infty} M_{\lambda_n}\psi_{\lambda_n} = \psi^* - ia\psi^*$ in $(C^{\alpha}(\overline{\Omega}))_{\mathbb{C}}$, by the continuity of $(L_{\lambda_*} - I)^{-1}$ from $(C^{\alpha}(\overline{\Omega}))_{\mathbb{C}}$ to $(C^{2,\alpha}(\overline{\Omega}))_{\mathbb{C}}$,

$$\lim_{n\to\infty}\psi_{\lambda_n} = -\lim_{n\to\infty}(L_{\lambda_*} - I)^{-1}M_{\lambda_n}\psi_{\lambda_n} = -(L_{\lambda_*} - I)^{-1}(\psi^* - ia\psi^*) \text{ in } (C^{2,\alpha}(\overline{\Omega}))_{\mathbb{C}}.$$

This, together with $\lim_{n\to\infty}\psi_{\lambda_n} = \psi^*$ in $(C^{2,\alpha}(\overline{\Omega}))_{\mathbb{C}}$, gives $L_{\lambda_*}\psi^* - ia\psi^* = 0$. As $\|\psi_{\lambda_n}\|_{\mathbb{X}_{\mathbb{C}}} = \|\phi\|_{\mathbb{X}_{\mathbb{C}}}$ and $\psi^* \neq 0$, we see that ia is an eigenvalue of L_{λ_*}. This implies $a = 0$ and $\psi^* = c\phi$ with $|c| = 1$ based on 0 being the principal eigenvalue. Moreover, $c = 1$ due to (3.9). The claim is shown.

Step 3. We show that $\frac{\nu_\lambda}{|t_\lambda|}$ remains bounded for $\lambda \in \Lambda$ as $\lambda \to \lambda_*$. Without loss of generality, we present the proof for the case $\displaystyle\int_\Omega f'_u(x,0)\phi^2\phi^*\,\mathrm{d}x \neq 0$, as that for the case $\displaystyle\int_\Omega f'_u(x,0)\phi^2\phi^*\,\mathrm{d}x = 0$ follows from an analogous argument. Note that for $\displaystyle\int_\Omega f'_u(x,0)\phi^2\phi^*\,\mathrm{d}x \neq 0$, proving the boundedness of $\frac{\nu_\lambda}{|t_\lambda|}$ is equivalent to proving that of $\frac{\nu_\lambda}{|\lambda-\lambda_*|}$. Choose a sequence $\{\lambda_l\}_{l=1}^{\infty} \subset \Lambda$ with $\lim_{l\to\infty}\lambda_l = \lambda_*$. To analyze the asymptotic behavior, we define the auxiliary function $m_1(t_\lambda, \lambda)$ by

$$m_1(t_\lambda, \lambda) := \begin{cases} \dfrac{f(x, t_\lambda\phi + h(t_\lambda\phi, \lambda)) - m(x)}{\lambda - \lambda_*} & \text{if } \lambda \neq \lambda_*, \\ f'_u(x,0)\beta_{\lambda_*}\phi(x) & \text{if } \lambda = \lambda_*. \end{cases}$$

Based on $\langle\phi^*, L_{\lambda_*}\psi_{\lambda_l}\rangle = \langle L^*_{\lambda_*}\phi^*, \psi_{\lambda_l}\rangle = 0$, we substitute u_{λ_l} and $m_1(t_{\lambda_l}, \lambda_l)$ into (3.8), and then multiply the resultant by ϕ^* to get

$$\begin{aligned} \frac{i\nu_{\lambda_l}}{\lambda_l - \lambda_*}\int_\Omega \psi_{\lambda_l}\phi^*\,\mathrm{d}x = &\int_\Omega m(x)\psi_{\lambda_l}\phi^*\,\mathrm{d}x + \lambda_l\int_\Omega m_1(t_{\lambda_l}, \lambda_l)\psi_{\lambda_l}\phi^*\,\mathrm{d}x \\ &+ \lambda_l e^{-i\theta_{\lambda_l}}\int_\Omega f'_u(x, u_{\lambda_l})\left(\beta_{\lambda_l}\phi + \frac{h(t_{\lambda_l}\phi, \lambda_l)}{\lambda_l - \lambda_*}\right)\psi_{\lambda_l}\phi^*\,\mathrm{d}x. \end{aligned} \tag{3.14}$$

From Step 2 and Theorem 3.1, we have $\lim_{\lambda\to\lambda_*}\psi_\lambda = \phi$ in $(C^{2,\alpha}(\overline{\Omega}))_{\mathbb{C}}$

$$\text{and } \lim_{\lambda\to\lambda_*}\frac{h(t_\lambda\phi(x), \lambda)}{\lambda - \lambda_*} = 0 \text{ in } (C^{\alpha}(\overline{\Omega}))_{\mathbb{C}}.$$

Letting $l \to \infty$ in (3.14) yields

$$\lim_{l\to\infty} \sup \frac{\nu_{\lambda_l}}{|\lambda_l - \lambda_*|} \le \frac{3\int_\Omega m(x)\phi\phi^* \,\mathrm{d}x}{\int_\Omega \phi\phi^* \,\mathrm{d}x}.$$

This completes the proof. □

Lemma 3.4. *If $0 \in \sigma(\mathcal{A}_{\tau,\lambda})$ for all $(\tau,\lambda) \in \mathbb{R}_+ \times \Lambda$, then $\int_\Omega f'_u(x,0)\phi^2\phi^* \,\mathrm{d}x = 0$.*

Proof. By contradiction, assume $\int_\Omega f'_u(x,0)\phi^2\phi^* \,\mathrm{d}x \neq 0$. Choose a sequence $\{\lambda'_n\}$ with $\lim_{n\to\infty} \lambda'_n = \lambda_*$. If 0 is an eigenvalue of $\mathcal{A}_{\tau,\lambda}$ for all $\tau \ge 0$, then there exists $\mu = 0$ for (3.7). We substitute $\psi_{\lambda'_n} = r_{\lambda'_n}\phi + w_{\lambda'_n}$ into (3.7) and multiply the resultant by ϕ^* to have

$$\frac{1}{\lambda'_n - \lambda_*}\left\{\begin{array}{l}\displaystyle\int_\Omega \left[(\lambda'_n - \lambda_*)\phi^*(x)\left[m(x) + \lambda m_1(t_{\lambda'_n}, \lambda'_n)\right](r_{\lambda'_n}\phi(x) + w_{\lambda'_n})\right. \\ \left. +\lambda'_n t_{\lambda'_n}\phi(x) f'_u(x, u_{\lambda'_n})(r_{\lambda'_n}\phi(x) + w_{\lambda'_n})\phi^*(x)\right] \mathrm{d}x + o(\lambda'_n - \lambda_*)\end{array}\right\} = 0,$$

where $u_{\lambda'_n} = t_{\lambda'_n}\phi + h\left(t_{\lambda'_n}\phi, \lambda'_n\right)$.

With $n \to \infty$, we obviously have $r_{\lambda'_n} \to 1$, $\|w_{\lambda'_n}\|_{\mathbb{X}_\mathbb{C}} \to 0$, and $\frac{t_{\lambda'_n}}{\lambda'_n - \lambda_*} \to \beta_{\lambda_*}$. Then

$$\int_\Omega m(x)\phi\phi^* \,\mathrm{d}x + 2\lambda_*\beta_{\lambda_*}\int_\Omega f'_u(x,0)\phi^2\phi^* \,\mathrm{d}x = 0.$$

We substitute β_{λ_*} given by (3.4) into the above equality to get

$$\begin{aligned}&\int_\Omega m(x)\phi\phi^* \,\mathrm{d}x + 2\lambda_*\left(\frac{\int_\Omega m\phi\phi^* \,\mathrm{d}x}{-\lambda_*\int_\Omega f'_u(x,0)\phi^2\phi^* \,\mathrm{d}x}\right)\int_\Omega f'_u(x,0)\phi^2\phi^* \,\mathrm{d}x \\ &= \int_\Omega m(x)\phi\phi^* \,\mathrm{d}x - 2\int_\Omega m(x)\phi\phi^* \,\mathrm{d}x = -\int_\Omega m(x)\phi\phi^* \,\mathrm{d}x = 0,\end{aligned}$$

which contradicts Lemma 2.2. Thus, $\int_\Omega f'_u(x,0)\phi^2\phi^* \,\mathrm{d}x = 0$. □

Next, we intend to use the implicit function theorem to prove that (3.8) is solvable.

Since our primary interest lies in the occurrence of Hopf bifurcation, we must ensure that 0 is not an eigenvalue of the infinitesimal generator $\mathcal{A}_{\tau,\lambda}$. In view of Lemma 3.4, we assume $\int_\Omega f'_u(x,0)\phi^2\phi^* \,\mathrm{d}x \neq 0$ throughout the following analysis. For the boundedness of $\frac{\nu}{|\lambda-\lambda_*|}$, we set $\nu = |\lambda - \lambda_*|\hat{h}$. Noting $u_\lambda(x) = t_\lambda\phi(x) + h(t_\lambda\phi(x), \lambda)$, we focus on (3.8) with $\nu > 0$, $\theta \in [0, 2\pi)$, and $\psi \in \mathbb{X}_\mathbb{C}\backslash\{0\}$ satisfying (3.9) to get

$$\begin{aligned}[L_{\lambda_*} + (\lambda - \lambda_*)m(x) + \lambda(f(x,u_\lambda) - m(x)) - i|\lambda - \lambda_*|\hat{h}](r\phi(x) + w(x)) + \\ \lambda(t_\lambda\phi(x) + h(t_\lambda\phi(x), \lambda))f'_u(x,u_\lambda)(r\phi(x) + w(x))e^{-i\theta} = 0.\end{aligned} \tag{3.15}$$

We multiply (3.15) by ϕ^* to have

$$\int_\Omega \phi^*(x)[L_{\lambda_*} + (\lambda-\lambda_*)m(x) + \lambda(f(x,u_\lambda) - m(x)) - i|\lambda-\lambda_*|\hat{h}](r\phi(x)+w(x))+ \\ \lambda(t_\lambda\phi(x) + h(t_\lambda\phi(x),\lambda))f'_u(x,u_\lambda)(r\phi(x)+w(x))e^{-i\theta}\phi^*(x)\mathrm{d}x = 0.$$

Recalling $\langle\phi^*, L_{\lambda_*}(r\phi+w)\rangle = 0$ and dividing both sides of the above integral equation by $\lambda-\lambda_*$, for $\lambda\in\Lambda_{2\epsilon}$ with $\lambda\to\lambda_*$, we obtain

$$\int_\Omega \phi^*(x)[m(x) + \lambda f'_u(x,0)\beta_\lambda\phi(x) - i\hat{h}]\phi(x) + \lambda\beta_\lambda f'_u(x,0)\phi^2(x)e^{-i\theta}\phi^*(x)\mathrm{d}x = 0.$$

Separating the real and imaginary parts of the above equation, we get

$$\begin{cases} -\lambda_*\beta_{\lambda_*}\sin\theta\displaystyle\int_\Omega f'_u(x,0)\,\phi^2\phi^*\,\mathrm{d}x = \hat{h}\int_\Omega \phi\phi^*\,\mathrm{d}x, \\ \lambda_*\beta_{\lambda_*}\cos\theta\displaystyle\int_\Omega f'_u(x,0)\,\phi^2\phi^*\,\mathrm{d}x = 0. \end{cases}$$

This, combined with (3.4), implies

$$\theta = \frac{\pi}{2} := \theta_2, \hat{h} = \hat{h}_{\lambda_*} = \frac{\int_\Omega m(x)\phi\phi^*\,\mathrm{d}x}{\int_\Omega \phi\phi^*\,\mathrm{d}x}. \tag{3.16}$$

Similarly, if $\lambda\in\Lambda_{1\epsilon}$ with $\lambda\to\lambda_*$, we get

$$\theta = \frac{3\pi}{2} := \theta_1, \hat{h} = \hat{h}_{\lambda_*} = \frac{\int_\Omega m(x)\phi\phi^*\,\mathrm{d}x}{\int_\Omega \phi\phi^*\,\mathrm{d}x}. \tag{3.17}$$

Without loss of generality, we only consider the case of $\lambda\in\Lambda_{2\epsilon}$ as the other case $\lambda\in\Lambda_{1\epsilon}$ can be dealt with similarly.

Lemma 3.5. *Assume that* $\displaystyle\int_\Omega f'_u(x,0)\phi^2\phi^*\,\mathrm{d}x \neq 0$. *Then there exists a continuously differentiable unique mapping* $\lambda\mapsto(w_\lambda, r_\lambda, \hat{h}_\lambda, \theta_\lambda)$ *from* $\Lambda_{2\epsilon}$ *to* $(X_1)_{\mathbb{C}}\times\mathbb{R}^3$ *such that* $G(w_\lambda, r_\lambda, \hat{h}_\lambda, \theta_\lambda, \lambda) = 0$. *Moreover, if* $\lambda\in\Lambda_{2\epsilon}$ *and* $(w^\lambda, r^\lambda, \hat{h}^\lambda, \theta^\lambda)$ *solve the equation*

$$\begin{cases} G(w, r, \hat{h}, \theta, \lambda) = 0, \\ w\in(X_1)_{\mathbb{C}}, \hat{h} > 0, r\geq 0, \theta\in[0, 2\pi), \end{cases} \tag{3.18}$$

then $(w^\lambda, r^\lambda, \hat{h}^\lambda, \theta^\lambda) = (w_\lambda, r_\lambda, \hat{h}_\lambda, \theta_\lambda)$. *Here* $G(w, r, \hat{h}, \theta, \lambda) : (X_1)_{\mathbb{C}}\times[0,\infty)\times(0,\infty)\times[0,2\pi)\times\mathbb{R}\to(Y_1)_{\mathbb{C}}\times\mathbb{C}\times\mathbb{R}$ *is defined by*

$$G(w, r, \hat{h}, \theta, \lambda) = (g_1, g_2, g_3)^T,$$

where

$$\begin{cases} g_1(w,r,\hat{h},\theta,\lambda) & := L_{\lambda_*}w + (\lambda-\lambda_*)[m(x)+\lambda m_1(t_\lambda,\lambda) - i\hat{h}](r\phi(x)+w) \\ & \quad +\lambda(t_\lambda\phi(x)+h(t_\lambda\phi(x),\lambda))f_u'(x,u_\lambda)(r\phi(x)+w)e^{-i\theta} \\ & \quad -(\lambda-\lambda_*)g_2(w,r,\hat{h},\theta,\lambda)\phi(x), \\ g_2(w,r,\hat{h},\theta,\lambda) & := \int_\Omega \phi^*(x)[m(x)+\lambda m_1(t_\lambda,\lambda) - i\hat{h}](r\phi(x)+w)\mathrm{d}x \\ & \quad +\lambda\beta_\lambda\int_\Omega \phi^*(x)f_u'(x,u_\lambda)\phi(x)(r\phi(x)+w)e^{-i\theta}\mathrm{d}x, \\ g_3(w,r) & := \|r\phi+w\|^2_{\mathbb{X}_\mathbb{C}} - \|\phi\|^2_{\mathbb{X}_\mathbb{C}}. \end{cases}$$

Proof. Obviously, $G(w_{\lambda_*}, r_{\lambda_*}, \hat{h}_{\lambda_*}, \theta_{\lambda_*}, \lambda_*) = G(0,1,\hat{h}_{\lambda_*},\pi/2,\lambda_*) = 0$. We define $T = (T_1,T_2,T_3)^T$ as the Fréchet derivative of G with respect to $(w,r,\hat{h},\theta)$ at $(0,1,\hat{h}_{\lambda_*},\pi/2,\lambda_*)$. A direct computation shows that $T : (X_1)_\mathbb{C}\times\mathbb{R}^3 \to (Y_1)_\mathbb{C}\times\mathbb{C}\times\mathbb{R}$ is given by

$$\begin{cases} T_1(\chi,\kappa,\zeta,\vartheta) & = L_{\lambda_*}\chi, \\ T_2(\chi,\kappa,\zeta,\vartheta) & = \displaystyle\int_\Omega \phi^*(x)[m(x)+\lambda_*\beta_{\lambda_*}f_u'(x,0)\,\phi(x) - i\lambda_*\beta_{\lambda_*}f_u'(x,0)\,\phi(x) \\ & \quad -i\hat{h}_{\lambda_*}](\kappa\phi(x)+\chi(x))\,\mathrm{d}x - i\zeta\int_\Omega \phi(x)\phi^*(x)\,\mathrm{d}x \\ & \quad -\lambda_*\beta_{\lambda_*}\vartheta\int_\Omega f_u'(x,0)\,\phi^2(x)\phi^*(x)\,\mathrm{d}x, \\ T_3(\chi,\kappa,\zeta,\vartheta) & = \langle\phi,\chi\rangle_{\mathbb{X}_\mathbb{C}} + \langle\chi,\phi\rangle_{\mathbb{X}_\mathbb{C}} + 2\kappa\|\phi\|^2_{\mathbb{X}_\mathbb{C}}. \end{cases}$$

Since T is a bijection from $(X_1)_\mathbb{C}\times\mathbb{R}^3$ to $(Y_1)_\mathbb{C}\times\mathbb{C}\times\mathbb{R}$, the implicit function theorem guarantees the existence of a continuously differentiable map $\lambda\mapsto(w_\lambda,r_\lambda,\hat{h}_\lambda,\theta_\lambda)$ from $\Lambda_{2\epsilon}$ to $(X_1)_\mathbb{C}\times\mathbb{R}^3$ such that $(w_\lambda,r_\lambda,\hat{h}_\lambda,\theta_\lambda)$ solves (3.18).

To establish uniqueness, assume that $(w^\lambda,r^\lambda,\hat{h}^\lambda,\theta^\lambda)$ is another solution of (3.18). We show that

$$(w^\lambda,r^\lambda,\hat{h}^\lambda,\theta^\lambda)\to(w_{\lambda_*},r_{\lambda_*},\hat{h}_{\lambda_*},\theta_{\lambda_*}),$$

as $\lambda\to\lambda_*$ in $\mathbb{X}_\mathbb{C}\times\mathbb{R}^3$. Note that $\lim_{\lambda\to\lambda_*}r^\lambda = r_{\lambda_*} = 1$ and $\lim_{\lambda\to\lambda_*}w^\lambda = w_{\lambda_*} = 0$ in $(C^{2,\alpha}(\overline{\Omega}))_\mathbb{C}$. Furthermore, because the sequences $\{\hat{h}^\lambda\}$ and $\{\theta^\lambda\}$ are bounded, for any sequence $\{\lambda_n\}$ with $\lambda_n\to\lambda_*$, there exists a subsequence (still denoted as $\{\lambda_n\}$) such that

$$\lim_{n\to\infty}\hat{h}^{\lambda_n} = \hat{h}^{\lambda_*} \text{ and } \lim_{n\to\infty}\theta^{\lambda_n} = \theta^{\lambda_*}.$$

Taking the limit of $G(w^{\lambda_n},r^{\lambda_n},\hat{h}^{\lambda_n},\theta^{\lambda_n},\lambda_n) = 0$ with $\lambda_n\to\lambda_*$, we get

$$G(w_{\lambda_*},r_{\lambda_*},\hat{h}^{\lambda_*},\theta^{\lambda_*},\lambda_*) = 0.$$

This, combined with (3.16), implies

$$(\hat{h}^{\lambda_*},\theta^{\lambda_*}) = (\hat{h}_{\lambda_*},\theta_{\lambda_*}).$$

We complete the proof. □

For $\nu > 0$, $\theta \in [0, 2\pi)$, and $\psi \in \mathbb{X}_{\mathbb{C}}$ satisfying (3.9), (ν, θ, ψ) solves (3.8) if and only if (3.18) admits a solution $(w, r, \hat{h}, \theta)$. From Lemmas 3.4 and 3.5 we derive the following result.

Theorem 3.6. *Assume that* $\int_\Omega f_u'(x,0)\phi^2\phi^* \,\mathrm{d}x \neq 0$. *Then there exists* $\epsilon > 0$ *such that, for each* $\lambda \in \Lambda$, *system* (1.2) *has exactly one spatially nonhomogeneous steady-state solution. Its associated infinitesimal generator* $\mathcal{A}_{\tau,\lambda}$ *has a pair of purely imaginary eigenvalues* $\pm i\nu_\lambda$ *if and only if*

$$\nu = \nu_\lambda = |\lambda - \lambda_*|\hat{h}_\lambda, \tau = \tau_{\lambda,n} = \frac{\theta_\lambda + 2n\pi}{\nu_\lambda}, n \in \mathbb{N}_0 = \{0, 1, 2, 3, \dots\}, \tag{3.19}$$

and a corresponding eigenfunction is $\psi = c\psi_\lambda$. *Here* ψ_λ *is given by* (3.9), *and* $\hat{h}_\lambda$, θ_λ *are defined in Lemma 3.5 such that*

$$\lim_{\lambda\to\lambda_*} \theta_\lambda = \begin{cases} \pi/2 & \text{if } \lambda \in \Lambda_{2\epsilon}, \\ 3\pi/2 & \text{if } \lambda \in \Lambda_{1\epsilon}. \end{cases}$$

Remark: From the expression of $\tau_{\lambda,n}$ in Theorem 3.6, we observe that $\nu_\lambda = O(|\lambda - \lambda_*|)$ as $\lambda \to \lambda_*$. Consequently, the bifurcation values of the delay satisfy the following condition

$$\lim_{\lambda\to\lambda_*} \tau_{\lambda,0} = \infty. \tag{3.20}$$

This result highlights that the Hopf bifurcation near λ_* is characteristic of large delay dynamics, which is consistent with the findings in [1, 7].

To determine the transversality condition and the direction of the bifurcation, we consider the adjoint operator of $\Delta(\lambda, i\nu_\lambda, \tau_{\lambda,n})$. Using a similar argument as above, we have the following result.

Theorem 3.7. *Let* $\Delta^*(\lambda, i\nu_\lambda, \tau_{\lambda,n})$ *be the adjoint operator of* $\Delta(\lambda, i\nu_\lambda, \tau_{\lambda,n})$. *Then*

$$\begin{aligned} &\Delta^*(\lambda, \mu, \tau)\psi^* := \\ &\nabla \cdot [d(x)\nabla\psi^*] + \vec{b}(x) \cdot \nabla\psi^* + \lambda f\left(x, u_\lambda\right)\psi^* + \lambda u_\lambda f_u'\left(x, u_\lambda\right)\psi^* e^{\mu\tau} + \mu\psi^* \end{aligned}$$

and the kernel space $\mathscr{N}\left(\Delta^*(\lambda, i\nu_\lambda, \tau_{\lambda,n})\right) = \operatorname{span}\{\psi_\lambda^*\}$. *Moreover, up to a scalar factor,* ψ_λ^* *can be represented as*

$$\begin{cases} \psi_\lambda^* = \widetilde{r}_\lambda\phi^* + \widetilde{w}_\lambda, \quad \widetilde{r}_\lambda \geq 0,\ \widetilde{w}_\lambda \in (\widetilde{X}_1)_{\mathbb{C}}, \\ \|\psi_\lambda^*\|_{\mathbb{X}_{\mathbb{C}}} = \|\phi^*\|_{\mathbb{X}_{\mathbb{C}}}, \end{cases} \tag{3.21}$$

and it satisfies $\lim_{\lambda\to\lambda_*}\psi_\lambda^*=\phi^*$ *in* $(C^{2,\alpha}(\overline{\Omega}))_{\mathbb{C}}$, *where* ϕ^*, α *are defined in Lemma 2.1 and* $(\mathbf{H}_2)$, *respectively.*

Now, we give some estimates to support the simplicity of $i\nu_\lambda$.

Lemma 3.8. *Assume that* $\int_\Omega f_u'(x,0)\phi^2\phi^*\,\mathrm{d}x\neq 0$. *For* $n\in\mathbb{N}_0$, *we define*

$$S_n(\lambda):=\int_\Omega \psi_\lambda\overline{\psi_\lambda^*}dx+\lambda\tau_{\lambda,n}e^{-i\theta_\lambda}\int_\Omega f_u'\left(x,u_\lambda\right)u_\lambda\psi_\lambda\overline{\psi_\lambda^*}dx,$$

where ψ_λ *and* ψ_λ^* *are depicted in* (3.9) *and* (3.21), *respectively. Then* $S_n(\lambda)\neq 0$ *for* $\lambda\in\Lambda$.

Proof. We only consider $\lambda\in\Lambda_{2\epsilon}$ (for $\lambda\in\Lambda_{1\epsilon}$, the proof is similar). It follows from Lemma 3.5 and Theorem 3.6 that $\theta_\lambda\to\pi/2$, $\tau_{\lambda,n}(\lambda-\lambda_*)\to(\frac{\pi}{2}+2n\pi)/h_{\lambda_*}$, and $\psi_\lambda\to\phi$ in $\mathbb{X}_{\mathbb{C}}$ as $\lambda\to\lambda_*$. These, combined with (3.4), produce

$$\begin{aligned}&\lim_{\lambda\to\lambda_*}S_n(\lambda)\\&=\int_\Omega\phi\phi^*\,\mathrm{d}x-\frac{i\beta_{\lambda_*}\lambda_*}{\hat{h}_{\lambda_*}}\left(\frac{\pi}{2}+2n\pi\right)\int_\Omega f_u'(x,0)\phi^2\phi^*\,\mathrm{d}x\\&=\left[1+i(\frac{\pi}{2}+2n\pi)\right]\int_\Omega\phi\phi^*\,\mathrm{d}x\neq 0.\end{aligned}\tag{3.22}$$

This completes the proof. □

Now we show that the imaginary eigenvalue iν is simple in the following theorem.

Theorem 3.9. *Assume that* $\lambda\in\Lambda$. *Then* $\mu=i\nu_\lambda$ *is a simple eigenvalue of* $\mathcal{A}_{\tau_{\lambda,n},\lambda}$ *for* $n\in\mathbb{N}_0$.

Proof. It follows from Theorem 3.6 that $\mathscr{N}[\mathcal{A}_{\tau_{\lambda,n},\lambda}-i\nu_\lambda]=\mathrm{Span}[e^{i\nu_\lambda\widetilde{\theta}}\psi_\lambda]$, where $\widetilde{\theta}\in[-\tau_{\lambda,n},0]$ and ψ_λ is defined in Theorem 3.6. If $\phi_1\in[\mathcal{A}_{\tau_{\lambda,n},\lambda}-i\nu_\lambda]^2$, then

$$[\mathcal{A}_{\tau_{\lambda,n},\lambda}-i\nu_\lambda]\phi_1\in\mathscr{N}[\mathcal{A}_{\tau_{\lambda,n},\lambda}-i\nu_\lambda]=\mathrm{Span}[e^{i\nu_\lambda\widetilde{\theta}}\psi_\lambda].$$

Therefore, there exists a constant a such that

$$[\mathcal{A}_{\tau_{\lambda,n},\lambda}-i\nu_\lambda]\phi_1=ae^{i\nu_\lambda\widetilde{\theta}}\psi_\lambda,$$

which, combined with (3.6), yields

$$\dot{\phi}_1(\widetilde{\theta})=i\nu_\lambda\phi_1(\widetilde{\theta})+ae^{i\nu_\lambda\widetilde{\theta}}\psi_\lambda,\ \widetilde{\theta}\in[-\tau_{\lambda,n},0]\tag{3.23}$$

with

$$\dot{\phi_1}(0) = \nabla \cdot \left[d(x)\nabla\phi_1(0) - \vec{b}(x)\phi_1(0) \right] + \lambda f\left(x, u_\lambda\right)\phi_1(0) + \lambda f'_u\left(x, u_\lambda\right) u_\lambda \phi_1(-\tau_{\lambda,n}). \tag{3.24}$$

Solving the differential equation (3.23), we have

$$\phi_1(\widetilde{\theta}) = \phi_1(0)e^{i\nu_\lambda \widetilde{\theta}} + a\widetilde{\theta}e^{i\nu_\lambda \widetilde{\theta}}\psi_\lambda. \tag{3.25}$$

Due to $\dot{\phi_1}(0) = i\nu_\lambda\phi_1(0) + a\psi_\lambda$, we exploit (3.23)-(3.25) to get

$$\begin{aligned} \Delta\left(\lambda, i\nu_\lambda, \tau_{\lambda,n}\right)\phi_1(0) =& \nabla \cdot \left[d(x)\nabla\phi_1(0) - \vec{b}(x)\phi_1(0) \right] + \lambda f\left(x, u_\lambda\right)\phi_1(0) \\ &+ \lambda f'_u\left(x, u_\lambda\right) u_\lambda \phi_1(0)e^{-i\nu_\lambda\tau_{\lambda,n}} - i\nu_\lambda\phi_1(0) \\ =& a\psi_\lambda + a\lambda\tau_{\lambda,n} f'_u\left(x, u_\lambda\right) u_\lambda \psi_\lambda e^{-i\nu_\lambda\tau_{\lambda,n}}. \end{aligned} \tag{3.26}$$

Recalling $e^{-i\nu_\lambda\tau_{\lambda,n}} = e^{-i\theta_\lambda}$ from Theorem 3.6 and $\Delta^*\left(\lambda, i\nu_\lambda, \tau_{\lambda,n}\right)\psi^*_\lambda = 0$, we obviously have

$$\begin{aligned} 0 &= \langle \Delta^*\left(\lambda, i\nu_\lambda, \tau_{\lambda,n}\right)\psi^*_\lambda, \phi_1(0)\rangle = \langle \psi^*_\lambda, \Delta\left(\lambda, i\nu_\lambda, \tau_{\lambda,n}\right)\phi_1(0)\rangle \\ &= aS_n(\lambda), \end{aligned}$$

which implies $a = 0$ because of $S_n(\lambda) \neq 0$ from Lemma 3.8. Therefore,

$$\mathscr{N}[\mathcal{A}_{\tau_{\lambda,n},\lambda} - i\nu_\lambda]^j = \mathscr{N}[\mathcal{A}_{\tau_{\lambda,n},\lambda} - i\nu_\lambda], \quad j = 2, 3, \ldots, \quad n \in \mathbb{N}_0,$$

and hence $\mu = i\nu_\lambda$ is a simple eigenvalue of $\mathcal{A}_{\tau_{\lambda,n},\lambda}$ for $n \in \mathbb{N}_0$. □

3.3 Stability and Hopf Bifurcation

In what follows, we only consider the case $\displaystyle\int_\Omega f'_u(x,0)\phi^2\phi^* \,\mathrm{d}x \neq 0$. We first analyze the stability of the steady-state solution u_λ of (1.2) for $\tau = 0$ and have the following result.

Proposition 3.10. *If $\displaystyle\int_\Omega f'_u(x,0)\phi^2\phi^* \,\mathrm{d}x \neq 0$, then there exists $\epsilon > 0$ such that for each $\lambda \in \Lambda_{2\epsilon}$, all the eigenvalues of $\mathcal{A}_{0,\lambda}$ have negative real parts. Consequently, for $\tau = 0$ and $\lambda \in \Lambda_{2\epsilon}$, the steady-state solution u_λ of (1.2) is locally asymptotically stable.*

Proof. Let $\{\lambda'_n\}_{n=1}^\infty \subseteq \Lambda_{2\epsilon}$ be any sequence with $\lim_{n\to\infty}\lambda'_n = \lambda_*$. For each λ'_n, denote $\mu_{\lambda'_n}$ as the principal eigenvalue of the following problem,

$$\mu\psi = \nabla \cdot [d(x)\nabla\psi - \vec{b}(x)\psi] + \lambda'_n f\left(x, u_{\lambda'_n}\right)\psi + \lambda'_n u_{\lambda'_n} f'_u\left(x, u_{\lambda'_n}\right)\psi, \tag{3.27}$$

with $\|\psi_{\lambda_n'}\|_{\mathbb{Y}_{\mathbb{C}}} = 1$. For each $n \geq 1$, $\psi_{\lambda_n'} = r_{\lambda_n'}\phi + w_{\lambda_n'}$ with $r_{\lambda_n'} \to 1$, $\left\|w_{\lambda_n'}\right\|_{\mathbb{Y}_{\mathbb{C}}} \to 0$ as $n \to \infty$. we substitute $u_{\lambda_n'} = t_{\lambda_n'}\phi + h\left(t_{\lambda_n'}\phi, \lambda_n'\right)$, $\psi_{\lambda_n'} = r_{\lambda_n'}\phi + w_{\lambda_n'}$, and $\mu = \mu_{\lambda_n'}$ into (3.27) and then multiply the resultant by $\phi^*(x)$ to have

$$\frac{1}{\lambda_n' - \lambda_*}\left\{\begin{array}{l}\displaystyle\int_\Omega\left[(\lambda_n' - \lambda_*)\phi^*(x)\left[m(x) + \lambda_n' m_1(t_{\lambda_n'}, \lambda_n') - \frac{\mu_{\lambda_n'}}{\lambda_n' - \lambda_*}\right](r_{\lambda_n'}\phi(x) + w_{\lambda_n'})\right. \\ \left. + \lambda_n' t_{\lambda_n'}\phi(x) f_u'(x, u_{\lambda_n'})(r_{\lambda_n'}\phi(x) + w_{\lambda_n'})\phi^*(x)\right]\mathrm{d}x + o(\lambda_n' - \lambda_*)\end{array}\right\} = 0.$$

Obviously,

$$\lim_{n\to\infty}\frac{\mu_{\lambda_n'}}{\lambda_* - \lambda_n'} = \hat{h}_{\lambda_*} > 0.$$

Then for $\lambda_* - \lambda_n' < 0$, we have $\mathrm{Re}\left(\mu_{\lambda_n'}\right) < 0$ for sufficiently large n. Thus, there exists $\epsilon > 0$ to guarantee that all the eigenvalues of $\mathcal{A}_{0,\lambda}$ have negative real parts for $\lambda \in \Lambda_{2\epsilon}$. □

We can use a similar proof to that of Proposition 3.10 to obtain the following result.

Proposition 3.11. *If* $\int_\Omega f_u'(x,0)\phi^2\phi^*\,\mathrm{d}x \neq 0$, *then there exists* $\epsilon > 0$ *such that, for each* $\lambda \in \Lambda_{1\epsilon}$, *the infinitesimal generator* $\mathcal{A}_{0,\lambda}$ *has at least one eigenvalue with a positive real part. Hence, for* $\tau = 0$ *and* $\lambda \in \Lambda_{1\epsilon}$, *the steady-state solution* u_λ *of* (1.2) *is unstable.*

Next, we establish the following transversality condition to support the existence of Hopf bifurcations.

Theorem 3.12. *Assume* $\int_\Omega f_u'(x,0)\phi^2\phi^*\,\mathrm{d}x \neq 0$ *and* $\lambda \in \Lambda$. *Then, for each fixed* $n \in \mathbb{N}_0$, *there exists a unique continuously differentiable mapping* $\tau \mapsto (\mu(\tau), \psi(\tau))$ *from a neighborhood* O_n *of* $\tau_{\lambda,n}$ *into* $D_n \times H_n$ *with* $\mu(\tau_{\lambda,n}) = i\nu_\lambda$ *and* $\psi(\tau_{\lambda,n}) = \psi_\lambda$. *Moreover, for* $\tau \in O_n$, $\mu(\tau)$ *is the unique eigenvalue of* $\mathcal{A}_{\tau,\lambda}$ *in* D_n *with* $\Delta(\lambda, \mu(\tau), \tau)\psi(\tau) = 0$. *Surely*

$$Re(\frac{\mathrm{d}\mu(\tau_{\lambda,n})}{\mathrm{d}\tau}) > 0, n \in \mathbb{N}_0.$$

Proof. For $\mu = i\nu_\lambda$ being a simple eigenvalue of $\mathcal{A}_{\tau,\lambda}$, the implicit function theorem ensures that the eigenvalue branch $\mu(\tau)$ can be uniquely extended in a neighborhood of $\tau_{\lambda,n}$. Differentiating $\Delta(\lambda, \mu(\tau), \tau)\psi(\tau) = 0$ with respect to τ and evaluating at $\tau = \tau_{\lambda,n}$ yield

$$\begin{aligned}&\frac{d\mu(\tau_{\lambda,n})}{d\tau}\left[-\psi_\lambda - \lambda\tau_{\lambda,n} f_u'\left(x, u_\lambda\right) u_\lambda \psi_\lambda e^{-i\theta_\lambda}\right] \\ &+ \Delta(\lambda, i\nu_\lambda, \tau_{\lambda,n})\frac{d\psi(\tau_{\lambda,n})}{d\tau} - i\nu_\lambda\lambda f_u'\left(x, u_\lambda\right) u_\lambda \psi_\lambda e^{-i\theta_\lambda} = 0.\end{aligned} \tag{3.28}$$

Note that

$$\left\langle \psi_\lambda^*, \Delta(\lambda, i\nu_\lambda, \tau_{\lambda,n}) \frac{d\psi(\tau_{\lambda,n})}{d\tau} \right\rangle = \left\langle \Delta^*(\lambda, i\nu_\lambda, \tau_{\lambda,n}) \psi_\lambda^*, \frac{d\psi(\tau_{\lambda,n})}{d\tau} \right\rangle = 0. \tag{3.29}$$

Then, exploiting ψ_λ^* to make inner product with (3.28), we get

$$\begin{aligned}
\frac{\mathrm{d}\mu(\tau_{\lambda,n})}{\mathrm{d}\tau} &= -\frac{\left\langle \psi_\lambda^*, i\nu_\lambda \lambda f_u'(x, u_\lambda)\, u_\lambda \psi_\lambda e^{-i\theta_\lambda} \right\rangle}{\left\langle \psi_\lambda^*, \psi_\lambda + \lambda \tau_{\lambda,n} f_u'(x, u_\lambda)\, u_\lambda \psi_\lambda e^{-i\theta_\lambda} \right\rangle} \\
&= -\frac{i\nu_\lambda \lambda e^{-i\theta_\lambda} \int_\Omega f_u'(x, u_\lambda)\, u_\lambda \psi_\lambda(x) \overline{\psi_\lambda^*}(x) \mathrm{d}x}{\int_\Omega \overline{\psi_\lambda^*}(x) \psi_\lambda(x) \mathrm{d}x + \lambda \tau_{n,\lambda} e^{-i\theta_\lambda} \int_\Omega f_u'(x, u_\lambda)\, u_\lambda \psi_\lambda(x) \overline{\psi_\lambda^*}(x) \mathrm{d}x} \\
&= -\frac{1}{|S_n(\lambda)|^2} \left[i\nu_\lambda \lambda e^{-i\theta_\lambda} \iint_{\Omega\times\Omega} f_u'(x, u_\lambda)\, u_\lambda \psi_\lambda(x) \overline{\psi_\lambda^*}(x) \psi_\lambda^*(y) \overline{\psi}_\lambda(y) \mathrm{d}x \mathrm{d}y \right. \\
&\quad \left. + i\nu_\lambda \lambda^2 \tau_{n,\lambda} \left| \int_\Omega f_u'(x, u_\lambda)\, u_\lambda \psi_\lambda(x) \overline{\psi_\lambda^*}(x) \mathrm{d}x \right|^2 \right].
\end{aligned}$$

This, combined with (3.16) and the expression of u_λ, θ_λ, ν_λ, and ψ_λ, induces

$$\lim_{\lambda\to\lambda_*} \frac{1}{(\lambda-\lambda_*)^2} \mathrm{Re}\left(\frac{d\mu(\tau_{\lambda,n})}{d\tau}\right) = \frac{\hat{h}_{\lambda_*}^2}{\lim_{\lambda\to\lambda_*} |S_n(\lambda)|^2} \left(\int_\Omega \phi\phi^* \,\mathrm{d}x \right)^2 > 0.$$

□

Based on Theorems 3.6, 3.9, 3.12, and Propositions 3.10, 3.11, we have the following result on the distribution of eigenvalues of $\mathcal{A}_{\tau,\lambda}$.

Theorem 3.13. *If $\int_\Omega f_u'(x,0)\phi^2\phi^* \,\mathrm{d}x \neq 0$, then there exists a constant $\epsilon > 0$ such that the following two statements hold.*

(i) For each fixed $\lambda \in \Lambda_{1\epsilon}$, the infinitesimal generator $\mathcal{A}_{\tau,\lambda}$ has at least an eigenvalue with positive real part for all $\tau \geq 0$.

(ii) For each fixed $\lambda \in \Lambda_{2\epsilon}$, the infinitesimal generator $\mathcal{A}_{\tau,\lambda}$ only has eigenvalues with negative real parts when $\tau \in [0, \tau_{\lambda,0})$, and exactly $2(n+1)$ eigenvalues with positive real parts when $\tau \in (\tau_{\lambda,n}, \tau_{\lambda,n+1}]$, $n \in \mathbb{N}_0$.

In summary, we have obtained the following results on the stability and Hopf bifurcation of the steady-state solution u_λ.

Theorem 3.14. *If $\int_\Omega f_u'(x,0)\phi^2\phi^* \,\mathrm{d}x \neq 0$, then there exists a constant $\epsilon > 0$ such that*

(i) for $\lambda \in \Lambda_{1\epsilon}$, the steady-state solution u_λ of (1.2) is locally unstable for all $\tau \geq 0$;

(ii) for $\lambda \in \Lambda_{2\epsilon}$, the steady-state solution u_λ of (1.2) is locally asymptotically stable for $\tau \in [0, \tau_{\lambda,0})$. Moreover, for $\tau = \tau_{\lambda,n}$ ($n \in \mathbb{N}_0$), (1.2) undergoes a Hopf bifurcation at the positive steady state u_λ.

4 The direction of Hopf bifurcation

In this section, by virtue of the center manifold theorem and the normal form theory, we investigate the direction of Hopf bifurcation and analyze the stability of bifurcating periodic solutions for (1.2). Although the computational framework is similar to those in [4,16–18,20], the non-self-adjointness of the elliptic operator along with the generality of the nonlinear terms brings significant mathematical difficulties in computing the center manifold. We demonstrate how to overcome these difficulties in the subsequent analysis.

We first transform the steady-state solution u_λ and the bifurcation parameter $\tau_{\lambda,n}$ to the origin via the translations $U(t) = u(\cdot,t) - u_\lambda$ and $\tau = \tau_{\lambda,n} + \gamma$. Thus, $\gamma = 0$ corresponds to the Hopf bifurcation values. Rescaling time by $t = \tau_{\lambda,n}\tilde{t}$ to normalize the delay and dropping the tilde sign, we can rewrite system (1.2) as

$$\frac{dU(t)}{dt} = \tau_{\lambda,n} L_0(U_t) + J(U_t, \gamma), \tag{4.1}$$

where $U_t \in \mathcal{C} := C([-1,0], \mathbb{Y})$ is defined by $U_t(\varsigma) = U(t+\varsigma)$ for $\varsigma \in [-1,0]$. The linear operator $L_0 : \mathcal{C} \to \mathbb{Y}$ and the nonlinear term $J : \mathcal{C} \times \mathbb{R} \to \mathbb{Y}$ are defined respectively by

$$L_0(U_t) = \nabla \cdot [d(x)\nabla U_t(0) - \vec{b}(x)U_t(0)] + \lambda f(x, u_\lambda)U_t(0) + \lambda u_\lambda f'_u(x, u_\lambda)U_t(-1), \tag{4.2}$$

$$\begin{aligned} J(U_t, \gamma) = \gamma L_0(U_t) + \lambda(\gamma + \tau_{\lambda,n})\Big[&(U_t(0) + u_\lambda) f(x, U_t(-1) + u_\lambda) \\ &- u_\lambda f(x, u_\lambda) - f(x, u_\lambda)U_t(0) - u_\lambda f'_u(x, u_\lambda)U_t(-1)\Big]. \end{aligned} \tag{4.3}$$

Then the solution semigroup of the linearized equation $\frac{dU(t)}{dt} = \tau_{\lambda,n} L_0(U_t)$ has the infinitesimal generator $\mathcal{A}_{\tau_{\lambda,n}}$ satisfying $\mathcal{A}_{\tau_{\lambda,n}}\Psi = \dot{\Psi}$ with the domain

$$\mathscr{D}(\mathcal{A}_{\tau_{\lambda,n}}) = \left\{ \Psi \in \mathcal{C}_{\mathbb{C}} \cap \mathcal{C}^1_{\mathbb{C}} : \Psi(0) \in \mathbb{X}_{\mathbb{C}}, \dot{\Psi}(0) = \tau_{\lambda,n} L_0(\Psi) \right\},$$

where $\mathcal{C}^1_{\mathbb{C}} := C^1([-1,0], \mathbb{Y}_{\mathbb{C}})$. Hence, we rewrite (4.1) in the following abstract form,

$$\frac{dU_t}{dt} = \mathcal{A}_{\tau_{\lambda,n}} U_t + X_0 J(U_t, \gamma),$$

where

$$X_0(\varsigma) = \begin{cases} 0 & \varsigma \in [-1, 0), \\ I & \varsigma = 0. \end{cases}$$

According to Theorem 3.14, after time rescaling, the infinitesimal generator $\mathcal{A}_{\tau_{\lambda,n}}$ has exactly one pair of simple purely imaginary eigenvalues $\pm i\nu_\lambda \tau_{\lambda,n}$. A corresponding

eigenfunction of $i\nu_\lambda\tau_{\lambda,n}$ is $\psi_\lambda e^{i\nu_\lambda\tau_{\lambda,n}\varsigma}$ for $\varsigma\in[-1,0]$, where ψ_λ is given by (3.9). We denote it as $p(\varsigma)$. Correspondingly, a eigenfunction associated with $-i\nu_\lambda\tau_{\lambda,n}$ is $\bar{p}(\varsigma)$.

To compute the normal form coefficients on the center manifold, it is necessary to consider the formal adjoint operator of the infinitesimal generator $\mathcal{A}_{\tau_{\lambda,n}}$. To this end, following the framework in [19, 20], we introduce the bilinear form $\langle\langle\cdot,\cdot\rangle\rangle$ for $\Psi\in\mathcal{C}_{\mathbb{C}}$ and $\Psi^*\in\mathcal{C}^*_{\mathbb{C}}:=C([0,1],\mathbb{Y}_{\mathbb{C}})$ as follows:

$$\langle\langle\Psi^*,\Psi\rangle\rangle=\langle\Psi^*(0),\Psi(0)\rangle+\lambda\tau_{\lambda,n}\int_{-1}^{0}\langle\Psi^*(s+1),u_\lambda f'_u\left(x,u_\lambda\right)\Psi(s)\rangle\,ds.$$

Lemma 4.1. *Define the operator* $\mathcal{A}^*_{\tau_{\lambda,n}}:\mathscr{D}(\mathcal{A}^*_{\tau_{\lambda,n}})\to\mathcal{C}^*_{\mathbb{C}}$ *by* $\mathcal{A}^*_{\tau_{\lambda,n}}\Psi^*(s)=-\dot{\Psi}^*(s)$ *for* $s\in(0,1]$, *with the domain*

$$\begin{aligned}\mathscr{D}\left(\mathcal{A}^*_{\tau_{\lambda,n}}\right)=\Big\{&\Psi^*\in\mathcal{C}^*_{\mathbb{C}}\cap(\mathcal{C}^*_{\mathbb{C}})^1:\Psi^*(0)\in\mathbb{X}_{\mathbb{C}},-\dot{\Psi}^*(0)=\tau_{\lambda,n}\nabla\cdot[d(x)\nabla\Psi^*(0)]\\&+\tau_{\lambda,n}\vec{b}(x)\cdot\nabla\Psi^*(0)+\lambda\tau_{\lambda,n}f\left(x,u_\lambda\right)\Psi^*(0)+\lambda\tau_{\lambda,n}f'_u\left(x,u_\lambda\right)u_\lambda\Psi^*(1)\Big\},\end{aligned}$$

where $(\mathcal{C}^*_{\mathbb{C}})^1=C^1([0,1],\mathbb{Y}_{\mathbb{C}})$. *Then* $\mathcal{A}^*_{\tau_{\lambda,n}}$ *and* $\mathcal{A}_{\tau_{\lambda,n}}$ *satisfy*

$$\langle\langle\mathcal{A}^*_{\tau_{\lambda,n}}\Psi^*,\Psi\rangle\rangle=\langle\langle\Psi^*,\mathcal{A}_{\tau_{\lambda,n}}\Psi\rangle\rangle\ \text{ for }\Psi\in\mathscr{D}(\mathcal{A}_{\tau_{\lambda,n}})\text{ and }\Psi^*\in\mathscr{D}(\mathcal{A}^*_{\tau_{\lambda,n}}).$$

Proof. For $\Psi\in\mathscr{D}(\mathcal{A}_{\tau_{\lambda,n}})$ and $\Psi^*\in\mathscr{D}(\mathcal{A}^*_{\tau_{\lambda,n}})$,

$$\begin{aligned}\langle\langle\Psi^*,\mathcal{A}_{\tau_{\lambda,n}}\Psi\rangle\rangle=&\left\langle\Psi^*(0),(\mathcal{A}_{\tau_{\lambda,n}}\Psi)(0)\right\rangle+\lambda\tau_{\lambda,n}\int_{-1}^{0}\left\langle\Psi^*(s+1),u_\lambda f'_u\left(x,u_\lambda\right)\dot{\Psi}(s)\right\rangle ds\\=&\left\langle\Psi^*(0),\tau_{\lambda,n}\nabla\cdot[d(x)\nabla\Psi(0)-\vec{b}(x)\Psi(0)]\right\rangle\\&+\langle\Psi^*(0),\lambda\tau_{\lambda,n}f\left(x,u_\lambda\right)\Psi(0)+\lambda\tau_{\lambda,n}f'_u\left(x,u_\lambda\right)u_\lambda\Psi(-1)\rangle\\&+\lambda\tau_{\lambda,n}\left[\langle\Psi^*(s+1),u_\lambda f'_u\left(x,u_\lambda\right)\Psi(s)\rangle\right]_{-1}^{0}\\&-\lambda\tau_{\lambda,n}\int_{-1}^{0}\left\langle\dot{\Psi}^*(s+1),u_\lambda f'_u\left(x,u_\lambda\right)\Psi(s)\right\rangle ds\\=&\left\langle(\mathcal{A}^*_{\tau_{\lambda,n}}\Psi^*)(0),\Psi(0)\right\rangle+\lambda\tau_{\lambda,n}\int_{-1}^{0}\left\langle-\dot{\Psi}^*(s+1),u_\lambda f'_u\left(x,u_\lambda\right)\Psi(s)\right\rangle ds\\=&\langle\langle\mathcal{A}^*_{\tau_{\lambda,n}}\Psi^*,\Psi\rangle\rangle.\end{aligned}$$

□

To utilize the center manifold theory, we first decompose the space $\mathcal{C}_{\mathbb{C}}$ as $\mathcal{C}_{\mathbb{C}}=P\oplus Q$. Here the center subspace P is given by $P=\operatorname{span}\{p(\varsigma),\bar{p}(\varsigma)\}$ and Q is defined by

$$Q=\left\{\Psi\in\mathcal{C}_{\mathbb{C}}:\langle\langle\Psi^*,\Psi\rangle\rangle=0\text{ for all }\Psi^*\in P^*\right\},$$

where $P^* = \text{span}\{q(s), \overline{q}(s)\}$ is the formal adjoint subspace of P. Here $q(s) = \psi_\lambda^* e^{i\nu_\lambda \tau_{\lambda,n} s}$ is an eigenfunction of $\mathcal{A}^*_{\tau_{\lambda,n}}$ corresponding to $-i\nu_\lambda\tau_{\lambda,n}$ for $s \in [0,1]$ and ψ_λ^* is given in (3.21). Let $\Phi_p = (p(\varsigma), \overline{p}(\varsigma))$ and $\Psi_P = \left(\frac{1}{\overline{S_n(\lambda)}} q(s), \frac{1}{S_n(\lambda)} \overline{q}(s)\right)^T$ with $S_n(\lambda)$ in Lemma 3.8. One can verify that $\langle\langle \Psi_p, \Phi_p \rangle\rangle = I$, where I is the 2×2 identity matrix.

To obtain the flow of (4.1) on the center manifold, we first define the complex coordinate $z(t)$ on the center subspace as

$$z(t) = \langle\langle \frac{1}{\overline{S_n(\lambda)}} q(s), U_t \rangle\rangle \tag{4.4}$$

and its component in the complementary subspace as

$$w(t) = U_t - \Phi_p \cdot (z(t), \overline{z}(t))^T. \tag{4.5}$$

According to the center manifold theory [8], on the local center manifold, $w(t)$ can be represented as $w(z, \bar{z})$, where

$$w(z, \bar{z}) = w_{20}(\varsigma)\frac{z^2}{2} + w_{11}(\varsigma) z\overline{z} + w_{02}(\varsigma)\frac{\overline{z}^2}{2} + \cdots. \tag{4.6}$$

Consequently, the flow of (4.1) restricted to the center manifold is determined by the following equation

$$\begin{aligned}
\dot{z}(t) =& \frac{d}{dt}\langle\langle \frac{1}{\overline{S_n(\lambda)}} q(s), U_t \rangle\rangle \\
=& \frac{1}{S_n(\lambda)} \langle\langle q(s), \mathcal{A}_{\tau_{\lambda,n}} U_t \rangle\rangle + \frac{1}{S_n(\lambda)} \langle\langle q(s), X_0 J(U_t, 0) \rangle\rangle \\
=& i\nu_\lambda \tau_{\lambda,n} z(t) + \frac{1}{S_n(\lambda)} \left\langle q(0), J\left(\Phi_p(z(t), \overline{z}(t))^T + w(z(t), \overline{z}(t)), 0\right)\right\rangle \\
\triangleq& i\nu_\lambda \tau_{\lambda,n} z(t) + g(z, \overline{z}),
\end{aligned}$$

where

$$g(z, \bar{z}) = g_{20}\frac{z^2}{2} + g_{11} z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2} + \cdots. \tag{4.7}$$

After some calculations, we have

$$\begin{aligned}
g_{20} &= \frac{2\lambda\tau_{\lambda,n}}{S_n(\lambda)} \int_\Omega \left[f'_u(x, u_\lambda)\, \psi_\lambda^2 \overline{\psi_\lambda^*} e^{-i\nu_\lambda\tau_{\lambda,n}} + \frac{1}{2} u_\lambda f''_{uu}(x, u_\lambda)\, \psi_\lambda^2 \overline{\psi_\lambda^*} e^{-2i\nu_\lambda\tau_{\lambda,n}} \right] \mathrm{d}x, \\
g_{11} &= \frac{\lambda\tau_{\lambda,n}}{S_n(\lambda)} \int_\Omega \left[f'_u(x, u_\lambda)\, (e^{i\nu_\lambda\tau_{\lambda,n}} + e^{-i\nu_\lambda\tau_{\lambda,n}}) |\psi_\lambda|^2 \overline{\psi_\lambda^*} + u_\lambda f''_{uu}(x, u_\lambda)\, |\psi_\lambda|^2 \overline{\psi_\lambda^*} \right] \mathrm{d}x, \\
g_{02} &= \frac{2\lambda\tau_{\lambda,n}}{S_n(\lambda)} \int_\Omega \left[f'_u(x, u_\lambda)\, \overline{\psi}_\lambda^2 \overline{\psi_\lambda^*} e^{i\nu_\lambda\tau_{\lambda,n}} + \frac{1}{2} u_\lambda f''_{uu}(x, u_\lambda)\, \overline{\psi}_\lambda^2 \overline{\psi_\lambda^*} e^{2i\nu_\lambda\tau_{\lambda,n}} \right] \mathrm{d}x,
\end{aligned}$$

$$
\begin{aligned}
g_{21} = \frac{2\lambda\tau_{\lambda,n}}{S_n(\lambda)} \int\limits_{\Omega} \overline{\psi_\lambda^*} \Big\{ & \frac{1}{2} u_\lambda f''_{uu}(x,u_\lambda)\, w_{20}(-1) \overline{\psi}_\lambda e^{i\nu_\lambda \tau_{\lambda,n}} + u_\lambda f''_{uu}(x,u_\lambda)\, \psi_\lambda e^{-i\nu_\lambda\tau_{\lambda,n}} w_{11}(-1) \\
& + f'_u(x,u_\lambda) \left[\psi_\lambda w_{11}(-1) + w_{11}(0)\psi_\lambda e^{-i\nu_\lambda\tau_{\lambda,n}} + \frac{1}{2} w_{20}(0)\overline{\psi}_\lambda e^{i\nu_\lambda\tau_{\lambda,n}} + \frac{1}{2}\overline{\psi}_\lambda w_{20}(-1) \right] \\
& + \frac{1}{2} f''_{uu}(x,u_\lambda) \left[2\psi_\lambda^2 \overline{\psi}_\lambda + \overline{\psi}_\lambda (\psi_\lambda e^{-i\nu_\lambda\tau_{\lambda,n}})^2 \right] \\
& + \frac{1}{2} u_\lambda f'''_{uuu}(x,u_\lambda) (\psi_\lambda e^{-i\nu_\lambda\tau_{\lambda,n}})^2 \overline{\psi}_\lambda e^{i\nu_\lambda\tau_{\lambda,n}} \Big\} \mathrm{d}x,
\end{aligned} \tag{4.8}
$$

where, as in [2, 18],

$$
w_{20}(\varsigma) = \frac{i g_{20}}{\nu_\lambda \tau_{\lambda,n}} p(\varsigma) + \frac{i\overline{g}_{02}}{3\nu_\lambda\tau_{\lambda,n}} \overline{p}(\varsigma) + E_\lambda e^{2i\nu_\lambda\tau_{\lambda,n}\varsigma}, \tag{4.9}
$$

$$
w_{11}(\varsigma) = -\frac{i g_{11}}{\nu_\lambda \tau_{\lambda,n}} p(\varsigma) + \frac{i\overline{g}_{11}}{\nu_\lambda\tau_{\lambda,n}} \overline{p}(\varsigma) + F_\lambda, \tag{4.10}
$$

with

$$
E_\lambda = -2\lambda\Delta(\lambda, 2i\nu_\lambda, \tau_{\lambda,n})^{-1} \left(f'_u(x,u_\lambda)\, \psi_\lambda^2 e^{-i\nu_\lambda\tau_{\lambda,n}} + \frac{1}{2} u_\lambda f''_{uu}(x,u_\lambda)\, \psi_\lambda^2 e^{-2i\nu_\lambda\tau_{\lambda,n}} \right), \tag{4.11}
$$

$$
F_\lambda = -\lambda\Delta(\lambda, 0, \tau_{\lambda,n})^{-1} \left(f'_u(x,u_\lambda) (e^{i\nu_\lambda\tau_{\lambda,n}} + e^{-i\nu_\lambda\tau_{\lambda,n}}) |\psi_\lambda|^2 + u_\lambda f''_{uu}(x,u_\lambda) |\psi_\lambda|^2 \right). \tag{4.12}
$$

In order to compute w_{20} and w_{11}, we provide estimates for E_λ and F_λ as follows.

Lemma 4.2. *Assume $\lambda \in \Lambda_{2\epsilon}$. For the above E_λ and F_λ, we have*

$$
E_\lambda = \frac{1}{\lambda - \lambda_*}(c_\lambda \phi + \eta_\lambda) \ \textit{and}\ F_\lambda = \frac{\tilde{\eta}_\lambda}{\lambda - \lambda_*}, \tag{4.13}
$$

where $c_\lambda \in \mathbb{C}$, and η_λ, $\tilde{\eta}_\lambda \in (X_1)_{\mathbb{C}}$ satisfy

$$
\lim_{\lambda\to\lambda_*} c_\lambda = \frac{2i}{\beta_{\lambda_*}(2i-1)}, \lim_{\lambda\to\lambda_*} \|\eta_\lambda\|_{\mathbb{X}_{\mathbb{C}}} = 0, \ \textit{and}\ \lim_{\lambda\to\lambda_*} \|\tilde{\eta}_\lambda\|_{\mathbb{X}_{\mathbb{C}}} = 0. \tag{4.14}
$$

Proof. We first derive the estimate for E_λ. The estimate for F_λ can be obtained in a similar manner and we omit the detail here. To analyze the singular behavior near the critical value λ_*, we introduce the auxiliary function $\hat{\psi} := (\lambda - \lambda_*)E_\lambda$. It follows from (4.11) that $\hat{\psi}$ satisfies the following equation:

$$
\Delta(\lambda, 2i\nu_\lambda, \tau_{\lambda,n})\hat{\psi} = -2\lambda(\lambda - \lambda_*) \left(f'_u(x,u_\lambda)\psi_\lambda^2 e^{-i\nu_\lambda\tau_{\lambda,n}} + \frac{1}{2} u_\lambda f''_{uu}(x,u_\lambda)\psi_\lambda^2 e^{-2i\nu_\lambda\tau_{\lambda,n}} \right). \tag{4.15}
$$

To resolve the singularity as $\lambda \to \lambda_*$, based on the space decomposition, we set $\hat{\psi} = c\phi + \eta$ with $c \in \mathbb{C}$ and $\eta \in (X_1)_{\mathbb{C}}$. Replacing $\hat{\psi}$ by $c\phi + \eta$, we substitute $u_\lambda(x) =$

$t_\lambda\phi(x)+h(t_\lambda\phi(x),\lambda)$ and $\nu_\lambda=(\lambda-\lambda_*)\hat{h}_\lambda$ into (4.15) to have

$$\begin{aligned}
&L_{\lambda_*}\eta+(\lambda-\lambda_*)[m(x)+\lambda m_1(t_\lambda,\lambda)-2i\hat{h}](c\phi+\eta)\\
&\quad+\lambda(t_\lambda\phi+h(t_\lambda\phi,\lambda))f_u'(x,u_\lambda)(c\phi+\eta)e^{-2i\theta}\\
&=-2\lambda(\lambda-\lambda_*)\left(f_u'(x,u_\lambda)\psi_\lambda^2e^{-i\theta}+\frac{1}{2}u_\lambda f_{uu}''(x,u_\lambda)\psi_\lambda^2e^{-2i\theta}\right).
\end{aligned}$$

One can see that $\hat{\psi}$ solves (4.15) if and only if the pair (η,c) satisfies the following system

$$\mathcal{F}(\eta,c,\lambda)=0, \tag{4.16}$$

where $\mathcal{F}=(f_1,f_2)^T:(X_1)_{\mathbb{C}}\times\mathbb{C}\times\mathbb{R}\to(Y_1)_{\mathbb{C}}\times\mathbb{C}$, moreover,

$$\begin{cases}
f_1(\eta,c,\lambda) & :=L_{\lambda_*}\eta+(\lambda-\lambda_*)[m(x)+\lambda m_1(t_\lambda,\lambda)-2i\hat{h}_\lambda](c\phi+\eta)+\\
& \lambda(t_\lambda\phi+h(t_\lambda\phi,\lambda))f_u'\left(x,u_\lambda\right)(c\phi+\eta)e^{-2i\theta}\\
& +2\lambda(\lambda-\lambda_*)(f_u'\left(x,u_\lambda\right)\psi_\lambda^2e^{-i\theta}+\tfrac{1}{2}u_\lambda f_{uu}''\left(x,u_\lambda\right)\psi_\lambda^2e^{-2i\theta})\\
& -(\lambda-\lambda_*)f_2(\eta,c,\lambda),\\
f_2(\eta,c,\lambda) & :=\int_\Omega[m(x)+\lambda m_1(t_\lambda,\lambda)-2i\hat{h}_\lambda](c\phi+\eta)\phi^*+\\
& \lambda(\beta_{\lambda_*}\phi+\tfrac{h(t_\lambda\phi,\lambda)}{\lambda-\lambda_*})f_u'\left(x,u_\lambda\right)(c\phi+\eta)e^{-2i\theta}\phi^*\\
& +2\lambda(f_u'\left(x,u_\lambda\right)\psi_\lambda^2e^{-i\theta}+\tfrac{1}{2}u_\lambda f_{uu}''\left(x,u_\lambda\right)\psi_\lambda^2e^{-2i\theta})\phi^*\,\mathrm{d}x.
\end{cases}$$

At the critical value $\lambda=\lambda_*$, the equation $f_1(\eta,c,\lambda_*)=L_{\lambda_*}\eta=0$ implies that $\eta=0$. By substituting $\eta=0$ into $f_2=0$ and utilizing

$$\beta_{\lambda_*}=-\frac{\int_\Omega m\phi\phi^*\,\mathrm{d}x}{\lambda_*\int_\Omega f_u'(x,0)\phi^2\phi^*\,\mathrm{d}x},\theta_{\lambda_*}=\frac{\pi}{2},\hat{h}_{\lambda_*}=\frac{\int_\Omega m\phi\phi^*\,\mathrm{d}x}{\int_\Omega\phi\phi^*\,\mathrm{d}x},$$

one finds that $f_2(0,c,\lambda_*)=0$ is satisfied if and only if

$$c=\frac{2i}{\beta_{\lambda_*}(2i-1)}=:c_{\lambda_*}.$$

The Fréchet derivative of $\mathcal{F}$ with respect to (η,c) at the point $(0,c_{\lambda_*},\lambda_*)$, denoted as $\widetilde{T}:(X_1)_{\mathbb{C}}\times\mathbb{C}\to(Y_1)_{\mathbb{C}}\times\mathbb{C}$, is given by

$$\widetilde{T}=(\widetilde{T}_1,\widetilde{T}_2)^T,$$

where

$$\begin{cases}
\widetilde{T}_1[\chi,\kappa] & =L_{\lambda_*}\chi,\\
\widetilde{T}_2[\chi,\kappa] & =\int_\Omega\Big[m(x)\phi^*+\lambda m_1(0,\lambda_*)\phi^*-2i\hat{h}_{\lambda_*}\phi^*\\
& \quad+\lambda\beta_{\lambda_*}\phi\ f_u'(x,0)e^{-2i\theta_{\lambda_*}}\phi^*\Big](\kappa\phi+\chi)\ \mathrm{d}x.
\end{cases}$$

Since $\widetilde{T}$ is a bijection from $(X_1)_{\mathbb{C}} \times \mathbb{C}$ to $(Y_1)_{\mathbb{C}} \times \mathbb{C}$, the implicit function theorem ensures that there is a continuously differentiable mapping $\lambda \mapsto (\eta_\lambda, c_\lambda)$ such that $\mathcal{F}(\eta_\lambda, c_\lambda, \lambda) = 0$ for $\lambda \in \Lambda_{2\epsilon}$ and $(\eta_\lambda, c_\lambda) = (\eta_{\lambda_*}, c_{\lambda_*})$ for $\lambda = \lambda_*$. Hence,

$$\lim_{\lambda \to \lambda_*} c_\lambda = \frac{2i}{\beta_{\lambda_*}(2i-1)} \text{ and } \lim_{\lambda \to \lambda_*} \eta_\lambda = 0 \text{ in } \mathbb{X}_{\mathbb{C}}.$$

□

Taking a similar argument as above, we have the following result for $\lambda \in \Lambda_{1\epsilon}$.

Lemma 4.3. *Assume $\lambda \in \Lambda_{1\epsilon}$. For the above E_λ and F_λ, we have*

$$E_\lambda = \frac{1}{\lambda - \lambda_*}(c_\lambda \phi + \eta_\lambda) \text{ and } F_\lambda = \frac{\tilde{\eta}_\lambda}{\lambda - \lambda_*}, \tag{4.17}$$

where $c_\lambda \in \mathbb{C}$, and η_λ, $\tilde{\eta}_\lambda \in (X_1)_{\mathbb{C}}$ satisfy

$$\lim_{\lambda \to \lambda_*} c_\lambda = \frac{2i}{\beta_{\lambda_*}(2i+1)}, \lim_{\lambda \to \lambda_*} \|\eta_\lambda\|_{\mathbb{X}_{\mathbb{C}}} = 0, \text{ and } \lim_{\lambda \to \lambda_*} \|\tilde{\eta}_\lambda\|_{\mathbb{X}_{\mathbb{C}}} = 0. \tag{4.18}$$

It is well known that the following quantities determine the direction and stability of bifurcated periodic orbits (see [8, 18, 20]):

$$\begin{aligned} C_1(0) &= \frac{i}{2\nu_\lambda \tau_{\lambda,n}} \left(g_{11} g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3} \right) + \frac{g_{21}}{2}, \\ \mu_2 &= -\frac{\mathrm{Re}(C_1(0))}{\mathrm{Re}(\mu'(\tau_{\lambda,n}))}, \\ \beta_2 &= 2\mathrm{Re}(C_1(0)). \end{aligned}$$

Precisely,

1. μ_2 determines the direction of the Hopf bifurcation: if $\mu_2 > 0$ (respectively, $\mu_2 < 0$), then the bifurcating periodic solutions exist for $\tau > \tau_{\lambda,n}$ (respectively, $\tau < \tau_{\lambda,n}$), where the bifurcation is called forward (respectively, backward);

2. β_2 determines the stability of bifurcated periodic solutions: the bifurcated periodic solutions are orbitally asymptotically stable (respectively, unstable) on the center manifold if $\beta_2 < 0$ (respectively, $\beta_2 > 0$).

Finally, we have the following results on the Hopf bifurcation.

Theorem 4.4. *If $\int_\Omega f'_u(x,0)\phi^2\phi^* \,\mathrm{d}x \neq 0$, then there exists $\epsilon > 0$ such that for each $(n, \lambda) \in \mathbb{N}_0 \times \Lambda$, a branch of spatially nonhomogeneous periodic orbits of (1.2) emerges from $(\tau, u) = (\tau_{\lambda,n}, u_\lambda)$. Moreover, the direction of the Hopf bifurcation at $\tau = \tau_{\lambda,n}$ is forward. Specially,*

(i) if $\lambda \in \Lambda_{2\epsilon}$, the bifurcated periodic solutions emerging from $\tau = \tau_{\lambda,0}$ are locally asymptotically stable;

(ii) if $\lambda \in \Lambda_{1\epsilon}$, the bifurcated periodic solutions are locally unstable.

Proof. First, we consider the case $\lambda \in \Lambda_{2\epsilon}$. As $\lambda \to \lambda_*$, we obtain the limits for the following essential quantities:

$$\begin{aligned}
&\lim_{\lambda\to\lambda_*} S_n(\lambda) = \left[1 + i\left(\frac{\pi}{2} + 2n\pi\right)\right] \int_\Omega \phi\phi^* \,\mathrm{d}x,\\
&\lim_{\lambda\to\lambda_*} (\lambda - \lambda_*) F_\lambda = 0,\\
&\lim_{\lambda\to\lambda_*} \tau_{\lambda,n}(\lambda - \lambda_*) = \frac{1}{\hat{h}_{\lambda_*}}\left(\frac{\pi}{2} + 2n\pi\right),\\
&\lim_{\lambda\to\lambda_*} (\lambda - \lambda_*) E_\lambda = \frac{2i}{\beta_{\lambda_*}(2i-1)}\phi.
\end{aligned}$$

Then we substitute the above limits into the expressions for g_{ij} given in (4.8). A direct calculation yields

$$\begin{aligned}
&\lim_{\lambda\to\lambda_*} (\lambda - \lambda_*) g_{20} = \frac{2i(\pi + 4n\pi)}{\beta_{\lambda_*}(2 + i(\pi + 4n\pi))},\\
&\lim_{\lambda\to\lambda_*} (\lambda - \lambda_*) g_{11} = 0,\\
&\lim_{\lambda\to\lambda_*} (\lambda - \lambda_*) g_{02} = \frac{-2i(\pi + 4n\pi)}{\beta_{\lambda_*}(2 + i(\pi + 4n\pi))},\\
&\lim_{\lambda\to\lambda_*} (\lambda - \lambda_*)^2 g_{21} = \frac{2(\pi + 4n\pi)(1 - 3i)}{\beta_{\lambda_*}^2(10 + 5i(\pi + 4n\pi))} + \frac{8i\pi(1 + 4n)}{3\beta_{\lambda_*}^2|2 + i(\pi + 4n\pi)|^2}.
\end{aligned}$$

Hence, $\lim_{\lambda\to\lambda_*} \mathrm{Re}((\lambda - \lambda_*)^2 g_{21}) < 0$ and $\lim_{\lambda\to\lambda_*} \mathrm{Re}((\lambda - \lambda_*)^2 C_1(0)) < 0$.

For $\lambda \in \Lambda_{1\epsilon}$, by a similar discussion, we have

$$\begin{aligned}
&\lim_{\lambda\to\lambda_*} S_n(\lambda) = \left[1 + i\left(\frac{3\pi}{2} + 2n\pi\right)\right] \int_\Omega \phi\phi^* \,\mathrm{d}x,\\
&\lim_{\lambda\to\lambda_*} (\lambda - \lambda_*) F_\lambda = 0,\\
&\lim_{\lambda\to\lambda_*} \tau_{\lambda,n}(\lambda - \lambda_*) = \frac{-\left(\frac{3\pi}{2} + 2n\pi\right)}{\hat{h}_{\lambda_*}},\\
&\lim_{\lambda\to\lambda_*} \nu_\lambda \tau_{\lambda,n} = \frac{3\pi}{2} + 2n\pi,\\
&\lim_{\lambda\to\lambda_*} (\lambda - \lambda_*) E_\lambda = \frac{2i}{\beta_{\lambda_*}(2i+1)}\phi,\\
&\lim_{\lambda\to\lambda_*} (\lambda - \lambda_*) g_{20} = \frac{2i(3\pi + 4n\pi)}{\beta_{\lambda_*}(2 + i(3\pi + 4n\pi))},
\end{aligned}$$

$$\lim_{\lambda\to\lambda_*}(\lambda-\lambda_*)g_{11}=0,$$
$$\lim_{\lambda\to\lambda_*}(\lambda-\lambda_*)g_{02}=\frac{-2i(3\pi+4n\pi)}{\beta_{\lambda_*}(2+i(3\pi+4n\pi))},$$
$$\lim_{\lambda\to\lambda_*}(\lambda-\lambda_*)^2g_{21}=\frac{2(3\pi+4n\pi)(-1-3i)}{\beta_{\lambda_*}^2(10+5i(3\pi+4n\pi))}+\frac{8i\pi(3+4n)}{3\beta_{\lambda_*}^2|2+i(3\pi+4n\pi)|^2}.$$

There also exists $\lim_{\lambda\to\lambda_*}\mathrm{Re}((\lambda-\lambda_*)^2g_{21})<0$ and $\lim_{\lambda\to\lambda_*}\mathrm{Re}((\lambda-\lambda_*)^2C_1(0))<0$.

Based on the discussion for $\lambda\in\Lambda_{2\epsilon}$ and $\lambda\in\Lambda_{1\epsilon}$, we declare that the direction of the Hopf bifurcation is forward and the bifurcating periodic orbits are orbitally asymptotically stable on the center manifold. However, according to Theorem 3.14, for these two cases, the stabilities of the spatially nonhomogeneous steady state u_λ are different. This leads to a distinction for the overall stability of the manifold, yielding opposite conclusions for $\lambda\in\Lambda_{2\epsilon}$ and $\lambda\in\Lambda_{1\epsilon}$. □

5 Two Illustrative Examples

In this section, we conduct numerical simulations to validate our theoretical results. We first consider a food-limited model. As the logistic growth has been thoroughly analyzed [4], we focus on the model originally proposed by Davidson and Gourley [41],

$$\frac{\partial v(x,t)}{\partial t}=-d\frac{\partial^2 v(x,t)}{\partial x^2}+rv(x,t)\frac{k-v(x,t-\tau)}{k+cv(x,t-\tau)},\tag{5.1}$$

where $r>0$ is the growth rate, $k>0$ is the carrying capacity, $c>0$ is a constant, and r/c represents the mass replacement at saturation. To incorporate population movement toward favorable habitats, we introduce a general advection term. After non-dimensionalization, the model is read as

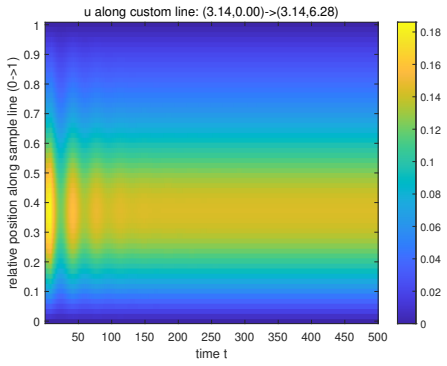
$$\begin{cases}u_t=\nabla\cdot[d(x)\nabla u-\vec{b}(x)u]+\lambda u(x,t)\frac{1-u(x,t-\tau)}{1+cu(x,t-\tau)}, & x\in\Omega,\ t>0,\\ u(x,t)=0, & x\in\partial\Omega,\ t>0,\end{cases}\tag{5.2}$$

with the initial condition

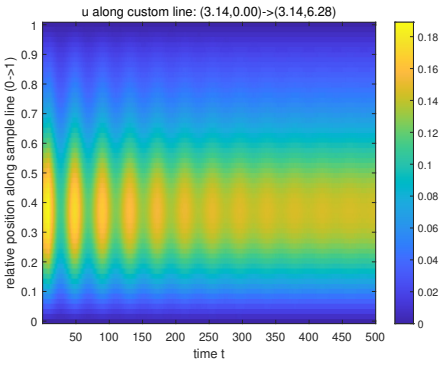
$$u(x,t)=\eta(x,t),x\in\Omega,t\in[-\tau,0].\tag{5.3}$$

To illustrate the dynamics, we set the domain and the initial condition and then fix some parameters as follows:

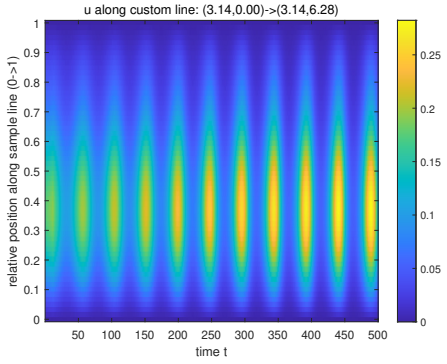
$$\begin{aligned}&\Omega=\left\{(x,y)\in\mathbb{R}^2\mid(x-\pi)^2+(y-\pi)^2\le\pi^2\right\},\eta(x,t)=0.1,\\ &d(x,y)=1+0.1x+0.1y,\vec{b}(x,y)=\left(\frac{\cos x}{\sin x+2},\frac{\cos y}{\sin y+2}\right)^T,c=0.5.\end{aligned}\tag{5.4}$$



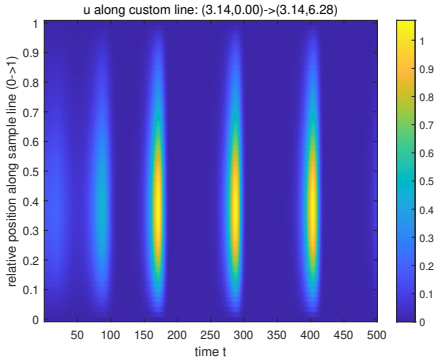
(a) $\tau = 8$



(b) $\tau = 10$



(c) $\tau = 12$



(d) $\tau = 20$

Figure 1: Numerical simulations for system (5.2) with $\lambda = 1.0$. (a) $\tau = 8$: convergence to steady state. (b) $\tau = 10$: oscillatory convergence. (c)-(d) $\tau = 12, 20$: emergence of stable periodic solutions.

Based on $f(0) = m(x) = 1$, the principal eigenvalue λ_*, determined by the linearization of system (5.2) at the trivial equilibrium $u = 0$, is approximately 0.9 and $f'(0) < 0$. Thus, we know that (5.2) has a positive steady state u_λ for $\lambda \in \Lambda_{2\epsilon}$ due to Theorem 3.1.

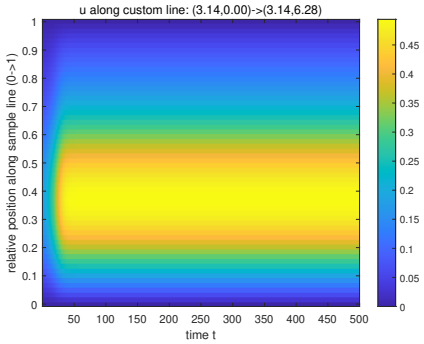
Fix $\lambda = 1.0 > \lambda_* \approx 0.9$. The simulations are presented in Figure 1. For small delays ($\tau = 8, 10$), the solution approaches the positive steady state, although noticeable oscillations appear at $\tau = 10$. As τ increases further to 12 and 20, the steady state loses stability, and the solution converges to a stable time-periodic solution. The period and amplitude of these oscillations increase as τ increases. This phenomenon aligns perfectly with our theoretical prediction. In other words, the increase of τ drives the system toward criticality, beyond which a forward Hopf bifurcation generates stable periodic solutions.

Next, we build the following model incorporating the weak Allee effect to illustrate

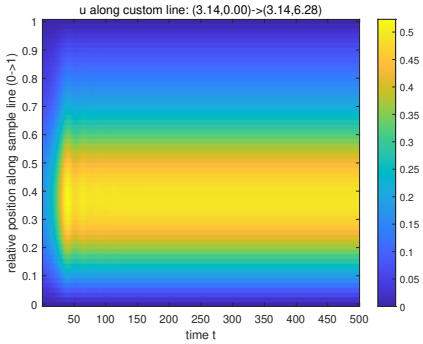
our theoretical results,

$$u_t = \nabla \cdot [d(x)\nabla u - \vec{b}(x)u] + 2\lambda u(x,t)(1 - u(x,t-\tau))\,(u(x,t-\tau) + 0.5)\,. \tag{5.5}$$

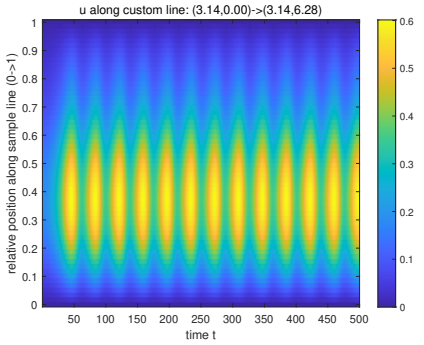
For convenience, we choose the same spatial domain, initial condition, and parameters as specified in (5.4). Still this model satisfies $f(0) = m(x) = 1$. Thus the critical value remains $\lambda_* \approx 0.9$. However, as $f'(0) > 0$, the positive non-trivial steady state exists for $\lambda \in \Lambda_{1\epsilon}$. Fix $\lambda = 0.8 < \lambda_* \approx 0.9$. Our analysis indicates that the bifurcated periodic solutions are locally unstable and thus it is practically difficult to capture them in numerical simulations. However, as illustrated in Figure 3, numerical simulations reveal that stable periodic patterns can still emerge as the delay τ increases beyond the bifurcation threshold $\tau_{\lambda,n}$ (here, $\tau_{\lambda,n}$ lies between 5.5 and 9.5). For population models with a weak Allee effect, it is well known that there often exist bistable phenomena or global attractors, so it should be emphasized that this numerical observation does not contradict our local stability results. We hypothesize that this phenomenon is attributed to the existence of global attractors or additional stable branches not captured by the local analysis around the small solution branch. Actually, for this case with general advection, these numerical observations are consistent with findings in [7].



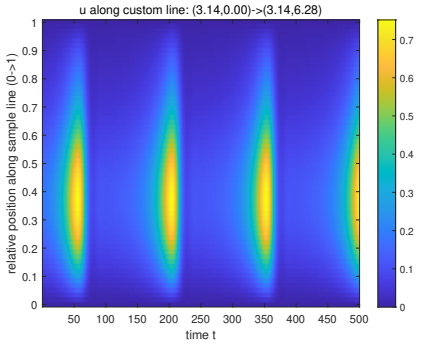
(a) $\tau = 1.5$



(b) $\tau = 5.5$



(c) $\tau = 9.5$



(d) $\tau = 15.5$

Figure 3: Simulations for the weak Allee effect model with $\lambda = 0.8$. (a) Convergence to the steady state. (b)–(c) Emergence of periodic solutions. (d) Asymptotic stability of the periodic orbit for large delay.

6 Summary and Conclusion

In this paper, we investigate the dynamics of a class of reaction-diffusion population models incorporating a general advection term and a time-delayed per capita growth rate. Based on the principal eigenvalue λ_* of the associated non-self-adjoint elliptic operator, our study centers on the existence, stability, and Hopf bifurcation of spatially nonhomogeneous steady states when the parameter λ lies in Λ, i.e., the ϵ-neighborhood of λ_*.

By employing the Lyapunov-Schmidt reduction method, we first establish the existence of spatially nonhomogeneous steady states for $\lambda \in \Lambda$. Subsequently, spectral analysis indicates that the time delay τ plays a critical role for losing the stability of steady states. Specifically, we demonstrate that as λ approaches λ_*, the critical delay threshold required to trigger instability tends to infinity (i.e., $\lim_{\lambda\to\lambda_*} \tau_{\lambda,0} = \infty$), which implies that the system becomes increasingly robust near λ_* even with bigger delay.

Furthermore, the center manifold theorem and the normal form theory allow us to derive explicit formulas for determining the direction of Hopf bifurcation and the stability of bifurcated periodic orbits. The analysis indicates that the dynamics depends critically on the form of the function f, i.e., the per capita growth rate. We build two examples to support the theoretical results. For the logistic-type growth, because of $\int_\Omega f'_u(x,0)\phi^2\phi^* \,\mathrm{d}x < 0$, the positive steady state exists for $\lambda \in \Lambda_{2\epsilon}$ and remains locally stable for small delays. Moreover, a forward Hopf bifurcation occurs and induces the emergence of stable periodic orbits when the delay increases. For the weak Allee effect growth, because of $\int_\Omega f'_u(x,0)\phi^2\phi^* \,\mathrm{d}x > 0$, the positive steady state is unstable for $\lambda \in \Lambda_{1\epsilon}$. Consequently, the bifurcated periodic solutions are likewise unstable.

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