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Article

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Posted Date: July 10th, 2026

DOI: <https://doi.org/10.21203/rs.3.rs-9021226/v1>

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Additional Declarations: No competing interests reported.

Modeling the Complex Dynamics of Monkeypox and Typhoid Fever Co-Infection Using a Novel Fractal-Fractional Approach

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Abstract

In this study presents a novel fractal-fractional order mathematical model for monkey pox and typhoid fever co-infection dynamics, employing fractal-fractional derivatives to capture memory effects and multi-scale heterogeneities inherent in dual pathogen transmission systems. Unlike conventional integer-order models, the fractal-fractional framework integrates fractional calculus, which accounts for non-local dynamics and long-term dependencies in disease progression, with fractal geometry, which represents self-similar patterns in population mixing and co-infection transmission pathways. This dual approach effectively captures the complex, non-linear nature of monkeypox-typhoid co-infection spread, where historical infection states and heterogeneous contact patterns significantly influence current epidemiological outcomes. Analytical results establish solution positivity, boundedness, and derive the basic reproduction numbers for both typhoid fever (R_{0T}) and monkeypox (R_{0m}), with stability analysis demonstrating global asymptotic stability at endemic equilibria when reproduction numbers exceed unity. The monkeypox-free equilibrium and typhoid-free equilibrium are analyzed to determine conditions for disease elimination. Sensitivity analysis identifies critical parameters including typhoid transmission rate, monkeypox transmission rate, relative infectiousness parameters for co-infected individuals, recovery rates, and treatment rates as primary drivers of epidemic dynamics. Numerical simulations, implemented via fractal-fractional computational schemes, reveal that varying

fractional orders (α) and fractal dimensions (σ) significantly influence population oscillatory behaviors, with lower fractional orders producing more pronounced memory effects and sustained fluctuations in co-infected compartments. The model demonstrates zoonotic transmission patterns characteristic of monkey pox while capturing the environmental persistence of typhoid fever. Results show that monkey pox co-infection substantially increases typhoid morbidity and mortality rates, while typhoid episodes accelerate monkey pox disease progression and viral shedding. The model demonstrates superior performance over integer-order frameworks in capturing real-world epidemic complexities, providing enhanced predictive capabilities for intervention strategies targeting monkey pox-typhoid co-infection control. Findings highlight the critical importance of integrated zoonotic control measures, sanitation improvements, and coordinated vaccination approaches for managing dual pathogen epidemics in regions with high typhoid endemicity and emerging monkey pox outbreaks.

Keywords: Fractal-Fractional Modeling, Sensitivity analysis, Data fitting, Numerical Simulations, Basic Reproduction Number S

1.0 Introduction

Monkeypox (MPX) is a contagious zoonotic disease caused by the monkeypox virus, a double-stranded DNA virus belonging to the genus *Orthopoxvirus* within the family *Poxviridae* [1]. First identified in captive monkeys in 1958, the disease is primarily found in tropical rainforest regions of Central and West Africa [2]. The virus exists in two distinct genetic clades: clade I (formerly Congo Basin clade) and clade II (formerly West African clade), corresponding to their geographical distribution [3]. Natural reservoirs include various rodent species such as Gambian pouched rats (*Cricetomys gambianus*), dormice (*Graphiurus* spp.), and African squirrels (*Heliosciurus* and *Funisciurus* species) [4, 5]. The clinical presentation of monkeypox includes fever, headache, muscle aches, backache, swollen lymph nodes, and exhaustion, with lymphadenopathy serving as a distinguishing feature from smallpox and chickenpox [6]. Following a characteristic incubation period ranging from 7 to 14 days, patients develop a distinctive rash that progresses through macular, papular, vesicular, and pustular stages before crusting and desquamation, with the illness typically lasting 2 to 4 weeks [7, 8]. Transmission occurs through multiple pathways: zoonotic transmission from infected animals to humans via direct contact with blood, bodily fluids, or cutaneous/mucosal lesions; human-to-human transmission through respiratory droplets during prolonged face-to-face contact; direct contact with body fluids, lesion material, or contaminated materials such as bedding; and vertical transmission from mother to fetus through the placenta [9, 10]. The case fatality ratio historically ranges from 1% to 10%, with clade I associated with higher mortality rates than clade II [11].

Typhoid fever, caused by the bacterium *Salmonella enterica* serotype Typhi (*S. Typhi*), represents a significant global health burden, particularly in low- and middle-income countries with inadequate water, sanitation, and hygiene (WASH) infrastructure [12]. Unlike other *Salmonella* strains that infect multiple animal species, *S. Typhi* is exclusively a human pathogen, with infected individuals and asymptomatic chronic carriers serving as the sole reservoirs [13, 14]. In 2015, approximately 12.5 million new cases were reported globally, resulting in an estimated 149,000 deaths, representing a decline from 181,000 deaths in 1990 due to improved sanitation and antibiotic availability [15, 16]. The disease is particularly prevalent in South-central and Southeast

Asia, with children and adolescents aged 5 to 19 years bearing the highest disease burden [17]. Transmission occurs predominantly through the fecal-oral route, with ingestion of food or water contaminated by feces or urine from infected individuals or chronic carriers [18]. Asymptomatic carriers, defined as individuals who continue to excrete *S. Typhi* in their stool for more than one year following acute infection, play a critical epidemiological role as persistent sources of transmission [19]. The clinical presentation follows a characteristic three-week progression: during the first week, patients experience gradually rising fever with relative bradycardia (Faget sign), malaise, headache, and cough, with approximately 25% developing epistaxis and abdominal pain [20]. Laboratory findings reveal leukopenia, eosinopenia, and relative lymphocytosis, with blood cultures confirming *S. Typhi* presence while Widal tests typically remain negative in this early phase [21]. The second week manifests with sustained high fever around 40°C (104°F), persistent bradycardia, delirium alternating between calm and agitated states (hence the historical designation "nervous fever"), rose spots on the lower chest and abdomen in approximately one-third of cases, hepatosplenomegaly, and abdominal distension with right lower quadrant tenderness [22]. During the third week, untreated cases may develop serious complications including intestinal hemorrhage from congested Peyer's patches, potentially fatal intestinal perforation in the distal ileum, and respiratory complications such as pneumonia and acute bronchitis [23]. Without treatment, the case fatality rate ranges from 10% to 20%, but timely antibiotic therapy reduces mortality to less than 1% [24]. Treatment typically involves fluoroquinolone antibiotics such as ciprofloxacin in regions without widespread resistance, or third-generation cephalosporins like ceftriaxone or cefotaxime where fluoroquinolone resistance is prevalent [25, 26]. Approximately 2% to 5% of infected individuals become chronic carriers, harboring bacteria in the biliary tract even after symptom resolution, complicating disease control efforts [27].

The co-infection of monkeypox and typhoid fever represents a particularly concerning scenario in endemic regions where both pathogens circulate simultaneously. Both diseases share common initial symptoms including fever, chills, fatigue, exhaustion, and muscle aches, which can complicate early differential diagnosis [28]. The zoonotic nature of monkeypox, combined with the environmental persistence of typhoid through contaminated water and food sources, creates overlapping transmission pathways in communities with inadequate sanitation infrastructure [29]. Individuals simultaneously infected with both pathogens face compounded morbidity, with monkeypox-induced immunosuppression potentially exacerbating typhoid disease progression, while the systemic inflammatory response to typhoid may increase susceptibility to secondary monkeypox infection [30]. The co-circulation of these pathogens poses significant public health challenges, particularly in sub-Saharan African regions where limited healthcare infrastructure, poor sanitation, and frequent human-animal contact create ideal conditions for dual pathogen transmission [31].

Mathematical modeling has become an indispensable tool for understanding infectious disease transmission dynamics, evaluating intervention strategies, and informing public health policy decisions. Recent developments have demonstrated the power of compartmental models, network-based approaches, and advanced computational methods in capturing complex epidemiological phenomena. Acheneje et al. [32] developed a mathematical model for COVID-19 and monkeypox co-infection with optimal control strategies and cost-benefit analysis, demonstrating that combined intervention approaches significantly reduce disease burden. Acheneje et al. [33] utilized the Laplace-Adomian Decomposition method to derive approximate solutions for fractional-order

COVID-19 and monkeypox co-infection dynamics, revealing that increasing effective treatment capacity significantly alleviates disease burdens in human populations. Agahiu et al. [34] applied the Laplace-Adomian decomposition method to fractional-order mathematical models for Zika and Chikungunya virus co-transmission, providing insights into dual arboviral infection dynamics. Bolaji et al. [35] presented a model for controlling monkeypox transmission dynamics in sub-Saharan Africa, identifying key intervention parameters for disease management. Odiba et al. [36] developed a compartmental deterministic epidemiological model with non-linear differential equations for analyzing COVID-19, HIV, and monkeypox co-infection dynamics, demonstrating global asymptotic stability of disease-free equilibrium when reproduction numbers fall below unity. Omede et al. [37] conducted mathematical analysis on vertical and horizontal transmission dynamics of HIV and Zika virus co-infection, incorporating both transmission pathways in their modeling framework. Omede et al. [38] performed mathematical analysis of two-strain tuberculosis model dynamics with exogenous re-infection, revealing complex strain competition patterns. Odionyenma et al. [39] analyzed a model to control the co-dynamics of Chlamydia and Gonorrhea using Caputo fractional derivatives, demonstrating superior performance of fractional-order formulations. Bolaji et al. [40] modeled the transmission dynamics of the Omicron variant of COVID-19 in densely populated Lagos, Nigeria, providing insights for urban outbreak management. Omede et al. [41] developed a mathematical model for the third wave of COVID-19 with optimal control strategies for reducing disease burden in Nigeria, employing Pontryagin's maximum principle to identify effective intervention combinations.

Oname et al. [42] developed a comprehensive co-infection model examining the concurrent circulation of COVID-19, Zika, chikungunya, and dengue viruses, incorporating symptom similarity and co-infection dynamics while identifying backward bifurcation patterns under specific conditions and demonstrating through sensitivity analysis and optimal control that preventive measures against COVID-19 significantly reduce co-infection burdens. Peter et al. [43] analyzed monkeypox transmission dynamics using a deterministic mathematical model with optimal control, establishing that disease-free equilibrium becomes locally and globally asymptotically stable when $R_0 < 1$, and demonstrating through cost-effectiveness analysis that preventive strategies targeting rodent-to-human transmission represent the most economical intervention approach. Marcus et al. [44] investigated monkeypox and HIV/AIDS co-infection dynamics through a 13-compartment deterministic model, conducting comprehensive analyses including equilibrium point determination, non-negativity verification, basic reproduction number calculation, invariant region identification, and stability assessment, while revealing forward bifurcation patterns and identifying critical basic reproduction number thresholds for disease management. Atokolo et al. [45] formulated mathematical models for COVID-19 transmission and control, determining disease-free and endemic equilibria with stability analysis, calculating basic reproduction numbers via the next-generation method, and demonstrating that when $R_0 < 1$ the disease vanishes while persistence occurs when $R_0 > 1$, with control strategies focusing on quarantine rates for exposed individuals, isolation rates for infected individuals, and treatment rates. Okyere and Ackora-Prah [46] examined monkeypox transmission in Ghana using Atangana-Baleanu fractional derivatives in the Caputo sense, computing a basic reproduction number of $R_0 = 0.1940$ and demonstrating through numerical simulations that the fractional operator significantly impacts each population category, with findings indicating potential pandemic elimination within initial days when the fractional order $\alpha = 0.2$. Tilahun et al. [47] introduced a

five-compartment mathematical model for typhoid fever dynamics (susceptible, infected, carrier, recovered, and bacterial population), exploring optimal control strategies incorporating sanitation, vaccination, and treatment, conducting sensitivity assessments to evaluate control measure impacts, and concluding that combined treatment and prevention approaches represent cost-effective strategies for disease eradication. Peter et al. [48] employed the variational iteration method to model typhoid fever transmission dynamics, categorizing the human population into susceptible, infected carrier, infected, and recovered compartments with direct transmission from infected to susceptible individuals, incorporating an environmental bacteria compartment to reflect contamination from food, water, and beverages, and comparing results with the classical fourth-order Runge-Kutta method to ensure model accuracy. Nthiiri et al. [49] developed a mathematical model focusing on typhoid fever dynamics with emphasis on protection as a control strategy, demonstrating through numerical simulations that augmenting protection measures significantly reduces disease prevalence, highlighting the potential impact of proactive intervention strategies in disease control. Stephen and Nyerere [50] proposed a framework evaluating educational campaign effectiveness in managing typhoid fever transmission within communities, revealing that awareness-raising through educational initiatives could potentially decrease transmission rates by over 40%, underscoring the importance of public health interventions in infectious disease control. Mohamed et al. [51] examined monkeypox virus spread dynamics across diverse regions encompassing human-animal host interactions through a continuous-time mathematical framework, introducing optimal strategies involving awareness campaigns, security measures, and health interventions to curtail transmission while minimizing human infections and eliminating the virus in animals, utilizing the discrete-time Pontryagin maximum principle to determine optimal controls through iterative approaches, with numerical simulations demonstrating the effectiveness of strategies emphasizing vulnerable individual protection, prevention of contact with infected counterparts, and promotion of quarantine facilities.

The fractal-fractional derivative approach represents a significant advancement in mathematical epidemiology, combining fractional calculus with fractal geometry to capture memory effects and multi-scale heterogeneities in disease transmission. Jan et al. [52] utilized fractional-order derivatives with the Atangana-Baleanu operator to study syphilis dynamics, demonstrating that fractional derivatives provide more precise information about disease transmission compared to integer-order models. Farman et al. [53] presented a fractal-fractional sex-structured syphilis model with three stages and loss of immunity, incorporating advanced fractional operators to capture disease progression complexities. Farman et al. [54] performed computational analysis and chaos control of a fractional-order syphilis model, revealing intricate dynamical behaviors including chaotic regimes and demonstrating control strategies for stabilizing disease dynamics. Abdulwahab et al. [55] proposed a novel sex-structured model incorporating congenital infection and disability compartments using the Atangana-Baleanu-Caputo fractional derivative, demonstrating superior performance in capturing real-world epidemic complexities compared to classical integer-order formulations. Acheneje et al. [33] introduced a fractional-order compartmental model examining COVID-19 and monkeypox co-infection transmission dynamics using the Laplace-Adomian Decomposition method, deriving approximate solutions demonstrating convergence to exact values, utilizing real-life data with MATLAB `fmincon` algorithm for parameter estimation, and revealing that increasing effective treatment capacity significantly alleviates disease burdens. These fractal-fractional approaches have demonstrated superior capabilities in modeling infectious disease dynamics by accounting for non-local

interactions, long-term memory effects, and heterogeneous population structures that conventional integer-order models fail to capture.

Despite extensive mathematical modeling efforts on typhoid fever and monkeypox individually, and emerging research on various disease co-infections, a significant gap persists in the literature regarding the application of fractal-fractional derivative frameworks to monkeypox-typhoid fever co-infection dynamics. While fractional-order models have been successfully applied to single-pathogen systems such as syphilis, COVID-19, and monkeypox alone, and integer-order models have examined co-infections involving monkeypox with HIV/AIDS or COVID-19, no study has yet integrated the fractal-fractional methodology—which uniquely captures both memory effects through fractional derivatives and self-similar transmission patterns through fractal dimensions—to analyze the complex interplay between these two epidemiologically distinct pathogens. Furthermore, existing co-infection models predominantly employ classical integer-order differential equations that fail to account for the non-local dynamics, long-term dependencies in disease progression, and heterogeneous contact patterns that characterize real-world dual pathogen transmission systems, particularly in resource-limited settings where monkeypox and typhoid fever co-circulate. This research bridges these critical gaps by developing the first comprehensive fractal-fractional order mathematical model for monkeypox-typhoid fever co-infection, incorporating zoonotic transmission pathways for monkeypox, environmental persistence mechanisms for typhoid fever, treatment intervention strategies, vaccination dynamics, and rigorous stability analysis of disease-free, monkeypox-free, typhoid-free, and endemic equilibria, thereby providing enhanced predictive capabilities and more realistic intervention strategy assessments for regions grappling with simultaneous monkeypox emergence and endemic typhoid burden.

The remaining part of this manuscript is structured as follows; In section 2, we described how the model was formulated, section 3 presents the preliminaries of the fractal fractional method used in analyzing the co-infection of monkeypox and Typhoid fever, section 4 entails the numerical schemes of the fractal fractional order system as well as the numerical simulations of the model and section 5 is the concluding remark of the work.

2.0 Model formulation

2.1 Model Assumptions

The following fundamental assumptions underpin the formulation of the fractal-fractional order monkeypox-typhoid fever co-infection model:

- (1) **Homogeneous Population Mixing:** The susceptible human population mixes homogeneously with infected individuals across all infectious compartments (monkeypox-infected, typhoid-infected, and co-infected), with uniform contact rates regardless of demographic, behavioral, or spatial heterogeneities, thereby simplifying the force of infection calculation while maintaining epidemiological validity for community-level transmission dynamics [56].
- (2) **Permanent Immunity from Recovery:** Recovered individuals from monkeypox, typhoid fever, or co-infection acquire complete and permanent immunity against

reinfection with the respective pathogen(s), transitioning to a permanently recovered compartment without possibility of returning to susceptible status, consistent with observations of robust immunological memory following natural infection with orthopoxviruses and typhoid fever [57, 58].

- (3) **Disease-Induced Mortality:** Natural death rates remain constant across all compartments at rate μ_h for humans and μ_r for rodents, while additional disease-induced mortality occurs specifically in monkeypox-infected (δ_1), typhoid-infected (δ_3), and co-infected (δ_4) compartments, with higher mortality rates in co-infected individuals reflecting compounded pathogen burden and immunological stress, assuming earlier disease stages do not significantly elevate mortality risk compared to baseline natural death rates [59, 60].
- (4) **Imperfect Vaccination Efficacy:** Vaccination against both monkeypox and typhoid fever provides partial protection that reduces susceptibility to infection but does not confer complete immunity, with vaccinated individuals susceptible to breakthrough infections at reduced rates and experiencing waning immunity over time at rate γ_1 , reflecting real-world observations of imperfect vaccine effectiveness and time-dependent immunity decline for both monkeypox vaccines (approximately 85% efficacy) and typhoid vaccines (varying between 50-80% efficacy depending on formulation) [61, 62].

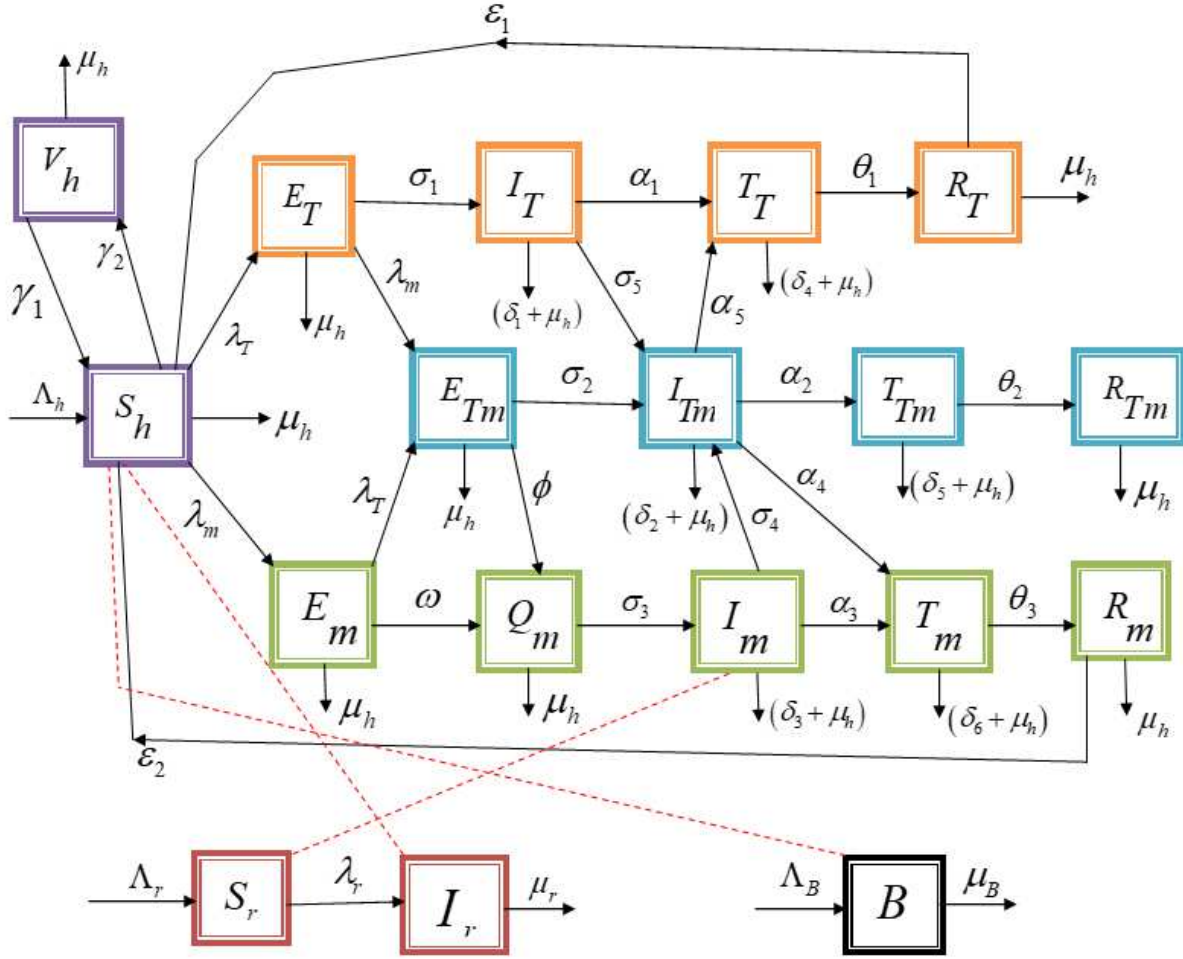


Figure 1: Schematic Diagram of the Monkeypox – Typhoid fever co-infection model

Figure 1 illustrates the schematic representation of the Monkeypox–Typhoid fever co-infection model [69, 71]. It shows the compartments representing different population groups, such as susceptible, exposed, infected, and recovered individuals for each disease, as well as transitions between these states.

2.2 Model Description

In modeling the dynamics of the co-infection of Monkey pox and Typhoid fever , the population is divided into eighteen compartments. Susceptible individuals (S_h), these are the individuals who are free of the infection; Exposed individuals to monkey pox only (E_m); Quarantined human

population of monkey pox only (Q_m), Infected human population with monkey pox only as (I_m), Isolated humans for treatment of monkey pox only (T_m), Population of Exposed humans to Typhoid fever only disease (E_T), Population of Exposed individuals to both monkey pox and Typhoid fever diseases (E_{TM}), Infected human population with Typhoid fever only (I_T), Infected human population with both monkey pox and Typhoid fever diseases (I_{TM}), Humans on Typhoid fever only treatment population (T_T), Humans on both monkey pox and Typhoid fever treatment population (T_{MT}), Recovered human population from monkey pox only (R_M), Recovered human population from Typhoid fever only (R_T), Recovered human population from both monkey pox and Typhoid fever (R_{MT}), vaccinated human population against both diseases (V_h), susceptible rodent population (S_r) and infected rodent population. The susceptible humans are recruited at the rate of Λ_h , susceptible rodents are recruited at the rate of Λ_r while the susceptible Bacteria are recruited at the rate of Λ_B , Contact rate between the susceptible humans and infected human population with Typhoid fever only, Contact rate between the susceptible humans and human population on Typhoid fever only treatment, Contact rate between the susceptible humans and infected human population with both Monkey pox and Typhoid fever, Contact rate between the susceptible humans and human population on both Monkey pox and Typhoid fever treatment, Contact rate between the susceptible humans and infected human population with Monkey pox only, Contact rate between the susceptible humans and human population on Monkey pox only treatment, Contact rate between the susceptible humans and infected Rodents population with Monkey pox only, Contact rate between the susceptible Rodents and infected humans with Monkey pox only, are $\psi_1, \psi_2, \psi_3, \psi_4, \beta_1, \beta_2, \beta_3$ and β_r respectively.

Natural death rate of human population, Rodent population and vector population are μ_h, μ_r and μ_b respectively. Vaccination rate of susceptible human population against both Monkey pox and Typhoid fever γ_2 , The vaccine wane after some times at the rate γ_1 , Progression rate from Exposed human population to Monkey pox only infected human population σ_1 , Exposed human population to monkey pox only are quarantined at the rate ω , Exposed human population to both monkey pox and Typhoid fever are quarantined at the rate ϕ , Exposed human population to both monkey pox and Typhoid fever become infected with both monkey pox and Typhoid fever at the rate σ_2 , Progression rate from Quarantined human population of Monkey pox only to infected human population with Monkey pox only σ_3 , Infected human population with monkey pox only progresses to both monkey pox and Typhoid fever infected human population at the rate σ_4 , Infected human population with Typhoid fever only progresses to both monkey pox and Typhoid fever infected human population at the rate σ_5 , Humans infected with typhoid fever only are treated at the rate of α_1 , Humans infected with monkey pox only are treated at the rate of α_3 , Humans infected with both monkey pox and Typhoid fever are treated at the rate of α_2 , Humans infected with both monkey pox and Typhoid fever receive monkey pox only treatment at the rate

of α_4 , Humans infected with both monkey pox and Typhoid fever receive Typhoid fever only treatment at the rate of α_5 , Recovery rate due to treatment of Monkey pox only θ_3 , Recovery rate due to treatment of Typhoid fever only θ_1 , Recovery rate due to treatment of humans from both monkey pox and Typhoid fever θ_2 , Disease induced death rate of Monkey pox only infected humans, Disease induced death rate of Typhoid fever only infected humans, Disease induced death rate of humans on Monkey pox only treatment, Disease induced death rate of humans on Typhoid fever only treatment, Disease induced death rate of humans infected with both monkey pox and Typhoid fever, Disease induced death rate of humans on both monkey pox and Typhoid fever treatment, are $\delta_1, \delta_3, \delta_1, \delta_6, \delta_4, \delta_2$ and δ_5 respectively. The rate at which both monkey pox only and Typhoid fever only Recovered human population become susceptible again are ε_1 and ε_2 respectively.

In light of the description of the model above, the differential equations for the co-infection of Monkeypox and Typhoid fever is given below as:

$$\begin{aligned}
\frac{dS_h}{dt} &= \Lambda_h + \varepsilon_1 R_T + \varepsilon_2 R_m + \gamma_1 V_h - \lambda_T S_h - \lambda_m S_h - (\gamma_2 + \mu_h) S_h, \\
\frac{dE_T}{dt} &= \lambda_T S_h - \lambda_m E_T - (\sigma_1 + \mu_h) E_T, \\
\frac{dE_m}{dt} &= \lambda_m S_h - \lambda_T E_m - (\omega + \mu_h) E_m, \\
\frac{dE_{Tm}}{dt} &= \lambda_m E_T + \lambda_T E_m - (\sigma_2 + \phi + \mu_h) E_{Tm}, \\
\frac{dQ_m}{dt} &= \omega E_m + \phi E_{Tm} - (\sigma_3 + \mu_h) Q_m, \\
\frac{dI_T}{dt} &= \sigma_1 E_T - (\alpha_1 + \sigma_5 + \delta_1 + \mu_h) I_T, \\
\frac{dI_m}{dt} &= \sigma_3 Q_m - (\alpha_3 + \sigma_4 + \delta_3 + \mu_h) I_m, \\
\frac{dI_{Tm}}{dt} &= \sigma_2 E_{Tm} + \sigma_4 I_m + \sigma_5 I_T - (\alpha_2 + \alpha_4 + \alpha_5 + \delta_2 + \mu_h) I_{Tm}, \\
\frac{dT_T}{dt} &= \alpha_1 I_T + \alpha_5 I_{Tm} - (\theta_1 + \delta_4 + \mu_h) T_T, \\
\frac{dT_m}{dt} &= \alpha_3 I_m + \alpha_4 I_{Tm} - (\theta_3 + \delta_6 + \mu_h) T_m, \\
\frac{dT_{Tm}}{dt} &= \alpha_2 I_{Tm} - (\theta_2 + \delta_5 + \mu_h) T_{Tm},
\end{aligned} \tag{1}$$

$$\frac{dV_h}{dt} = \gamma_2 S_h - (\gamma_1 + \mu_h) V_h,$$

$$\frac{dR_T}{dt} = \theta_1 T_T - (\varepsilon_1 + \mu_h) R_T,$$

$$\frac{dR_m}{dt} = \theta_3 T_m - (\varepsilon_3 + \mu_h) R_m,$$

$$\frac{dR_{Tm}}{dt} = \theta_2 T_m - \mu_h R_{Tm},$$

$$\frac{dS_r}{dt} = \Lambda_r - \lambda_r S_r - \mu_r S_r,$$

$$\frac{dI_r}{dt} = \lambda_r S_r - \mu_r I_r,$$

$$\frac{dB}{dt} = \Lambda_B - \mu_B B.$$

Where

$$\lambda_T = \frac{(\psi_1 I_T + \psi_2 T_T + \psi_3 I_{Tm} + \psi_4 T_{Tm})}{N_h},$$

$$\lambda_m = \frac{(\beta_1 I_m + \beta_2 T_m + \beta_3 I_r + \psi_3 I_{Tm} + \psi_4 T_{Tm})}{N_h}$$

$$\lambda_r = \frac{\beta_r I_m S_r}{N_h}$$

Table 1: MODEL VARIABLES AND PARAMETERS DESCRIPTIONS

Variables	Descriptions
S_h	Susceptible humans to both monkey pox and Typhoid fever
E_M	Exposed human population to monkey pox only
E_T	Exposed human population to Typhoid fever only
E_{TM}	Exposed human population to both monkey pox and Typhoid fever
Q_M	Population of humans Quarantined of monkey pox only
I_M	Humans infected with monkey pox only
I_T	Humans infected with Typhoid fever only

I_{TM}	Humans infected with both monkey pox and Typhoid fever
T_M	Humans isolated for monkey pox only treatment
T_T	Humans on Typhoid fever only treatment
T_{TM}	Humans on treatment of both monkey pox and Typhoid fever
V_h	Vaccinated Humans against both monkey pox and Typhoid fever
R_M	Recovered humans from monkey pox only
R_T	Recovered humans from Typhoid fever only
R_{TM}	Recovered humans from both monkey pox and Typhoid fever
B	Bacteria population
S_r	Susceptible Rodent population to monkey pox only
I_r	Infected Rodent with monkey pox only
Parameters	Descriptions
Λ_h	Recruitment rate of human population
Λ_B	Recruitment rate of Bacteria population
Λ_r	Recruitment rate of Rodent population
ψ_1	Contact rate between the susceptible humans and infected human population with Typhoid fever only
ψ_2	Contact rate between the susceptible humans and human population on Typhoid fever only treatment
ψ_3	Contact rate between the susceptible humans and infected human population with both Monkey pox and Typhoid fever
ψ_4	Contact rate between the susceptible humans and human population on both Monkey pox and Typhoid fever treatment
β_1	Contact rate between the susceptible humans and infected human population with Monkey pox only
β_2	Contact rate between the susceptible humans and human population on Monkey pox only treatment
β_3	Contact rate between the susceptible humans and infected Rodents population with Monkey pox only
β_r	Contact rate between the susceptible Rodents and infected humans with Monkey pox only
μ_h	Natural death rate of human population
μ_B	Natural death rate of Bacteria population
μ_r	Natural death rate of Rodent population.
γ_2	Vaccination rate of susceptible human population against both diseases
γ_1	Waning rate of vaccine
σ_1	Progression rate from Exposed human population to Monkey pox only infected human population

ω	The rate at which Exposed human population to both monkey pox and Typhoid fever are quarantined
ϕ	The rate at which Exposed human population to both monkey pox and Typhoid fever become infected with both monkey pox and Typhoid fever
σ_2	Progression rate from Exposed human population to both Monkey pox and Typhoid fever to infected human population both Monkey pox and Typhoid fever
σ_3	Progression rate from Quarantined human population of Monkey pox only to infected human population with Monkey pox only
σ_4	The rate at which infected human population with monkey pox only progresses to both monkey pox and Typhoid fever infected human population
σ_5	The rate at which infected human population with Typhoid fever only progresses to both monkey pox and Typhoid fever infected human population
α_1	Treatment rate of monkey pox only infected humans
α_2	Treatment rate of both Monkey pox and Typhoid fever infected humans
α_3	Treatment rate of Typhoid fever only infected humans
α_4	The rate at which Humans infected with both monkey pox and Typhoid fever receive monkey pox only treatment
α_5	The rate at which Humans infected with both monkey pox and Typhoid fever receive Typhoid fever only treatment
θ_1	Recovery rate due to treatment of Typhoid fever only
θ_2	Recovery rate due to treatment of humans from both monkey pox and Typhoid fever
θ_3	Recovery rate due to treatment of Monkey pox only
δ_1	Disease induced death rate of Typhoid fever only infected humans
δ_2	Disease induced death rate of humans on both monkey pox and Typhoid fever treatment
δ_3	Disease induced death rate of humans on Monkey pox only treatment
δ_4	Disease induced death rate of humans on Typhoid fever only treatment
δ_5	Disease induced death rate of humans on both monkey pox and Typhoid fever treatment
δ_6	Disease induced death rate of humans on Monkey pox only treatment

Table 1 presents the variables and parameters used in the Monkeypox–Typhoid fever co-infection model [72, 73]. It defines each model variable (e.g., susceptible, exposed, infected, recovered compartments) and explains the meaning, units, and roles of all parameters (e.g., transmission rates, recovery rates, co-infection factors) used in the mathematical formulation.

3.0 Preliminaries

3.1 Background materials

The manuscript relies on Fractal fractional operator definitions which we will explain first. Let us define (α) to represent fractional-order while the symbol (σ) stands for fractal-dimension. When $x(t)$ remain continuous it also has fractal differentiability on (a,b) with (σ) , these definitions find their description in (Rahman 2022) as well as (Atagana et al. 2020).

Definition 1

The Caputo fractional differential operator makes use of the Mittag-Leffler Kernel to define the fractal fractional derivative of functions $x(t)$ at order (α) while operating on dimension (σ) as the follows:

$$FFHD_{0,t}^{\alpha,\sigma} x(t) = \frac{AB(\alpha)}{1-\alpha} \int_a^t E_{\sigma} \left[\frac{-\alpha}{1-\alpha} (t-s)^{\alpha} \right] \frac{dy(s)}{dS^{\sigma}} ds, \quad (2)$$

$$t \geq a \geq 0,$$

Where

$\theta, \sigma \in (0,1]$, $AB(\alpha) = 1 - \alpha + \alpha / \Gamma(\alpha)$ is the normalization function with $AB(0) = AB(1) = 1$ and

$$d_x(t) / d_t \sigma = \lim_{s \rightarrow t} (x(s) - x(t)) / (s^{\sigma} - t^{\sigma}).$$

Moreover, the Mittag-Leffler function is defined by;

$$E_{\alpha}(Z) = \sum_{k=0}^{\infty} Z^k \Gamma(\alpha k + 1), Z, \alpha \in C, \alpha > 0,$$

with C is the set the of complex number.

[63].

Definition 2

We present the equation for fractal fractional integration with the Mittag-Leffler Kernel in the Caputo sense of order (α) for dimension (σ) ;

$$FFHD_{0,t}^{\alpha,\sigma} x(t) = \frac{\sigma(1-\theta)t^{\sigma-1}x(t)}{AB(\alpha)} + \frac{\alpha\sigma}{AB(\alpha)} \int_a^t (t-s)^{\alpha-1} (3)$$

$$S^{\sigma-1}x(s)ds, t \geq a \geq 0. \quad (\text{Atagana et al. 2020, Granas and Dugundji 2003})$$

Definition 3

$x \in C((a,b), R)$, The fractal Laplace transform of order (α) is given as follows

$${}^F L_p^{\sigma} x(t) = \int_0^{\infty} x(t) \exp(-pt) t^{\sigma-1} dt \quad [63]$$

(4)

Lemma 1

Fractal fractional initial value problem is as follows:

$$\begin{cases} FFHD_{0,t}^{\alpha,\sigma} x(t) = G(t, x(t)), & t \in [0, T], \alpha, \sigma \in (0, 1] \\ x(0) = x_0. \end{cases} \quad (5)$$

The problem in (4) can be expressed as [42]

$$\begin{cases} FFHD_{0,t}^{\alpha,\sigma} x(t) = \sigma t^{\sigma-1} G(t, x(t)), & t \in [0, T], \alpha, \sigma \in (0, 1] \\ x(0) = x_0. \end{cases} \quad (6)$$

The corresponding integral equation defines the solution of problem (5) is as follow [64]:

$$x(t) = x(0) + \frac{\sigma t^{\alpha-1} (1-\rho) G(t, x(t))}{AB(\rho)} + \frac{\sigma \alpha}{AB(\alpha) \Gamma(\alpha)} \int_0^t S^{\sigma-1} (t-s)^{\alpha-1} G(S, x(s)) dS. \quad (7)$$

Several Lemmas became necessary for analyzing both uniqueness and existence in the proposed model.

Lemma 2

(Banach fixed point theorem)

The Banach space X contains a closed subset $D \subset X$, $D \neq \emptyset$ within it.

The operator P yields exactly one fixed point in D when it meets the contraction condition $P: D \rightarrow D$ [65].

]

Lemma 3

(Krasnoselskii's fixed point theorem)

The set D consists of every element from Y forming both a non empty subset and a closed as well as convex space [65, 66].

Let P and Q represent the two operators such that,

- $PU + QV \in D$, whenever $U, V \in D$,
- Q is compact and continuous
- P denotes the contraction mapping. then, there exist $k \in D$ such $Z = PK + QK$.

The new application utilizes the fractal fractional derivative operator to advance the Monkeypox – typhoid fever co-infection model (2.7) within the Mittag-Leffler Kernel framework $\alpha \in (0, 1]$ and $\sigma \in [0, 1]$, $FFHD_{0,t}^{\alpha,\sigma}$, as follows;

$$\begin{aligned}
FFHD_{0,t}^{\alpha,\sigma} S_h(t) &= G_1 \begin{pmatrix} t, S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), \\ T_T(t), T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), \\ B(t) \end{pmatrix}, \\
FFHD_{0,t}^{\alpha,\sigma} E_T(t) &= G_2 \begin{pmatrix} t, S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), \\ T_T(t), T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), \\ B(t) \end{pmatrix}, \\
FFHD_{0,t}^{\alpha,\sigma} E_m(t) &= G_3 \begin{pmatrix} t, S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), \\ T_T(t), T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), \\ B(t) \end{pmatrix}, \\
FFHD_{0,t}^{\alpha,\sigma} E_{Tm}(t) &= G_4 \begin{pmatrix} t, S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), \\ T_T(t), T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), \\ B(t) \end{pmatrix}, \\
FFHD_{0,t}^{\alpha,\sigma} Q_m(t) &= G_5 \begin{pmatrix} t, S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), \\ T_T(t), T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), \\ B(t) \end{pmatrix}, \\
FFHD_{0,t}^{\alpha,\sigma} I_T(t) &= G_6 \begin{pmatrix} t, S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), \\ T_T(t), T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), \\ B(t) \end{pmatrix}, \\
FFHD_{0,t}^{\alpha,\sigma} I_m(t) &= G_7 \begin{pmatrix} t, S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), \\ T_T(t), T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), \\ B(t) \end{pmatrix}, \\
FFHD_{0,t}^{\alpha,\sigma} I_{Tm}(t) &= G_8 \begin{pmatrix} t, S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), \\ T_T(t), T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), \\ B(t) \end{pmatrix}, \\
FFHD_{0,t}^{\alpha,\sigma} T_T(t) &= G_9 \begin{pmatrix} t, S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), \\ T_T(t), T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), \\ B(t) \end{pmatrix}, \\
FFHD_{0,t}^{\alpha,\sigma} T_m(t) &= G_{10} \begin{pmatrix} t, S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), \\ T_T(t), T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), \\ B(t) \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
FFHD_{0,t}^{\alpha,\sigma} T_{Tm}(t) &= G_{11} \begin{pmatrix} t, S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), \\ T_T(t), T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), \\ B(t) \end{pmatrix}, \\
FFHD_{0,t}^{\alpha,\sigma} V_h(t) &= G_{12} \begin{pmatrix} t, S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), \\ T_T(t), T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), \\ B(t) \end{pmatrix}, \\
FFHD_{0,t}^{\alpha,\sigma} R_T(t) &= G_{13} \begin{pmatrix} t, S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), \\ T_T(t), T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), \\ B(t) \end{pmatrix}, \\
FFHD_{0,t}^{\alpha,\sigma} R_m(t) &= G_{14} \begin{pmatrix} t, S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), \\ T_T(t), T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), \\ B(t) \end{pmatrix}, \\
FFHD_{0,t}^{\alpha,\sigma} R_{Tm}(t) &= G_{15} \begin{pmatrix} t, S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), \\ T_T(t), T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), \\ B(t) \end{pmatrix}, \\
FFHD_{0,t}^{\alpha,\sigma} S_r(t) &= G_{16} \begin{pmatrix} t, S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), \\ T_T(t), T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), \\ B(t) \end{pmatrix}, \\
FFHD_{0,t}^{\alpha,\sigma} I_r(t) &= G_{17} \begin{pmatrix} t, S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), \\ T_T(t), T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), \\ B(t) \end{pmatrix}, \\
FFHD_{0,t}^{\alpha,\sigma} B(t) &= G_{18} \begin{pmatrix} t, S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), \\ T_T(t), T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), \\ B(t) \end{pmatrix}.
\end{aligned}$$

The functions G_i of the developed model follow the sequence for $i = 1, 2, 3, \dots, 18$ below:

$$G_1 = \Lambda_h + \varepsilon_1 R_T + \varepsilon_2 R_m + \varepsilon_3 R_{Tm} + \gamma_1 V_h - \lambda_T S_h - \lambda_m S_h - (\gamma_2 + \mu_h) S_h,$$

$$G_2 = \lambda_T S_h - \lambda_m E_T - (\sigma_1 + \mu_h) E_T,$$

$$G_3 = \lambda_m S_h - \lambda_T E_m - (\omega + \mu_h) E_m,$$

$$G_4 = \lambda_m E_T + \lambda_T E_m - (\sigma_2 + \phi + \mu_h) E_{Tm},$$

$$\begin{aligned}
G_5 &= \omega E_m + \phi E_{Tm} - (\sigma_3 + \mu_h) Q_m, \\
G_6 &= \sigma_1 E_T - (\alpha_1 + \sigma_5 + \delta_1 + \mu_h) I_T, \\
G_7 &= \sigma_3 Q_m - (\alpha_3 + \sigma_4 + \delta_3 + \mu_h) I_m, \\
G_8 &= \sigma_2 E_{Tm} + \sigma_4 I_m + \sigma_5 I_T - (\alpha_2 + \alpha_4 + \alpha_5 + \delta_2 + \mu_h) I_{Tm}, \\
G_9 &= \alpha_1 I_T + \alpha_5 I_{Tm} - (\theta_1 + \delta_4 + \mu_h) T_T, \\
G_{10} &= \alpha_3 I_m + \alpha_4 I_{Tm} - (\theta_3 + \delta_6 + \mu_h) T_m, \\
G_{11} &= \alpha_2 I_{Tm} - (\theta_2 + \delta_5 + \mu_h) T_{Tm}, \\
G_{12} &= \gamma_2 S_h - (\gamma_1 + \mu_h) V_h, \\
G_{13} &= \theta_1 T_T - (\varepsilon_1 + \mu_h) R_T, \\
G_{14} &= \theta_3 T_m - (\varepsilon_3 + \mu_h) R_m, \\
G_{15} &= \theta_2 T_{Tm} - (\varepsilon_3 + \mu_h) R_{Tm}, \\
G_{16} &= \Lambda_r - \lambda_r S_r - \mu_r S_r, \\
G_{17} &= \lambda_r S_r - \mu_r I_r, \\
G_{18} &= \Lambda_B - \mu_B B.
\end{aligned}$$

With

$$\begin{aligned}
&S_{h0} \geq 0, E_{T0} \geq 0, E_{0m} \geq 0, E_{Tm0} \geq 0, Q_{m0} \geq 0, I_{T0} \geq 0, I_{m0} \geq 0, I_{Tm0} \geq 0, T_{T0} \geq 0, T_{m0} \geq 0, \\
&T_{Tm0} \geq 0, V_{h0} \geq 0, R_{T0} \geq 0, R_{m0} \geq 0, R_{Tm0} \geq 0, S_r \geq 0, I_r \geq 0, B_0 \geq 0.
\end{aligned}$$

The fractal fractional FF-Monkey pox and Typhoid fever co-infection model exists under (8).

We observed that if we let $\alpha = 1$ in the FF-Monkey pox and Typhoid fever co-infection model transforms into the (7) to create an integer-order model system.

The model gets simplified into the fractional Monkey pox and Typhoid fever co-infection model by letting $\sigma = 1$.

3.2 Model Analysis

3.3 Positivity of the model solution

$$R_+^{18} = \left\{ G \in R_+^{18} : G \geq 0 \text{ and } \left(\begin{array}{l} S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), \\ T_T(t), T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), \\ B(t) \end{array} \right)^T \right\}$$

The vector transform is expressed by $(.)^T$.

Theorem 1

G satisfies boundedness in R_+^{18} in addition to being non-negative for the solution of the FF-Monkey pox and Typhoid fever co-infection model described by (8).

Proof.

for $t \in (0, \infty)$, Existence and uniqueness of the FF-Monkey pox and Typhoid fever co-infection model (8) is obtained.

Consequently, The non-negative region R_+^{18} represents a positive invariant region according to the presented content. Our research used the FF-Monkey pox and Typhoid fever co-infection model described in (8) as the basis for our analysis;

$$FFHD_{0,t}^{\alpha,\sigma} S_h(t) = \Lambda_h + \varepsilon_1 R_T + \varepsilon_2 R_m + \varepsilon_3 R_{Tm} + \gamma_1 V_h \geq 0,$$

$$FFHD_{0,t}^{\alpha,\sigma} E_T(t) = \lambda_T S_h \geq 0,$$

$$FFHD_{0,t}^{\alpha,\sigma} E_m(t) = \lambda_m S_h \geq 0,$$

$$FFHD_{0,t}^{\alpha,\sigma} E_{Tm}(t) = \lambda_m E_T + \lambda_T E_m \geq 0,$$

$$FFHD_{0,t}^{\alpha,\sigma} Q_m(t) = \omega E_m + \phi E_{Tm} \geq 0,$$

$$FFHD_{0,t}^{\alpha,\sigma} I_T(t) = \sigma_1 E_T \geq 0,$$

$$FFHD_{0,t}^{\alpha,\sigma} I_m(t) = \sigma_3 Q_m \geq 0,$$

$$FFHD_{0,t}^{\alpha,\sigma} I_{Tm}(t) = \sigma_2 E_{Tm} + \sigma_4 I_m + \sigma_5 I_T \geq 0,$$

$$FFHD_{0,t}^{\alpha,\sigma} T_T(t) = \alpha_1 I_T + \alpha_5 I_{Tm} \geq 0,$$

$$FFHD_{0,t}^{\alpha,\sigma} T_m(t) = \alpha_3 I_m + \alpha_4 I_{Tm} \geq 0,$$

$$FFHD_{0,t}^{\alpha,\sigma} T_{Tm}(t) = \alpha_2 I_{Tm} \geq 0,$$

$$FFHD_{0,t}^{\alpha,\sigma} V_h(t) = \gamma_2 S_h \geq 0,$$

$$FFHD_{0,t}^{\alpha,\sigma} R_T(t) = \theta_1 T_T \geq 0,$$

$$FFHD_{0,t}^{\alpha,\sigma} R_m(t) = \theta_3 T_m \geq 0,$$

$$FFHD_{0,t}^{\alpha,\sigma} R_{Tm}(t) = \theta_2 T_{Tm} \geq 0,$$

If

$$(S_{h0}, E_{T0}, E_{m0}, E_{Tm0}, Q_{m0}, I_{T0}, I_{m0}, I_{Tm0}, T_{T0}, T_{m0}, T_{Tm0}, V_{h0}, R_{T0}, R_{m0}, R_{Tm0}, S_{r0}, I_{r0}, B_0) \in R_+^{18},$$

then, from the time interval from (10) solution G remains prevented from escaping through the hyper planes. Consequently, R_+^{18} is a positively invariant.

$$N_h = S_h + E_T + E_m, E_{Tm} + Q_m + I_T + I_m + I_{Tm} + T_T + T_m + T_{Tm} + V_h + R_T + R_m + R_{Tm},$$

$$N_r = S_r + I_r, \text{ and } N_B = B.$$

So from the FF-Monkey pox and Typhoid fever co-infection model(8) we obtained:

$$\left\{ FFHD_{0,t}^{\alpha,\sigma} N_h(t) = \Lambda_h - \mu_h N_h \right\}, \left\{ FFHD_{0,t}^{\alpha,\sigma} N_r(t) = \Lambda_r - \mu_r N_r \right\} \text{ and}$$

$$\left\{ FFHD_{0,t}^{\alpha,\sigma} N_B(t) = \Lambda_B - \mu_B N_B \right\} \quad (11)$$

Using the fractal Laplace transform Eq. (3), we conclude that,

$$N_h(t) \leq \frac{\Lambda_h}{\mu_h} \text{ as } t \rightarrow \infty, N_r(t) \leq \frac{\Lambda_r}{\mu_r} \text{ as } t \rightarrow \infty, \quad \text{And} \quad N_B(t) \leq \frac{\Lambda_B}{\mu_B} \text{ as } t \rightarrow \infty.$$

So we now have the biological range of values of the FF-Monkey pox and Typhoid fever co-infection model (8) as follows;

$$\left\{ (S_h, E_T, E_m, E_{Tm}, Q_m, I_T, I_m, I_{Tm}, T_T, T_m, T_{Tm}, V_h, R_T, R_m, R_{Tm}, S_r, I_r, B) \in R^{18} \right\} \text{ and } N_h(t) \leq \frac{\Lambda_h}{\mu_h} .$$

$$\left\{ : S_h, E_T, E_m, E_{Tm}, Q_m, I_T, I_m, I_{Tm}, T_T, T_m, T_{Tm}, V_h, R_T, R_m, R_{Tm}, S_r, I_r, B \geq 0 \right\}$$

$$\text{Similarly, } N_r(t) \leq \frac{\Lambda_r}{\mu_r} \text{ and } N_B(t) \leq \frac{\Lambda_B}{\mu_B} .$$

This shows that the Monkey pox and Typhoid fever co-infection model is biologically feasible.

3.4. Monkey pox sub-model

$$\frac{dS_h}{dt} = \Lambda_h + \varepsilon_2 R_m + \gamma_1 V_h - \lambda_m S_h - (\gamma_2 + \mu_h) S_h,$$

$$\frac{dE_m}{dt} = \lambda_m S_h - (\omega + \mu_h) E_m,$$

$$\frac{dQ_m}{dt} = \omega E_m - (\sigma_3 + \mu_h) Q_m,$$

$$\frac{dI_m}{dt} = \sigma_3 Q_m - (\alpha_3 + \delta_3 + \mu_h) I_m,$$

$$\frac{dT_m}{dt} = \alpha_3 I_m - (\theta_3 + \delta_6 + \mu_h) T_m,$$

$$\frac{dV_h}{dt} = \gamma_2 S_h - (\gamma_2 + \mu_h) V_h,$$

$$\frac{dR_m}{dt} = \theta_3 T_m - (\varepsilon_2 + \mu_h) R_m,$$

$$\frac{dS_r}{dt} = \Lambda_r - \lambda_r S_r - \mu_r S_r,$$

$$\frac{dI_r}{dt} = \lambda_r S_r - \mu_r I_r.$$

Where $\lambda_m = \frac{(\beta_1 I_m + \beta_2 T_m + \beta_r I_r)}{N_h}$ and $\lambda_r = \frac{\beta_r I_m}{N_h}$.

4.4.1 Equilibrium points and their stabilities

The FF-Monkey pox and Typhoid fever co-infection model demonstrates two steady states which occur when the right side of equivalence (8) displays zero value. Then, we obtain;

3.4.1 The Disease free Equilibrium point of Monkey pox and Typhoid fever co-infection

The disease-free equilibrium exists as a state with zero illness symptoms in human populations.

$$(S_h, E_T, E_m, E_{Tm}, Q_m, I_T, I_m, I_{Tm}, T_T, T_m, T_{Tm}, V_h, R_T, R_m, R_{Tm}, S_r, I_r, B) \\ = \left(\frac{\Lambda_h (\gamma_1 + \mu_h)}{\mu_h (\gamma_2 + \gamma_1 + \mu_h)}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \frac{\gamma_2 \Lambda_h}{\mu_h (\gamma_2 + \gamma_1 + \mu_h)}, 0, 0, 0, \frac{\Lambda_r}{\mu_r}, 0, \frac{\Lambda_B}{\mu_B} \right). \quad (12)$$

3.5 Monkey pox Basic Reproduction Number

Basic Reproduction number defines the complete number of infections which spread from one infected human to susceptible population. The determination of Basic Reproduction number happens through next generation method $R_0^T = \rho FV^{-1}$ by using F as a non-negative matrix with other transition terms V and determining the largest Eigen value ρ .

Monkey pox sub-model Basic Reproduction Number

$$F_M = \begin{pmatrix} 0 & 0 & \frac{\beta_1 (\gamma_1 + \mu_h)}{\gamma_2 + \gamma_1 + \mu_h} & \frac{\beta_2 (\gamma_1 + \mu_h)}{\gamma_2 + \gamma_1 + \mu_h} & \frac{\beta_r (\gamma_1 + \mu_h)}{\gamma_2 + \gamma_1 + \mu_h} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\beta_r \Lambda_r \mu_h}{\Lambda_h \mu_r} & 0 & 0 \end{pmatrix},$$

$$V_M = \begin{pmatrix} P_2 & 0 & 0 & 0 & 0 \\ -\omega & P_3 & 0 & 0 & 0 \\ 0 & -\sigma_3 & P_4 & 0 & 0 \\ 0 & 0 & -\alpha_3 & P_5 & 0 \\ 0 & 0 & 0 & 0 & P_9 \end{pmatrix},$$

$$V_M^{-1} = \begin{pmatrix} \frac{1}{P_2} & 0 & 0 & 0 & 0 \\ \frac{\omega}{P_2 P_3} & \frac{1}{P_3} & 0 & 0 & 0 \\ \frac{\sigma_3 \omega}{P_2 P_3 P_4} & \frac{\sigma_3}{P_3 P_4} & \frac{1}{P_4} & 0 & 0 \\ \frac{\alpha_3 \sigma_3 \omega}{P_2 P_3 P_4 P_5} & \frac{\alpha_3 \sigma_3}{P_3 P_4 P_5} & \frac{\alpha_3}{P_4 P_5} & \frac{1}{P_5} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{A_9} \end{pmatrix},$$

$$F_M V_M^{-1} = \begin{pmatrix} \frac{\beta_1 (\gamma_1 + \mu_h) \sigma_3 \omega}{(\gamma_2 + \gamma_1 + \mu_h) P_4 P_3 P_2} + \frac{\beta_2 (\gamma_1 + \mu_h) \alpha_3 \sigma_3 \omega}{(\gamma_2 + \gamma_1 + \mu_h) P_4 P_3 P_2 P_5} & \frac{\beta_1 (\gamma_1 + \mu_h) \sigma_3}{(\gamma_2 + \gamma_1 + \mu_h) P_4 P_3} + \frac{\beta_2 (\gamma_1 + \mu_h) \alpha_3 \sigma_3}{(\gamma_2 + \gamma_1 + \mu_h) P_4 P_3 P_5} & \frac{\beta_1 (\gamma_1 + \mu_h)}{(\gamma_2 + \gamma_1 + \mu_h) P_4} + \frac{\beta_2 (\gamma_1 + \mu_h) \alpha_3}{(\gamma_2 + \gamma_1 + \mu_h) P_4 P_5} & \frac{\beta_2 (\gamma_1 + \mu_h)}{(\gamma_2 + \gamma_1 + \mu_h) P_5} & \frac{\beta_1 (\gamma_1 + \mu_h)}{(\gamma_2 + \gamma_1 + \mu_h) P_9} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\beta_r \Lambda_r \mu_h \sigma_3 \omega}{\Lambda_h \mu_r P_3 P_2 P_4} & \frac{\beta_r \Lambda_r \mu_h \sigma_3}{\Lambda_h \mu_r P_3 P_4} & \frac{\beta_r \Lambda_r \mu_h}{\Lambda_h \mu_r P_4} & 0 & 0 \end{pmatrix},$$

Taking the characteristic polynomial of the $F_M V_M^{-1}$, we have:

$$\lambda^5 - W_1 \lambda^4 - W_2 \lambda^3 = 0$$

We now factorizing we have:

$$\lambda^3 (\lambda^2 - W_1 \lambda - W_2) = 0,$$

$$\lambda^2 - W_1 \lambda - W_2 = 0,$$

solving the quadratic equation using the formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, we obtained:

$$\lambda = \frac{1}{2} (W_1 + \sqrt{W_1^2 + 4W_2}), \text{ and } \lambda = \frac{1}{2} (W_1 - \sqrt{W_1^2 + 4W_2}).$$

This implies that $R_0^M = \frac{1}{2} (W_1 + \sqrt{W_1^2 + 4W_2})$. which is the dominant Eigen value.

Where

$$W_1 = \frac{(\gamma_1 + \mu_h) \sigma_3 \omega (\beta_1 P_5 + \beta_2 \alpha_3)}{(\gamma_2 + \gamma_1 + \mu_h) P_4 P_3 P_2 P_5},$$

$$W_2 = \frac{\beta_r^2 \Lambda_r \mu_h \sigma_3 \omega (\gamma_1 + \mu_h)}{\Lambda_h \mu_r P_3 P_2 P_4 (\gamma_2 + \gamma_1 + \mu_h) P_9}.$$

(13)

$$P_1 = (\gamma_2 + \mu_h), P_2 = (\omega + \mu_h), P_3 = (\sigma_3 + \mu_h), P_4 = (\alpha_3 + \delta_3 + \mu_h),$$

$$P_5 = (\theta_3 + \delta_6 + \mu_h), P_6 = (\gamma_1 + \mu_h), K_7 = (\varepsilon_2 + \mu_h), K_8 = \mu_r, K_9 = \mu_r.$$

3.6 Monkey pox sub-model Endemic Equilibrium point

Endemic Equilibrium signifies the permanent existence of Monkey pox among human population.

$$S_h^* = -\frac{A_7 A_6 A_5 A_4 A_3 A_2 \Lambda_h}{\omega A_6 \alpha_3 \partial_2 \lambda_m \sigma_3 \theta_3 - A_1 A_2 A_3 A_4 A_5 A_6 A_7 - A_2 A_3 A_4 A_5 A_6 A_7 \lambda_m + A_2 A_3 A_4 A_5 A_7 \gamma_1 \gamma_2},$$

$$E_m^* = -\frac{\Lambda_h A_3 A_4 A_5 A_6 A_7 \lambda_m}{\omega A_6 \alpha_3 \partial_2 \lambda_m \sigma_3 \theta_3 - A_1 A_2 A_3 A_4 A_5 A_6 A_7 - A_2 A_3 A_4 A_5 A_6 A_7 \lambda_m + A_2 A_3 A_4 A_5 A_7 \gamma_1 \gamma_2},$$

$$Q_m^* = -\frac{\omega \Lambda_h A_4 A_5 A_6 A_7 \lambda_m}{\omega A_6 \alpha_3 \partial_2 \lambda_m \sigma_3 \theta_3 - A_1 A_2 A_3 A_4 A_5 A_6 A_7 - A_2 A_3 A_4 A_5 A_6 A_7 \lambda_m + A_2 A_3 A_4 A_5 A_7 \gamma_1 \gamma_2},$$

$$I_m^* = -\frac{\Lambda_h \omega A_5 A_6 A_7 \lambda_m \sigma_3}{\omega A_6 \alpha_3 \partial_2 \lambda_m \sigma_3 \theta_3 - A_1 A_2 A_3 A_4 A_5 A_6 A_7 - A_2 A_3 A_4 A_5 A_6 A_7 \lambda_m + A_2 A_3 A_4 A_5 A_7 \gamma_1 \gamma_2},$$

$$T_m^* = -\frac{\Lambda_h \omega A_6 A_7 \lambda_m \sigma_3 \alpha_3}{\omega A_6 \alpha_3 \partial_2 \lambda_m \sigma_3 \theta_3 - A_1 A_2 A_3 A_4 A_5 A_6 A_7 - A_2 A_3 A_4 A_5 A_6 A_7 \lambda_m + A_2 A_3 A_4 A_5 A_7 \gamma_1 \gamma_2},$$

$$V_m^* = -\frac{\gamma_2 A_7 A_5 A_4 A_3 A_2 \Lambda_h}{\omega A_6 \alpha_3 \partial_2 \lambda_m \sigma_3 \theta_3 - A_1 A_2 A_3 A_4 A_5 A_6 A_7 - A_2 A_3 A_4 A_5 A_6 A_7 \lambda_m + A_2 A_3 A_4 A_5 A_7 \gamma_1 \gamma_2},$$

$$R_m^* = -\frac{\theta_3 \alpha_3 \sigma_3 \lambda_m A_6 \Lambda_h \omega}{\omega A_6 \alpha_3 \partial_2 \lambda_m \sigma_3 \theta_3 - A_1 A_2 A_3 A_4 A_5 A_6 A_7 - A_2 A_3 A_4 A_5 A_6 A_7 \lambda_m + A_2 A_3 A_4 A_5 A_7 \gamma_1 \gamma_2},$$

$$S_r^* = 2\Lambda_r^2 (\omega P_6 \alpha_3 \partial_2 \lambda_m \sigma_3 \theta_3 - P_1 P_2 P_3 P_4 P_5 P_6 P_7 - P_2 P_3 P_4 P_5 P_6 \lambda_m P_7 + P_2 P_3 P_4 P_5 \gamma_1 \gamma_2 P_7)$$

$$- \omega \Lambda_h P_5 P_6 P_9 \beta_r \lambda_m \sigma_3 P_7 + 2\omega \Lambda_r P_6 P_8 \alpha_3 \partial_2 \lambda_m \sigma_3 \theta_3 - \omega \Lambda_r P_6 P_9 \alpha_3 \partial_2 \lambda_m \sigma_3 \theta_3$$

$$- 2\Lambda_r P_1 P_2 P_3 P_4 P_5 P_6 P_8 P_7 + \Lambda_r P_1 P_2 P_3 P_4 P_5 P_6 P_9 P_7 - 2\Lambda_r P_2 P_3 P_4 P_5 P_6 P_8 \lambda_m P_7$$

$$+ \Lambda_r P_2 P_3 P_4 P_5 P_6 P_9 \lambda_m + 2\Lambda_r P_2 P_3 P_4 P_5 P_8 \gamma_1 \gamma_2 P_7 - \Lambda_r P_2 P_3 P_4 P_5 P_9 \gamma_1 \gamma_2 P_7$$

$$+ \omega^2 \Lambda_h^2 P_5^2 P_6^2 P_9 \beta_r^2 \lambda_m^2 \sigma_3^2 P_7^2,$$

$$I_r^* = -\left((P_4 P_2 ((-\lambda_m - P_1) P_6 + \gamma_1 \gamma_2) P_3 \Lambda_r + \omega \beta_r \sigma_3 \Lambda_h \lambda_m P_6) P_5 P_7 + \omega \Lambda_r P_6 \alpha_3 \partial_2 \lambda_m \sigma_3 \theta_3 \right)^2 P_9$$

$$- 4\lambda_m \sigma_3 \omega \beta_r P_6 P_5 \Lambda_r \Lambda_h P_8 (P_4 P_2 ((-\lambda_m - P_1) P_6 + \gamma_1 \gamma_2) P_5 P_3 P_7 + \omega P_6 \alpha_3 \partial_2 \lambda_m \sigma_3 \theta_3).$$

Substituting into the force of infection:

$$\lambda_m = \frac{(\beta_1 I_m + \beta_2 T_m + \beta_3 I_r)}{N_h} \text{ and } \lambda_r = \frac{\beta_r I_m}{N_h}.$$

We have:

$$\begin{aligned} Q_1 = & \omega^2 A_4 A_5 A_6 A_7 \Lambda_h P_4 P_5 P_6 P_8 P_9 P_7 + \omega^2 A_4 A_5 A_6 A_7 \Lambda_h P_5 P_6 P_8 P_9 \sigma_3 P_7 + \omega^2 A_4 A_5 A_6 A_7 \Lambda_h P_6 P_8 P_9 \alpha_3 \sigma_3 \theta_3 \\ & + \omega^2 A_4 A_5 A_6 A_7 \Lambda_h P_6 P_8 P_9 \alpha_3 \sigma_3 P_7 + \omega^2 A_5 A_6 A_7 \Lambda_h P_4 P_5 P_6 P_8 P_9 \sigma_3 P_7 + \omega^2 A_5 A_6 A_7 \Lambda_h P_4 P_5 P_6 P_9 \beta_r \sigma_3 P_7 \\ & + \omega^2 A_5 A_6 A_7 \Lambda_h P_5 P_6 P_8 P_9 \sigma_3^2 P_7 + \omega^2 A_5 A_6 A_7 \Lambda_h P_5 P_6 P_9 \beta_r \sigma_3^2 P_7 + \omega^2 A_5 A_6 A_7 \Lambda_h P_6 P_8 P_9 \alpha_3 \sigma_3^2 \theta_3 \\ & + \omega^2 A_5 A_6 A_7 \Lambda_h P_6 P_8 P_9 \alpha_3 \sigma_3^2 P_7 + \omega^2 A_5 A_6 A_7 \Lambda_h P_6 P_9 \alpha_3 \beta_r \sigma_3^2 \theta_3 + \omega^2 A_5 A_6 A_7 \Lambda_h P_6 P_9 \alpha_3 \beta_r \sigma_3^2 P_7 \\ & + \omega^2 A_6 A_7 \Lambda_h P_4 P_5 P_6 P_8 P_9 \alpha_3 \sigma_3 P_7 + \omega^2 A_6 A_7 \Lambda_h P_5 P_6 P_8 P_9 \alpha_3 \sigma_3^2 P_7 + \omega^2 A_6 A_7 \Lambda_h P_6 P_8 P_9 \alpha_3^2 \sigma_3^2 \theta_3 \\ & + \omega^2 A_6 A_7 \Lambda_h P_6 P_8 P_9 \alpha_3^2 \sigma_3^2 P_7 + \omega^2 A_6 \Lambda_h P_4 P_5 P_6 P_8 P_9 \alpha_3 \sigma_3 \theta_3 P_7 + \omega^2 A_6 \Lambda_h P_5 P_6 P_8 P_9 \alpha_3 \sigma_3^2 \theta_3 P_7 \\ & + \omega^2 A_6 \Lambda_h P_6 P_8 P_9 \alpha_3^2 \sigma_3^2 \theta_3^2 + \omega^2 A_6 \Lambda_h P_6 P_8 P_9 \alpha_3^2 \sigma_3^2 \theta_3 P_7, \end{aligned}$$

$$\begin{aligned} Q_2 = & -\omega^2 A_4 A_5 A_6 A_7 \Lambda_h P_5 P_6 P_8 P_9 \beta_1 \sigma_3 P_7 - \omega^2 A_4 A_5 A_6 A_7 \Lambda_h P_6 P_8 P_9 \alpha_3 \beta_2 \sigma_3 P_7 \\ & - \omega^2 A_5 A_6 A_7 \Lambda_h P_5 P_6 P_8 P_9 \beta_1 \sigma_3^2 P_7 - \omega^2 A_5 A_6 A_7 \Lambda_h P_5 P_6 P_9 \beta_1 \beta_r \sigma_3^2 P_7 \\ & - \omega^2 A_5 A_6 A_7 \Lambda_h P_6 P_8 P_9 \alpha_3 \beta_2 \sigma_3^2 P_7 - \omega^2 A_5 A_6 A_7 \Lambda_h P_6 P_9 \alpha_3 \beta_2 \beta_r \sigma_3^2 P_7 \\ & + \omega^2 A_5 A_6 A_7 \Lambda_h P_6 \alpha_3 \beta_3 \beta_r \sigma_3^2 \theta_3 - \omega^2 A_6 A_7 \Lambda_h P_5 P_6 P_8 P_9 \alpha_3 \beta_1 \sigma_3^2 P_7 \\ & - \omega^2 A_6 A_7 \Lambda_h P_6 P_8 P_9 \alpha_3^2 \beta_2 \sigma_3^2 P_7 - \omega^2 A_6 \Lambda_h P_5 P_6 P_8 P_9 \alpha_3 \beta_1 \sigma_3^2 \theta_3 P_7 \\ & - \omega^2 A_6 \Lambda_h P_6 P_8 P_9 \alpha_3^2 \beta_2 \sigma_3^2 \theta_3 P_7, \end{aligned}$$

$$Q_3 = P_1 P_2 \left(1 - (R_0^m)^2 \right).$$

This implies that the above model (1) has a stable endemic equilibrium point.

3.7 Monkey pox sub-model Sensitivity Analysis

Sensitivity analysis help to determine which population disease transmission parameters create positive effects. It can be determine using: $S_x^{R_0} = \left(\frac{\partial R_0}{\partial x} \right) \left(\frac{x}{R_0} \right).$

$$\begin{aligned} S_{\omega}^{R_0^M} &= 0.245762, S_{\theta_3}^{R_0^M} = 0.170559, S_{\sigma_3}^{R_0^M} = 0.877612, S_{\beta_1}^{R_0^M} = 0.00262876, S_{\beta_2}^{R_0^M} = 0.0157568, \\ S_{\alpha_3}^{R_0^M} &= 0.133877, S_{\delta_3}^{R_0^M} = 0.13725020, S_{\delta_6}^{R_0^M} = 0.0396335, S_{\Lambda_h}^{R_0^M} = -0.1200525, S_{\Lambda_r}^{R_0^M} = 0.07200, \\ S_{\mu_r}^{R_0^M} &= -0.00046, S_{\mu_h}^{R_0^M} = -0.00564. \end{aligned}$$

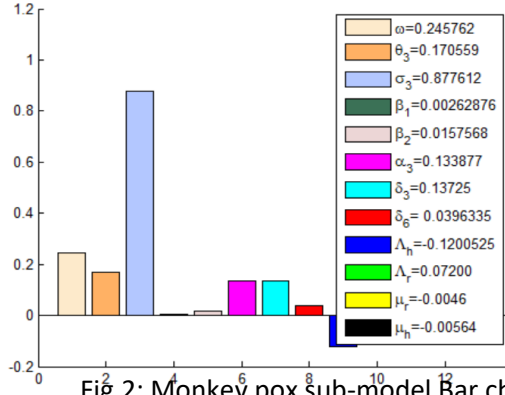


Fig.2: Monkey pox sub-model Bar chart

3.8 Interpretation of Monkey pox Sensitivity Bar chart

The above figure 2 displays the sensitivity indices of basic reproduction number for Monkey pox model diseases [70, 72]. Positive indices for FF-Monkey pox model parameters determine their capability to escalate disease transmission when their values rise. An increase in value for any parameter changes the basic reproduction number in a similar direction. When disease-burden reducing parameters increase in value they function as disease protective elements that lead to decreased basic reproduction numbers.

3.9 Typhoid fever sub-model

$$\frac{dS_h}{dt} = \Lambda_h + \varepsilon_1 R_T + \gamma_1 V_h - \lambda_T S_h - (\gamma_2 + \mu_h) S_h,$$

$$\frac{dE_T}{dt} = \lambda_T S_h - (\sigma_1 + \mu_h) E_T,$$

$$\frac{dI_T}{dt} = \sigma_1 E_T - (\alpha_1 + \delta_1 + \mu_h) I_T,$$

$$\frac{dT_T}{dt} = \alpha_1 I_T - (\theta_1 + \delta_4 + \mu_h) T_T,$$

$$\frac{dV_h}{dt} = \gamma_2 S_h - (\gamma_2 + \mu_h) V_h,$$

$$\frac{dR_T}{dt} = \theta_1 T_T - (\varepsilon_1 + \mu_h) R_T.$$

$$\text{Where } \lambda_i = \frac{(\psi_1 I_T + \psi_2 T_T + \psi_3 B)}{N_h}.$$

3.9.1. Typhoid fever sub-model Basic Reproduction Number

$$F_T = \begin{pmatrix} 0 & \frac{\psi_1(\gamma_1 + \mu_h)}{\gamma_2 + \gamma_1 + \mu_h} & \frac{\psi_2(\gamma_1 + \mu_h)}{\gamma_2 + \gamma_1 + \mu_h} & \frac{\psi_5(\gamma_1 + \mu_h)}{\gamma_2 + \gamma_1 + \mu_h} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$V_T = \begin{pmatrix} K_2 & 0 & 0 & 0 \\ -\sigma_1 & K_3 & 0 & 0 \\ 0 & -\alpha_1 & K_4 & 0 \\ 0 & 0 & 0 & K_7 \end{pmatrix},$$

$$V_T^{-1} = \begin{pmatrix} \frac{1}{K_2} & 0 & 0 & 0 \\ \frac{\sigma_1}{K_3 K_2} & \frac{1}{K_3} & 0 & 0 \\ \frac{\alpha_1 \sigma_1}{K_3 K_2 K_4} & \frac{\alpha_1}{K_3 K_4} & \frac{1}{K_4} & 0 \\ 0 & 0 & 0 & \frac{1}{K_7} \end{pmatrix},$$

$$FV_T^{-1} = \begin{pmatrix} \frac{\psi_1(\gamma_1 + \mu_h)\sigma_1}{(\gamma_2 + \gamma_1 + \mu_h)K_2 K_3} + \frac{\psi_2(\gamma_1 + \mu_h)\alpha_1 \sigma_1}{(\gamma_2 + \gamma_1 + \mu_h)K_2 K_3 K_4} & \frac{\psi_1(\gamma_1 + \mu_h)}{(\gamma_2 + \gamma_1 + \mu_h)K_3} + \frac{\psi_2(\gamma_1 + \mu_h)\alpha_1}{(\gamma_2 + \gamma_1 + \mu_h)K_3 K_4} & \frac{\psi_2(\gamma_1 + \mu_h)}{(\gamma_2 + \gamma_1 + \mu_h)K_4} & \frac{\psi_5(\gamma_1 + \mu_h)}{(\gamma_2 + \gamma_1 + \mu_h)K_7} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\text{Eigen value} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{\sigma_1(K_4 \gamma_1 \psi_1 + K_4 \mu_h \psi_1 + \alpha_1 \gamma_1 \psi_2 + \alpha_1 \mu_h \psi_2)}{(\gamma_2 + \gamma_1 + \mu_h)K_2 K_3 K_4} \end{pmatrix},$$

$$R_0^T = \frac{\sigma_1(K_4 \gamma_1 \psi_1 + K_4 \mu_h \psi_1 + \alpha_1 \gamma_1 \psi_2 + \alpha_1 \mu_h \psi_2)}{(\gamma_2 + \gamma_1 + \mu_h)K_2 K_3 K_4}.$$

Where

$$K_1 = (\gamma_2 + \mu_h), K_2 = (\sigma_1 + \mu_h), K_3 = (\alpha_1 + \delta_1 + \mu_h), K_4 = (\theta_1 + \delta_4 + \mu_h), K_5 = (\gamma_1 + \mu_h), \\ K_6 = (\varepsilon_1 + \mu_h), K_7 = \mu_B.$$

3.10 Endemic Equilibrium point of Typhoid fever sub-model

$$S_h^* = \frac{\Lambda_h K_2 K_3 K_4 K_5}{((\lambda_t + K_1) K_5 - \gamma_1 \gamma_2) K_4 K_3 K_2 - \alpha_1 \dot{\lambda}_t \sigma_1 \theta_1},$$

$$E_t^* = \frac{\Lambda_h K_3 K_4 K_5 \lambda_t}{K_2 K_3 K_4 (K_1 K_5 + K_5 \lambda_t - \gamma_1 \gamma_2) - \alpha_1 \dot{\lambda}_t \sigma_1 \theta_1},$$

$$I_t^* = \frac{\Lambda_h K_4 K_5 \lambda_t \sigma_1}{K_2 K_3 K_4 (K_1 K_5 + K_5 \lambda_t - \gamma_1 \gamma_2) - \alpha_1 \dot{\lambda}_t \sigma_1 \theta_1},$$

$$T_t^* = \frac{\Lambda_h K_5 \lambda_t \sigma_1 \alpha_1}{K_2 K_3 K_4 (K_1 K_5 + K_5 \lambda_t - \gamma_1 \gamma_2) - \alpha_1 \dot{\lambda}_t \sigma_1 \theta_1},$$

$$V_h^* = \frac{\Lambda_h K_2 K_3 K_4 \gamma_2}{K_2 K_3 K_4 (K_1 K_5 + K_5 \lambda_t - \gamma_1 \gamma_2) - \alpha_1 \dot{\lambda}_t \sigma_1 \theta_1},$$

$$R_T^* = -\frac{\theta_1 \alpha_1 \sigma_1 \lambda_t \Lambda_h}{-K_1 K_2 K_3 K_4 K_5 - K_2 K_3 K_4 K_5 \lambda_t + K_2 K_3 K_4 \gamma_1 \gamma_2 + \alpha_1 \dot{\lambda}_t \sigma_1 \theta_1},$$

$$B^* = \frac{\Lambda_b}{K_6}.$$

Substituting into the force of infection: $\lambda_T = \frac{(\psi_1 I_T + \psi_2 T_T + \psi_3 B)}{N_h}$. we have:

Where $Q_1 = (K_3 K_4 K_5 K_6 \Lambda_h + K_4 K_5 K_6 \Lambda_h \sigma_1 + K_5 K_6 \Lambda_h \alpha_1 \sigma_1 + K_6 \Lambda_h \alpha_1 \sigma_1 \theta_1)$,

$$Q_2 = K_5 K_6 \left(1 - \frac{\sigma_1 (K_4 \gamma_1 \psi_1 + K_4 \mu_h \psi_1 + \alpha_1 \gamma_1 \psi_2 + \alpha_1 \mu_h \psi_2)}{(\gamma_2 + \gamma_1 + \mu_h) K_2 K_3 K_4} \right),$$

$$Q_2 = K_5 K_6 (1 - R_0^T).$$

This implies that the endemic of Typhoid fever sub-model is stable.

3.11. Sensitivity Analysis of Typhoid Fever sub-model

$$S_{\sigma_1}^{R_0^T} = 1,$$

$$S_{\theta_4}^{R_0^T} = -\frac{\alpha_1 \psi_2 \theta_4}{((\theta_4 + \delta_4 + \mu_h) \psi_1 + \psi_2 \alpha_1) (\theta_4 + \delta_4 + \mu_h)} = -0.05432,$$

$$S_{\delta_4}^{R_0^T} = -\frac{\alpha_1 \psi_2 \delta_4}{((\theta_4 + \delta_4 + \mu_h) \psi_1 + \psi_2 \alpha_1)(\theta_4 + \delta_4 + \mu_h)} = -0.6812,$$

$$S_{\gamma_1}^{R_0^T} = \frac{\gamma_1 \gamma_2}{(\gamma_1 + \mu_h)(\gamma_2 + \gamma_1 + \mu_h)} = 0.1267,$$

$$S_{\gamma_2}^{R_0^T} = -\frac{(2\gamma_2 + \gamma_1 + 2\mu_h)\gamma_2}{(\gamma_2 + \mu_h)(\gamma_2 + \gamma_1 + \mu_h)} = -0.0741,$$

$$S_{\psi_1}^{R_0^T} = \frac{((\theta_4 + \delta_4 + \mu_h)\gamma_1 + (\theta_4 + \delta_4 + \mu_h)\mu_h)\psi_1}{(\theta_4 + \delta_4 + \mu_h)\gamma_1\psi_1 + (\theta_4 + \delta_4 + \mu_h)\mu_h\psi_1 + \alpha_1\gamma_1\psi_2 + \alpha_1\mu_h\psi_2} = 0.8483,$$

$$S_{\psi_2}^{R_0^T} = \frac{(\alpha_1\gamma_1 + \alpha_1\mu_h)\psi_2}{(\theta_4 + \delta_4 + \mu_h)\gamma_1\psi_1 + (\theta_4 + \delta_4 + \mu_h)\mu_h\psi_1 + \alpha_1\gamma_1\psi_2 + \alpha_1\mu_h\psi_2} = 0.5902,$$

$$S_{\alpha_1}^{R_0^T} = -\frac{((\theta_4 + \delta_4 + \mu_h)\psi_1 - \psi_2(\mu_h + \delta_1))\alpha_1}{((\theta_4 + \delta_4 + \mu_h)\psi_1 + \psi_2\alpha_1)(\alpha_1 + \delta_1 + \mu_h)} = -0.4965,$$

$$S_{\delta_1}^{R_0^T} = -\frac{\delta_1}{\alpha_1 + \delta_1 + \mu_h} = -0.0587.$$

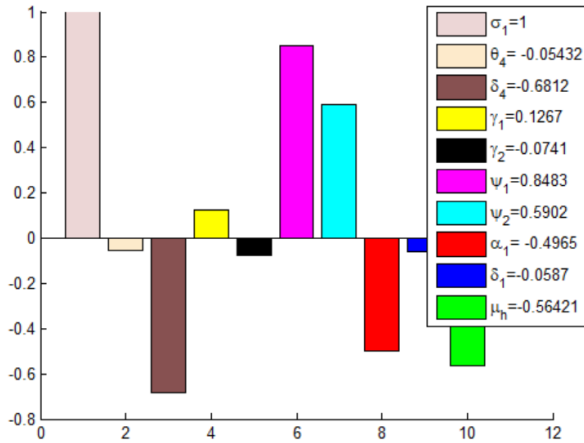


Fig.3: Typhoid fever sub-model Bar chart

3.11.1 Interpretation of Typhoid fever Sensitivity Bar chart

The above figure 3 displays the sensitivity indices of basic reproduction number for Typhoid fever and Typhoid fever co-infection model diseases [72, 73]. Positive indices for FF- Typhoid fever model parameters determine their capability to escalate disease transmission when their values rise. An increase in value for any parameter changes the basic reproduction number in a similar

direction. When disease-burden reducing parameters increase in value they function as disease protective elements that lead to decreased basic reproduction numbers

3.12 Qualitative Analysis of the Fractal fractional Monkey pox and co-infection model

For the FF-Monkey pox and Typhoid fever co-infection model (8) to demonstrate its qualitative results we need this specified Banach space. $\tau = [0, T]$ of all continuous real-valued functions denoted as:

$Y = C(\tau \times R^{18}, R)$ equipped with the norm

$$\begin{aligned} \|x\| &= \|(S_h, E_T, E_m, E_{Tm}, Q_m, I_T, I_m, I_{Tm}, T_T, T_m, T_{Tm}, V_h, R_T, R_m, R_{Tm}, S_r, I_r, B)\| \\ &= \|S_h\| + \|E_T\| + \|E_m\| + \|E_{Tm}\| + \|Q_m\| + \|I_T\| + \|I_m\| + \|I_{Tm}\| + \|T_T\| \\ &\quad + \|T_m\| + \|T_{Tm}\| + \|V_h\| + \|R_T\| + \|R_m\| + \|R_{Tm}\| + \|S_r\| + \|I_r\| + \|B\|. \end{aligned}$$

$$\|S_h\| = \sup_{t \in \tau} |S_h(t)| = P_{h1},$$

$$\|E_T\| = \sup_{t \in \tau} |E_T(t)| = P_{h2},$$

$$\|E_m\| = \sup_{t \in \tau} |E_m(t)| = P_{h3},$$

$$\|E_{Tm}\| = \sup_{t \in \tau} |E_{Tm}(t)| = P_{h4},$$

$$\|Q_m\| = \sup_{t \in \tau} |Q_m(t)| = P_{h5},$$

$$\|I_T\| = \sup_{t \in \tau} |I_T(t)| = P_{h6},$$

$$\|I_m\| = \sup_{t \in \tau} |I_m(t)| = P_{h7},$$

$$\|I_{Tm}\| = \sup_{t \in \tau} |I_{Tm}(t)| = P_{h8},$$

$$\|T_T\| = \sup_{t \in \tau} |T_T(t)| = P_{h9},$$

$$\|T_m\| = \sup_{t \in \tau} |T_m(t)| = P_{h10},$$

$$\|T_{Tm}\| = \sup_{t \in \tau} |T_{Tm}(t)| = P_{h11},$$

$$\|V_h\| = \sup_{t \in \tau} |V_h(t)| = P_{h12},$$

$$\|R_T\| = \sup_{t \in \tau} |R_T(t)| = P_{h13},$$

$$\|R_m\| = \sup_{t \in \tau} |R_m(t)| = P_{h14},$$

$$\|R_{Tm}\| = \sup_{t \in \tau} |R_{Tm}(t)| = P_{h15},$$

$$\|S_r\| = \text{Sup}_{t \in \tau} |S_r(t)| = P_{h16},$$

$$\|I_r\| = \text{Sup}_{t \in \tau} |I_r(t)| = P_{h17},$$

$$\|B\| = \text{Sup}_{t \in \tau} |B(t)| = P_{h18}.$$

Where $(S_h, E_T, E_m, E_{Tm}, Q_m, I_T, I_m, I_{Tm}, T_T, T_m, T_{Tm}, V_h, R_T, R_m, R_{Tm}, S_r, I_r, B) \in X$.

$G \in X$ and $x \in C(\tau, R)$, Then, the FF-Monkey pox and Typhoid fever co-infection model (8) can be expressed as;

$$\begin{cases} {}^{FFHD}_{0,t}^{\alpha, \sigma} x(t) = G(t, x(t)), t \in [0, T], \alpha, \sigma \in (0, 1] \\ x(0) = x_0 \end{cases} \quad (26)$$

$$\text{Where } x(t) = \begin{pmatrix} S_h(t) \\ E_T(t) \\ E_m(t) \\ E_{Tm}(t) \\ Q_m(t) \\ I_T(t) \\ I_m(t) \\ I_{Tm}(t) \\ T_T(t) \\ T_m(t) \\ T_{Tm}(t) \\ V_h(t) \\ R_T(t) \\ R_m(t) \\ R_{Tm}(t) \\ S_r(t) \\ I_r(t) \\ B(t) \end{pmatrix}, x(0) = \begin{pmatrix} S_h(0) \\ E_T(0) \\ E_m(0) \\ E_{Tm}(0) \\ Q_m(0) \\ I_T(0) \\ I_m(0) \\ I_{Tm}(0) \\ T_T(0) \\ T_m(0) \\ T_{Tm}(0) \\ V_h(0) \\ R_T(0) \\ R_m(0) \\ R_{Tm}(0) \\ S_r(0) \\ I_r(0) \\ B(0) \end{pmatrix}, G(t, x(t)) = \begin{pmatrix} G_1(t, S_h(t)) \\ G_2(t, E_T(t)) \\ G_3(t, E_m(t)) \\ G_4(t, E_{Tm}(t)) \\ G_5(t, Q_m(t)) \\ G_6(t, I_T(t)) \\ G_7(t, I_m(t)) \\ G_8(t, I_{Tm}(t)) \\ G_9(t, T_T(t)) \\ G_{10}(t, T_m(t)) \\ G_{11}(t, T_{Tm}(t)) \\ G_{12}(t, V_h(t)) \\ G_{13}(t, R_T(t)) \\ G_{14}(t, R_m(t)) \\ G_{15}(t, R_{Tm}(t)) \\ G_{16}(t, S_r(t)) \\ G_{17}(t, I_r(t)) \\ G_{18}(t, B(t)) \end{pmatrix}. \quad (27)$$

The solution of problem (25) corresponds to this integral equation when $G_i, i = (1, 2, \dots, 18)$ are defined as (8);

$$x(t) = x(0) + \frac{\sigma t^{\sigma-1} (1-\alpha) G(t, x(t))}{AB(\alpha)} + \frac{\sigma \alpha}{AB\alpha\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} G(s, x(s)) ds, \quad (28)$$

We simplify the calculations by introducing the following symbols in this work.

$$\begin{aligned} G_1(t, S_h(t)) &= G_1 \left(t, S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), T_T(t), \right. \\ &\quad \left. T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), B(t) \right), \\ G_2(t, E_T(t)) &= G_2 \left(t, S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), T_T(t), \right. \\ &\quad \left. T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), B(t) \right), \\ G_3(t, E_m(t)) &= G_3 \left(t, S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), T_T(t), \right. \\ &\quad \left. T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), B(t) \right), \\ G_4(t, E_{Tm}(t)) &= G_4 \left(t, S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), T_T(t), \right. \\ &\quad \left. T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), B(t) \right), \\ G_5(t, Q_m(t)) &= G_5 \left(t, S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), T_T(t), \right. \\ &\quad \left. T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), B(t) \right), \\ G_6(t, I_T(t)) &= G_6 \left(t, S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), T_T(t), \right. \\ &\quad \left. T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), B(t) \right), \\ G_7(t, I_m(t)) &= G_7 \left(t, S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), T_T(t), \right. \\ &\quad \left. T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), B(t) \right), \\ G_8(t, I_{Tm}(t)) &= G_8 \left(t, S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), T_T(t), \right. \\ &\quad \left. T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), B(t) \right), \\ G_9(t, T_T(t)) &= G_9 \left(t, S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), T_T(t), \right. \\ &\quad \left. T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), B(t) \right), \\ G_{10}(t, T_m(t)) &= G_{10} \left(t, S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), T_T(t), \right. \\ &\quad \left. T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), B(t) \right), \\ G_{11}(t, T_{Tm}(t)) &= G_{11} \left(t, S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), T_T(t), \right. \\ &\quad \left. T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), B(t) \right), \\ G_{12}(t, V_h(t)) &= G_{12} \left(t, S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), T_T(t), \right. \\ &\quad \left. T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), B(t) \right), \end{aligned}$$

$$\begin{aligned}
G_{13}(t, R_T(t)) &= G_{13} \begin{pmatrix} t, S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), T_T(t), \\ T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), B(t) \end{pmatrix}, \\
G_{14}(t, R_m(t)) &= G_{14} \begin{pmatrix} t, S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), T_T(t), \\ T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), B(t) \end{pmatrix}, \\
G_{15}(t, R_{Tm}(t)) &= G_{15} \begin{pmatrix} t, S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), T_T(t), \\ T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), B(t) \end{pmatrix}, \\
G_{16}(t, S_r(t)) &= G_{16} \begin{pmatrix} t, S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), T_T(t), \\ T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), B(t) \end{pmatrix}, \\
G_{17}(t, I_r(t)) &= G_{17} \begin{pmatrix} t, S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), T_T(t), \\ T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), B(t) \end{pmatrix}, \\
G_{18}(t, B(t)) &= G_{18} \begin{pmatrix} t, S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), T_T(t), \\ T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), B(t) \end{pmatrix}.
\end{aligned}$$

The FF-Monkey pox and Typhoid fever co-infection model (8) represents the following set of Volterra integral equations according to Lemma 1.

$$\begin{aligned}
S_h(t) &= S_h(0) + \frac{\sigma t^{\sigma-1}(1-\alpha)G_1(t, S_h(t))}{AB(\alpha)} + \frac{\sigma\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t S_h^{\sigma-1}(t-s)^{\alpha-1} G_1(s, S_h(s)) ds, \\
E_T(t) &= E_T(0) + \frac{\sigma t^{\sigma-1}(1-\alpha)G_2(t, E_T(t))}{AB(\alpha)} + \frac{\sigma\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t E_T^{\sigma-1}(t-s)^{\alpha-1} G_2(s, E_T(s)) ds, \\
E_m(t) &= E_m(0) + \frac{\sigma t^{\sigma-1}(1-\alpha)G_3(t, E_m(t))}{AB(\alpha)} + \frac{\sigma\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t E_m^{\sigma-1}(t-s)^{\alpha-1} G_3(s, E_m(s)) ds, \\
E_{Tm}(t) &= E_{Tm}(0) + \frac{\sigma t^{\sigma-1}(1-\alpha)G_4(t, E_{Tm}(t))}{AB(\alpha)} + \frac{\sigma\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t E_{Tm}^{\sigma-1}(t-s)^{\alpha-1} G_4(s, E_{Tm}(s)) ds, \\
Q_m(t) &= Q_m(0) + \frac{\sigma t^{\sigma-1}(1-\alpha)G_5(t, Q_m(t))}{AB(\alpha)} + \frac{\sigma\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t Q_m^{\sigma-1}(t-s)^{\alpha-1} G_5(s, Q_m(s)) ds, \\
I_T(t) &= I_T(0) + \frac{\sigma t^{\sigma-1}(1-\alpha)G_6(t, I_T(t))}{AB(\alpha)} + \frac{\sigma\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t I_T^{\sigma-1}(t-s)^{\alpha-1} G_6(s, I_T(s)) ds,
\end{aligned}$$

$$\begin{aligned}
I_m(t) &= I_m(0) + \frac{\sigma t^{\sigma-1}(1-\alpha)G_7(t, I_m(t))}{AB(\alpha)} + \frac{\sigma\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t I_m^{\sigma-1}(t-s)^{\alpha-1} G_7(s, I_m(s)) ds, \\
I_{Tm}(t) &= I_{Tm}(0) + \frac{\sigma t^{\sigma-1}(1-\alpha)G_8(t, I_{Tm}(t))}{AB(\alpha)} + \frac{\sigma\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t I_{Tm}^{\sigma-1}(t-s)^{\alpha-1} G_8(s, I_{Tm}(s)) ds, \\
T_T(t) &= T_T(0) + \frac{\sigma t^{\sigma-1}(1-\alpha)G_9(t, T_T(t))}{AB(\alpha)} + \frac{\sigma\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t T_T^{\sigma-1}(t-s)^{\alpha-1} G_9(s, T_T(s)) ds, \\
T_m(t) &= T_m(0) + \frac{\sigma t^{\sigma-1}(1-\alpha)G_{10}(t, T_m(t))}{AB(\alpha)} + \frac{\sigma\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t T_m^{\sigma-1}(t-s)^{\alpha-1} G_{10}(s, T_m(s)) ds, \\
T_{Tm}(t) &= T_{Tm}(0) + \frac{\sigma t^{\sigma-1}(1-\alpha)G_{11}(t, T_{Tm}(t))}{AB(\alpha)} + \frac{\sigma\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t T_{Tm}^{\sigma-1}(t-s)^{\alpha-1} G_{11}(s, T_{Tm}(s)) ds, \\
V_h(t) &= V_h(0) + \frac{\sigma t^{\sigma-1}(1-\alpha)G_{12}(t, V_h(t))}{AB(\alpha)} + \frac{\sigma\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t V_h^{\sigma-1}(t-s)^{\alpha-1} G_{12}(s, V_h(s)) ds, \\
R_T(t) &= R_T(0) + \frac{\sigma t^{\sigma-1}(1-\alpha)G_{13}(t, R_T(t))}{AB(\alpha)} + \frac{\sigma\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t R_T^{\sigma-1}(t-s)^{\alpha-1} G_{13}(s, R_T(s)) ds, \\
R_m(t) &= R_m(0) + \frac{\sigma t^{\sigma-1}(1-\alpha)G_{14}(t, R_m(t))}{AB(\alpha)} + \frac{\sigma\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t R_m^{\sigma-1}(t-s)^{\alpha-1} G_{14}(s, R_m(s)) ds, \\
R_{Tm}(t) &= R_{Tm}(0) + \frac{\sigma t^{\sigma-1}(1-\alpha)G_{15}(t, R_{Tm}(t))}{AB(\alpha)} + \frac{\sigma\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t R_{Tm}^{\sigma-1}(t-s)^{\alpha-1} G_{15}(s, R_{Tm}(s)) ds, \\
S_r(t) &= S_r(0) + \frac{\sigma t^{\sigma-1}(1-\alpha)G_{16}(t, S_r(t))}{AB(\alpha)} + \frac{\sigma\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t S_r^{\sigma-1}(t-s)^{\alpha-1} G_{16}(s, S_r(s)) ds, \\
I_r(t) &= I_r(0) + \frac{\sigma t^{\sigma-1}(1-\alpha)G_{17}(t, I_r(t))}{AB(\alpha)} + \frac{\sigma\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t I_r^{\sigma-1}(t-s)^{\alpha-1} G_{17}(s, I_r(s)) ds, \\
B(t) &= B(0) + \frac{\sigma t^{\sigma-1}(1-\alpha)G_{18}(t, B(t))}{AB(\alpha)} + \frac{\sigma\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t B^{\sigma-1}(t-s)^{\alpha-1} G_{18}(s, B(s)) ds.
\end{aligned}$$

In view of (30), we define an operator,

$\tau : X \rightarrow X$, where

$$\tau = (\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6, \tau_7, \tau_8, \tau_9, \tau_{10}, \tau_{11}, \tau_{13}, \tau_{14}, \tau_{15}, \tau_{16}, \tau_{17}, \tau_{18}),$$

$$\begin{aligned}
(\tau_1, S_h)(t) &= S_h(0) + \frac{\sigma t^{\sigma-1}(1-\alpha)G_1(t, S_h(t))}{AB(\alpha)} + \frac{\sigma\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t S_h^{\sigma-1}(t-s)^{\rho-1} G_1(s, S_h(s)) ds, \\
(\tau_2, E_T)(t) &= E_T(0) + \frac{\sigma t^{\sigma-1}(1-\alpha)G_2(t, E_T(t))}{AB(\alpha)} + \frac{\sigma\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t E_T^{\sigma-1}(t-s)^{\rho-1} G_2(s, E_T(s)) ds, \\
(\tau_3, E_m)(t) &= E_m(0) + \frac{\sigma t^{\sigma-1}(1-\alpha)G_3(t, E_m(t))}{AB(\alpha)} + \frac{\sigma\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t E_m^{\sigma-1}(t-s)^{\rho-1} G_3(s, E_m(s)) ds, \\
(\tau_4, E_{Tm})(t) &= E_{Tm}(0) + \frac{\sigma t^{\sigma-1}(1-\alpha)G_4(t, E_{Tm}(t))}{AB(\alpha)} + \frac{\sigma\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t E_{Tm}^{\sigma-1}(t-s)^{\rho-1} G_4(s, E_{Tm}(s)) ds, \\
(\tau_5, Q_m)(t) &= Q_m(0) + \frac{\sigma t^{\sigma-1}(1-\alpha)G_5(t, Q_m(t))}{AB(\alpha)} + \frac{\sigma\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t Q_m^{\sigma-1}(t-s)^{\rho-1} G_5(s, Q_m(s)) ds, \\
(\tau_6, I_T)(t) &= I_T(0) + \frac{\sigma t^{\sigma-1}(1-\alpha)G_6(t, I_T(t))}{AB(\alpha)} + \frac{\sigma\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t I_T^{\sigma-1}(t-s)^{\rho-1} G_6(s, I_T(s)) ds, \\
(\tau_7, I_m)(t) &= I_m(0) + \frac{\sigma t^{\sigma-1}(1-\alpha)G_7(t, I_m(t))}{AB(\alpha)} + \frac{\sigma\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t I_m^{\sigma-1}(t-s)^{\rho-1} G_7(s, I_m(s)) ds, \\
(\tau_8, I_{Tm})(t) &= I_{Tm}(0) + \frac{\sigma t^{\sigma-1}(1-\alpha)G_8(t, I_{Tm}(t))}{AB(\alpha)} + \frac{\sigma\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t I_{Tm}^{\sigma-1}(t-s)^{\rho-1} G_8(s, I_{Tm}(s)) ds, \\
(\tau_9, T_T)(t) &= T_T(0) + \frac{\sigma t^{\sigma-1}(1-\alpha)G_9(t, T_T(t))}{AB(\alpha)} + \frac{\sigma\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t T_T^{\sigma-1}(t-s)^{\rho-1} G_9(s, T_T(s)) ds, \\
(\tau_{10}, T_m)(t) &= T_m(0) + \frac{\sigma t^{\sigma-1}(1-\alpha)G_{10}(t, T_m(t))}{AB(\alpha)} + \frac{\sigma\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t T_m^{\sigma-1}(t-s)^{\rho-1} G_{10}(s, T_m(s)) ds, \\
(\tau_{11}, T_{Tm})(t) &= T_{Tm}(0) + \frac{\sigma t^{\sigma-1}(1-\alpha)G_{11}(t, T_{Tm}(t))}{AB(\alpha)} + \frac{\sigma\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t T_{Tm}^{\sigma-1}(t-s)^{\rho-1} G_{11}(s, T_{Tm}(s)) ds, \\
(\tau_{12}, V_h)(t) &= V_h(0) + \frac{\sigma t^{\sigma-1}(1-\alpha)G_{12}(t, V_h(t))}{AB(\alpha)} + \frac{\sigma\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t V_h^{\sigma-1}(t-s)^{\rho-1} G_{12}(s, V_h(s)) ds, \\
(\tau_{13}, R_T)(t) &= R_T(0) + \frac{\sigma t^{\sigma-1}(1-\alpha)G_{13}(t, R_T(t))}{AB(\alpha)} + \frac{\sigma\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t R_T^{\sigma-1}(t-s)^{\rho-1} G_{13}(s, R_T(s)) ds, \\
(\tau_{14}, R_m)(t) &= R_m(0) + \frac{\sigma t^{\sigma-1}(1-\alpha)G_{14}(t, R_m(t))}{AB(\alpha)} + \frac{\sigma\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t R_m^{\sigma-1}(t-s)^{\rho-1} G_{14}(s, R_m(s)) ds, \\
(\tau_{15}, R_{Tm})(t) &= R_{Tm}(0) + \frac{\sigma t^{\sigma-1}(1-\alpha)G_{15}(t, R_{Tm}(t))}{AB(\alpha)} + \frac{\sigma\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t R_{Tm}^{\sigma-1}(t-s)^{\rho-1} G_{15}(s, R_{Tm}(s)) ds,
\end{aligned}$$

$$\begin{aligned}
(\tau_{16}, S_r)(t) &= S_r(0) + \frac{\sigma t^{\sigma-1}(1-\alpha)G_{16}(t, S_r(t))}{AB(\alpha)} + \frac{\sigma\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t S_r^{\sigma-1}(t-s)^{\rho-1} G_{16}(s, S_r(s)) ds, \\
(\tau_{17}, I_r)(t) &= I_r(0) + \frac{\sigma t^{\sigma-1}(1-\alpha)G_{17}(t, I_r(t))}{AB(\alpha)} + \frac{\sigma\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t I_r^{\sigma-1}(t-s)^{\rho-1} G_{17}(s, I_r(s)) ds, \\
(\tau_{18}, B)(t) &= B(0) + \frac{\sigma t^{\sigma-1}(1-\alpha)G_{18}(t, B(t))}{AB(\alpha)} + \frac{\sigma\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t B^{\sigma-1}(t-s)^{\rho-1} G_{18}(s, B(s)) ds.
\end{aligned}
\tag{30}$$

We transform the examined model into a fixed point problem for fixed point theory investigation. ($x = \tau x$), This will apply to create a fixed theory.

Our next section demonstrates the FF-Monkey pox and Typhoid fever co-infection model (8) achieves a single solution.

Theorem 2

Let $G \in Y$ satisfying the assumption below B_1 .

There exist a constant

$$L_{\max} = \max \{L_1, L_2, L_3, L_4, L_5, L_6, L_7, L_8, L_9, L_{10}, L_{11}, L_{12}, L_{13}, L_{14}, L_{15}, L_{16}, L_{17}, L_{18}\},$$

such that,

$$\begin{aligned}
& \left| G \begin{pmatrix} t, S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), T_T(t), \\ T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), B(t) \end{pmatrix} \right. \\
& \left. - G \begin{pmatrix} t, S_h^*(t), E_T^*(t), E_m^*(t), E_{Tm}^*(t), Q_m^*(t), I_T^*(t), I_m^*(t), I_{Tm}^*(t), T_T^*(t), \\ T_m^*(t), T_{Tm}^*(t), V_h^*(t), R_T^*(t), R_m^*(t), R_{Tm}^*(t), S_r^*(t), I_r^*(t), B^*(t) \end{pmatrix} \right| \\
& \leq L_{\max} \begin{pmatrix} |S_h(t) - S_h^*(t)| + |E_T(t) - E_T^*(t)| + |E_m(t) - E_m^*(t)| + |E_{Tm}(t) - E_{Tm}^*(t)| \\ + |Q_m(t) - Q_m^*(t)| + |I_T(t) - I_T^*(t)| + |I_m(t) - I_m^*(t)| + |I_{Tm}(t) - I_{Tm}^*(t)| \\ + |T_T(t) - T_T^*(t)| + |T_m(t) - T_m^*(t)| + |T_{Tm}(t) - T_{Tm}^*(t)| + |V_h(t) - V_h^*(t)| \\ + |R_T(t) - R_T^*(t)| + |R_m(t) - R_m^*(t)| + |R_{Tm}(t) - R_{Tm}^*(t)| + |S_r(t) - S_r^*(t)| \\ + |I_r(t) - I_r^*(t)| + |B(t) - B^*(t)| \end{pmatrix}.
\end{aligned}$$

(32)

For any

$$\begin{aligned} & \left(S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), T_T(t), \right. \\ & \left. T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), B(t) \right) \in X \text{ and } t \in \tau. \\ & \left(\sigma(1-\alpha)T_{\min}^{\alpha-1} + \frac{\sigma T^{\alpha+\sigma-1}\Gamma(\sigma+1)}{\Gamma(\sigma+1)} \right) \frac{L_{\max}}{AB\sigma} < 1, \end{aligned} \quad (33)$$

Then, FF- Monkey pox and Typhoid fever co-infection model (8) has a unique solution.

Proof

Let D_{K1} be a bounded, closed and convex subset.

$$\text{Where } D = \left\{ \begin{aligned} & \left(S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), T_T(t), \right. \\ & \left. T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), B(t) \right) \in X \\ & : \left\| \begin{pmatrix} S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), T_T(t), \\ T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), B(t) \end{pmatrix} \right\| \leq K_1 \end{aligned} \right\}$$

with a radius;

$$\begin{aligned} & \alpha_{\max} + \left(\sigma(1-\alpha)T_{\min}^{\alpha-1} + \frac{\alpha T^{\alpha+\sigma-1}\Gamma(\sigma+1)}{\Gamma(\alpha+\sigma)} \right) \frac{G_{\max}^*}{AB(\alpha)} \\ & 1 - \left(\sigma(1-\sigma)\alpha T_{\min}^{\alpha-1} + \frac{\rho T^{\alpha+\alpha-1}\Gamma(\sigma+1)}{\Gamma(\alpha+\sigma)} \right) \frac{L_{\max}^*}{AB(\alpha)} \end{aligned} \quad (34)$$

$$\text{Where } \alpha_{\max} = \max \left\{ \begin{pmatrix} S_{h0}(t), E_{T0}(t), E_{m0}(t), E_{Tm0}(t), Q_{m0}(t), I_{T0}(t), I_{m0}(t), I_{Tm0}(t), T_{T0}(t), \\ T_{m0}(t), T_{Tm0}(t), V_{h0}(t), R_{T0}(t), R_{m0}(t), R_{Tm0}(t), S_{r0}(t), I_{r0}(t), B_0(t) \end{pmatrix} \right\},$$

$$G_{\max}^* = \max \{ G_1^*, G_2^*, G_3^*, G_4^*, G_5^*, G_6^*, G_7^*, G_8^*, G_9^*, G_{10}^*, G_{11}^*, G_{12}^*, G_{13}^*, G_{14}^*, G_{15}^*, G_{16}^*, G_{17}^*, G_{18}^* \},$$

and let

$$\sup_{t \in \tau} |G_i(s, 0)| = G_i^* < +\infty, i = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18.$$

First we show that, $\tau D_{K1} \subset D_{K1}$.

$$\text{For any } \left(\begin{pmatrix} S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), T_T(t), \\ T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), B(t) \end{pmatrix} \right) \in D_{K1}, t \in \tau,$$

we obtained that,

$$|(\tau_1 S_h)(t)| \leq |S_h(0)| + \sigma t^{\alpha-1} (1-\alpha) |G_1(t, S_h(t))| + \frac{\alpha \sigma}{AB(\alpha) \Gamma \alpha} \int_0^t S_h^{\sigma-1}(t-s)^{\alpha-1} |G_1(s, S_h(s))| ds,$$

$$\begin{aligned}
&\leq |S_h(0)| + \frac{\sigma t^{\alpha-1}(1-\alpha)}{AB(\alpha)} \left[L_1 \left(\begin{aligned} &|S_h(t)| + |E_T(t)| + |E_m(t)| + |E_{Tm}(t)| + |Q_m(t)| + |I_T(t)| \\ &+ |I_m(t)| + |I_{Tm}(t)| + |T_T(t)| + |T_m(t)| + |T_{Tm}(t)| + |V_h(t)| \\ &+ |R_T(t)| + |R_m(t)| + |R_{Tm}(t)| + |S_r(t)| + |I_r(t)| + |B(t)| \end{aligned} \right) + G_1^* \right] \\
&+ \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \int_0^t S^{\sigma-1}(t-s)^{\alpha-1} \left[L_1 \left(\begin{aligned} &|S_h(s)| + |E_T(s)| + |E_m(s)| + |E_{Tm}(s)| + |Q_m(s)| \\ &+ |I_T(s)| + |I_m(s)| + |I_{Tm}(s)| + |T_T(s)| + |T_m(s)| \\ &+ |T_{Tm}(s)| + |V_h(s)| + |R_T(s)| + |R_m(s)| + |R_{Tm}(s)| \\ &+ |S_r(s)| + |I_r(s)| + |B(s)| \end{aligned} \right) + G_1^* \right] ds, \quad (35) \\
&\leq S_{h0} + \left(\frac{\sigma(1-\alpha)T_{\min}^{\alpha-1}}{AB(\alpha)} + \frac{\sigma T^{\alpha+\sigma-1}\Gamma(\sigma+1)}{AB(\alpha)\Gamma(\alpha+\sigma)} \right) \\
&\left[L_1 \left(\begin{aligned} &\|S_h\| + \|E_T\| + \|E_m\| + \|E_{Tm}\| + \|Q_m\| + \|I_T\| + \|I_m\| + \|I_{Tm}\| + \|T_T\| \\ &+ \|T_m\| + \|T_{Tm}\| + \|V_h\| + \|R_T\| + \|R_m\| + \|R_{Tm}\| + \|S_r\| + \|I_r\| + \|B\| \end{aligned} \right) + G_1^* \right].
\end{aligned}$$

We now obtain:

$$\|\tau_1 S_h\| \leq S_{h0} + \left(\frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\rho)} + \frac{\alpha T^{\alpha+\sigma-1}\Gamma(\sigma+1)}{AB(\alpha)\Gamma(\alpha+\sigma)} \right) [L_1 K_1 + G_1^*],$$

(36)

Similarly, we have:

$$\begin{aligned}
\|\tau_2 E_T\| &\leq E_{T0} + \left(\frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)} + \frac{\alpha T^{\alpha+\sigma-1}\Gamma(\sigma+1)}{AB(\alpha)\Gamma(\alpha+\sigma)} \right) [L_2 K_2 + G_2^*], \\
\|\tau_3 E_m\| &\leq E_{m0} + \left(\frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)} + \frac{\alpha T^{\alpha+\sigma-1}\Gamma(\sigma+1)}{AB(\alpha)\Gamma(\alpha+\sigma)} \right) [L_3 K_3 + G_3^*], \\
\|\tau_4 E_{Tm}\| &\leq E_{Tm0} + \left(\frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)} + \frac{\alpha T^{\alpha+\sigma-1}\Gamma(\sigma+1)}{AB(\alpha)\Gamma(\alpha+\sigma)} \right) [L_4 K_4 + G_4^*], \\
\|\tau_5 Q_m\| &\leq Q_{m0} + \left(\frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)} + \frac{\alpha T^{\alpha+\sigma-1}\Gamma(\sigma+1)}{AB(\alpha)\Gamma(\alpha+\sigma)} \right) [L_5 K_5 + G_5^*], \\
\|\tau_6 I_T\| &\leq I_{T0} + \left(\frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)} + \frac{\alpha T^{\alpha+\sigma-1}\Gamma(\sigma+1)}{AB(\alpha)\Gamma(\alpha+\sigma)} \right) [L_6 K_6 + G_6^*], \\
\|\tau_7 I_m\| &\leq I_{m0} + \left(\frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)} + \frac{\alpha T^{\alpha+\sigma-1}\Gamma(\sigma+1)}{AB(\alpha)\Gamma(\alpha+\sigma)} \right) [L_7 K_7 + G_7^*], \\
\|\tau_8 I_{Tm}\| &\leq I_{Tm0} + \left(\frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)} + \frac{\alpha T^{\alpha+\sigma-1}\Gamma(\sigma+1)}{AB(\alpha)\Gamma(\alpha+\sigma)} \right) [L_8 K_8 + G_8^*],
\end{aligned}$$

$$\begin{aligned}
\|\tau_9 T_T\| &\leq I_{Tm0} + \left(\frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)} + \frac{\alpha T^{\alpha+\sigma-1}\Gamma(\sigma+1)}{AB(\alpha)\Gamma(\alpha+\sigma)} \right) [L_9 K_9 + G_9^*], \\
\|\tau_{10} T_m\| &\leq T_{m0} + \left(\frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)} + \frac{\alpha T^{\alpha+\sigma-1}\Gamma(\sigma+1)}{AB(\alpha)\Gamma(\alpha+\sigma)} \right) [L_{10} K_{10} + G_{10}^*], \\
\|\tau_{11} T_{Tm}\| &\leq T_{Tm0} + \left(\frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)} + \frac{\alpha T^{\alpha+\sigma-1}\Gamma(\sigma+1)}{AB(\alpha)\Gamma(\alpha+\sigma)} \right) [L_{11} K_{11} + G_{11}^*], \\
\|\tau_{12} V_h\| &\leq V_{h0} + \left(\frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)} + \frac{\alpha T^{\alpha+\sigma-1}\Gamma(\sigma+1)}{AB(\alpha)\Gamma(\alpha+\sigma)} \right) [L_{12} K_{12} + G_{12}^*], \\
\|\tau_{13} R_T\| &\leq R_{T0} + \left(\frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)} + \frac{\alpha T^{\alpha+\sigma-1}\Gamma(\sigma+1)}{AB(\alpha)\Gamma(\alpha+\sigma)} \right) [L_{13} K_{13} + G_{13}^*], \\
\|\tau_{14} R_m\| &\leq R_{m0} + \left(\frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)} + \frac{\alpha T^{\alpha+\sigma-1}\Gamma(\sigma+1)}{AB(\alpha)\Gamma(\alpha+\sigma)} \right) [L_{14} K_{14} + G_{14}^*], \\
\|\tau_{15} R_{Tm}\| &\leq R_{Tm0} + \left(\frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)} + \frac{\alpha T^{\alpha+\sigma-1}\Gamma(\sigma+1)}{AB(\alpha)\Gamma(\alpha+\sigma)} \right) [L_{15} K_{15} + G_{15}^*], \\
\|\tau_{15} R_{Tm}\| &\leq R_{Tm0} + \left(\frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)} + \frac{\alpha T^{\alpha+\sigma-1}\Gamma(\sigma+1)}{AB(\alpha)\Gamma(\alpha+\sigma)} \right) [L_{15} K_{15} + G_{15}^*], \\
\|\tau_{16} S_r\| &\leq S_{r0} + \left(\frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)} + \frac{\alpha T^{\alpha+\sigma-1}\Gamma(\sigma+1)}{AB(\alpha)\Gamma(\alpha+\sigma)} \right) [L_{16} K_{16} + G_{16}^*], \\
\|\tau_{17} I_r\| &\leq I_{r0} + \left(\frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)} + \frac{\alpha T^{\alpha+\sigma-1}\Gamma(\sigma+1)}{AB(\alpha)\Gamma(\alpha+\sigma)} \right) [L_{17} K_{17} + G_{17}^*], \\
\|\tau_{18} B\| &\leq B_0 + \left(\frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)} + \frac{\alpha T^{\alpha+\sigma-1}\Gamma(\sigma+1)}{AB(\alpha)\Gamma(\alpha+\sigma)} \right) [L_{18} K_{18} + G_{18}^*].
\end{aligned}$$

From (36) - (37), it implies that, $\tau D_{K1} \subset DK_{K1}$.

Secondly, we show that τ is a contraction.

We assume that,

$$\begin{aligned}
&\left(S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), T_T(t), \right. \\
&\quad \left. T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), B(t) \right) \in D_{K1} \\
&\text{and} \left(S_h^*(t), E_T^*(t), E_m^*(t), E_{Tm}^*(t), Q_m^*(t), I_T^*(t), I_m^*(t), I_{Tm}^*(t), T_T^*(t), \right. \\
&\quad \left. T_m^*(t), T_{Tm}^*(t), V_h^*(t), R_T^*(t), R_m^*(t), R_{Tm}^*(t), S_r^*(t), I_r^*(t), B^*(t) \right) \in D_{K1}
\end{aligned}$$

We have:

$$\left| (\tau_1 S_h)(t) - (\tau_1 S_h^*)(t) \right|$$

$$\begin{aligned}
&\leq \frac{\sigma t^{\sigma-1}(1-\alpha)}{AB(\alpha)} \left| G_1(t, S_h(t) - G_1(t, S_h^*(t))) \right| + \frac{\sigma \alpha}{AB(\alpha) \Gamma(\alpha)} \int_0^t S_h^{\alpha-1}(t-s)^{\alpha-1} \left| G_1(s, S_h(s) - G_1(s, S_h^*(s))) \right| ds, \\
&\leq \frac{\sigma t^{\sigma-1}(1-\alpha)}{AB(\alpha)} L_1 \left[\begin{aligned} &\left(|S_h(t) - S_h^*(t)| + |E_T(t) - E_T^*(t)| + |E_m(t) - E_m^*(t)| + |E_{Tm}(t) - E_{Tm}^*(t)| \right. \\ &+ |Q_m(t) - Q_m^*(t)| + |I_T(t) - I_T^*(t)| + |I_m(t) - I_m^*(t)| + |I_{Tm}(t) - I_{Tm}^*(t)| \\ &+ |T_T(t) - T_T^*(t)| + |T_m(t) - T_m^*(t)| + |T_{Tm}(t) - T_{Tm}^*(t)| + |V_h(t) - R_h^*(t)| \\ &+ |R_T(t) - R_T^*(t)| + |R_m(t) - R_m^*(t)| + |R_{Tm}(t) - R_{Tm}^*(t)| + |S_r(t) - S_r^*(t)| \\ &\left. + |I_r(t) - I_r^*(t)| + |B(t) - B^*(t)| \right) \end{aligned} \right] \\
&+ \frac{\sigma \alpha}{AB(\alpha) \Gamma(\alpha)} \int_0^t S^{\sigma-1}(t-s)^{\alpha-1} L_1 \left[\begin{aligned} &\left(|S_h(s) - S_h^*(s)| + |E_T(s) - E_T^*(s)| + |E_m(s) - E_m^*(s)| + |E_{Tm}(s) - E_{Tm}^*(s)| \right. \\ &+ |Q_m(s) - Q_m^*(s)| + |I_T(s) - I_T^*(s)| + |I_m(s) - I_m^*(s)| + |I_{Tm}(s) - I_{Tm}^*(s)| \\ &+ |T_T(s) - T_T^*(s)| + |T_m(s) - T_m^*(s)| + |T_{Tm}(s) - T_{Tm}^*(s)| + |V_h(s) - R_h^*(s)| \\ &+ |R_T(s) - R_T^*(s)| + |R_m(s) - R_m^*(s)| + |R_{Tm}(s) - R_{Tm}^*(s)| + |S_r(s) - S_r^*(s)| \\ &\left. + |I_r(s) - I_r^*(s)| + |B(s) - B^*(s)| \right) \end{aligned} \right] ds \\
&\leq \left(\frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)} + \frac{\sigma T^{\alpha+\sigma-1} + \Gamma(\sigma+1)}{\Gamma(\sigma+\alpha)} \right) \frac{L_1}{AB(\alpha)} \left[\begin{aligned} &\left(\|S_h(t) - S_h^*(t)\| + \|E_T(t) - E_T^*(t)\| + \|E_m(t) - E_m^*(t)\| + \|E_{Tm}(t) - E_{Tm}^*(t)\| \right. \\ &+ \|Q_m(t) - Q_m^*(t)\| + \|I_T(t) - I_T^*(t)\| + \|I_m(t) - I_m^*(t)\| + \|I_{MC}(t) - I_{MC}^*(t)\| \\ &+ \|T_C(t) - T_C^*(t)\| + \|I_{TM}(t) - I_{TM}^*(t)\| + \|T_T(t) - T_T^*(t)\| + \|T_m(t) - T_m^*(t)\| \\ &+ \|T_m(t) - T_m^*(t)\| + \|R_T(t) - R_T^*(t)\| + \|R_m(t) - R_m^*(t)\| + \|R_{Tm}(t) - R_{Tm}^*(t)\| \\ &\left. + \|S_r(t) - S_r^*(t)\| + \|I_r(t) - I_r^*(t)\| + \|B(t) - B^*(t)\| \right) \end{aligned} \right].
\end{aligned} \tag{38}$$

Then,

$$\begin{aligned}
&\left| (\tau_1 S_h)(t) - (\tau_1 S_h^*)(t) \right| \leq \left(\frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)} + \frac{\sigma T^{\alpha+\sigma-1} + \Gamma(\sigma+1)}{\Gamma(\sigma+\alpha)} \right) \frac{L_1}{AB(\alpha)} \left[\begin{aligned} &\left(\|S_h(t) - S_h^*(t)\| + \|E_T(t) - E_T^*(t)\| + \|E_m(t) - E_m^*(t)\| \right. \\ &+ \|E_{Tm}(t) - E_{Tm}^*(t)\| + \|Q_m(t) - Q_m^*(t)\| + \|I_T(t) - I_T^*(t)\| \\ &+ \|I_m(t) - I_m^*(t)\| + \|I_{MC}(t) - I_{MC}^*(t)\| + \|T_C(t) - T_C^*(t)\| \\ &+ \|I_{TM}(t) - I_{TM}^*(t)\| + \|T_T(t) - T_T^*(t)\| + \|T_m(t) - T_m^*(t)\| \\ &+ \|T_m(t) - T_m^*(t)\| + \|R_T(t) - R_T^*(t)\| + \|R_m(t) - R_m^*(t)\| \\ &+ \|R_{Tm}(t) - R_{Tm}^*(t)\| + \|S_r(t) - S_r^*(t)\| + \|I_r(t) - I_r^*(t)\| \\ &\left. + \|B(t) - B^*(t)\| \right) \end{aligned} \right],
\end{aligned} \tag{39}$$

Similarly, we have:

$$\begin{aligned}
\left| \begin{array}{l} (\tau_2 E_T)(t) \\ -(\tau_2 E_T^*)(t) \end{array} \right| &\leq \left(\frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)} + \frac{\alpha T^{\alpha+\sigma-1} + \Gamma(\sigma+1)}{\Gamma(\sigma+\alpha)} \right) \frac{L_2}{AB(\alpha)} \left(\begin{array}{l} \|S_h(t) - S_h^*(t)\| + \|E_T(t) - E_T^*(t)\| + \|E_m(t) - E_m^*(t)\| \\ + \|E_{Tm}(t) - E_{Tm}^*(t)\| + \|Q_m(t) - Q_m^*(t)\| + \|I_T(t) - I_T^*(t)\| \\ + \|I_m(t) - I_m^*(t)\| + \|I_{MC}(t) - I_{MC}^*(t)\| + \|T_C(t) - T_C^*(t)\| \\ + \|I_{TM}(t) - I_{TM}^*(t)\| + \|T_T(t) - T_T^*(t)\| + \|T_m(t) - T_m^*(t)\| \\ + \|T_m(t) - T_m^*(t)\| + \|R_T(t) - R_T^*(t)\| + \|R_m(t) - R_m^*(t)\| \\ + \|R_{Tm}(t) - R_{Tm}^*(t)\| + \|S_r(t) - S_r^*(t)\| + \|I_r(t) - I_r^*(t)\| \\ + \|B(t) - B^*(t)\| \end{array} \right), \\
\left| \begin{array}{l} (\tau_3 E_m)(t) \\ -(\tau_3 E_m^*)(t) \end{array} \right| &\leq \left(\frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)} + \frac{\alpha T^{\alpha+\sigma-1} + \Gamma(\sigma+1)}{\Gamma(\sigma+\alpha)} \right) \frac{L_3}{AB(\alpha)} \left(\begin{array}{l} \|S_h(t) - S_h^*(t)\| + \|E_T(t) - E_T^*(t)\| + \|E_m(t) - E_m^*(t)\| \\ + \|E_{Tm}(t) - E_{Tm}^*(t)\| + \|Q_m(t) - Q_m^*(t)\| + \|I_T(t) - I_T^*(t)\| \\ + \|I_m(t) - I_m^*(t)\| + \|I_{MC}(t) - I_{MC}^*(t)\| + \|T_C(t) - T_C^*(t)\| \\ + \|I_{TM}(t) - I_{TM}^*(t)\| + \|T_T(t) - T_T^*(t)\| + \|T_m(t) - T_m^*(t)\| \\ + \|T_m(t) - T_m^*(t)\| + \|R_T(t) - R_T^*(t)\| + \|R_m(t) - R_m^*(t)\| \\ + \|R_{Tm}(t) - R_{Tm}^*(t)\| + \|S_r(t) - S_r^*(t)\| + \|I_r(t) - I_r^*(t)\| \\ + \|B(t) - B^*(t)\| \end{array} \right), \\
\left| \begin{array}{l} (\tau_4 E_{Tm})(t) \\ -(\tau_4 E_{Tm}^*)(t) \end{array} \right| &\leq \left(\frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)} + \frac{\alpha T^{\alpha+\sigma-1} + \Gamma(\sigma+1)}{\Gamma(\sigma+\alpha)} \right) \frac{L_4}{AB(\alpha)} \left(\begin{array}{l} \|S_h(t) - S_h^*(t)\| + \|E_T(t) - E_T^*(t)\| + \|E_m(t) - E_m^*(t)\| \\ + \|E_{Tm}(t) - E_{Tm}^*(t)\| + \|Q_m(t) - Q_m^*(t)\| + \|I_T(t) - I_T^*(t)\| \\ + \|I_m(t) - I_m^*(t)\| + \|I_{MC}(t) - I_{MC}^*(t)\| + \|T_C(t) - T_C^*(t)\| \\ + \|I_{TM}(t) - I_{TM}^*(t)\| + \|T_T(t) - T_T^*(t)\| + \|T_m(t) - T_m^*(t)\| \\ + \|T_m(t) - T_m^*(t)\| + \|R_T(t) - R_T^*(t)\| + \|R_m(t) - R_m^*(t)\| \\ + \|R_{Tm}(t) - R_{Tm}^*(t)\| + \|S_r(t) - S_r^*(t)\| + \|I_r(t) - I_r^*(t)\| \\ + \|B(t) - B^*(t)\| \end{array} \right), \\
\left| \begin{array}{l} (\tau_5 Q_m)(t) \\ -(\tau_5 Q_m^*)(t) \end{array} \right| &\leq \left(\frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)} + \frac{\alpha T^{\alpha+\sigma-1} + \Gamma(\sigma+1)}{\Gamma(\sigma+\alpha)} \right) \frac{L_5}{AB(\alpha)} \left(\begin{array}{l} \|S_h(t) - S_h^*(t)\| + \|E_T(t) - E_T^*(t)\| + \|E_m(t) - E_m^*(t)\| \\ + \|E_{Tm}(t) - E_{Tm}^*(t)\| + \|Q_m(t) - Q_m^*(t)\| + \|I_T(t) - I_T^*(t)\| \\ + \|I_m(t) - I_m^*(t)\| + \|I_{MC}(t) - I_{MC}^*(t)\| + \|T_C(t) - T_C^*(t)\| \\ + \|I_{TM}(t) - I_{TM}^*(t)\| + \|T_T(t) - T_T^*(t)\| + \|T_m(t) - T_m^*(t)\| \\ + \|T_m(t) - T_m^*(t)\| + \|R_T(t) - R_T^*(t)\| + \|R_m(t) - R_m^*(t)\| \\ + \|R_{Tm}(t) - R_{Tm}^*(t)\| + \|S_r(t) - S_r^*(t)\| + \|I_r(t) - I_r^*(t)\| \\ + \|B(t) - B^*(t)\| \end{array} \right),
\end{aligned}$$

$$\begin{aligned}
\left| \begin{pmatrix} (\tau_6 I_T)(t) \\ -(\tau_6 I_T^*)(t) \end{pmatrix} \right| &\leq \left(\frac{\frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)}}{+\frac{\alpha T^{\alpha+\sigma-1} + \Gamma(\sigma+1)}{\Gamma(\sigma+\alpha)}} \right) \frac{L_6}{AB(\alpha)} \left(\begin{aligned} &\|S_h(t) - S_h^*(t)\| + \|E_T(t) - E_T^*(t)\| + \|E_m(t) - E_m^*(t)\| \\ &+ \|E_{Tm}(t) - E_{Tm}^*(t)\| + \|Q_m(t) - Q_m^*(t)\| + \|I_T(t) - I_T^*(t)\| \\ &+ \|I_m(t) - I_m^*(t)\| + \|I_{MC}(t) - I_{MC}^*(t)\| + \|T_C(t) - T_C^*(t)\| \\ &+ \|I_{TM}(t) - I_{TM}^*(t)\| + \|T_T(t) - T_T^*(t)\| + \|T_m(t) - T_m^*(t)\| \\ &+ \|T_m(t) - T_m^*(t)\| + \|R_T(t) - R_T^*(t)\| + \|R_m(t) - R_m^*(t)\| \\ &+ \|R_{Tm}(t) - R_{Tm}^*(t)\| + \|S_r(t) - S_r^*(t)\| + \|I_r(t) - I_r^*(t)\| \\ &+ \|B(t) - B^*(t)\| \end{aligned} \right), \\
\left| \begin{pmatrix} (\tau_7 I_m)(t) \\ -(\tau_7 I_m^*)(t) \end{pmatrix} \right| &\leq \left(\frac{\frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)}}{+\frac{\alpha T^{\alpha+\sigma-1} + \Gamma(\sigma+1)}{\Gamma(\sigma+\alpha)}} \right) \frac{L_7}{AB(\alpha)} \left(\begin{aligned} &\|S_h(t) - S_h^*(t)\| + \|E_T(t) - E_T^*(t)\| + \|E_m(t) - E_m^*(t)\| \\ &+ \|E_{Tm}(t) - E_{Tm}^*(t)\| + \|Q_m(t) - Q_m^*(t)\| + \|I_T(t) - I_T^*(t)\| \\ &+ \|I_m(t) - I_m^*(t)\| + \|I_{MC}(t) - I_{MC}^*(t)\| + \|T_C(t) - T_C^*(t)\| \\ &+ \|I_{TM}(t) - I_{TM}^*(t)\| + \|T_T(t) - T_T^*(t)\| + \|T_m(t) - T_m^*(t)\| \\ &+ \|T_m(t) - T_m^*(t)\| + \|R_T(t) - R_T^*(t)\| + \|R_m(t) - R_m^*(t)\| \\ &+ \|R_{Tm}(t) - R_{Tm}^*(t)\| + \|S_r(t) - S_r^*(t)\| + \|I_r(t) - I_r^*(t)\| \\ &+ \|B(t) - B^*(t)\| \end{aligned} \right), \\
\left| \begin{pmatrix} (\tau_8 I_{Tm})(t) \\ -(\tau_8 I_{Tm}^*)(t) \end{pmatrix} \right| &\leq \left(\frac{\frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)}}{+\frac{\alpha T^{\alpha+\sigma-1} + \Gamma(\sigma+1)}{\Gamma(\sigma+\alpha)}} \right) \frac{L_8}{AB(\alpha)} \left(\begin{aligned} &\|S_h(t) - S_h^*(t)\| + \|E_T(t) - E_T^*(t)\| + \|E_m(t) - E_m^*(t)\| \\ &+ \|E_{Tm}(t) - E_{Tm}^*(t)\| + \|Q_m(t) - Q_m^*(t)\| + \|I_T(t) - I_T^*(t)\| \\ &+ \|I_m(t) - I_m^*(t)\| + \|I_{MC}(t) - I_{MC}^*(t)\| + \|T_C(t) - T_C^*(t)\| \\ &+ \|I_{TM}(t) - I_{TM}^*(t)\| + \|T_T(t) - T_T^*(t)\| + \|T_m(t) - T_m^*(t)\| \\ &+ \|T_m(t) - T_m^*(t)\| + \|R_T(t) - R_T^*(t)\| + \|R_m(t) - R_m^*(t)\| \\ &+ \|R_{Tm}(t) - R_{Tm}^*(t)\| + \|S_r(t) - S_r^*(t)\| + \|I_r(t) - I_r^*(t)\| \\ &+ \|B(t) - B^*(t)\| \end{aligned} \right), \\
\left| \begin{pmatrix} (\tau_9 T_T)(t) \\ -(\tau_9 T_T^*)(t) \end{pmatrix} \right| &\leq \left(\frac{\frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)}}{+\frac{\alpha T^{\alpha+\sigma-1} + \Gamma(\sigma+1)}{\Gamma(\sigma+\alpha)}} \right) \frac{L_9}{AB(\alpha)} \left(\begin{aligned} &\|S_h(t) - S_h^*(t)\| + \|E_T(t) - E_T^*(t)\| + \|E_m(t) - E_m^*(t)\| \\ &+ \|E_{Tm}(t) - E_{Tm}^*(t)\| + \|Q_m(t) - Q_m^*(t)\| + \|I_T(t) - I_T^*(t)\| \\ &+ \|I_m(t) - I_m^*(t)\| + \|I_{MC}(t) - I_{MC}^*(t)\| + \|T_C(t) - T_C^*(t)\| \\ &+ \|I_{TM}(t) - I_{TM}^*(t)\| + \|T_T(t) - T_T^*(t)\| + \|T_m(t) - T_m^*(t)\| \\ &+ \|T_m(t) - T_m^*(t)\| + \|R_T(t) - R_T^*(t)\| + \|R_m(t) - R_m^*(t)\| \\ &+ \|R_{Tm}(t) - R_{Tm}^*(t)\| + \|S_r(t) - S_r^*(t)\| + \|I_r(t) - I_r^*(t)\| \\ &+ \|B(t) - B^*(t)\| \end{aligned} \right),
\end{aligned}$$

$$\begin{aligned}
\left| \begin{pmatrix} (\tau_{10} T_m)(t) \\ -(\tau_{10} T_m^*)(t) \end{pmatrix} \right| &\leq \left(\frac{\frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)}}{+\frac{\alpha T^{\alpha+\sigma-1} + \Gamma(\sigma+1)}{\Gamma(\sigma+\alpha)}} \right) \frac{L_{11}}{AB(\alpha)} \left(\begin{aligned} &\|S_h(t) - S_h^*(t)\| + \|E_T(t) - E_T^*(t)\| + \|E_m(t) - E_m^*(t)\| \\ &+ \|E_{Tm}(t) - E_{Tm}^*(t)\| + \|Q_m(t) - Q_m^*(t)\| + \|I_T(t) - I_T^*(t)\| \\ &+ \|I_m(t) - I_m^*(t)\| + \|I_{MC}(t) - I_{MC}^*(t)\| + \|T_C(t) - T_C^*(t)\| \\ &+ \|I_{TM}(t) - I_{TM}^*(t)\| + \|T_T(t) - T_T^*(t)\| + \|T_m(t) - T_m^*(t)\| \\ &+ \|T_m(t) - T_m^*(t)\| + \|R_T(t) - R_T^*(t)\| + \|R_m(t) - R_m^*(t)\| \\ &+ \|R_{Tm}(t) - R_{Tm}^*(t)\| + \|S_r(t) - S_r^*(t)\| + \|I_r(t) - I_r^*(t)\| \\ &+ \|B(t) - B^*(t)\| \end{aligned} \right), \\
\left| \begin{pmatrix} (\tau_{11} T_m)(t) \\ -(\tau_{11} T_m^*)(t) \end{pmatrix} \right| &\leq \left(\frac{\frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)}}{+\frac{\alpha T^{\alpha+\sigma-1} + \Gamma(\sigma+1)}{\Gamma(\sigma+\alpha)}} \right) \frac{L_{11}}{AB(\alpha)} \left(\begin{aligned} &\|S_h(t) - S_h^*(t)\| + \|E_T(t) - E_T^*(t)\| + \|E_m(t) - E_m^*(t)\| \\ &+ \|E_{Tm}(t) - E_{Tm}^*(t)\| + \|Q_m(t) - Q_m^*(t)\| + \|I_T(t) - I_T^*(t)\| \\ &+ \|I_m(t) - I_m^*(t)\| + \|I_{MC}(t) - I_{MC}^*(t)\| + \|T_C(t) - T_C^*(t)\| \\ &+ \|I_{TM}(t) - I_{TM}^*(t)\| + \|T_T(t) - T_T^*(t)\| + \|T_m(t) - T_m^*(t)\| \\ &+ \|T_m(t) - T_m^*(t)\| + \|R_T(t) - R_T^*(t)\| + \|R_m(t) - R_m^*(t)\| \\ &+ \|R_{Tm}(t) - R_{Tm}^*(t)\| + \|S_r(t) - S_r^*(t)\| + \|I_r(t) - I_r^*(t)\| \\ &+ \|B(t) - B^*(t)\| \end{aligned} \right), \\
\left| \begin{pmatrix} (\tau_{12} V_h)(t) \\ -(\tau_{12} V_h^*)(t) \end{pmatrix} \right| &\leq \left(\frac{\frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)}}{+\frac{\alpha T^{\alpha+\sigma-1} + \Gamma(\sigma+1)}{\Gamma(\sigma+\alpha)}} \right) \frac{L_{12}}{AB(\alpha)} \left(\begin{aligned} &\|S_h(t) - S_h^*(t)\| + \|E_T(t) - E_T^*(t)\| + \|E_m(t) - E_m^*(t)\| \\ &+ \|E_{Tm}(t) - E_{Tm}^*(t)\| + \|Q_m(t) - Q_m^*(t)\| + \|I_T(t) - I_T^*(t)\| \\ &+ \|I_m(t) - I_m^*(t)\| + \|I_{MC}(t) - I_{MC}^*(t)\| + \|T_C(t) - T_C^*(t)\| \\ &+ \|I_{TM}(t) - I_{TM}^*(t)\| + \|T_T(t) - T_T^*(t)\| + \|T_m(t) - T_m^*(t)\| \\ &+ \|T_m(t) - T_m^*(t)\| + \|R_T(t) - R_T^*(t)\| + \|R_m(t) - R_m^*(t)\| \\ &+ \|R_{Tm}(t) - R_{Tm}^*(t)\| + \|S_r(t) - S_r^*(t)\| + \|I_r(t) - I_r^*(t)\| \\ &+ \|B(t) - B^*(t)\| \end{aligned} \right), \\
\left| \begin{pmatrix} (\tau_{13} R_T)(t) \\ -(\tau_{13} R_T^*)(t) \end{pmatrix} \right| &\leq \left(\frac{\frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)}}{+\frac{\alpha T^{\alpha+\sigma-1} + \Gamma(\sigma+1)}{\Gamma(\sigma+\alpha)}} \right) \frac{L_{14}}{AB(\alpha)} \left(\begin{aligned} &\|S_h(t) - S_h^*(t)\| + \|E_T(t) - E_T^*(t)\| + \|E_m(t) - E_m^*(t)\| \\ &+ \|E_{Tm}(t) - E_{Tm}^*(t)\| + \|Q_m(t) - Q_m^*(t)\| + \|I_T(t) - I_T^*(t)\| \\ &+ \|I_m(t) - I_m^*(t)\| + \|I_{MC}(t) - I_{MC}^*(t)\| + \|T_C(t) - T_C^*(t)\| \\ &+ \|I_{TM}(t) - I_{TM}^*(t)\| + \|T_T(t) - T_T^*(t)\| + \|T_m(t) - T_m^*(t)\| \\ &+ \|T_m(t) - T_m^*(t)\| + \|R_T(t) - R_T^*(t)\| + \|R_m(t) - R_m^*(t)\| \\ &+ \|R_{Tm}(t) - R_{Tm}^*(t)\| + \|S_r(t) - S_r^*(t)\| + \|I_r(t) - I_r^*(t)\| \\ &+ \|B(t) - B^*(t)\| \end{aligned} \right),
\end{aligned}$$

$$\begin{aligned}
\left| \begin{aligned} &(\tau_{14} R_m)(t) \\ &-(\tau_{14} R_m^*)(t) \end{aligned} \right| &\leq \left(\frac{\frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)}}{+\frac{\alpha T^{\alpha+\sigma-1} + \Gamma(\sigma+1)}{\Gamma(\sigma+\alpha)}} \right) \frac{L_{14}}{AB(\alpha)} \left(\begin{aligned} &\|S_h(t) - S_h^*(t)\| + \|E_T(t) - E_T^*(t)\| + \|E_m(t) - E_m^*(t)\| \\ &+ \|E_{Tm}(t) - E_{Tm}^*(t)\| + \|Q_m(t) - Q_m^*(t)\| + \|I_T(t) - I_T^*(t)\| \\ &+ \|I_m(t) - I_m^*(t)\| + \|I_{MC}(t) - I_{MC}^*(t)\| + \|T_C(t) - T_C^*(t)\| \\ &+ \|I_{TM}(t) - I_{TM}^*(t)\| + \|T_T(t) - T_T^*(t)\| + \|T_m(t) - T_m^*(t)\| \\ &+ \|T_m(t) - T_m^*(t)\| + \|R_T(t) - R_T^*(t)\| + \|R_m(t) - R_m^*(t)\| \\ &+ \|R_{Tm}(t) - R_{Tm}^*(t)\| + \|S_r(t) - S_r^*(t)\| + \|I_r(t) - I_r^*(t)\| \\ &+ \|B(t) - B^*(t)\| \end{aligned} \right), \\
\left| \begin{aligned} &(\tau_{15} R_{Tm})(t) \\ &-(\tau_{15} R_{Tm}^*)(t) \end{aligned} \right| &\leq \left(\frac{\frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)}}{+\frac{\alpha T^{\alpha+\sigma-1} + \Gamma(\sigma+1)}{\Gamma(\sigma+\alpha)}} \right) \frac{L_{15}}{AB(\alpha)} \left(\begin{aligned} &\|S_h(t) - S_h^*(t)\| + \|E_T(t) - E_T^*(t)\| + \|E_m(t) - E_m^*(t)\| \\ &+ \|E_{Tm}(t) - E_{Tm}^*(t)\| + \|Q_m(t) - Q_m^*(t)\| + \|I_T(t) - I_T^*(t)\| \\ &+ \|I_m(t) - I_m^*(t)\| + \|I_{MC}(t) - I_{MC}^*(t)\| + \|T_C(t) - T_C^*(t)\| \\ &+ \|I_{TM}(t) - I_{TM}^*(t)\| + \|T_T(t) - T_T^*(t)\| + \|T_m(t) - T_m^*(t)\| \\ &+ \|T_m(t) - T_m^*(t)\| + \|R_T(t) - R_T^*(t)\| + \|R_m(t) - R_m^*(t)\| \\ &+ \|R_{Tm}(t) - R_{Tm}^*(t)\| + \|S_r(t) - S_r^*(t)\| + \|I_r(t) - I_r^*(t)\| \\ &+ \|B(t) - B^*(t)\| \end{aligned} \right), \\
\left| \begin{aligned} &(\tau_{16} S_r)(t) \\ &-(\tau_{16} S_r^*)(t) \end{aligned} \right| &\leq \left(\frac{\frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)}}{+\frac{\alpha T^{\alpha+\sigma-1} + \Gamma(\sigma+1)}{\Gamma(\sigma+\alpha)}} \right) \frac{L_{16}}{AB(\alpha)} \left(\begin{aligned} &\|S_h(t) - S_h^*(t)\| + \|E_T(t) - E_T^*(t)\| + \|E_m(t) - E_m^*(t)\| \\ &+ \|E_{Tm}(t) - E_{Tm}^*(t)\| + \|Q_m(t) - Q_m^*(t)\| + \|I_T(t) - I_T^*(t)\| \\ &+ \|I_m(t) - I_m^*(t)\| + \|I_{MC}(t) - I_{MC}^*(t)\| + \|T_C(t) - T_C^*(t)\| \\ &+ \|I_{TM}(t) - I_{TM}^*(t)\| + \|T_T(t) - T_T^*(t)\| + \|T_m(t) - T_m^*(t)\| \\ &+ \|T_m(t) - T_m^*(t)\| + \|R_T(t) - R_T^*(t)\| + \|R_m(t) - R_m^*(t)\| \\ &+ \|R_{Tm}(t) - R_{Tm}^*(t)\| + \|S_r(t) - S_r^*(t)\| + \|I_r(t) - I_r^*(t)\| \\ &+ \|B(t) - B^*(t)\| \end{aligned} \right),
\end{aligned}$$

$$\begin{aligned}
\left| \begin{aligned} &(\tau_{17} I_r)(t) \\ &-(\tau_{17} I_r^*)(t) \end{aligned} \right| &\leq \left(\frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)} + \frac{\alpha T^{\alpha+\sigma-1} + \Gamma(\sigma+1)}{\Gamma(\sigma+\alpha)} \right) \frac{L_{17}}{AB(\alpha)} \left(\begin{aligned} &\|S_h(t) - S_h^*(t)\| + \|E_T(t) - E_T^*(t)\| + \|E_m(t) - E_m^*(t)\| \\ &+ \|E_{Tm}(t) - E_{Tm}^*(t)\| + \|Q_m(t) - Q_m^*(t)\| + \|I_T(t) - I_T^*(t)\| \\ &+ \|I_m(t) - I_m^*(t)\| + \|I_{MC}(t) - I_{MC}^*(t)\| + \|T_C(t) - T_C^*(t)\| \\ &+ \|I_{TM}(t) - I_{TM}^*(t)\| + \|T_T(t) - T_T^*(t)\| + \|T_m(t) - T_m^*(t)\| \\ &+ \|T_m(t) - T_m^*(t)\| + \|R_T(t) - R_T^*(t)\| + \|R_m(t) - R_m^*(t)\| \\ &+ \|R_{Tm}(t) - R_{Tm}^*(t)\| + \|S_r(t) - S_r^*(t)\| + \|I_r(t) - I_r^*(t)\| \\ &+ \|B(t) - B^*(t)\| \end{aligned} \right) \\
\left| \begin{aligned} &(\tau_{18} B)(t) \\ &-(\tau_{18} B^*)(t) \end{aligned} \right| &\leq \left(\frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)} + \frac{\alpha T^{\alpha+\sigma-1} + \Gamma(\sigma+1)}{\Gamma(\sigma+\alpha)} \right) \frac{L_{18}}{AB(\alpha)} \left(\begin{aligned} &\|S_h(t) - S_h^*(t)\| + \|E_T(t) - E_T^*(t)\| + \|E_m(t) - E_m^*(t)\| \\ &+ \|E_{Tm}(t) - E_{Tm}^*(t)\| + \|Q_m(t) - Q_m^*(t)\| + \|I_T(t) - I_T^*(t)\| \\ &+ \|I_m(t) - I_m^*(t)\| + \|I_{MC}(t) - I_{MC}^*(t)\| + \|T_C(t) - T_C^*(t)\| \\ &+ \|I_{TM}(t) - I_{TM}^*(t)\| + \|T_T(t) - T_T^*(t)\| + \|T_m(t) - T_m^*(t)\| \\ &+ \|T_m(t) - T_m^*(t)\| + \|R_T(t) - R_T^*(t)\| + \|R_m(t) - R_m^*(t)\| \\ &+ \|R_{Tm}(t) - R_{Tm}^*(t)\| + \|S_r(t) - S_r^*(t)\| + \|I_r(t) - I_r^*(t)\| \\ &+ \|B(t) - B^*(t)\| \end{aligned} \right).
\end{aligned} \tag{40}$$

Since

$$\tau = (\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6, \tau_7, \tau_8, \tau_9, \tau_{10}, \tau_{11}, \tau_{12}, \tau_{13}, \tau_{14}, \tau_{15}, \tau_{16}, \tau_{17}, \tau_{18}) \text{ and } L_{\max} > 0$$

with the result obtained from (39)-(40), we have:

$$\begin{aligned}
&\left\| \begin{aligned} &\tau(S_h, E_T, E_m, E_{Tm}, Q_m, I_T, I_m, I_{Tm}, T_T, T_m, T_{Tm}, V_h, R_T, R_m, R_{Tm}, S_r, I_r, B) \\ &-\tau(S_h^*, E_T^*, E_m^*, E_{Tm}^*, Q_m^*, I_T^*, I_m^*, I_{Tm}^*, T_T^*, T_m^*, T_{Tm}^*, V_h^*, R_T^*, R_m^*, R_{Tm}^*, S_r^*, I_r^*, B^*) \end{aligned} \right\| \\
&\leq \left(\frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)} + \frac{\alpha T^{\alpha+\sigma-1} + \Gamma(\sigma+1)}{\Gamma(\sigma+\alpha)} \right) \frac{L_{\max}}{AB(\alpha)} \left(\begin{aligned} &\|S_h(t) - S_h^*(t)\| + \|E_T(t) - E_T^*(t)\| + \|E_m(t) - E_m^*(t)\| \\ &+ \|E_{Tm}(t) - E_{Tm}^*(t)\| + \|Q_m(t) - Q_m^*(t)\| + \|I_T(t) - I_T^*(t)\| \\ &+ \|I_m(t) - I_m^*(t)\| + \|I_{MC}(t) - I_{MC}^*(t)\| + \|T_C(t) - T_C^*(t)\| \\ &+ \|I_{TM}(t) - I_{TM}^*(t)\| + \|T_T(t) - T_T^*(t)\| + \|T_m(t) - T_m^*(t)\| \\ &+ \|T_m(t) - T_m^*(t)\| + \|R_T(t) - R_T^*(t)\| + \|R_m(t) - R_m^*(t)\| \\ &+ \|R_{Tm}(t) - R_{Tm}^*(t)\| + \|S_r(t) - S_r^*(t)\| + \|I_r(t) - I_r^*(t)\| \\ &+ \|B(t) - B^*(t)\| \end{aligned} \right).
\end{aligned}$$

By the condition (45), then τ is a contraction.

Lemma 2 establishes that model (8) of FF-Monkey pox and typhoid fever co-infection model has a solution.

Theorem 3

Let $G \in X$ satisfying the assumptions (B_1) in theorem 2 and (B_1) there exist constants, $g_{i,j} > 0, i = 0, 1, 2, \dots, 18, j = 0, 1, 2, \dots, 18$ such that,

$$\begin{aligned} & \left| G \begin{pmatrix} t, S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), T_T(t), \\ T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), B(t) \end{pmatrix} \right| \\ & \leq g_{0j} + g_{1j} |S_h(t)| + g_{2j} |E_T(t)| + g_{3j} |E_m(t)| + g_{4j} |E_{Tm}(t)| + g_{5j} |Q_m(t)| + g_{6j} |I_T(t)| + g_{7j} |I_m(t)| \\ & + g_{8j} |I_{Tm}(t)| + g_{9j} |T_T(t)| + g_{10j} |T_m(t)| + g_{11j} |T_{Tm}(t)| + g_{12j} |V_h(t)| + g_{13j} |R_T(t)| + g_{14j} |R_m(t)| \\ & + g_{15j} |R_{Tm}(t)| + g_{16j} |S_r(t)| + g_{17j} |I_r(t)| + g_{18j} |B(t)|. \end{aligned} \quad (41)$$

$$\text{If } \sigma(1-\alpha) T_{\min^{\sigma-1}} L_{\max} L_{AB(\alpha)}$$

Then, the FF-Monkey pox and Typhoid fever co-infection model (8) has at least one solution.

Proof

We define a set

$$D_{K2} = \left\{ \begin{pmatrix} S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), T_T(t), \\ T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), B(t) \end{pmatrix} \in X \right. \\ \left. : \left\| \begin{pmatrix} S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), T_T(t), \\ T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), B(t) \end{pmatrix} \right\| \leq K_2 \right\}.$$

Applying Eq. (31) - (37), we can define two operators $(Q, P): D_{K2} \rightarrow X$

$$Q = (Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, Q_7, Q_8, Q_9, Q_{10}, Q_{11}, Q_{12}, Q_{13}, Q_{14}, Q_{15}, Q_{16}, Q_{17}, Q_{18})$$

$$\text{and } P = (P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9, P_{10}, P_{11}, P_{12}, P_{13}, P_{14}, P_{15}, P_{16}, P_{17}, P_{18}).$$

The operator Q is defined by;

$$\begin{aligned} (Q_1, S_h)(t) &= S_h(0) + \frac{\sigma t^{\sigma-1} (1-\alpha) G_1(t, S_h(t))}{AB(\alpha)}, \\ (Q_2, E_T)(t) &= E_T(0) + \frac{\sigma t^{\sigma-1} (1-\alpha) G_2(t, E_T(t))}{AB(\alpha)}, \\ (Q_3, E_m)(t) &= E_m(0) + \frac{\sigma t^{\sigma-1} (1-\alpha) G_3(t, E_m(t))}{AB(\alpha)}, \\ (Q_4, E_{Tm})(t) &= E_{Tm}(0) + \frac{\sigma t^{\sigma-1} (1-\alpha) G_4(t, E_{Tm}(t))}{AB(\alpha)}, \\ (Q_5, Q_m)(t) &= Q_m(0) + \frac{\sigma t^{\sigma-1} (1-\alpha) G_5(t, Q_m(t))}{AB(\alpha)}, \end{aligned}$$

$$\begin{aligned}
(Q_6, I_T)(t) &= I_T(0) + \frac{\sigma t^{\sigma-1} (1-\alpha) G_6(t, I_T(t))}{AB(\alpha)}, \\
(Q_7, I_m)(t) &= I_m(0) + \frac{\sigma t^{\sigma-1} (1-\alpha) G_7(t, I_m(t))}{AB(\alpha)}, \\
(Q_8, I_{Tm})(t) &= I_{Tm}(0) + \frac{\sigma t^{\sigma-1} (1-\alpha) G_8(t, I_{Tm}(t))}{AB(\alpha)}, \\
(Q_9, T_T)(t) &= T_T(0) + \frac{\sigma t^{\sigma-1} (1-\alpha) G_9(t, T_T(t))}{AB(\alpha)}, \\
(Q_{10}, T_m)(t) &= T_m(0) + \frac{\sigma t^{\sigma-1} (1-\alpha) G_{10}(t, T_m(t))}{AB(\alpha)}, \\
(Q_{11}, T_{Tm})(t) &= T_{Tm}(0) + \frac{\sigma t^{\sigma-1} (1-\alpha) G_{11}(t, T_{Tm}(t))}{AB(\alpha)}, \\
(Q_{12}, V_h)(t) &= V_h(0) + \frac{\sigma t^{\sigma-1} (1-\alpha) G_{12}(t, V_h(t))}{AB(\alpha)}, \\
(Q_{13}, R_T)(t) &= R_T(0) + \frac{\sigma t^{\sigma-1} (1-\alpha) G_{13}(t, R_T(t))}{AB(\alpha)}, \\
(Q_{14}, R_m)(t) &= R_m(0) + \frac{\sigma t^{\sigma-1} (1-\alpha) G_{14}(t, R_m(t))}{AB(\alpha)}, \\
(Q_{15}, R_{Tm})(t) &= R_{Tm}(0) + \frac{\sigma t^{\sigma-1} (1-\alpha) G_{15}(t, R_{Tm}(t))}{AB(\alpha)}, \\
(Q_{16}, S_r)(t) &= S_r(0) + \frac{\sigma t^{\sigma-1} (1-\alpha) G_{16}(t, S_r(t))}{AB(\alpha)}, \\
(Q_{17}, I_r)(t) &= I_r(0) + \frac{\sigma t^{\sigma-1} (1-\alpha) G_{17}(t, I_r(t))}{AB(\alpha)}, \\
(Q_{18}, B)(t) &= B(0) + \frac{\sigma t^{\sigma-1} (1-\alpha) G_{18}(t, B(t))}{AB(\alpha)}.
\end{aligned}$$

And the operator P is defined by:

$$\begin{aligned}
(P_1, S_h)(t) &= \frac{\sigma \alpha}{AB(\alpha) \Gamma(\alpha)} \int_0^t S^{\sigma-1} (t-s)^{\alpha-1} G_1(s, S_h(t)) ds, \\
(P_2, E_T)(t) &= \frac{\sigma \alpha}{AB(\alpha) \Gamma(\alpha)} \int_0^t S^{\sigma-1} (t-s)^{\alpha-1} G_2(s, E_T(t)) ds,
\end{aligned}$$

$$(P_3, E_m)(t) = \frac{\sigma\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t S^{\sigma-1}(t-s)^{\alpha-1} G_3(s, E_m(t)) ds,$$

$$(P_4, E_{T_m})(t) = \frac{\sigma\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t S^{\sigma-1}(t-s)^{\alpha-1} G_4(s, E_{T_m}(t)) ds,$$

$$(P_5, Q_m)(t) = \frac{\sigma\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t S^{\sigma-1}(t-s)^{\alpha-1} G_5(s, Q_m(t)) ds,$$

$$(P_6, I_T)(t) = \frac{\sigma\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t S^{\sigma-1}(t-s)^{\alpha-1} G_6(s, I_T(t)) ds,$$

$$(P_7, I_m)(t) = \frac{\sigma\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t S^{\sigma-1}(t-s)^{\alpha-1} G_7(s, I_m(t)) ds,$$

$$(P_8, I_{T_m})(t) = \frac{\sigma\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t S^{\sigma-1}(t-s)^{\alpha-1} G_8(s, I_{T_m}(t)) ds,$$

$$(P_9, T_T)(t) = \frac{\sigma\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t S^{\sigma-1}(t-s)^{\alpha-1} G_9(s, T_T(t)) ds,$$

$$(P_{10}, T_m)(t) = \frac{\sigma\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t S^{\sigma-1}(t-s)^{\alpha-1} G_{10}(s, T_m(t)) ds,$$

$$(P_{11}, T_{T_m})(t) = \frac{\sigma\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t S^{\sigma-1}(t-s)^{\alpha-1} G_{11}(s, T_{T_m}(t)) ds,$$

$$(P_{12}, V_h)(t) = \frac{\sigma\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t S^{\sigma-1}(t-s)^{\alpha-1} G_{12}(s, V_h(t)) ds,$$

$$(P_{13}, R_T)(t) = \frac{\sigma\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t S^{\sigma-1}(t-s)^{\alpha-1} G_{13}(s, R_T(t)) ds,$$

$$(P_{14}, R_m)(t) = \frac{\sigma\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t S^{\sigma-1}(t-s)^{\alpha-1} G_{14}(s, R_m(t)) ds,$$

$$(P_{15}, R_{T_m})(t) = \frac{\sigma\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t S^{\sigma-1}(t-s)^{\alpha-1} G_{15}(s, R_{T_m}(t)) ds,$$

$$(P_{16}, S_r)(t) = \frac{\sigma\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t S^{\sigma-1}(t-s)^{\alpha-1} G_{16}(s, S_r(t)) ds,$$

$$(P_{17}, I_r)(t) = \frac{\sigma\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t S^{\sigma-1}(t-s)^{\alpha-1} G_{17}(s, I_r(t)) ds,$$

$$(P_{18}, B)(t) = \frac{\sigma\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t S^{\sigma-1}(t-s)^{\alpha-1} G_{18}(s, B(t)) ds.$$

Using (B_1) in theorem 2, for any

$$\left(S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), T_T(t), \right. \\ \left. T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), B(t) \right) \in D_{K1}$$

$$\text{and } \left(S_h^*(t), E_T^*(t), E_m^*(t), E_{Tm}^*(t), Q_m^*(t), I_T^*(t), I_m^*(t), I_{Tm}^*(t), T_T^*(t), \right. \\ \left. T_m^*(t), T_{Tm}^*(t), V_h^*(t), R_T^*(t), R_m^*(t), R_{Tm}^*(t), S_r^*(t), I_r^*(t), B^*(t) \right) \in D_{K1}$$

$$\|Q_1 S_h - Q_1 S_h^*\| \leq \frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)} \text{Sup} \left| G_1(t, S_h(t)) - G_1(t, S_h^*(t)) \right|,$$

$$\|Q_1 S_h - Q_1 S_h^*\| \leq \frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)} L_1 \left(\begin{array}{l} \|S_h(t) - S_h^*(t)\| + \|E_T(t) - E_T^*(t)\| + \|E_m(t) - E_m^*(t)\| \\ + \|E_{Tm}(t) - E_{Tm}^*(t)\| + \|Q_m(t) - Q_m^*(t)\| + \|I_T(t) - I_T^*(t)\| \\ + \|I_m(t) - I_m^*(t)\| + \|I_{MC}(t) - I_{MC}^*(t)\| + \|T_C(t) - T_C^*(t)\| \\ + \|I_{TM}(t) - I_{TM}^*(t)\| + \|T_T(t) - T_T^*(t)\| + \|T_m(t) - T_m^*(t)\| \\ + \|T_{Tm}(t) - T_{Tm}^*(t)\| + \|R_T(t) - R_T^*(t)\| + \|R_m(t) - R_m^*(t)\| \\ + \|R_{Tm}(t) - R_{Tm}^*(t)\| + \|S_r(t) - S_r^*(t)\| + \|I_r(t) - I_r^*(t)\| \\ + \|B(t) - B^*(t)\| \end{array} \right),$$

$$\begin{aligned}
\|Q_2 E_T - Q_2 E_T^*\| &\leq \frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)} L_2 \left(\begin{aligned} &\|S_h(t) - S_h^*(t)\| + \|E_T(t) - E_T^*(t)\| + \|E_m(t) - E_m^*(t)\| \\ &+ \|E_{Tm}(t) - E_{Tm}^*(t)\| + \|Q_m(t) - Q_m^*(t)\| + \|I_T(t) - I_T^*(t)\| \\ &+ \|I_m(t) - I_m^*(t)\| + \|I_{MC}(t) - I_{MC}^*(t)\| + \|T_C(t) - T_C^*(t)\| \\ &+ \|I_{TM}(t) - I_{TM}^*(t)\| + \|T_T(t) - T_T^*(t)\| + \|T_m(t) - T_m^*(t)\| \\ &+ \|T_m(t) - T_m^*(t)\| + \|R_T(t) - R_T^*(t)\| + \|R_m(t) - R_m^*(t)\| \\ &+ \|R_{Tm}(t) - R_{Tm}^*(t)\| + \|S_r(t) - S_r^*(t)\| + \|I_r(t) - I_r^*(t)\| \\ &+ \|B(t) - B^*(t)\| \end{aligned} \right), \\
\|Q_3 E_m - Q_3 E_m^*\| &\leq \frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)} L_3 \left(\begin{aligned} &\|S_h(t) - S_h^*(t)\| + \|E_T(t) - E_T^*(t)\| + \|E_m(t) - E_m^*(t)\| \\ &+ \|E_{Tm}(t) - E_{Tm}^*(t)\| + \|Q_m(t) - Q_m^*(t)\| + \|I_T(t) - I_T^*(t)\| \\ &+ \|I_m(t) - I_m^*(t)\| + \|I_{MC}(t) - I_{MC}^*(t)\| + \|T_C(t) - T_C^*(t)\| \\ &+ \|I_{TM}(t) - I_{TM}^*(t)\| + \|T_T(t) - T_T^*(t)\| + \|T_m(t) - T_m^*(t)\| \\ &+ \|T_m(t) - T_m^*(t)\| + \|R_T(t) - R_T^*(t)\| + \|R_m(t) - R_m^*(t)\| \\ &+ \|R_{Tm}(t) - R_{Tm}^*(t)\| + \|S_r(t) - S_r^*(t)\| + \|I_r(t) - I_r^*(t)\| \\ &+ \|B(t) - B^*(t)\| \end{aligned} \right), \\
\|Q_4 E_{Tm} - Q_4 E_{Tm}^*\| &\leq \frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)} L_4 \left(\begin{aligned} &\|S_h(t) - S_h^*(t)\| + \|E_T(t) - E_T^*(t)\| + \|E_m(t) - E_m^*(t)\| \\ &+ \|E_{Tm}(t) - E_{Tm}^*(t)\| + \|Q_m(t) - Q_m^*(t)\| + \|I_T(t) - I_T^*(t)\| \\ &+ \|I_m(t) - I_m^*(t)\| + \|I_{MC}(t) - I_{MC}^*(t)\| + \|T_C(t) - T_C^*(t)\| \\ &+ \|I_{TM}(t) - I_{TM}^*(t)\| + \|T_T(t) - T_T^*(t)\| + \|T_m(t) - T_m^*(t)\| \\ &+ \|T_m(t) - T_m^*(t)\| + \|R_T(t) - R_T^*(t)\| + \|R_m(t) - R_m^*(t)\| \\ &+ \|R_{Tm}(t) - R_{Tm}^*(t)\| + \|S_r(t) - S_r^*(t)\| + \|I_r(t) - I_r^*(t)\| \\ &+ \|B(t) - B^*(t)\| \end{aligned} \right),
\end{aligned}$$

$$\begin{aligned}
\|Q_5 Q_m - Q_5^* Q_m^*\| &\leq \frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)} L_5 \left(\begin{aligned} &\|S_h(t) - S_h^*(t)\| + \|E_T(t) - E_T^*(t)\| + \|E_m(t) - E_m^*(t)\| \\ &+ \|E_{Tm}(t) - E_{Tm}^*(t)\| + \|Q_m(t) - Q_m^*(t)\| + \|I_T(t) - I_T^*(t)\| \\ &+ \|I_m(t) - I_m^*(t)\| + \|I_{MC}(t) - I_{MC}^*(t)\| + \|T_C(t) - T_C^*(t)\| \\ &+ \|I_{TM}(t) - I_{TM}^*(t)\| + \|T_T(t) - T_T^*(t)\| + \|T_m(t) - T_m^*(t)\| \\ &+ \|T_m(t) - T_m^*(t)\| + \|R_T(t) - R_T^*(t)\| + \|R_m(t) - R_m^*(t)\| \\ &+ \|R_{Tm}(t) - R_{Tm}^*(t)\| + \|S_r(t) - S_r^*(t)\| + \|I_r(t) - I_r^*(t)\| \\ &+ \|B(t) - B^*(t)\| \end{aligned} \right), \\
\|Q_6 I_T - Q_6^* I_T^*\| &\leq \frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)} L_6 \left(\begin{aligned} &\|S_h(t) - S_h^*(t)\| + \|E_T(t) - E_T^*(t)\| + \|E_m(t) - E_m^*(t)\| \\ &+ \|E_{Tm}(t) - E_{Tm}^*(t)\| + \|Q_m(t) - Q_m^*(t)\| + \|I_T(t) - I_T^*(t)\| \\ &+ \|I_m(t) - I_m^*(t)\| + \|I_{MC}(t) - I_{MC}^*(t)\| + \|T_C(t) - T_C^*(t)\| \\ &+ \|I_{TM}(t) - I_{TM}^*(t)\| + \|T_T(t) - T_T^*(t)\| + \|T_m(t) - T_m^*(t)\| \\ &+ \|T_m(t) - T_m^*(t)\| + \|R_T(t) - R_T^*(t)\| + \|R_m(t) - R_m^*(t)\| \\ &+ \|R_{Tm}(t) - R_{Tm}^*(t)\| + \|S_r(t) - S_r^*(t)\| + \|I_r(t) - I_r^*(t)\| \\ &+ \|B(t) - B^*(t)\| \end{aligned} \right), \\
\|Q_7 I_m - Q_7^* I_m^*\| &\leq \frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)} L_7 \left(\begin{aligned} &\|S_h(t) - S_h^*(t)\| + \|E_T(t) - E_T^*(t)\| + \|E_m(t) - E_m^*(t)\| \\ &+ \|E_{Tm}(t) - E_{Tm}^*(t)\| + \|Q_m(t) - Q_m^*(t)\| + \|I_T(t) - I_T^*(t)\| \\ &+ \|I_m(t) - I_m^*(t)\| + \|I_{MC}(t) - I_{MC}^*(t)\| + \|T_C(t) - T_C^*(t)\| \\ &+ \|I_{TM}(t) - I_{TM}^*(t)\| + \|T_T(t) - T_T^*(t)\| + \|T_m(t) - T_m^*(t)\| \\ &+ \|T_m(t) - T_m^*(t)\| + \|R_T(t) - R_T^*(t)\| + \|R_m(t) - R_m^*(t)\| \\ &+ \|R_{Tm}(t) - R_{Tm}^*(t)\| + \|S_r(t) - S_r^*(t)\| + \|I_r(t) - I_r^*(t)\| \\ &+ \|B(t) - B^*(t)\| \end{aligned} \right),
\end{aligned}$$

$$\begin{aligned}
\|Q_8 I_{Tm} - Q_8^* I_{Tm}^*\| &\leq \frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)} L_8 \left(\begin{aligned} &\|S_h(t) - S_h^*(t)\| + \|E_T(t) - E_T^*(t)\| + \|E_m(t) - E_m^*(t)\| \\ &+ \|E_{Tm}(t) - E_{Tm}^*(t)\| + \|Q_m(t) - Q_m^*(t)\| + \|I_T(t) - I_T^*(t)\| \\ &+ \|I_m(t) - I_m^*(t)\| + \|I_{MC}(t) - I_{MC}^*(t)\| + \|T_C(t) - T_C^*(t)\| \\ &+ \|I_{TM}(t) - I_{TM}^*(t)\| + \|T_T(t) - T_T^*(t)\| + \|T_m(t) - T_m^*(t)\| \\ &+ \|T_m(t) - T_m^*(t)\| + \|R_T(t) - R_T^*(t)\| + \|R_m(t) - R_m^*(t)\| \\ &+ \|R_{Tm}(t) - R_{Tm}^*(t)\| + \|S_r(t) - S_r^*(t)\| + \|I_r(t) - I_r^*(t)\| \\ &+ \|B(t) - B^*(t)\| \end{aligned} \right), \\
\|Q_9 I_T - Q_9^* I_T^*\| &\leq \frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)} L_9 \left(\begin{aligned} &\|S_h(t) - S_h^*(t)\| + \|E_T(t) - E_T^*(t)\| + \|E_m(t) - E_m^*(t)\| \\ &+ \|E_{Tm}(t) - E_{Tm}^*(t)\| + \|Q_m(t) - Q_m^*(t)\| + \|I_T(t) - I_T^*(t)\| \\ &+ \|I_m(t) - I_m^*(t)\| + \|I_{MC}(t) - I_{MC}^*(t)\| + \|T_C(t) - T_C^*(t)\| \\ &+ \|I_{TM}(t) - I_{TM}^*(t)\| + \|T_T(t) - T_T^*(t)\| + \|T_m(t) - T_m^*(t)\| \\ &+ \|T_m(t) - T_m^*(t)\| + \|R_T(t) - R_T^*(t)\| + \|R_m(t) - R_m^*(t)\| \\ &+ \|R_{Tm}(t) - R_{Tm}^*(t)\| + \|S_r(t) - S_r^*(t)\| + \|I_r(t) - I_r^*(t)\| \\ &+ \|B(t) - B^*(t)\| \end{aligned} \right), \\
\|Q_{11} I_{Tm} - Q_{11}^* I_{Tm}^*\| &\leq \frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)} L_{11} \left(\begin{aligned} &\|S_h(t) - S_h^*(t)\| + \|E_T(t) - E_T^*(t)\| + \|E_m(t) - E_m^*(t)\| \\ &+ \|E_{Tm}(t) - E_{Tm}^*(t)\| + \|Q_m(t) - Q_m^*(t)\| + \|I_T(t) - I_T^*(t)\| \\ &+ \|I_m(t) - I_m^*(t)\| + \|I_{MC}(t) - I_{MC}^*(t)\| + \|T_C(t) - T_C^*(t)\| \\ &+ \|I_{TM}(t) - I_{TM}^*(t)\| + \|T_T(t) - T_T^*(t)\| + \|T_m(t) - T_m^*(t)\| \\ &+ \|T_m(t) - T_m^*(t)\| + \|R_T(t) - R_T^*(t)\| + \|R_m(t) - R_m^*(t)\| \\ &+ \|R_{Tm}(t) - R_{Tm}^*(t)\| + \|S_r(t) - S_r^*(t)\| + \|I_r(t) - I_r^*(t)\| \\ &+ \|B(t) - B^*(t)\| \end{aligned} \right),
\end{aligned}$$

$$\begin{aligned}
\|Q_{11}T_{Tm} - Q_{11}T_{Tm}^*\| &\leq \frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)} L_{11} \left(\begin{aligned} &\|S_h(t) - S_h^*(t)\| + \|E_T(t) - E_T^*(t)\| + \|E_m(t) - E_m^*(t)\| \\ &+ \|E_{Tm}(t) - E_{Tm}^*(t)\| + \|Q_m(t) - Q_m^*(t)\| + \|I_T(t) - I_T^*(t)\| \\ &+ \|I_m(t) - I_m^*(t)\| + \|I_{MC}(t) - I_{MC}^*(t)\| + \|T_C(t) - T_C^*(t)\| \\ &+ \|I_{TM}(t) - I_{TM}^*(t)\| + \|T_T(t) - T_T^*(t)\| + \|T_m(t) - T_m^*(t)\| \\ &+ \|T_m(t) - T_m^*(t)\| + \|R_T(t) - R_T^*(t)\| + \|R_m(t) - R_m^*(t)\| \\ &+ \|R_{Tm}(t) - R_{Tm}^*(t)\| + \|S_r(t) - S_r^*(t)\| + \|I_r(t) - I_r^*(t)\| \\ &+ \|B(t) - B^*(t)\| \end{aligned} \right), \\
\|Q_{12}V_h - Q_{12}V_h^*\| &\leq \frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)} L_{12} \left(\begin{aligned} &\|S_h(t) - S_h^*(t)\| + \|E_T(t) - E_T^*(t)\| + \|E_m(t) - E_m^*(t)\| \\ &+ \|E_{Tm}(t) - E_{Tm}^*(t)\| + \|Q_m(t) - Q_m^*(t)\| + \|I_T(t) - I_T^*(t)\| \\ &+ \|I_m(t) - I_m^*(t)\| + \|I_{MC}(t) - I_{MC}^*(t)\| + \|T_C(t) - T_C^*(t)\| \\ &+ \|I_{TM}(t) - I_{TM}^*(t)\| + \|T_T(t) - T_T^*(t)\| + \|T_m(t) - T_m^*(t)\| \\ &+ \|T_m(t) - T_m^*(t)\| + \|R_T(t) - R_T^*(t)\| + \|R_m(t) - R_m^*(t)\| \\ &+ \|R_{Tm}(t) - R_{Tm}^*(t)\| + \|S_r(t) - S_r^*(t)\| + \|I_r(t) - I_r^*(t)\| \\ &+ \|B(t) - B^*(t)\| \end{aligned} \right), \\
\|Q_{13}R_T - Q_{13}R_T^*\| &\leq \frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)} L_{13} \left(\begin{aligned} &\|S_h(t) - S_h^*(t)\| + \|E_T(t) - E_T^*(t)\| + \|E_m(t) - E_m^*(t)\| \\ &+ \|E_{Tm}(t) - E_{Tm}^*(t)\| + \|Q_m(t) - Q_m^*(t)\| + \|I_T(t) - I_T^*(t)\| \\ &+ \|I_m(t) - I_m^*(t)\| + \|I_{MC}(t) - I_{MC}^*(t)\| + \|T_C(t) - T_C^*(t)\| \\ &+ \|I_{TM}(t) - I_{TM}^*(t)\| + \|T_T(t) - T_T^*(t)\| + \|T_m(t) - T_m^*(t)\| \\ &+ \|T_m(t) - T_m^*(t)\| + \|R_T(t) - R_T^*(t)\| + \|R_m(t) - R_m^*(t)\| \\ &+ \|R_{Tm}(t) - R_{Tm}^*(t)\| + \|S_r(t) - S_r^*(t)\| + \|I_r(t) - I_r^*(t)\| \\ &+ \|B(t) - B^*(t)\| \end{aligned} \right),
\end{aligned}$$

$$\begin{aligned}
\|Q_{14}R_m - Q_{14}R_m^*\| &\leq \frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)} L_{14} \left(\begin{aligned} &\|S_h(t) - S_h^*(t)\| + \|E_T(t) - E_T^*(t)\| + \|E_m(t) - E_m^*(t)\| \\ &+ \|E_{Tm}(t) - E_{Tm}^*(t)\| + \|Q_m(t) - Q_m^*(t)\| + \|I_T(t) - I_T^*(t)\| \\ &+ \|I_m(t) - I_m^*(t)\| + \|I_{MC}(t) - I_{MC}^*(t)\| + \|T_C(t) - T_C^*(t)\| \\ &+ \|I_{TM}(t) - I_{TM}^*(t)\| + \|T_T(t) - T_T^*(t)\| + \|T_m(t) - T_m^*(t)\| \\ &+ \|T_m(t) - T_m^*(t)\| + \|R_T(t) - R_T^*(t)\| + \|R_m(t) - R_m^*(t)\| \\ &+ \|R_{Tm}(t) - R_{Tm}^*(t)\| + \|S_r(t) - S_r^*(t)\| + \|I_r(t) - I_r^*(t)\| \\ &+ \|B(t) - B^*(t)\| \end{aligned} \right), \\
\|Q_{15}R_{Tm} - Q_{15}R_{Tm}^*\| &\leq \frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)} L_{15} \left(\begin{aligned} &\|S_h(t) - S_h^*(t)\| + \|E_T(t) - E_T^*(t)\| + \|E_m(t) - E_m^*(t)\| \\ &+ \|E_{Tm}(t) - E_{Tm}^*(t)\| + \|Q_m(t) - Q_m^*(t)\| + \|I_T(t) - I_T^*(t)\| \\ &+ \|I_m(t) - I_m^*(t)\| + \|I_{MC}(t) - I_{MC}^*(t)\| + \|T_C(t) - T_C^*(t)\| \\ &+ \|I_{TM}(t) - I_{TM}^*(t)\| + \|T_T(t) - T_T^*(t)\| + \|T_m(t) - T_m^*(t)\| \\ &+ \|T_m(t) - T_m^*(t)\| + \|R_T(t) - R_T^*(t)\| + \|R_m(t) - R_m^*(t)\| \\ &+ \|R_{Tm}(t) - R_{Tm}^*(t)\| + \|S_r(t) - S_r^*(t)\| + \|I_r(t) - I_r^*(t)\| \\ &+ \|B(t) - B^*(t)\| \end{aligned} \right), \\
\|Q_{16}S_r - Q_{16}S_r^*\| &\leq \frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)} L_{16} \left(\begin{aligned} &\|S_h(t) - S_h^*(t)\| + \|E_T(t) - E_T^*(t)\| + \|E_m(t) - E_m^*(t)\| \\ &+ \|E_{Tm}(t) - E_{Tm}^*(t)\| + \|Q_m(t) - Q_m^*(t)\| + \|I_T(t) - I_T^*(t)\| \\ &+ \|I_m(t) - I_m^*(t)\| + \|I_{MC}(t) - I_{MC}^*(t)\| + \|T_C(t) - T_C^*(t)\| \\ &+ \|I_{TM}(t) - I_{TM}^*(t)\| + \|T_T(t) - T_T^*(t)\| + \|T_m(t) - T_m^*(t)\| \\ &+ \|T_m(t) - T_m^*(t)\| + \|R_T(t) - R_T^*(t)\| + \|R_m(t) - R_m^*(t)\| \\ &+ \|R_{Tm}(t) - R_{Tm}^*(t)\| + \|S_r(t) - S_r^*(t)\| + \|I_r(t) - I_r^*(t)\| \\ &+ \|B(t) - B^*(t)\| \end{aligned} \right),
\end{aligned}$$

$$\begin{aligned}
\|Q_{17}I_r - Q_{17}I_r^*\| &\leq \frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)} L_{17} \left(\begin{aligned} &\|S_h(t) - S_h^*(t)\| + \|E_T(t) - E_T^*(t)\| + \|E_m(t) - E_m^*(t)\| \\ &+ \|E_{Tm}(t) - E_{Tm}^*(t)\| + \|Q_m(t) - Q_m^*(t)\| + \|I_T(t) - I_T^*(t)\| \\ &+ \|I_m(t) - I_m^*(t)\| + \|I_{MC}(t) - I_{MC}^*(t)\| + \|T_C(t) - T_C^*(t)\| \\ &+ \|I_{TM}(t) - I_{TM}^*(t)\| + \|T_T(t) - T_T^*(t)\| + \|T_m(t) - T_m^*(t)\| \\ &+ \|T_m(t) - T_m^*(t)\| + \|R_T(t) - R_T^*(t)\| + \|R_m(t) - R_m^*(t)\| \\ &+ \|R_{Tm}(t) - R_{Tm}^*(t)\| + \|S_r(t) - S_r^*(t)\| + \|I_r(t) - I_r^*(t)\| \\ &+ \|B(t) - B^*(t)\| \end{aligned} \right), \\
\|Q_{18}B - Q_{18}B^*\| &\leq \frac{\sigma(1-\alpha)T_{\min}^{\sigma-1}}{AB(\alpha)} L_{18} \left(\begin{aligned} &\|S_h(t) - S_h^*(t)\| + \|E_T(t) - E_T^*(t)\| + \|E_m(t) - E_m^*(t)\| \\ &+ \|E_{Tm}(t) - E_{Tm}^*(t)\| + \|Q_m(t) - Q_m^*(t)\| + \|I_T(t) - I_T^*(t)\| \\ &+ \|I_m(t) - I_m^*(t)\| + \|I_{MC}(t) - I_{MC}^*(t)\| + \|T_C(t) - T_C^*(t)\| \\ &+ \|I_{TM}(t) - I_{TM}^*(t)\| + \|T_T(t) - T_T^*(t)\| + \|T_m(t) - T_m^*(t)\| \\ &+ \|T_m(t) - T_m^*(t)\| + \|R_T(t) - R_T^*(t)\| + \|R_m(t) - R_m^*(t)\| \\ &+ \|R_{Tm}(t) - R_{Tm}^*(t)\| + \|S_r(t) - S_r^*(t)\| + \|I_r(t) - I_r^*(t)\| \\ &+ \|B(t) - B^*(t)\| \end{aligned} \right). \quad (45)
\end{aligned}$$

From Eq. (45) with $L_{\max}$, it follows that,

$$\begin{aligned}
&\left\| Q(S_h, E_T, E_m, E_{Tm}, Q_m, I_T, I_m, I_{Tm}, T_T, T_m, T_{Tm}, V_h, R_T, R_m, R_{Tm}, S_r, I_r, B) \right. \\
&\quad \left. - \tau(S_h^*, E_T^*, E_m^*, E_{Tm}^*, Q_m^*, I_T^*, I_m^*, I_{Tm}^*, T_T^*, T_m^*, T_{Tm}^*, V_h^*, R_T^*, R_m^*, R_{Tm}^*, S_r^*, I_r^*, B^*) \right\| \\
&\leq \frac{\alpha(1-\sigma)T_{\min}^{\alpha-1}}{AB(\sigma)} L_{\max} \left\| \begin{aligned} &Q \left(\begin{aligned} &S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), T_T(t), \\ &T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), B(t) \end{aligned} \right) \\ &- \left(\begin{aligned} &S_h^*(t), E_T^*(t), E_m^*(t), E_{Tm}^*(t), Q_m^*(t), I_T^*(t), I_m^*(t), I_{Tm}^*(t), T_T^*(t), \\ &T_m^*(t), T_{Tm}^*(t), V_h^*(t), R_T^*(t), R_m^*(t), R_{Tm}^*(t), S_r^*(t), I_r^*(t), B^*(t) \end{aligned} \right) \end{aligned} \right\|.
\end{aligned}$$

Therefore, Q is a contraction.

The proof establishes continuous and compact properties of P resulting in completely continuous status.

Then, it is satisfactory to establish that P is bounded and equicontinuous.

Applying (B_2) , it is observable that P is continuous as Q is also continuous.

Consequently, for any $t \in \tau$, we obtained:

$$\|P_1 S_h\| \leq \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \sup_{t \in \tau} \left| \int_0^t S_h^{\sigma-1}(t-s)^{\alpha-1} G_1(s, S_h(s)) ds \right|,$$

$$\begin{aligned}
\|P_1 S_h\| &\leq \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \int_0^t S_h^{\sigma-1}(t-s)^{\sigma-1} \text{Sup}_{t \in \tau} |G_1(s, S_h(s))| ds, \\
&\leq \frac{\alpha T^{\alpha+\sigma-1}\Gamma(\sigma+1)}{AB(\alpha)\Gamma(\alpha+\sigma)} \left[\begin{aligned} &g_{01} + g_{11} \|S_h\| + g_{21} \|E_T\| + g_{31} \|E_m\| + g_{41} \|E_{Tm}\| + g_{51} \|Q_m\| + g_{61} \|I_T\| \\ &+ g_{71} \|I_m\| + g_{81} \|I_{Tm}\| + g_{91} \|T_T\| + g_{101} \|T_m\| + g_{111} \|T_{Tm}\| + g_{121} \|V_h\| \\ &+ g_{131} \|R_T\| + g_{141} \|R_m\| + g_{151} \|R_{Tm}\| + g_{161} \|S_r\| + g_{171} \|I_r\| + g_{181} \|B\| \end{aligned} \right]. \quad (46)
\end{aligned}$$

Then, P_1 is bounded.

Similarly, we have;

$$\begin{aligned}
\|P_2 E_T\| &\leq \frac{\alpha T^{\alpha+\sigma-1}\Gamma(\sigma+1)}{AB(\alpha)\Gamma(\alpha+\sigma)} \left[\begin{aligned} &g_{02} + g_{12} \|S_h\| + g_{22} \|E_T\| + g_{32} \|E_m\| + g_{42} \|E_{Tm}\| + g_{52} \|Q_m\| + g_{62} \|I_T\| \\ &+ g_{72} \|I_m\| + g_{82} \|I_{Tm}\| + g_{92} \|T_T\| + g_{102} \|T_m\| + g_{112} \|T_{Tm}\| + g_{122} \|V_h\| \\ &+ g_{132} \|R_T\| + g_{142} \|R_m\| + g_{152} \|R_{Tm}\| + g_{162} \|S_r\| + g_{172} \|I_r\| + g_{182} \|B\| \end{aligned} \right], \\
\|P_3 E_m\| &\leq \frac{\alpha T^{\alpha+\sigma-1}\Gamma(\sigma+1)}{AB(\alpha)\Gamma(\alpha+\sigma)} \left[\begin{aligned} &g_{03} + g_{13} \|S_h\| + g_{23} \|E_T\| + g_{33} \|E_m\| + g_{43} \|E_{Tm}\| + g_{53} \|Q_m\| + g_{63} \|I_T\| \\ &+ g_{73} \|I_m\| + g_{83} \|I_{Tm}\| + g_{93} \|T_T\| + g_{103} \|T_m\| + g_{113} \|T_{Tm}\| + g_{123} \|V_h\| \\ &+ g_{133} \|R_T\| + g_{143} \|R_m\| + g_{153} \|R_{Tm}\| + g_{163} \|S_r\| + g_{173} \|I_r\| + g_{183} \|B\| \end{aligned} \right], \\
\|P_4 E_{Tm}\| &\leq \frac{\alpha T^{\alpha+\sigma-1}\Gamma(\sigma+1)}{AB(\alpha)\Gamma(\alpha+\sigma)} \left[\begin{aligned} &g_{04} + g_{14} \|S_h\| + g_{24} \|E_T\| + g_{34} \|E_m\| + g_{44} \|E_{Tm}\| + g_{54} \|Q_m\| + g_{64} \|I_T\| \\ &+ g_{74} \|I_m\| + g_{84} \|I_{Tm}\| + g_{94} \|T_T\| + g_{104} \|T_m\| + g_{114} \|T_{Tm}\| + g_{124} \|V_h\| \\ &+ g_{134} \|R_T\| + g_{144} \|R_m\| + g_{154} \|R_{Tm}\| + g_{164} \|S_r\| + g_{174} \|I_r\| + g_{184} \|B\| \end{aligned} \right], \\
\|P_5 Q_m\| &\leq \frac{\alpha T^{\alpha+\sigma-1}\Gamma(\sigma+1)}{AB(\alpha)\Gamma(\alpha+\sigma)} \left[\begin{aligned} &g_{05} + g_{15} \|S_h\| + g_{25} \|E_T\| + g_{35} \|E_m\| + g_{45} \|E_{Tm}\| + g_{55} \|Q_m\| + g_{65} \|I_T\| \\ &+ g_{75} \|I_m\| + g_{85} \|I_{Tm}\| + g_{95} \|T_T\| + g_{105} \|T_m\| + g_{115} \|T_{Tm}\| + g_{125} \|V_h\| \\ &+ g_{135} \|R_T\| + g_{145} \|R_m\| + g_{155} \|R_{Tm}\| + g_{165} \|S_r\| + g_{175} \|I_r\| + g_{185} \|B\| \end{aligned} \right], \\
\|P_6 I_T\| &\leq \frac{\alpha T^{\alpha+\sigma-1}\Gamma(\sigma+1)}{AB(\alpha)\Gamma(\alpha+\sigma)} \left[\begin{aligned} &g_{06} + g_{16} \|S_h\| + g_{26} \|E_T\| + g_{36} \|E_m\| + g_{46} \|E_{Tm}\| + g_{56} \|Q_m\| + g_{66} \|I_T\| \\ &+ g_{76} \|I_m\| + g_{86} \|I_{Tm}\| + g_{96} \|T_T\| + g_{106} \|T_m\| + g_{116} \|T_{Tm}\| + g_{126} \|V_h\| \\ &+ g_{136} \|R_T\| + g_{146} \|R_m\| + g_{156} \|R_{Tm}\| + g_{166} \|S_r\| + g_{176} \|I_r\| + g_{186} \|B\| \end{aligned} \right], \\
\|P_7 I_m\| &\leq \frac{\alpha T^{\alpha+\sigma-1}\Gamma(\sigma+1)}{AB(\alpha)\Gamma(\alpha+\sigma)} \left[\begin{aligned} &g_{07} + g_{17} \|S_h\| + g_{27} \|E_T\| + g_{37} \|E_m\| + g_{47} \|E_{Tm}\| + g_{57} \|Q_m\| + g_{67} \|I_T\| \\ &+ g_{77} \|I_m\| + g_{87} \|I_{Tm}\| + g_{97} \|T_T\| + g_{107} \|T_m\| + g_{117} \|T_{Tm}\| + g_{127} \|V_h\| \\ &+ g_{137} \|R_T\| + g_{147} \|R_m\| + g_{157} \|R_{Tm}\| + g_{167} \|S_r\| + g_{177} \|I_r\| + g_{187} \|B\| \end{aligned} \right], \\
\|P_8 I_{Tm}\| &\leq \frac{\alpha T^{\alpha+\sigma-1}\Gamma(\sigma+1)}{AB(\alpha)\Gamma(\alpha+\sigma)} \left[\begin{aligned} &g_{08} + g_{18} \|S_h\| + g_{28} \|E_T\| + g_{38} \|E_m\| + g_{48} \|E_{Tm}\| + g_{58} \|Q_m\| + g_{68} \|I_T\| \\ &+ g_{78} \|I_m\| + g_{88} \|I_{Tm}\| + g_{98} \|T_T\| + g_{108} \|T_m\| + g_{118} \|T_{Tm}\| + g_{128} \|V_h\| \\ &+ g_{138} \|R_T\| + g_{148} \|R_m\| + g_{158} \|R_{Tm}\| + g_{168} \|S_r\| + g_{178} \|I_r\| + g_{188} \|B\| \end{aligned} \right], \\
\|P_9 T_T\| &\leq \frac{\alpha T^{\alpha+\sigma-1}\Gamma(\sigma+1)}{AB(\alpha)\Gamma(\alpha+\sigma)} \left[\begin{aligned} &g_{09} + g_{19} \|S_h\| + g_{29} \|E_T\| + g_{39} \|E_m\| + g_{49} \|E_{Tm}\| + g_{59} \|Q_m\| + g_{69} \|I_T\| \\ &+ g_{79} \|I_m\| + g_{89} \|I_{Tm}\| + g_{99} \|T_T\| + g_{109} \|T_m\| + g_{119} \|T_{Tm}\| + g_{129} \|V_h\| \\ &+ g_{139} \|R_T\| + g_{149} \|R_m\| + g_{159} \|R_{Tm}\| + g_{169} \|S_r\| + g_{179} \|I_r\| + g_{189} \|B\| \end{aligned} \right],
\end{aligned}$$

$$\begin{aligned}
\|P\| &= \max \left\{ \begin{aligned} &\|P_1\| + \|P_2\| + \|P_3\| + \|P_4\| + \|P_5\| + \|P_6\| + \|P_7\| + \|P_8\| + \|P_9\| \\ &+ \|P_{10}\| + \|P_{11}\| + \|P_{12}\| + \|P_{13}\| + \|P_{14}\| + \|P_{15}\| + \|P_{16}\| + \|P_{17}\| \\ &+ \|P_{18}\| \end{aligned} \right\}. \\
g_0^* &= \max \{g_{0j}^*\}, g_1^* = \max \{g_{1j}^*\}, g_2^* = \max \{g_{2j}^*\}, g_3^* = \max \{g_{3j}^*\}, g_4^* = \max \{g_{4j}^*\} \\
g_5^* &= \max \{g_{5j}^*\}, g_6^* = \max \{g_{6j}^*\}, g_7^* = \max \{g_{7j}^*\}, g_8^* = \max \{g_{8j}^*\}, g_9^* = \max \{g_{9j}^*\} \\
\text{With } g_{10}^* &= \max \{g_{10j}^*\}, g_{11}^* = \max \{g_{11j}^*\}, g_{12}^* = \max \{g_{12j}^*\}, g_{13}^* = \max \{g_{13j}^*\}, g_{14}^* = \max \{g_{14j}^*\} \\
g_{15}^* &= \max \{g_{15j}^*\}, g_{16}^* = \max \{g_{16j}^*\}, g_{17}^* = \max \{g_{17j}^*\}, g_{18}^* = \max \{g_{18j}^*\} \\
j &= 1, 2, 3, \dots, 18, \text{ it follows that:}
\end{aligned}$$

$$\begin{aligned}
&\left\| P \begin{pmatrix} S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), T_T(t), \\ T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), B(t) \end{pmatrix} \right\| \\
&\leq \frac{\alpha T^{\alpha+\sigma-1} \Gamma(\sigma+1)}{AB(\sigma) \Gamma(\alpha+\sigma)} \left[\begin{aligned} &g_0^* + g_1^* \|S_h\| + g_2^* \|E_T\| + g_3^* \|E_m\| + g_4^* \|E_{Tm}\| \\ &+ g_5^* \|Q_m\| + g_6^* \|I_T\| + g_7^* \|I_m\| + g_8^* \|I_{Tm}\| + g_9^* \|T_T\| \\ &+ g_{10}^* \|T_m\| + g_{11}^* \|T_{Tm}\| + g_{12}^* \|V_h\| + g_{13}^* \|R_T\| + g_{14}^* \|R_m\| \\ &+ g_{15}^* \|R_{Tm}\| + g_{16}^* \|S_r\| + g_{17}^* \|I_r\| + g_{18}^* \|B\| \end{aligned} \right]. \quad (48)
\end{aligned}$$

This means that, P is bounded.

We have to further prove that, P is equicontinuity.

If, $t_1, t_2 \in \tau$ with $0 \leq t_1 \leq t_2 \in \tau$

$$\text{We have, } |P_1 S_h(t_2) - P_1 S_h(t_1)| \leq \frac{\sigma \alpha}{AB(\alpha)} \left| \int_0^{t_2} S^{\alpha-1}(t_2-s)^{\alpha-1} ds - \int_0^{t_1} S_h^{\sigma-1}(t_1-s)^{\rho-1} ds \right|.$$

$$\begin{aligned}
|G_1(S_h(s))| &\leq \frac{\alpha \Gamma(\sigma+1)}{AB(\alpha) \Gamma(\alpha+\sigma)} \left| \begin{aligned} &t_1^{\alpha+\sigma-1} \\ &-t_2^{\alpha+\sigma-1} \\ &+ 2(t_2-t_1)^{\alpha+\sigma-1} \end{aligned} ds \right| \left[\begin{aligned} &g_{012} + g_{112} \|S_h\| + g_{212} \|E_T\| + g_{312} \|E_m\| \\ &+ g_{412} \|E_{Tm}\| + g_{512} \|Q_m\| + g_{612} \|I_T\| \\ &+ g_{712} \|I_m\| + g_{812} \|I_{Tm}\| + g_{912} \|T_T\| \\ &+ g_{1012} \|T_m\| + g_{1112} \|T_{Tm}\| + g_{1212} \|V_h\| \\ &+ g_{1312} \|R_T\| + g_{1412} \|R_m\| + g_{1512} \|R_{Tm}\| \\ &+ g_{1612} \|S_r\| + g_{1712} \|I_r\| + g_{1812} \|B\| \end{aligned} \right], \\
|(P_1 S_h)(t_2) - (P_1 S_h)(t_1)| &\leq \frac{\alpha \Gamma(\sigma+1)}{AB(\alpha) \Gamma(\alpha+\sigma)} \left| t_1^{\alpha+\sigma-1} - t_2^{\alpha+\sigma-1} + 2(t_2-t_1)^{\alpha+\sigma-1} \right| ds \left[g_{01} + K_1 \sum_{i=1}^{\infty} g_{i1} \right], \quad (49)
\end{aligned}$$

We now observed that, the right hand side of (49) is independent of

$$\begin{pmatrix} S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), T_T(t), \\ T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), B(t) \end{pmatrix}.$$

The definition indicates that these quantities fulfill the three criteria of boundedness, uniform continuity and compactness so they are completely continuous.

Similarly, we have the following inequality:

$$\begin{aligned}
|(P_2 E_T)(t_2) - (P_2 E_T)(t_1)| &\leq \frac{\alpha \Gamma(\sigma+1)}{AB(\alpha) \Gamma(\alpha+\sigma)} \left| t_1^{\alpha+\alpha-1} - t_2^{\alpha+\sigma-1} + 2(t_2 - t_1)^{\alpha+\sigma-1} \right| ds \left[g_{02} + K_2 \sum_{i=1}^{\infty} g_{i2} \right], \\
|(P_3 E_m)(t_2) - (P_3 E_m)(t_1)| &\leq \frac{\alpha \Gamma(\sigma+1)}{AB(\alpha) \Gamma(\alpha+\sigma)} \left| t_1^{\alpha+\alpha-1} - t_2^{\alpha+\sigma-1} + 2(t_2 - t_1)^{\alpha+\sigma-1} \right| ds \left[g_{03} + K_3 \sum_{i=1}^{\infty} g_{i3} \right], \\
|(P_4 E_{Tm})(t_2) - (P_4 E_{Tm})(t_1)| &\leq \frac{\alpha \Gamma(\sigma+1)}{AB(\alpha) \Gamma(\alpha+\sigma)} \left| t_1^{\alpha+\alpha-1} - t_2^{\alpha+\sigma-1} + 2(t_2 - t_1)^{\alpha+\sigma-1} \right| ds \left[g_{04} + K_4 \sum_{i=1}^{\infty} g_{i4} \right], \\
|(P_5 Q_m)(t_2) - (P_5 Q_m)(t_1)| &\leq \frac{\alpha \Gamma(\sigma+1)}{AB(\alpha) \Gamma(\alpha+\sigma)} \left| t_1^{\alpha+\alpha-1} - t_2^{\alpha+\sigma-1} + 2(t_2 - t_1)^{\alpha+\sigma-1} \right| ds \left[g_{05} + K_5 \sum_{i=1}^{\infty} g_{i5} \right], \\
|(P_6 I_T)(t_2) - (P_6 I_T)(t_1)| &\leq \frac{\alpha \Gamma(\sigma+1)}{AB(\alpha) \Gamma(\alpha+\sigma)} \left| t_1^{\alpha+\alpha-1} - t_2^{\alpha+\sigma-1} + 2(t_2 - t_1)^{\alpha+\sigma-1} \right| ds \left[g_{06} + K_6 \sum_{i=1}^{\infty} g_{i6} \right], \\
|(P_7 I_m)(t_2) - (P_7 I_m)(t_1)| &\leq \frac{\alpha \Gamma(\sigma+1)}{AB(\alpha) \Gamma(\alpha+\sigma)} \left| t_1^{\alpha+\alpha-1} - t_2^{\alpha+\sigma-1} + 2(t_2 - t_1)^{\alpha+\sigma-1} \right| ds \left[g_{07} + K_7 \sum_{i=1}^{\infty} g_{i7} \right], \\
|(P_8 I_{Tm})(t_2) - (P_8 I_{Tm})(t_1)| &\leq \frac{\alpha \Gamma(\sigma+1)}{AB(\alpha) \Gamma(\alpha+\sigma)} \left| t_1^{\alpha+\alpha-1} - t_2^{\alpha+\sigma-1} + 2(t_2 - t_1)^{\alpha+\sigma-1} \right| ds \left[g_{08} + K_8 \sum_{i=1}^{\infty} g_{i8} \right], \\
|(P_9 T_T)(t_2) - (P_9 T_T)(t_1)| &\leq \frac{\alpha \Gamma(\sigma+1)}{AB(\alpha) \Gamma(\alpha+\sigma)} \left| t_1^{\alpha+\alpha-1} - t_2^{\alpha+\sigma-1} + 2(t_2 - t_1)^{\alpha+\sigma-1} \right| ds \left[g_{09} + K_9 \sum_{i=1}^{\infty} g_{i9} \right], \\
\\
|(P_{10} T_m)(t_2) - (P_{10} T_m)(t_1)| &\leq \frac{\alpha \Gamma(\sigma+1)}{AB(\alpha) \Gamma(\alpha+\sigma)} \left| t_1^{\alpha+\alpha-1} - t_2^{\alpha+\sigma-1} + 2(t_2 - t_1)^{\alpha+\sigma-1} \right| ds \left[g_{010} + K_{10} \sum_{i=1}^{\infty} g_{i10} \right], \\
|(P_{11} T_{Tm})(t_2) - (P_{11} T_{Tm})(t_1)| &\leq \frac{\alpha \Gamma(\sigma+1)}{AB(\alpha) \Gamma(\alpha+\sigma)} \left| t_1^{\alpha+\alpha-1} - t_2^{\alpha+\sigma-1} + 2(t_2 - t_1)^{\alpha+\sigma-1} \right| ds \left[g_{011} + K_{11} \sum_{i=1}^{\infty} g_{i11} \right], \\
|(P_{12} V_m)(t_2) - (P_{12} V_m)(t_1)| &\leq \frac{\alpha \Gamma(\sigma+1)}{AB(\alpha) \Gamma(\alpha+\sigma)} \left| t_1^{\alpha+\alpha-1} - t_2^{\alpha+\sigma-1} + 2(t_2 - t_1)^{\alpha+\sigma-1} \right| ds \left[g_{012} + K_{12} \sum_{i=1}^{\infty} g_{i12} \right], \\
|(P_{13} R_T)(t_2) - (P_{13} R_T)(t_1)| &\leq \frac{\alpha \Gamma(\sigma+1)}{AB(\alpha) \Gamma(\alpha+\sigma)} \left| t_1^{\alpha+\alpha-1} - t_2^{\alpha+\sigma-1} + 2(t_2 - t_1)^{\alpha+\sigma-1} \right| ds \left[g_{013} + K_{13} \sum_{i=1}^{\infty} g_{i13} \right], \\
|(P_{14} R_m)(t_2) - (P_{14} R_m)(t_1)| &\leq \frac{\alpha \Gamma(\sigma+1)}{AB(\alpha) \Gamma(\alpha+\sigma)} \left| t_1^{\alpha+\alpha-1} - t_2^{\alpha+\sigma-1} + 2(t_2 - t_1)^{\alpha+\sigma-1} \right| ds \left[g_{014} + K_{14} \sum_{i=1}^{\infty} g_{i14} \right], \\
|(P_{15} R_{Tm})(t_2) - (P_{15} R_{Tm})(t_1)| &\leq \frac{\alpha \Gamma(\sigma+1)}{AB(\alpha) \Gamma(\alpha+\sigma)} \left| t_1^{\alpha+\alpha-1} - t_2^{\alpha+\sigma-1} + 2(t_2 - t_1)^{\alpha+\sigma-1} \right| ds \left[g_{015} + K_{15} \sum_{i=1}^{\infty} g_{i15} \right], \\
|(P_{16} S_r)(t_2) - (P_{16} S_r)(t_1)| &\leq \frac{\alpha \Gamma(\sigma+1)}{AB(\alpha) \Gamma(\alpha+\sigma)} \left| t_1^{\alpha+\alpha-1} - t_2^{\alpha+\sigma-1} + 2(t_2 - t_1)^{\alpha+\sigma-1} \right| ds \left[g_{016} + K_{16} \sum_{i=1}^{\infty} g_{i16} \right],
\end{aligned}$$

$$\begin{aligned}
|(P_{17}I_r)(t_2) - (P_{17}I_r)(t_1)| &\leq \frac{\alpha\Gamma(\sigma+1)}{AB(\alpha)\Gamma(\alpha+\sigma)} \left| t_1^{\alpha+\sigma-1} - t_2^{\alpha+\sigma-1} + 2(t_2 - t_1)^{\alpha+\sigma-1} \right| ds \left[g_{017} + K_{17} \sum_{i=1}^{\infty} g_{i17} \right], \\
|(P_{18}B)(t_2) - (P_{18}B)(t_1)| &\leq \frac{\alpha\Gamma(\sigma+1)}{AB(\alpha)\Gamma(\alpha+\sigma)} \left| t_1^{\alpha+\sigma-1} - t_2^{\alpha+\sigma-1} + 2(t_2 - t_1)^{\alpha+\sigma-1} \right| ds \left[g_{018} + K_{18} \sum_{i=1}^{\infty} g_{i18} \right].
\end{aligned}
\tag{50}$$

Similarly, the inequality from (49)-(50), that is

$$\begin{aligned}
&\left| Pi \begin{pmatrix} S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), T_T(t), \\ T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), B(t) \end{pmatrix} (t_2) \right. \\
&\left. - Pi \begin{pmatrix} S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), T_T(t), \\ T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), B(t) \end{pmatrix} (t_1) \right| \rightarrow 0 \text{ as } t_2 \rightarrow t_1,
\end{aligned}$$

for $i = 1, 2, 3, \dots, 18$, this yields that, $Pi, i = 1, 2, 3, \dots, 18$, Pi fulfills the requirements of being bounded, uniformly continuous and compact so it stands as completely continuous for $i = 1, 2, 3, \dots, 18$. By using Eq.(45) and making $\|P\|$ with g_i^* for $j = 0, 1, 2, \dots, 18$,

we have:

$$\begin{aligned}
&\left| Pi \begin{pmatrix} S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), T_T(t), \\ T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), B(t) \end{pmatrix} (t_2) \right. \\
&\left. - Pi \begin{pmatrix} S_h(t), E_T(t), E_m(t), E_{Tm}(t), Q_m(t), I_T(t), I_m(t), I_{Tm}(t), T_T(t), \\ T_m(t), T_{Tm}(t), V_h(t), R_T(t), R_m(t), R_{Tm}(t), S_r(t), I_r(t), B(t) \end{pmatrix} (t_1) \right| \rightarrow 0 \text{ as } t_2 \rightarrow t_1, \\
&\leq \frac{\alpha\Gamma(\sigma+1)}{AB(\alpha)\Gamma(\alpha+\sigma)} \left| t_1^{\alpha+\sigma-1} - t_2^{\alpha+\sigma-1} + 2(t_2 - t_1)^{\alpha+\sigma-1} \right| ds \left[g_{0i}^* + K_i \sum_{i=1}^{\infty} g_i^* \right].
\end{aligned}
\tag{51}$$

The conditions of boundedness together with uniform continuity and compactness outline that P represents a completely continuous operator. Therefore, P satisfies the criteria for having at least one solution. Based on the statement in Lemma 3 we can confirm the existence of at least one solution for the FF-Monkey pox and Typhoid fever co-infection model.

4.0 Numerical Schemes for the FF-Monkey pox and Chlamydia co-infection model

The second-order Adams-Bashforth algorithm will be used to create numerical solutions for FF-Monkey pox and Typhoid fever co-infection model (8).

We rewrite these integral equations (45), at $t = t_{n+1}$, this yields:

$$\begin{aligned}
S_h(t_{n+1}) &= S_{h0} + \sigma(1-\alpha)t_n^{\sigma-1}G_1(t_n, S_h(t_n)) + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \int_0^{t+1} (t_{n+1} - s)^{\alpha-1} S_h^{\sigma-1} G_1(s, S_h(s)) ds, \\
E_T(t_{n+1}) &= E_{T0} + \sigma(1-\alpha)t_n^{\sigma-1}G_2(t_n, E_T(t_n)) + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \int_0^{t+1} (t_{n+1} - s)^{\alpha-1} E_T^{\sigma-1} G_2(s, E_T(s)) ds, \\
E_m(t_{n+1}) &= E_{m0} + \sigma(1-\alpha)t_n^{\sigma-1}G_3(t_n, E_m(t_n)) + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \int_0^{t+1} (t_{n+1} - s)^{\alpha-1} E_m^{\sigma-1} G_3(s, E_m(s)) ds,
\end{aligned}$$

$$\begin{aligned}
E_{Tm}(t_{n+1}) &= E_{Tm0} + \sigma(1-\alpha)t_n^{\sigma-1}G_4(t_n, E_{Tm}(t_n)) + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \int_0^{t+1} (t_{n+1}-s)^{\alpha-1} E_{Tm}^{\sigma-1} G_4(s, E_{Tm}(s)) ds, \\
Q_m(t_{n+1}) &= Q_{m0} + \sigma(1-\alpha)t_n^{\sigma-1}G_5(t_n, Q_m(t_n)) + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \int_0^{t+1} (t_{n+1}-s)^{\alpha-1} Q_m^{\sigma-1} G_5(s, Q_m(s)) ds, \\
I_T(t_{n+1}) &= I_{T0} + \sigma(1-\alpha)t_n^{\sigma-1}G_6(t_n, I_T(t_n)) + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \int_0^{t+1} (t_{n+1}-s)^{\alpha-1} I_T^{\sigma-1} G_6(s, I_T(s)) ds, \\
I_m(t_{n+1}) &= I_{T0} + \sigma(1-\alpha)t_n^{\sigma-1}G_7(t_n, I_m(t_n)) + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \int_0^{t+1} (t_{n+1}-s)^{\alpha-1} I_m^{\sigma-1} G_7(s, I_m(s)) ds, \\
I_{Tm}(t_{n+1}) &= I_{Tm0} + \sigma(1-\alpha)t_n^{\sigma-1}G_8(t_n, I_{Tm}(t_n)) + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \int_0^{t+1} (t_{n+1}-s)^{\alpha-1} I_{Tm}^{\sigma-1} G_8(s, I_{Tm}(s)) ds, \\
T_T(t_{n+1}) &= T_{T0} + \sigma(1-\alpha)t_n^{\sigma-1}G_9(t_n, T_T(t_n)) + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \int_0^{t+1} (t_{n+1}-s)^{\alpha-1} T_T^{\sigma-1} G_9(s, T_T(s)) ds, \\
T_m(t_{n+1}) &= T_{m0} + \sigma(1-\alpha)t_n^{\sigma-1}G_{10}(t_n, T_m(t_n)) + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \int_0^{t+1} (t_{n+1}-s)^{\alpha-1} T_m^{\sigma-1} G_{10}(s, T_m(s)) ds, \\
T_{Tm}(t_{n+1}) &= T_{Tm0} + \sigma(1-\alpha)t_n^{\sigma-1}G_{11}(t_n, T_{Tm}(t_n)) + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \int_0^{t+1} (t_{n+1}-s)^{\alpha-1} T_{Tm}^{\sigma-1} G_{11}(s, T_{Tm}(s)) ds, \\
V_h(t_{n+1}) &= V_{h0} + \sigma(1-\alpha)t_n^{\sigma-1}G_{12}(t_n, V_h(t_n)) + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \int_0^{t+1} (t_{n+1}-s)^{\alpha-1} V_h^{\sigma-1} G_{12}(s, V_h(s)) ds, \\
R_T(t_{n+1}) &= R_{T0} + \sigma(1-\alpha)t_n^{\sigma-1}G_{13}(t_n, R_T(t_n)) + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \int_0^{t+1} (t_{n+1}-s)^{\alpha-1} R_T^{\sigma-1} G_{13}(s, R_T(s)) ds, \\
R_m(t_{n+1}) &= R_{m0} + \sigma(1-\alpha)t_n^{\sigma-1}G_{14}(t_n, R_m(t_n)) + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \int_0^{t+1} (t_{n+1}-s)^{\alpha-1} R_m^{\sigma-1} G_{14}(s, R_m(s)) ds, \\
R_{Tm}(t_{n+1}) &= R_{Tm0} + \sigma(1-\alpha)t_n^{\sigma-1}G_{15}(t_n, R_{Tm}(t_n)) + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \int_0^{t+1} (t_{n+1}-s)^{\alpha-1} R_{Tm}^{\sigma-1} G_{15}(s, R_{Tm}(s)) ds, \\
S_r(t_{n+1}) &= S_{r0} + \sigma(1-\alpha)t_n^{\sigma-1}G_{16}(t_n, S_r(t_n)) + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \int_0^{t+1} (t_{n+1}-s)^{\alpha-1} S_r^{\sigma-1} G_{16}(s, S_r(s)) ds, \\
I_r(t_{n+1}) &= I_{r0} + \sigma(1-\alpha)t_n^{\sigma-1}G_{17}(t_n, I_r(t_n)) + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \int_0^{t+1} (t_{n+1}-s)^{\alpha-1} I_r^{\sigma-1} G_{17}(s, I_r(s)) ds, \\
B(t_{n+1}) &= B_0 + \sigma(1-\alpha)t_n^{\sigma-1}G_{18}(t_n, B(t_n)) + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \int_0^{t+1} (t_{n+1}-s)^{\alpha-1} I_r^{\sigma-1} G_{18}(s, B(s)) ds.
\end{aligned}$$

So the numerical approximation of the integral (45) is obtained as:

$$\begin{aligned}
S_h(t_{n+1}) &= S_{h0} + \frac{\sigma(1-\alpha)t_n^{\sigma-1}G_1(t_n, S_h(t_n))}{AB(\alpha)} + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \sum_{j=1}^n \int_{t_j}^{t_{j+1}} (t_{j+1}-s)^{\alpha-1} S_h^{\sigma-1} G_1(t_n, S_h^j(s)) ds, \\
E_T(t_{n+1}) &= E_{T0} + \frac{\sigma(1-\alpha)t_n^{\sigma-1}G_2(t_n, E_T(t_n))}{AB(\alpha)} + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \sum_{j=1}^n \int_{t_j}^{t_{j+1}} (t_{j+1}-s)^{\alpha-1} E_T^{\sigma-1} G_2(t_n, E_T^j(s)) ds, \\
E_m(t_{n+1}) &= E_{m0} + \frac{\sigma(1-\alpha)t_n^{\sigma-1}G_3(t_n, E_m(t_n))}{AB(\alpha)} + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \sum_{j=1}^n \int_{t_j}^{t_{j+1}} (t_{j+1}-s)^{\alpha-1} E_m^{\sigma-1} G_3(t_n, E_m^j(s)) ds, \\
E_{Tm}(t_{n+1}) &= E_{Tm0} + \frac{\sigma(1-\alpha)t_n^{\sigma-1}G_4(t_n, E_{Tm}(t_n))}{AB(\alpha)} + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \sum_{j=1}^n \int_{t_j}^{t_{j+1}} (t_{j+1}-s)^{\alpha-1} E_{Tm}^{\sigma-1} G_4(t_n, E_{Tm}^j(s)) ds, \\
Q_m(t_{n+1}) &= Q_{m0} + \frac{\sigma(1-\alpha)t_n^{\sigma-1}G_5(t_n, Q_m(t_n))}{AB(\alpha)} + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \sum_{j=1}^n \int_{t_j}^{t_{j+1}} (t_{j+1}-s)^{\alpha-1} Q_m^{\sigma-1} G_5(t_n, Q_m^j(s)) ds, \\
I_T(t_{n+1}) &= I_{T0} + \frac{\sigma(1-\alpha)t_n^{\sigma-1}G_6(t_n, I_T(t_n))}{AB(\alpha)} + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \sum_{j=1}^n \int_{t_j}^{t_{j+1}} (t_{j+1}-s)^{\alpha-1} I_T^{\sigma-1} G_6(t_n, I_T^j(s)) ds, \\
I_m(t_{n+1}) &= I_{m0} + \frac{\sigma(1-\alpha)t_n^{\sigma-1}G_7(t_n, I_m(t_n))}{AB(\alpha)} + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \sum_{j=1}^n \int_{t_j}^{t_{j+1}} (t_{j+1}-s)^{\alpha-1} I_m^{\sigma-1} G_7(t_n, I_m^j(s)) ds, \\
I_{Tm}(t_{n+1}) &= I_{Tm0} + \frac{\sigma(1-\alpha)t_n^{\sigma-1}G_8(t_n, I_{Tm}(t_n))}{AB(\alpha)} + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \sum_{j=1}^n \int_{t_j}^{t_{j+1}} (t_{j+1}-s)^{\alpha-1} I_{Tm}^{\sigma-1} G_8(t_n, I_{Tm}^j(s)) ds, \\
T_T(t_{n+1}) &= T_{T0} + \frac{\sigma(1-\alpha)t_n^{\sigma-1}G_9(t_n, T_T(t_n))}{AB(\alpha)} + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \sum_{j=1}^n \int_{t_j}^{t_{j+1}} (t_{j+1}-s)^{\alpha-1} T_T^{\sigma-1} G_9(t_n, T_T^j(s)) ds, \\
T_m(t_{n+1}) &= T_{m0} + \frac{\sigma(1-\alpha)t_n^{\sigma-1}G_{10}(t_n, T_m(t_n))}{AB(\alpha)} + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \sum_{j=1}^n \int_{t_j}^{t_{j+1}} (t_{j+1}-s)^{\alpha-1} T_m^{\sigma-1} G_{10}(t_n, T_m^j(s)) ds, \\
T_{Tm}(t_{n+1}) &= T_{Tm0} + \frac{\sigma(1-\alpha)t_n^{\sigma-1}G_{11}(t_n, T_{Tm}(t_n))}{AB(\alpha)} + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \sum_{j=1}^n \int_{t_j}^{t_{j+1}} (t_{j+1}-s)^{\alpha-1} T_{Tm}^{\sigma-1} G_{11}(t_n, T_{Tm}^j(s)) ds, \\
V_h(t_{n+1}) &= V_{h0} + \frac{\sigma(1-\alpha)t_n^{\sigma-1}G_{12}(t_n, V_h(t_n))}{AB(\alpha)} + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \sum_{j=1}^n \int_{t_j}^{t_{j+1}} (t_{j+1}-s)^{\alpha-1} V_h^{\sigma-1} G_{12}(t_n, V_h^j(s)) ds, \\
R_T(t_{n+1}) &= R_{T0} + \frac{\sigma(1-\alpha)t_n^{\sigma-1}G_{13}(t_n, R_T(t_n))}{AB(\alpha)} + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \sum_{j=1}^n \int_{t_j}^{t_{j+1}} (t_{j+1}-s)^{\alpha-1} R_T^{\sigma-1} G_{13}(t_n, R_T^j(s)) ds, \\
R_m(t_{n+1}) &= R_{m0} + \frac{\sigma(1-\alpha)t_n^{\sigma-1}G_{14}(t_n, R_m(t_n))}{AB(\alpha)} + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \sum_{j=1}^n \int_{t_j}^{t_{j+1}} (t_{j+1}-s)^{\alpha-1} R_m^{\sigma-1} G_{14}(t_n, R_m^j(s)) ds, \\
R_{Tm}(t_{n+1}) &= R_{Tm0} + \frac{\sigma(1-\alpha)t_n^{\sigma-1}G_{15}(t_n, R_{Tm}(t_n))}{AB(\alpha)} + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \sum_{j=1}^n \int_{t_j}^{t_{j+1}} (t_{j+1}-s)^{\alpha-1} R_{Tm}^{\sigma-1} G_{15}(t_n, R_{Tm}^j(s)) ds,
\end{aligned}$$

$$\begin{aligned}
S_r(t_{n+1}) &= S_{r0} + \frac{\sigma(1-\alpha)t_n^{\sigma-1}G_{16}(t_n, S_r(t_n))}{AB(\alpha)} + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \sum_{j=1}^n \int_{t_j}^{t_{j+1}} (t_{j+1}-s)^{\alpha-1} S_r^{\sigma-1} G_{16}(t_n, S_r^j(s)) ds, \\
I_r(t_{n+1}) &= I_{r0} + \frac{\sigma(1-\alpha)t_n^{\sigma-1}G_{17}(t_n, I_r(t_n))}{AB(\alpha)} + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \sum_{j=1}^n \int_{t_j}^{t_{j+1}} (t_{j+1}-s)^{\alpha-1} I_r^{\sigma-1} G_{17}(t_n, I_r^j(s)) ds, \\
B(t_{n+1}) &= B_0 + \frac{\sigma(1-\alpha)t_n^{\sigma-1}G_{18}(t_n, B(t_n))}{AB(\alpha)} + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \sum_{j=1}^n \int_{t_j}^{t_{j+1}} (t_{j+1}-s)^{\alpha-1} B^{\sigma-1} G_{18}(t_n, B^j(s)) ds.
\end{aligned}$$

Two step Langrange's interpolation polynomials with $\Delta t = t_{j+1} - t_j$ function as an estimate for the integrand functions on $[t_j, t_{j+1}]$.

Thus, we obtained:

$$\begin{aligned}
S_h(t_{n+1}) &= S_{h0} + \frac{\sigma(1-\alpha)t_n^{\sigma-1}G_1(t_n, S_h(t_n))}{AB(\alpha)} + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \sum_{j=1}^n \int_{t_j}^{t_{j+1}} (t_{j+1}-s)^{\alpha-1} S_h^{\sigma-1} Z_j^S(s) ds, \\
E_T(t_{n+1}) &= E_{T0} + \frac{\sigma(1-\alpha)t_n^{\sigma-1}G_2(t_n, E_T(t_n))}{AB(\alpha)} + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \sum_{j=1}^n \int_{t_j}^{t_{j+1}} (t_{j+1}-s)^{\alpha-1} E_T^{\sigma-1} Z_j^S(s) ds, \\
E_m(t_{n+1}) &= E_{m0} + \frac{\sigma(1-\alpha)t_n^{\sigma-1}G_3(t_n, E_m(t_n))}{AB(\alpha)} + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \sum_{j=1}^n \int_{t_j}^{t_{j+1}} (t_{j+1}-s)^{\alpha-1} E_m^{\sigma-1} Z_j^S(s) ds, \\
E_{Tm}(t_{n+1}) &= E_{Tm0} + \frac{\sigma(1-\alpha)t_n^{\sigma-1}G_4(t_n, E_{Tm}(t_n))}{AB(\alpha)} + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \sum_{j=1}^n \int_{t_j}^{t_{j+1}} (t_{j+1}-s)^{\alpha-1} E_{Tm}^{\sigma-1} Z_j^S(s) ds, \\
E_{Tm}(t_{n+1}) &= E_{Tm0} + \frac{\sigma(1-\alpha)t_n^{\sigma-1}G_4(t_n, E_{Tm}(t_n))}{AB(\alpha)} + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \sum_{j=1}^n \int_{t_j}^{t_{j+1}} (t_{j+1}-s)^{\alpha-1} E_{Tm}^{\sigma-1} Z_j^S(s) ds, \\
Q_m(t_{n+1}) &= Q_{m0} + \frac{\sigma(1-\alpha)t_n^{\sigma-1}G_5(t_n, Q_m(t_n))}{AB(\alpha)} + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \sum_{j=1}^n \int_{t_j}^{t_{j+1}} (t_{j+1}-s)^{\alpha-1} Q_m^{\sigma-1} Z_j^S(s) ds, \\
I_T(t_{n+1}) &= I_{T0} + \frac{\sigma(1-\alpha)t_n^{\sigma-1}G_6(t_n, I_T(t_n))}{AB(\alpha)} + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \sum_{j=1}^n \int_{t_j}^{t_{j+1}} (t_{j+1}-s)^{\alpha-1} I_T^{\sigma-1} Z_j^S(s) ds, \\
I_m(t_{n+1}) &= I_{m0} + \frac{\sigma(1-\alpha)t_n^{\sigma-1}G_7(t_n, I_m(t_n))}{AB(\alpha)} + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \sum_{j=1}^n \int_{t_j}^{t_{j+1}} (t_{j+1}-s)^{\alpha-1} I_m^{\sigma-1} Z_j^S(s) ds, \\
I_{Tm}(t_{n+1}) &= I_{Tm0} + \frac{\sigma(1-\alpha)t_n^{\sigma-1}G_8(t_n, I_{Tm}(t_n))}{AB(\alpha)} + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \sum_{j=1}^n \int_{t_j}^{t_{j+1}} (t_{j+1}-s)^{\alpha-1} I_{Tm}^{\sigma-1} Z_j^S(s) ds, \\
T_T(t_{n+1}) &= T_{T0} + \frac{\sigma(1-\alpha)t_n^{\sigma-1}G_9(t_n, T_T(t_n))}{AB(\alpha)} + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \sum_{j=1}^n \int_{t_j}^{t_{j+1}} (t_{j+1}-s)^{\alpha-1} T_T^{\sigma-1} Z_j^S(s) ds, \\
T_m(t_{n+1}) &= T_{m0} + \frac{\sigma(1-\alpha)t_n^{\sigma-1}G_{10}(t_n, T_m(t_n))}{AB(\alpha)} + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \sum_{j=1}^n \int_{t_j}^{t_{j+1}} (t_{j+1}-s)^{\alpha-1} T_m^{\sigma-1} Z_j^S(s) ds,
\end{aligned}$$

$$\begin{aligned}
T_{Tm}(t_{n+1}) &= T_{Tm0} + \frac{\sigma(1-\alpha)t_n^{\sigma-1}G_{11}(t_n, T_{Tm}(t_n))}{AB(\alpha)} + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \sum_{j=1}^n \int_{t_j}^{t_{j+1}} (t_{j+1}-s)^{\alpha-1} T_{Tm}^{\sigma-1} Z_j^S(s) ds, \\
V_h(t_{n+1}) &= V_{h0} + \frac{\sigma(1-\alpha)t_n^{\sigma-1}G_{12}(t_n, V_h(t_n))}{AB(\alpha)} + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \sum_{j=1}^n \int_{t_j}^{t_{j+1}} (t_{j+1}-s)^{\alpha-1} V_h^{\sigma-1} Z_j^S(s) ds, \\
R_T(t_{n+1}) &= R_{T0} + \frac{\sigma(1-\alpha)t_n^{\sigma-1}G_{13}(t_n, R_T(t_n))}{AB(\alpha)} + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \sum_{j=1}^n \int_{t_j}^{t_{j+1}} (t_{j+1}-s)^{\alpha-1} R_T^{\sigma-1} Z_j^S(s) ds, \\
R_m(t_{n+1}) &= R_{m0} + \frac{\sigma(1-\alpha)t_n^{\sigma-1}G_{14}(t_n, R_m(t_n))}{AB(\alpha)} + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \sum_{j=1}^n \int_{t_j}^{t_{j+1}} (t_{j+1}-s)^{\alpha-1} R_m^{\sigma-1} Z_j^S(s) ds, \\
R_{Tm}(t_{n+1}) &= R_{Tm0} + \frac{\sigma(1-\alpha)t_n^{\sigma-1}G_{15}(t_n, R_{Tm}(t_n))}{AB(\alpha)} + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \sum_{j=1}^n \int_{t_j}^{t_{j+1}} (t_{j+1}-s)^{\alpha-1} R_{Tm}^{\sigma-1} Z_j^S(s) ds, \\
S_r(t_{n+1}) &= S_{r0} + \frac{\sigma(1-\alpha)t_n^{\sigma-1}G_{16}(t_n, S_r(t_n))}{AB(\alpha)} + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \sum_{j=1}^n \int_{t_j}^{t_{j+1}} (t_{j+1}-s)^{\alpha-1} S_r^{\sigma-1} Z_j^S(s) ds, \\
I_r(t_{n+1}) &= I_{r0} + \frac{\sigma(1-\alpha)t_n^{\sigma-1}G_{17}(t_n, I_r(t_n))}{AB(\alpha)} + \frac{\alpha\sigma}{AB(\alpha)\Gamma(\alpha)} \sum_{j=1}^n \int_{t_j}^{t_{j+1}} (t_{j+1}-s)^{\alpha-1} I_r^{\sigma-1} Z_j^S(s) ds.
\end{aligned}$$

Where,

$$\begin{aligned}
Z_j^{S_h}(s) &= \frac{s-t_{j-1}}{t_j-t_{j-1}} t_j^{\sigma-1} G_1(s, S_h^j(s)) - \frac{s-t_{j-1}}{t_j-t_{j-1}} t_{j-1}^{\sigma-1} G_1(s, S_h^{j-1}(s)), \\
Z_j^{E_T}(s) &= \frac{s-t_{j-1}}{t_j-t_{j-1}} t_j^{\sigma-1} G_2(s, E_T^j(s)) - \frac{s-t_{j-1}}{t_j-t_{j-1}} t_{j-1}^{\sigma-1} G_2(s, E_T^{j-1}(s)), \\
Z_j^{E_m}(s) &= \frac{s-t_{j-1}}{t_j-t_{j-1}} t_j^{\sigma-1} G_3(s, E_m^j(s)) - \frac{s-t_{j-1}}{t_j-t_{j-1}} t_{j-1}^{\sigma-1} G_3(s, E_m^{j-1}(s)), \\
Z_j^{E_{Tm}}(s) &= \frac{s-t_{j-1}}{t_j-t_{j-1}} t_j^{\sigma-1} G_4(s, E_{Tm}^j(s)) - \frac{s-t_{j-1}}{t_j-t_{j-1}} t_{j-1}^{\sigma-1} G_4(s, E_{Tm}^{j-1}(s)), \\
Z_j^{Q_m}(s) &= \frac{s-t_{j-1}}{t_j-t_{j-1}} t_j^{\sigma-1} G_5(s, Q_m^j(s)) - \frac{s-t_{j-1}}{t_j-t_{j-1}} t_{j-1}^{\sigma-1} G_5(s, Q_m^{j-1}(s)), \\
Z_j^{I_T}(s) &= \frac{s-t_{j-1}}{t_j-t_{j-1}} t_j^{\sigma-1} G_6(s, I_T^j(s)) - \frac{s-t_{j-1}}{t_j-t_{j-1}} t_{j-1}^{\sigma-1} G_6(s, I_T^{j-1}(s)), \\
Z_j^{I_m}(s) &= \frac{s-t_{j-1}}{t_j-t_{j-1}} t_j^{\sigma-1} G_7(s, I_m^j(s)) - \frac{s-t_{j-1}}{t_j-t_{j-1}} t_{j-1}^{\sigma-1} G_7(s, I_m^{j-1}(s)), \\
Z_j^{I_{Tm}}(s) &= \frac{s-t_{j-1}}{t_j-t_{j-1}} t_j^{\sigma-1} G_8(s, I_{Tm}^j(s)) - \frac{s-t_{j-1}}{t_j-t_{j-1}} t_{j-1}^{\sigma-1} G_8(s, I_{Tm}^{j-1}(s)), \\
Z_j^{T_r}(s) &= \frac{s-t_{j-1}}{t_j-t_{j-1}} t_j^{\sigma-1} G_9(s, T_r^j(s)) - \frac{s-t_{j-1}}{t_j-t_{j-1}} t_{j-1}^{\sigma-1} G_9(s, T_r^{j-1}(s)),
\end{aligned}$$

$$\begin{aligned}
Z_j^{T_m}(s) &= \frac{s-t_{j-1}}{t_j-t_{j-1}} t_j^{\sigma-1} G_{10}(s, T_m^j(s)) - \frac{s-t_{j-1}}{t_j-t_{j-1}} t_{j-1}^{\sigma-1} G_{10}(s, T_m^{j-1}(s)), \\
Z_j^{T_m}(s) &= \frac{s-t_{j-1}}{t_j-t_{j-1}} t_j^{\sigma-1} G_{11}(s, T_m^j(s)) - \frac{s-t_{j-1}}{t_j-t_{j-1}} t_{j-1}^{\sigma-1} G_{11}(s, T_m^{j-1}(s)), \\
Z_j^{V_h}(s) &= \frac{s-t_{j-1}}{t_j-t_{j-1}} t_j^{\sigma-1} G_{12}(s, V_h^j(s)) - \frac{s-t_{j-1}}{t_j-t_{j-1}} t_{j-1}^{\sigma-1} G_{12}(s, V_h^{j-1}(s)), \\
Z_j^{R_T}(s) &= \frac{s-t_{j-1}}{t_j-t_{j-1}} t_j^{\sigma-1} G_{13}(s, R_T^j(s)) - \frac{s-t_{j-1}}{t_j-t_{j-1}} t_{j-1}^{\sigma-1} G_{13}(s, R_T^{j-1}(s)), \\
Z_j^{R_m}(s) &= \frac{s-t_{j-1}}{t_j-t_{j-1}} t_j^{\sigma-1} G_{14}(s, R_m^j(s)) - \frac{s-t_{j-1}}{t_j-t_{j-1}} t_{j-1}^{\sigma-1} G_{14}(s, R_m^{j-1}(s)), \\
Z_j^{R_{Tm}}(s) &= \frac{s-t_{j-1}}{t_j-t_{j-1}} t_j^{\sigma-1} G_{15}(s, R_{Tm}^j(s)) - \frac{s-t_{j-1}}{t_j-t_{j-1}} t_{j-1}^{\sigma-1} G_{15}(s, R_{Tm}^{j-1}(s)), \\
Z_j^{S_r}(s) &= \frac{s-t_{j-1}}{t_j-t_{j-1}} t_j^{\sigma-1} G_{16}(s, S_r^j(s)) - \frac{s-t_{j-1}}{t_j-t_{j-1}} t_{j-1}^{\sigma-1} G_{16}(s, S_r^{j-1}(s)), \\
Z_j^{I_r}(s) &= \frac{s-t_{j-1}}{t_j-t_{j-1}} t_j^{\sigma-1} G_{17}(s, I_r^j(s)) - \frac{s-t_{j-1}}{t_j-t_{j-1}} t_{j-1}^{\sigma-1} G_{17}(s, I_r^{j-1}(s)), \\
Z_j^B(s) &= \frac{s-t_{j-1}}{t_j-t_{j-1}} t_j^{\sigma-1} G_{18}(s, B^j(s)) - \frac{s-t_{j-1}}{t_j-t_{j-1}} t_{j-1}^{\sigma-1} G_{18}(s, B^{j-1}(s)).
\end{aligned}$$

We finally solved the schemes needed for numerical calculations of the -Monkey pox and Typhoid fever co-infection model (8).

$$\begin{aligned}
S_h(t_{n+1}) &= S_{h0} + \sigma(1-\alpha)t_n^{\sigma-1}G_1(t_n, S_h(t_n)) + \frac{\sigma(\Delta t)^\alpha}{AB(\alpha)\Gamma(\alpha+2)} \sum_{j=1}^n \left[\begin{aligned} &t_j^{\sigma-1}G_1(s, S_h^j(s))X_1(n, j) \\ &-t_{j-1}^{\sigma-1}G_1(s, S_h^{j-1}(s))X_2(n, j) \end{aligned} \right], \\
E_T(t_{n+1}) &= E_{T0} + \sigma(1-\alpha)t_n^{\sigma-1}G_2(t_n, E_T(t_n)) + \frac{\sigma(\Delta t)^\alpha}{AB(\alpha)\Gamma(\alpha+2)} \sum_{j=1}^n \left[\begin{aligned} &t_j^{\sigma-1}G_2(s, E_T^j(s))X_1(n, j) \\ &-t_{j-1}^{\sigma-1}G_2(s, E_T^{j-1}(s))X_2(n, j) \end{aligned} \right], \\
E_m(t_{n+1}) &= E_{m0} + \sigma(1-\alpha)t_n^{\sigma-1}G_3(t_n, E_m(t_n)) + \frac{\sigma(\Delta t)^\alpha}{AB(\alpha)\Gamma(\alpha+2)} \sum_{j=1}^n \left[\begin{aligned} &t_j^{\sigma-1}G_3(s, E_m^j(s))X_1(n, j) \\ &-t_{j-1}^{\sigma-1}G_3(s, E_m^{j-1}(s))X_2(n, j) \end{aligned} \right], \\
E_{Tm}(t_{n+1}) &= E_{Tm0} + \sigma(1-\alpha)t_n^{\sigma-1}G_4(t_n, E_{Tm}(t_n)) + \frac{\sigma(\Delta t)^\alpha}{AB(\alpha)\Gamma(\alpha+2)} \sum_{j=1}^n \left[\begin{aligned} &t_j^{\sigma-1}G_4(s, E_{Tm}^j(s))X_1(n, j) \\ &-t_{j-1}^{\sigma-1}G_4(s, E_{Tm}^{j-1}(s))X_2(n, j) \end{aligned} \right], \\
Q_m(t_{n+1}) &= Q_{m0} + \sigma(1-\alpha)t_n^{\sigma-1}G_5(t_n, Q_m(t_n)) + \frac{\sigma(\Delta t)^\alpha}{AB(\alpha)\Gamma(\alpha+2)} \sum_{j=1}^n \left[\begin{aligned} &t_j^{\sigma-1}G_5(s, Q_m^j(s))X_1(n, j) \\ &-t_{j-1}^{\sigma-1}G_5(s, Q_m^{j-1}(s))X_2(n, j) \end{aligned} \right], \\
I_T(t_{n+1}) &= I_{T0} + \sigma(1-\alpha)t_n^{\sigma-1}G_6(t_n, I_T(t_n)) + \frac{\sigma(\Delta t)^\alpha}{AB(\alpha)\Gamma(\alpha+2)} \sum_{j=1}^n \left[\begin{aligned} &t_j^{\sigma-1}G_6(s, I_T^j(s))X_1(n, j) \\ &-t_{j-1}^{\sigma-1}G_6(s, I_T^{j-1}(s))X_2(n, j) \end{aligned} \right],
\end{aligned}$$

$$\begin{aligned}
I_m(t_{n+1}) &= I_{m0} + \sigma(1-\alpha)t_n^{\sigma-1}G_7(t_n, I_m(t_n)) + \frac{\sigma(\Delta t)^\alpha}{AB(\alpha)\Gamma(\alpha+2)} \sum_{j=1}^n \left[t_j^{\sigma-1}G_7(s, I_m^j(s))X_1(n, j) \right. \\
&\quad \left. - t_{j-1}^{\sigma-1}G_7(s, I_m^{j-1}(s))X_2(n, j) \right], \\
I_{Tm}(t_{n+1}) &= I_{Tm0} + \sigma(1-\alpha)t_n^{\sigma-1}G_8(t_n, I_{Tm}(t_n)) + \frac{\sigma(\Delta t)^\alpha}{AB(\alpha)\Gamma(\alpha+2)} \sum_{j=1}^n \left[t_j^{\sigma-1}G_8(s, I_{Tm}^j(s))X_1(n, j) \right. \\
&\quad \left. - t_{j-1}^{\sigma-1}G_8(s, I_{Tm}^{j-1}(s))X_2(n, j) \right], \\
T_T(t_{n+1}) &= T_{T0} + \sigma(1-\alpha)t_n^{\sigma-1}G_9(t_n, T_T(t_n)) + \frac{\sigma(\Delta t)^\alpha}{AB(\alpha)\Gamma(\alpha+2)} \sum_{j=1}^n \left[t_j^{\sigma-1}G_9(s, T_T^j(s))X_1(n, j) \right. \\
&\quad \left. - t_{j-1}^{\sigma-1}G_9(s, T_T^{j-1}(s))X_2(n, j) \right], \\
T_m(t_{n+1}) &= T_{m0} + \sigma(1-\alpha)t_n^{\sigma-1}G_{10}(t_n, T_m(t_n)) + \frac{\sigma(\Delta t)^\alpha}{AB(\alpha)\Gamma(\alpha+2)} \sum_{j=1}^n \left[t_j^{\sigma-1}G_{10}(s, T_m^j(s))X_1(n, j) \right. \\
&\quad \left. - t_{j-1}^{\sigma-1}G_{10}(s, T_m^{j-1}(s))X_2(n, j) \right], \\
T_{Tm}(t_{n+1}) &= T_{Tm0} + \sigma(1-\alpha)t_n^{\sigma-1}G_{11}(t_n, T_{Tm}(t_n)) + \frac{\sigma(\Delta t)^\alpha}{AB(\alpha)\Gamma(\alpha+2)} \sum_{j=1}^n \left[t_j^{\sigma-1}G_{11}(s, T_{Tm}^j(s))X_1(n, j) \right. \\
&\quad \left. - t_{j-1}^{\sigma-1}G_{11}(s, T_{Tm}^{j-1}(s))X_2(n, j) \right], \\
V_h(t_{n+1}) &= V_{h0} + \sigma(1-\alpha)t_n^{\sigma-1}G_{12}(t_n, V_h(t_n)) + \frac{\sigma(\Delta t)^\alpha}{AB(\alpha)\Gamma(\alpha+2)} \sum_{j=1}^n \left[t_j^{\sigma-1}G_{12}(s, V_h^j(s))X_1(n, j) \right. \\
&\quad \left. - t_{j-1}^{\sigma-1}G_{12}(s, V_h^{j-1}(s))X_2(n, j) \right], \\
R_T(t_{n+1}) &= R_{T0} + \sigma(1-\alpha)t_n^{\sigma-1}G_{13}(t_n, R_T(t_n)) + \frac{\sigma(\Delta t)^\alpha}{AB(\alpha)\Gamma(\alpha+2)} \sum_{j=1}^n \left[t_j^{\sigma-1}G_{13}(s, R_T^j(s))X_1(n, j) \right. \\
&\quad \left. - t_{j-1}^{\sigma-1}G_{13}(s, R_T^{j-1}(s))X_2(n, j) \right], \\
R_m(t_{n+1}) &= R_{m0} + \sigma(1-\alpha)t_n^{\sigma-1}G_{14}(t_n, R_m(t_n)) + \frac{\sigma(\Delta t)^\alpha}{AB(\alpha)\Gamma(\alpha+2)} \sum_{j=1}^n \left[t_j^{\sigma-1}G_{14}(s, R_m^j(s))X_1(n, j) \right. \\
&\quad \left. - t_{j-1}^{\sigma-1}G_{14}(s, R_m^{j-1}(s))X_2(n, j) \right], \\
R_{Tm}(t_{n+1}) &= R_{Tm0} + \sigma(1-\alpha)t_n^{\sigma-1}G_{15}(t_n, R_{Tm}(t_n)) + \frac{\sigma(\Delta t)^\alpha}{AB(\alpha)\Gamma(\alpha+2)} \sum_{j=1}^n \left[t_j^{\sigma-1}G_{15}(s, R_{Tm}^j(s))X_1(n, j) \right. \\
&\quad \left. - t_{j-1}^{\sigma-1}G_{15}(s, R_{Tm}^{j-1}(s))X_2(n, j) \right], \\
S_r(t_{n+1}) &= S_{r0} + \sigma(1-\alpha)t_n^{\sigma-1}G_{16}(t_n, S_r(t_n)) + \frac{\sigma(\Delta t)^\alpha}{AB(\alpha)\Gamma(\alpha+2)} \sum_{j=1}^n \left[t_j^{\sigma-1}G_{16}(s, S_r^j(s))X_1(n, j) \right. \\
&\quad \left. - t_{j-1}^{\sigma-1}G_{16}(s, S_r^{j-1}(s))X_2(n, j) \right], \\
I_r(t_{n+1}) &= I_{r0} + \sigma(1-\alpha)t_n^{\sigma-1}G_{17}(t_n, S_r(t_n)) + \frac{\sigma(\Delta t)^\alpha}{AB(\alpha)\Gamma(\alpha+2)} \sum_{j=1}^n \left[t_j^{\sigma-1}G_{17}(s, S_r^j(s))X_1(n, j) \right. \\
&\quad \left. - t_{j-1}^{\sigma-1}G_{17}(s, S_r^{j-1}(s))X_2(n, j) \right], \\
B(t_{n+1}) &= B_0 + \sigma(1-\alpha)t_n^{\sigma-1}G_{18}(t_n, B(t_n)) + \frac{\sigma(\Delta t)^\alpha}{AB(\alpha)\Gamma(\alpha+2)} \sum_{j=1}^n \left[t_j^{\sigma-1}G_{18}(s, B^j(s))X_1(n, j) \right. \\
&\quad \left. - t_{j-1}^{\sigma-1}G_{18}(s, B^{j-1}(s))X_2(n, j) \right].
\end{aligned}$$

Where

$$X_1(n, j) = (n+1-j)^{\alpha+1} - (n-j)^\alpha - (n-j)^\alpha (n-j+2+2\alpha) \quad (56)$$

And $X_2(n, j) = (n+1-j)^{\rho+1} - (n-j)^{\rho} - (n-j)^{\rho}(n-j+1+\rho)$.

Table 2: Monthly Typhoid Fever Cases in Guangxi Province, China (2005–2009). **Source:** <https://doi.org/10.1371/journal.pone.0063116>. PubMed ID: 23650546, PubMed Central: PMC3641111. Available at: <https://journals.plos.org/plosone/article?id=10.1371/journal.pone.0063116>.

Month	Month-Year	Cases	Year
1	January 2005	45	2005
2	February 2005	38	2005
3	March 2005	52	2005
4	April 2005	68	2005
5	May 2005	89	2005
6	June 2005	112	2005
7	July 2005	145	2005
8	August 2005	138	2005
9	September 2005	98	2005
10	October 2005	76	2005
11	November 2005	54	2005
12	December 2005	42	2005
13	January 2006	41	2006
14	February 2006	36	2006
15	March 2006	48	2006
16	April 2006	62	2006
17	May 2006	82	2006
18	June 2006	105	2006
19	July 2006	134	2006
20	August 2006	128	2006
21	September 2006	91	2006
22	October 2006	71	2006
23	November 2006	50	2006
24	December 2006	39	2006
25	January 2007	38	2007
26	February 2007	34	2007

27	March 2007	45	2007
28	April 2007	58	2007
29	May 2007	76	2007
30	June 2007	98	2007
31	July 2007	125	2007
32	August 2007	119	2007
33	September 2007	85	2007
34	October 2007	66	2007
35	November 2007	47	2007
36	December 2007	36	2007
37	January 2008	35	2008
38	February 2008	31	2008
39	March 2008	42	2008
40	April 2008	54	2008
41	May 2008	71	2008
42	June 2008	92	2008
43	July 2008	117	2008
44	August 2008	112	2008
45	September 2008	80	2008
46	October 2008	62	2008
47	November 2008	44	2008
48	December 2008	34	2008
49	January 2009	33	2009
50	February 2009	29	2009
51	March 2009	39	2009
52	April 2009	50	2009
53	May 2009	66	2009
54	June 2009	86	2009
55	July 2009	109	2009
56	August 2009	105	2009
57	September 2009	75	2009
58	October 2009	58	2009
59	November 2009	41	2009
60	December 2009	32	2009

Table 2 presents the monthly reported cases of typhoid fever in Guangxi, China, from 2005 to 2009, illustrating both seasonal and interannual variations in disease incidence. The data show a clear seasonal pattern, with typhoid fever cases generally increasing during the warmer and wetter months and declining during cooler periods, which is consistent with the known environmental and hygienic factors influencing typhoid transmission [73]. Across the five-year period, fluctuations in monthly case numbers are observed, reflecting changes in transmission intensity, population exposure, and possible improvements in public health interventions. Interannual

comparison suggests variations in overall disease burden from year to year, indicating that typhoid fever incidence in Guangxi was not constant during the study period.

Table 3 Monkey pox case data from October 2, 2022 to April 23, 2023

Date	Oct. 2	Oct. 9	Oct. 16	Oct. 23	Oct. 30	Nov. 6	Nov. 13	Nov. 20	Nov. 27	Dec. 4
Cases	56	25	49	22	31	19	4	14	21	22
Date	Dec. 11	Dec. 18	Dec. 25	Jan. 1	Jan. 8	Jan. 15	Jan. 23	Jan. 29	Feb. 5	Feb. 12
Cases	24	24	9	3	7	1	3	4	10	2
Date	Feb. 19	Feb. 26	Mar. 5	Mar. 12	Mar. 19	Mar. 26	Apr. 2	Apr. 9	Apr. 16	Apr. 23
Cases	7	2	2	5	4	7	3	11	10	3

Table 3 labeled exactly as Monkeypox case data from October 2, 2022 to April 23, 2023, Such tables typically appear within WHO situation report PDFs that include cumulative and weekly case counts by date [61].

Table 4: Parameter table of values

Parameters	Values	Sources
β_1	0.0246	Fitted
β_2	0.0217	Fitted
β_3	0.00224	Fitted
β_r	0.06	Assumed
δ_r	0.5	[48]
μ_r	0.002	[67]
δ_3	0.2	[32]
δ_6	0.01	Assumed
ω	0.043	Assumed
μ_h	0.007	Fitted
Λ_r	0.2	Assumed
α_2	0.5331	Fitted
α_3	0.02541	Fitted
θ_3	0.04527	Fitted
γ_2	0.01931	Fitted
γ_1	0.07720	Fitted
α_1	0.6217	Fitted
σ_1	0.5932	Fitted

δ_1	0.0067	Fitted
Λ_h	0.029	Assumed
ε_1	0.4251	Assumed
μ_h	0.00567	Assumed
ψ_1	0.7499	Fitted
ψ_2	0.0532	Fitted
ψ_3	0.005	Fitted

Table 4 lists the assigned to the Monkeypox–

co-infection model [73]. It includes transmission rates, recovery rates, co-infection coefficients, and other relevant parameters, providing a reference for simulations and analysis of the model's dynamics.

numerical values parameters of the Typhoid fever

4.1 Numerical Simulation

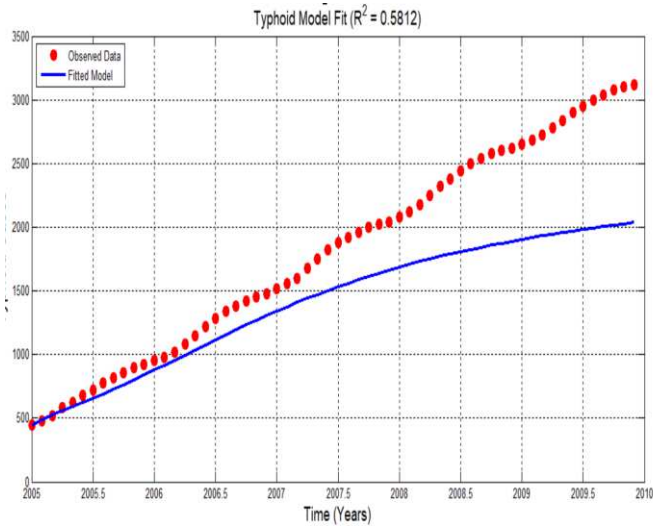


Fig.3c: Typhoid fever Data fitting

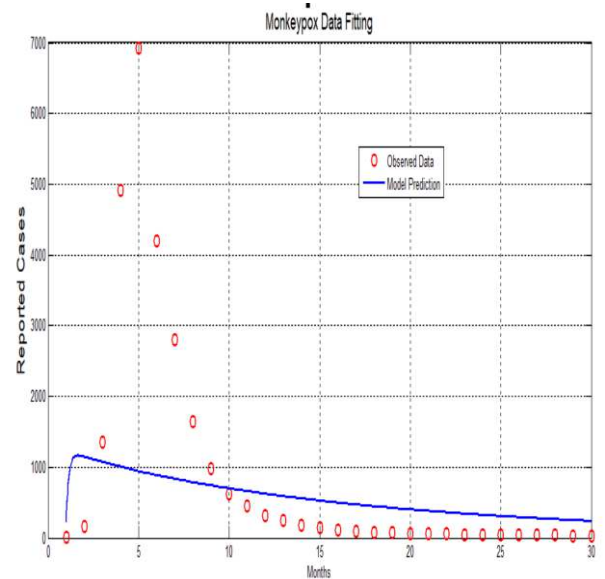


Fig.3d: monkey pox Data fitting

Figures 3c and 3d illustrate the data-fitting results for Typhoid fever and Monkeypox, respectively. In Fig. 3c, the model accurately captures the temporal trends of Typhoid fever infections, demonstrating its capability to reflect the observed dynamics in the population [68, 69]. Similarly, Fig. 3d shows that the model reliably fits the reported Monkeypox data, indicating its effectiveness in reproducing the spread patterns of this disease [69].

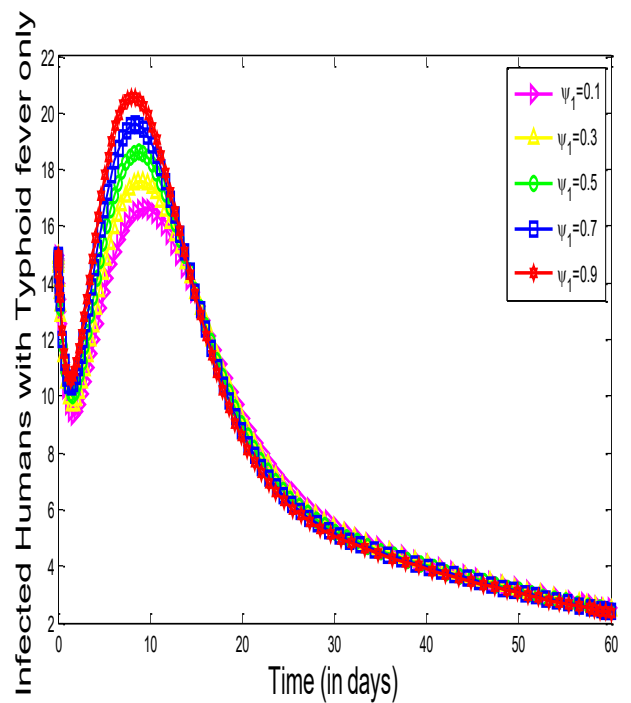
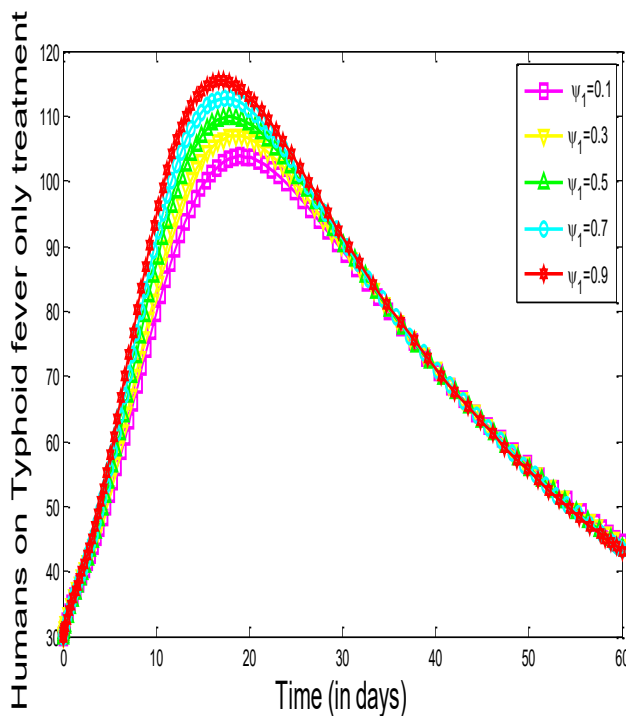


Fig.4a: Simulation of the effect of ψ_1 on humans on typhoid fever only treatment

Fig.4b: Simulation of the effect of ψ_1 on infected humans with typhoid fever only

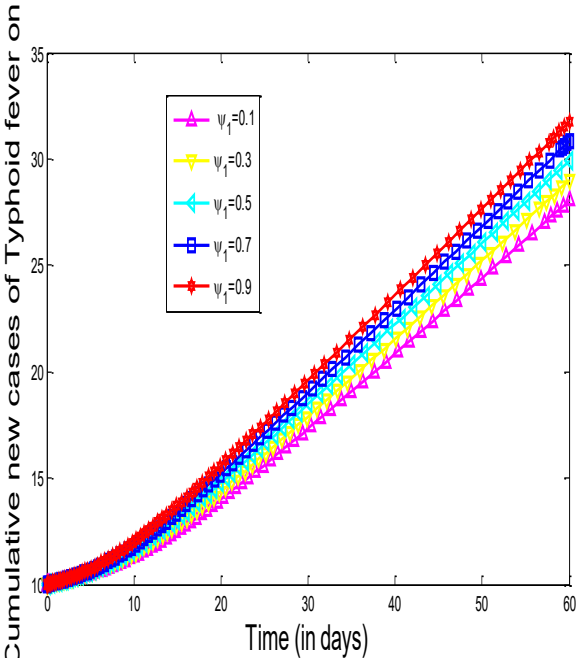
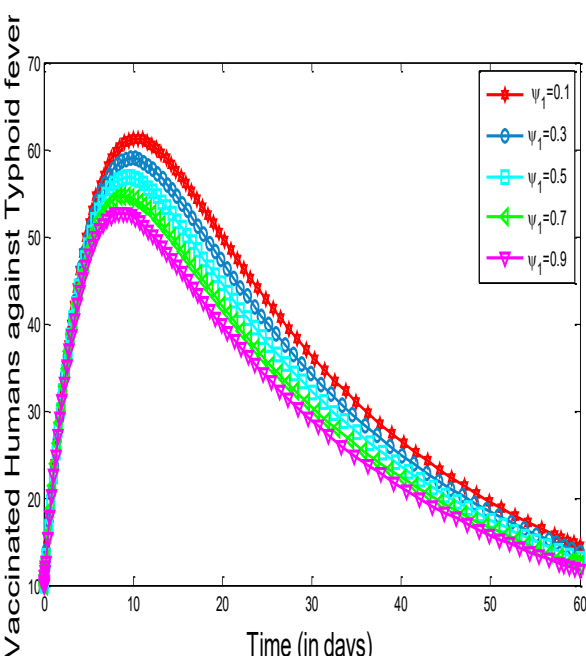


Fig.4c: Simulation of the effect of ψ_1 on vaccinated humans against typhoid fever only

Fig.4d: Simulation of the effect of ψ_1 on cumulative new cases of typhoid fever only

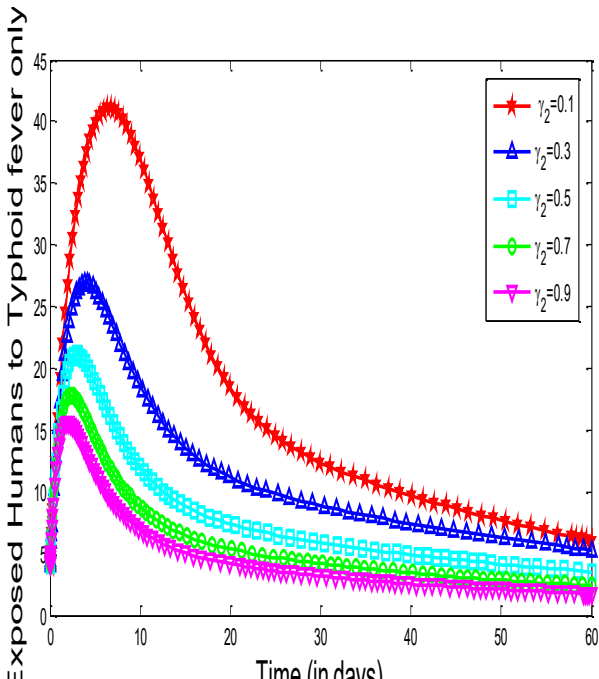
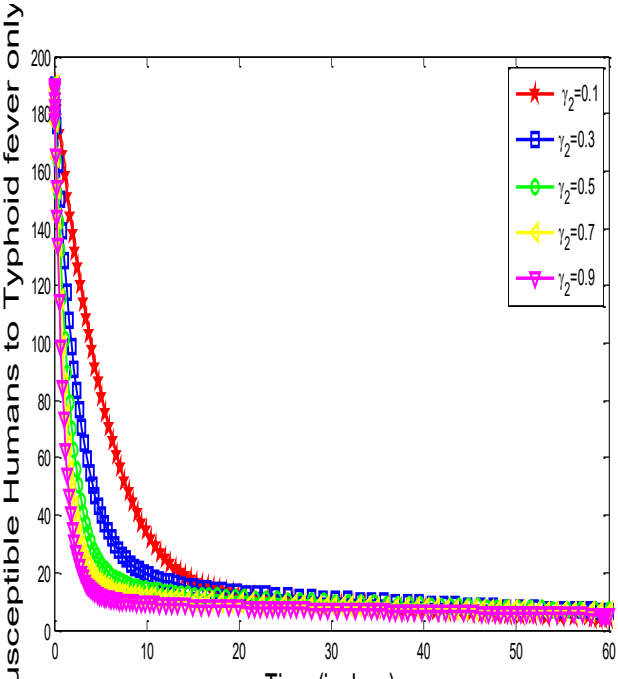


Fig.4e: Simulation of the effect of γ_2 on susceptible humans to typhoid fever only

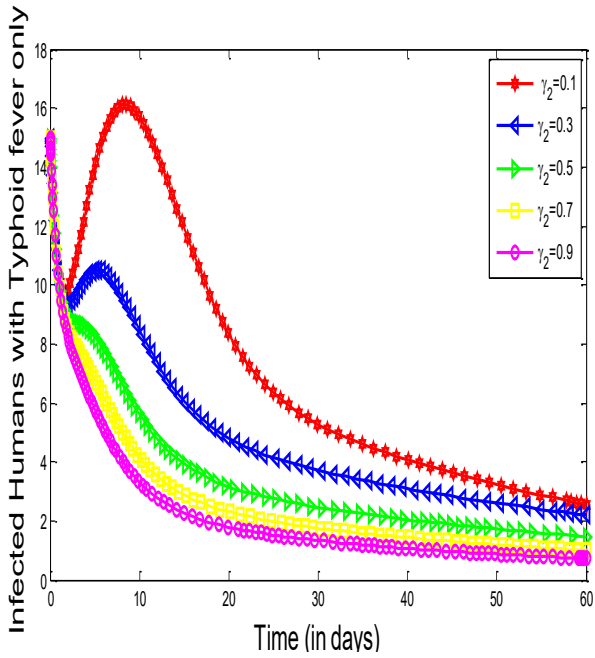


Fig.4g: Simulation of the effect of γ_2 on infected humans with typhoid fever only

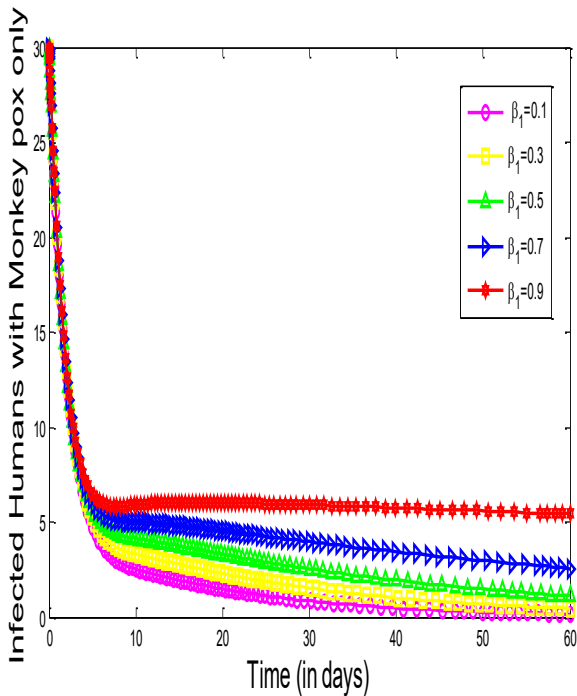


Fig.4f: Simulation of the effect of γ_2 on exposed humans to typhoid fever only

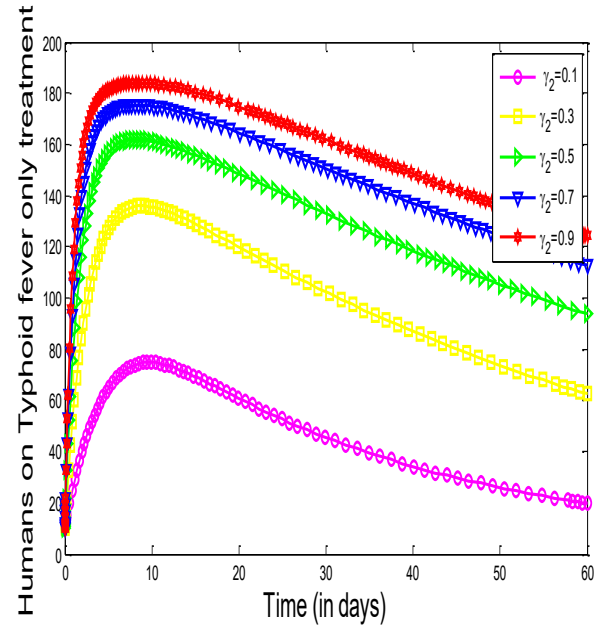


Fig.4h: Simulation of the effect of γ_2 on humans on typhoid fever only treatment

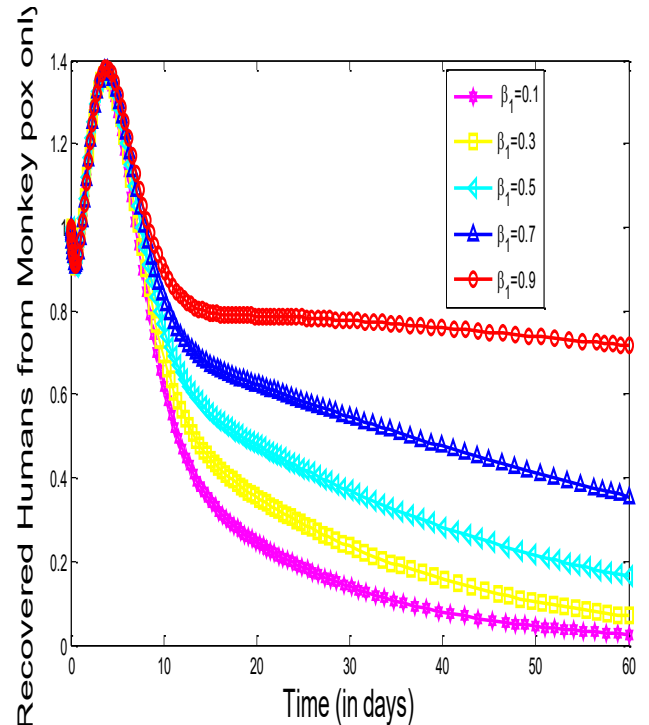


Fig.4i: Simulation of the effect of β_1 on infected humans with monkey pox only

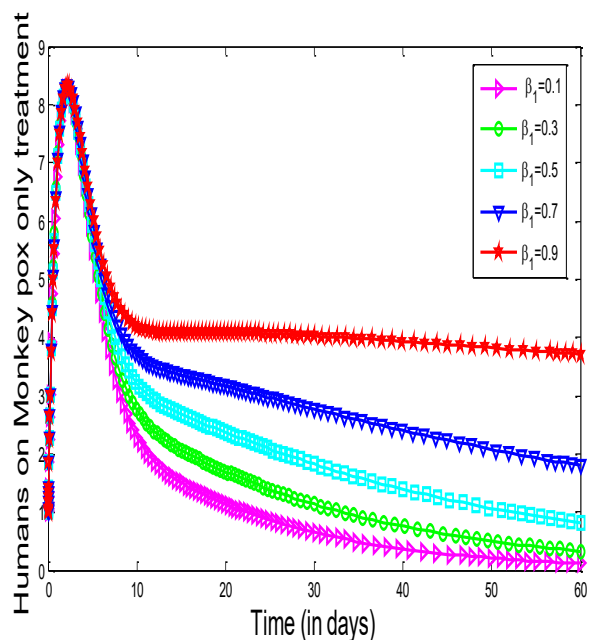


Fig.4k: Simulation of the effect of β_1 on humans on monkey pox only treatment

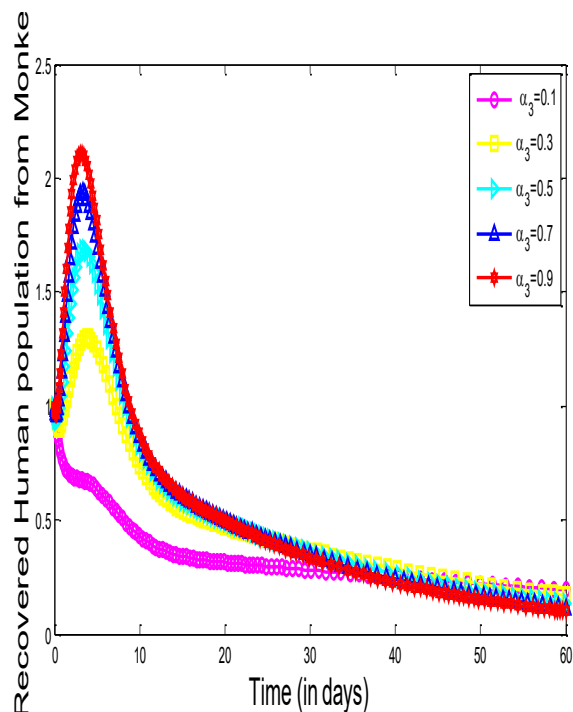


Fig.4j: Simulation of the effect of β_1 on Recovered humans from monkey pox only

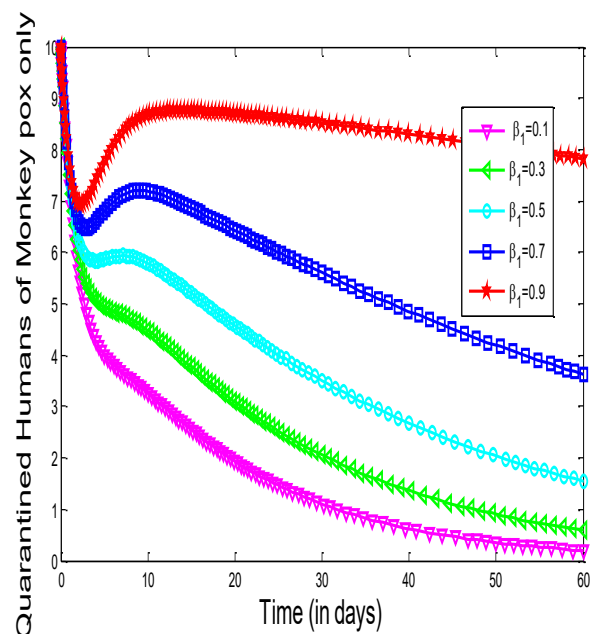


Fig.4l: Simulation of the effect of β_1 on humans Quarantined of monkey pox only

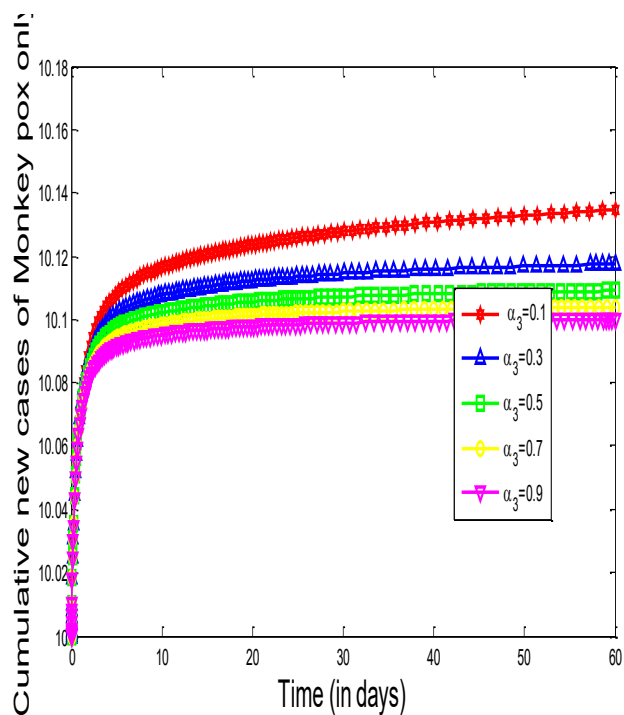


Fig.4m: Simulation of the effect of α_3 on humans
Recovered from monkey pox only

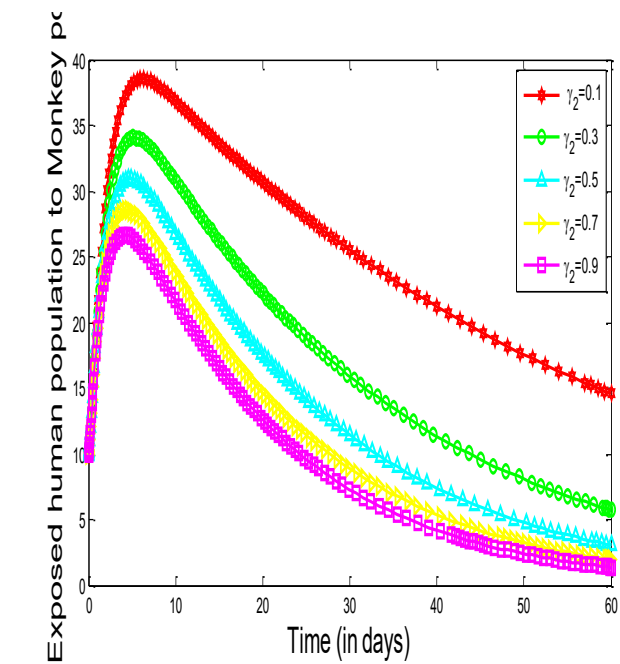


Fig.4o: Simulation of the effect of γ_2 on
exposed humans to monkey pox only

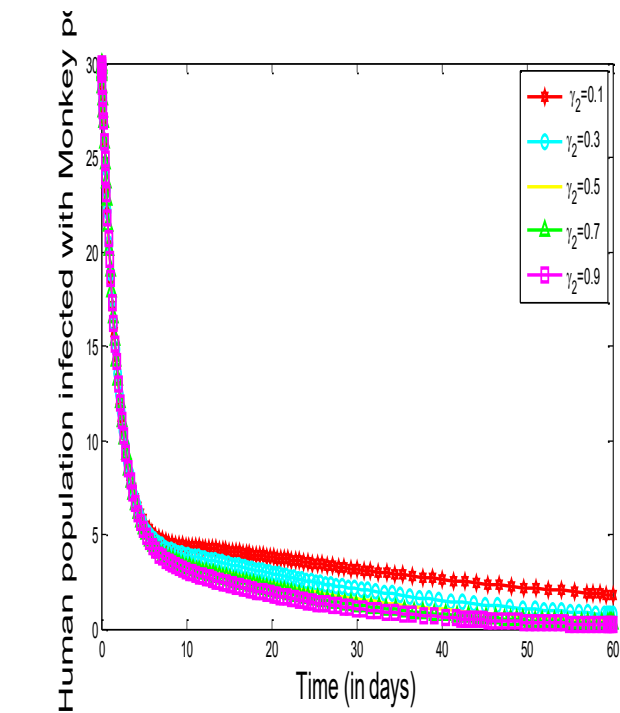


Fig.4n: Simulation of the effect of α_3 on humans’
cumulative new cases of monkey pox only

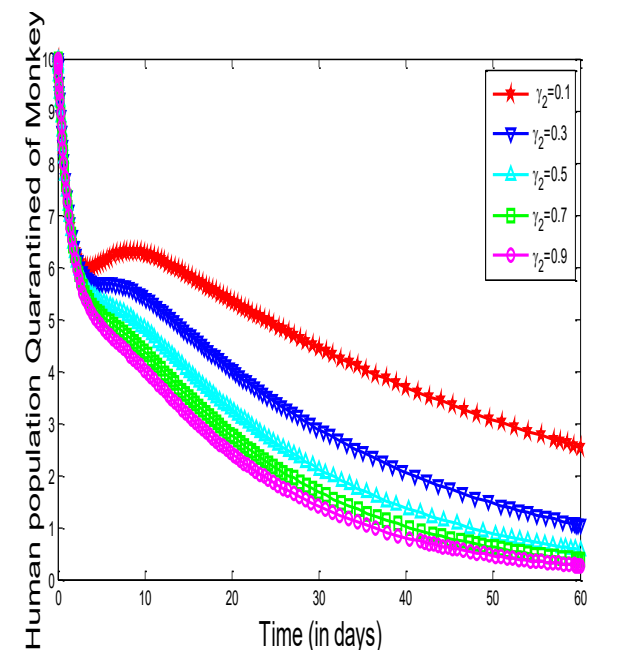


Fig.4p: Simulation of the effect of γ_2 on
quarantined humans of monkey pox only

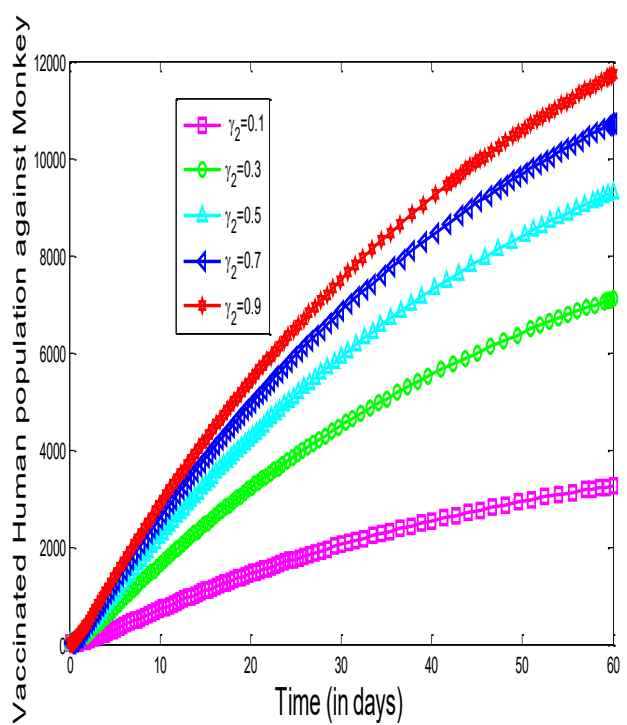


Fig.4q: Simulation of the effect of γ_2 on infected humans with monkey pox only

Fig.4r: Simulation of the effect of γ_2 on vaccinated humans against monkey pox only

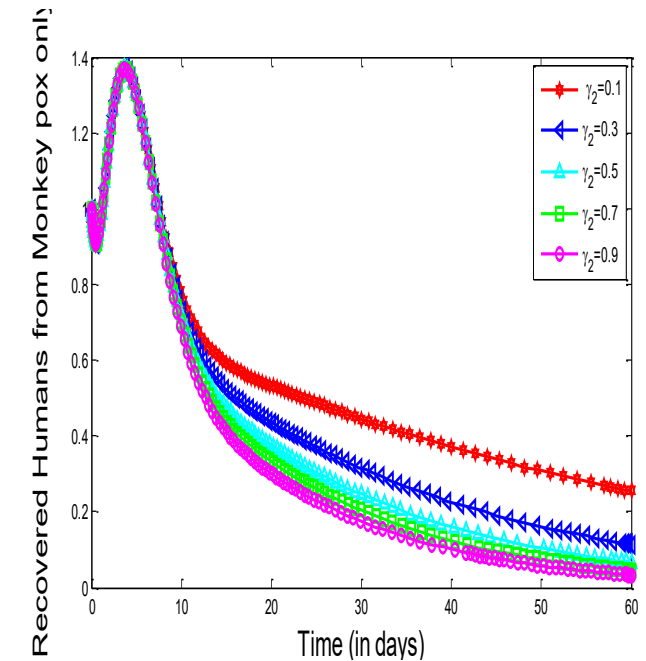


Fig.4s: Simulation of the effect of γ_2 on Recovered humans from monkey pox only

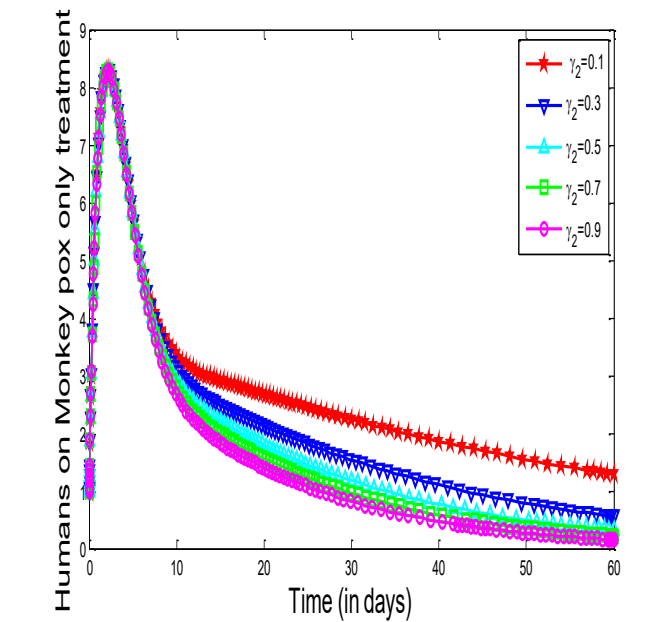


Fig.4t: Simulation of the effect of γ_2 on humans on monkey pox only treatment

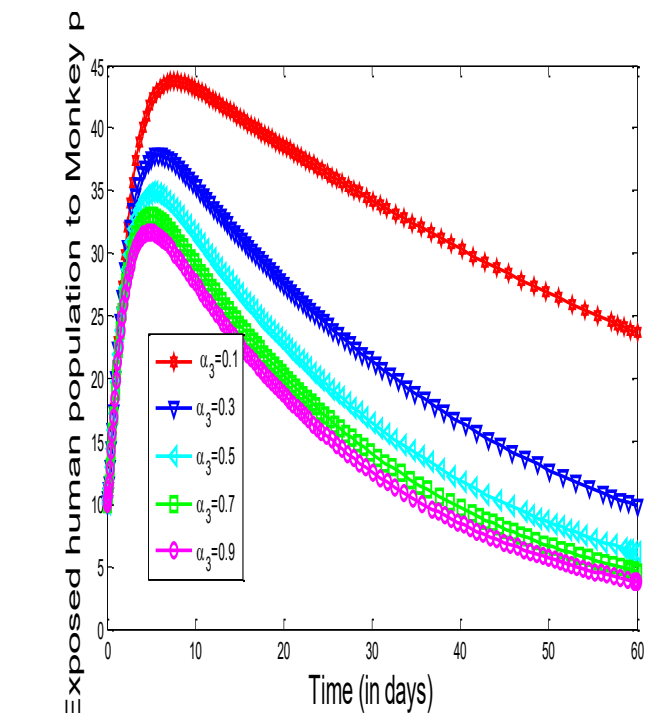
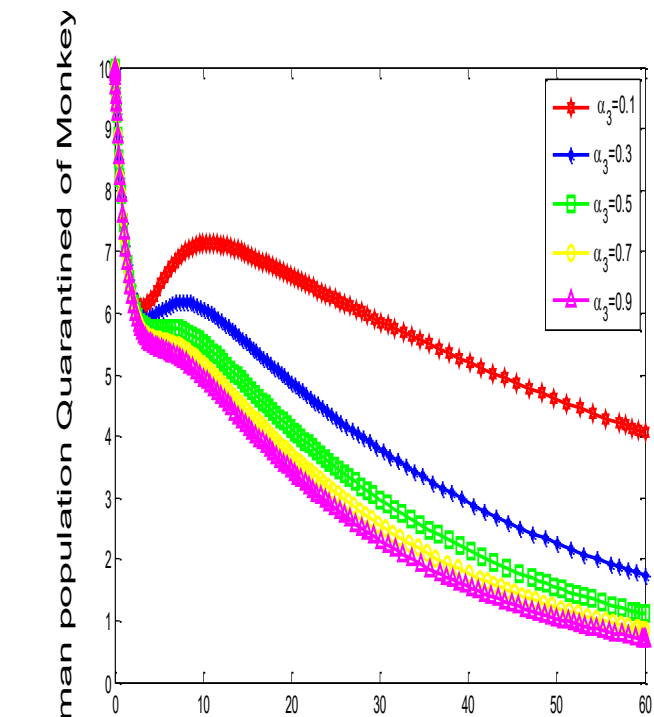


Fig.4u: Simulation of the effect of α_3 on



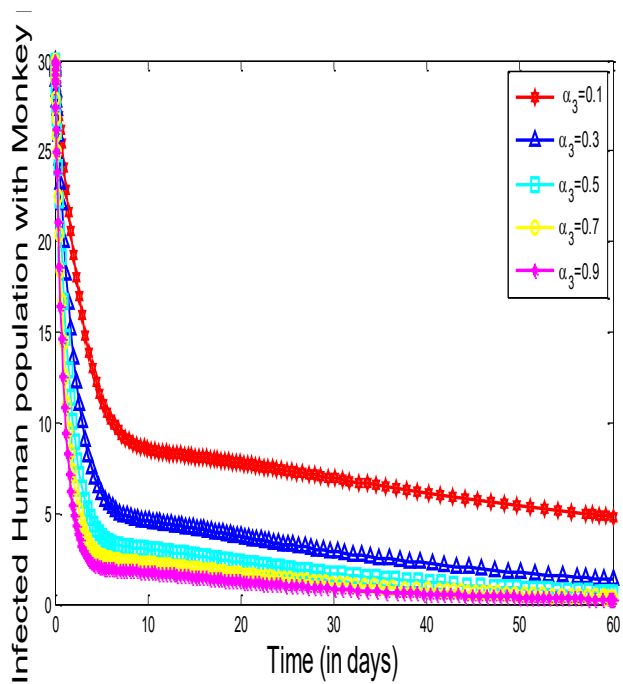


Fig.4w: Simulation of the effect of α_3 on infected humans with monkey pox only

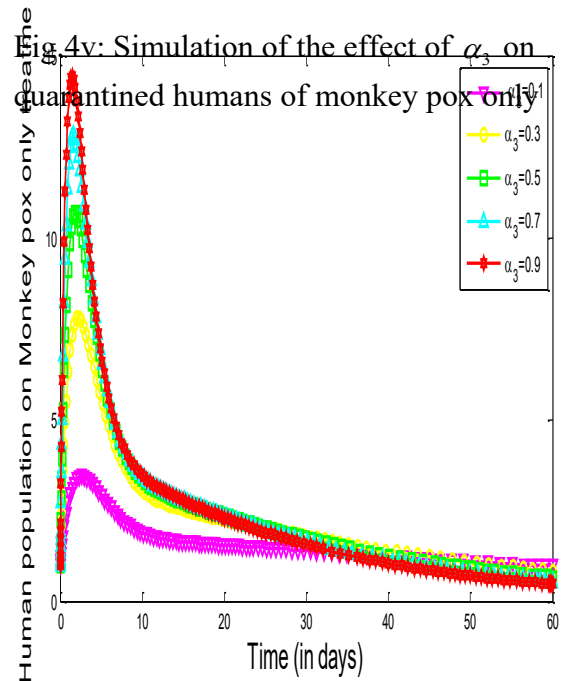


Fig.4x: Simulation of the effect of α_3 on humans on monkey pox only treatment

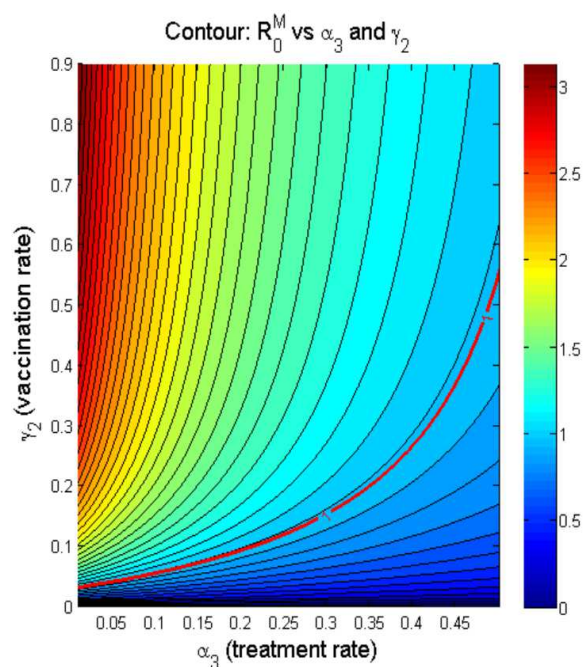
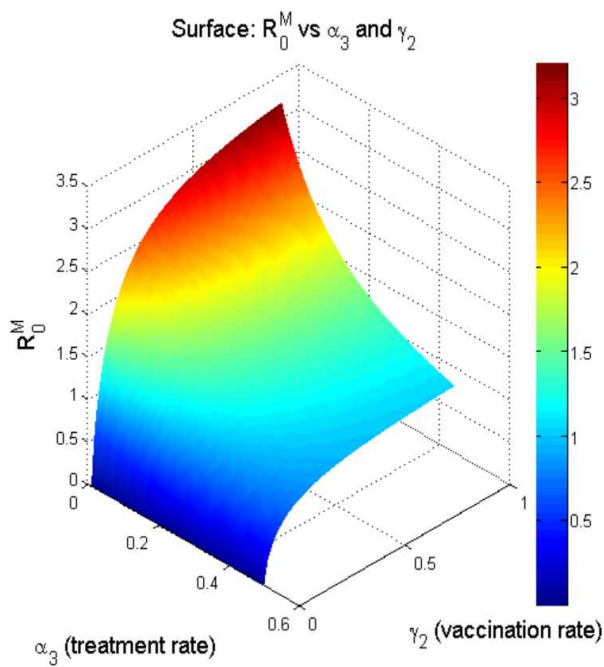


Fig.5a: Surface plot showing the effect of α_3 and γ_2 on R_0^M

Fig.5b: Contour plot showing the effect of α_3 and γ_2 on R_0^M

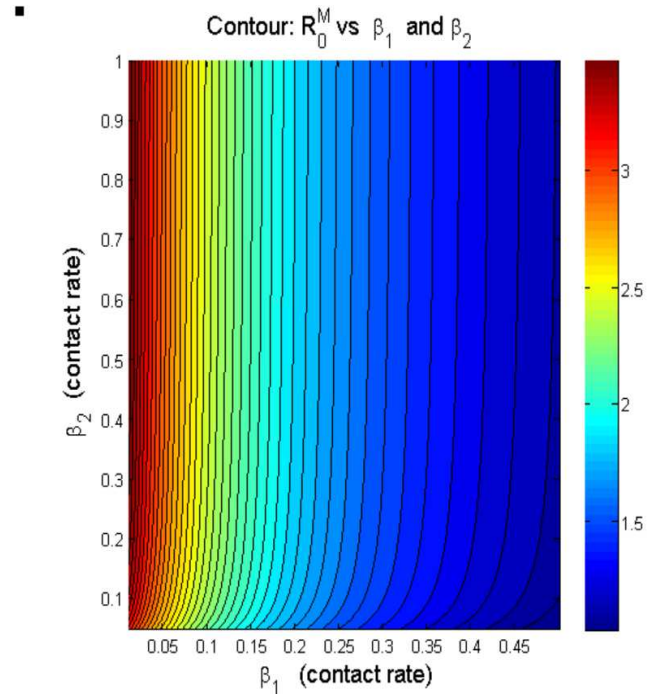
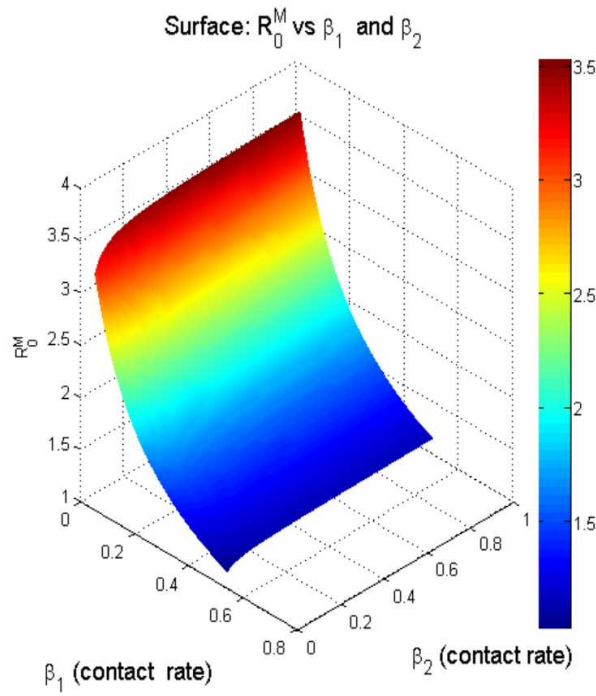


Fig.5c: Surface plot showing the effect of β_1 and β_2 on R_0^M

Fig.5d: Contour plot showing the effect of β_1 and β_2 on R_0^M

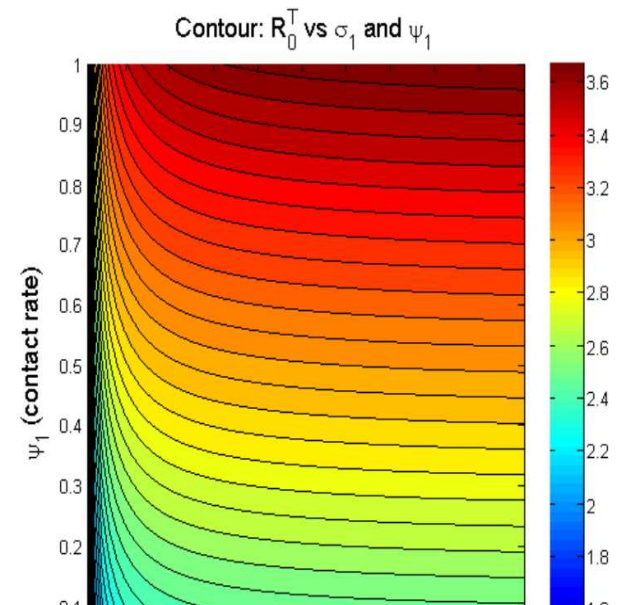
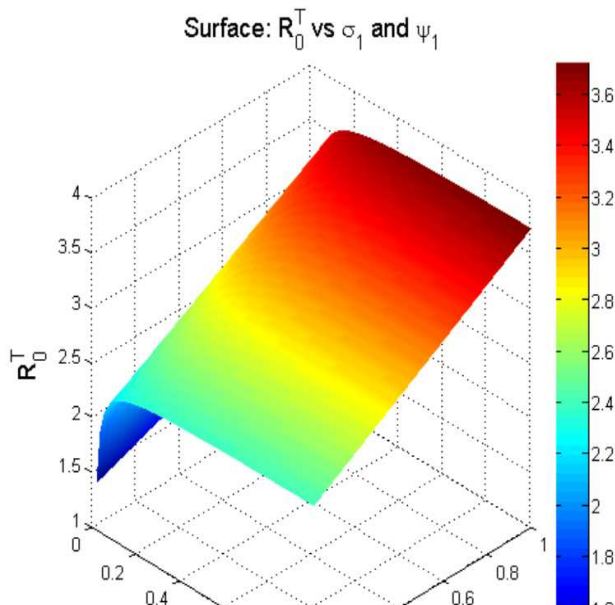


Fig.5e: Surface plot showing the effect of

σ_1 and ψ_1 on R_0^T

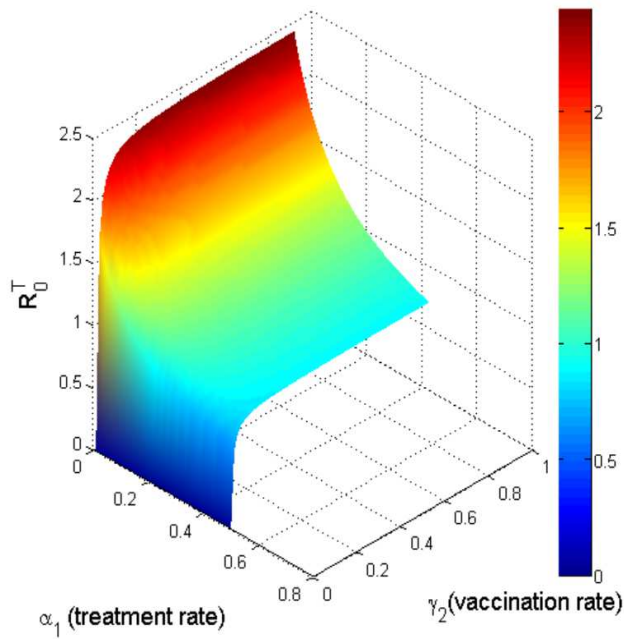


Fig.5g: Surface plot showing the effect of σ_1 and γ_2 on R_0^T

Fig.5f: Contour plot showing the effect of

σ_1 and ψ_1 on R_0^T

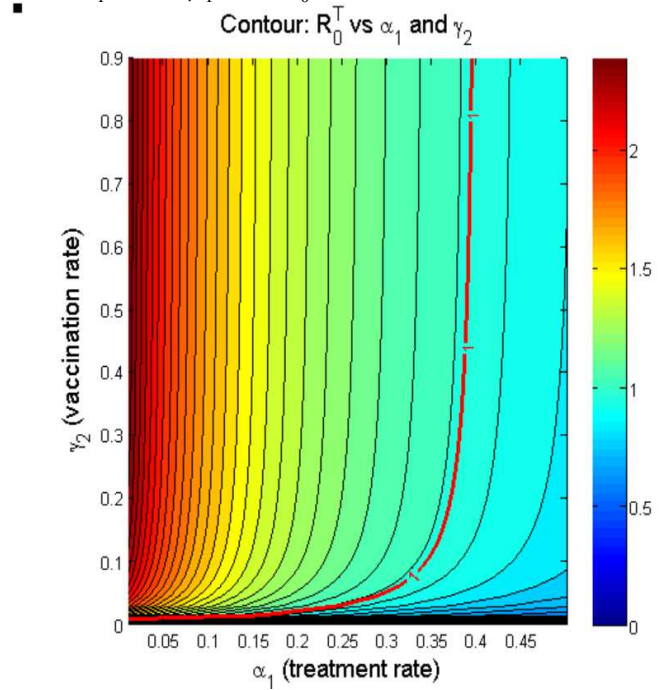


Fig.5g: Contour plot showing the effect of σ_1 and γ_2 on R_0^T

Figure (4a) shows simulation of the effect of contact rate (ψ_1) on human population on treatment of typhoid fever only. It can be observed that higher contact rate (ψ_1) leads to the increase in the population of human on treatment of typhoid fever only.

Figure (4b) shows simulation of the effect of contact rate (ψ_1) on infected human population with typhoid fever only. It can be observed that higher contact rate (ψ_1) leads to the increase in the population of human infected with typhoid fever only.

Figure (4c) shows simulation of the effect of contact rate (ψ_1) on vaccinated human population against typhoid fever only. It can be observed that increasing the contact rate (ψ_1) leads to the decrease in the population of human vaccinated against typhoid fever only. Figure (4d) shows simulation of the effect of contact rate (ψ_1) on cumulative new cases of typhoid fever only. It can be observed that increasing the contact rate (ψ_1) leads to the increase in the cumulative new cases of typhoid fever only. Figure (4e) shows simulation of the effect of vaccination rate (γ_2) on susceptible human population to typhoid fever only. It can be observed that increasing the vaccination rate (γ_2) leads to the decrease in the susceptible human population to typhoid fever only. Figure (4f) shows simulation of the effect of vaccination rate (γ_2) on exposed human population to typhoid fever only. It can be observed that increasing the vaccination rate (γ_2) leads to the decrease in the exposed human population to typhoid fever only. Figure (4g) shows simulation of the effect of vaccination rate (γ_2) on infected human population with typhoid fever only. It can be observed that increasing the vaccination rate (γ_2) leads to the decrease in the infected human population with typhoid fever only. Figure (4h) shows simulation of the effect of vaccination rate (γ_2) on human population on typhoid fever only treatment. It can be observed that increasing the vaccination rate (γ_2) leads to the increase in the population of human on typhoid fever only treatment. Figure (4i) shows simulation of the effect of contact rate (β_1) on infected human population with monkey pox only. It can be observed that increasing the contact rate (β_1) leads to the increase in the population of human infected with monkey pox only. Figure (4j) shows simulation of the effect of contact rate (β_1) on recovered human population from monkey pox only. It can be observed that increasing the contact rate (β_1) leads to the increase in the population of recovered human from monkey pox only. Figure (4k) shows simulation of the effect of contact rate (β_1) on human population on monkey pox only treatment. It can be observed that increasing the contact rate (β_1) leads to the increase in the population of human monkey pox only treatment. Figure (4l) shows simulation of the effect of contact rate (β_1) on quarantined human population of monkey pox only. It can be observed that increasing the contact rate (β_1) leads to the increase in the population of quarantined human of monkey pox only. Figure (4m) shows simulation of the effect of treatment rate (α_3) on recovered human population from monkey pox only. It can be observed that increasing the treatment rate (α_3) leads to the increase in the population of recovered human from monkey pox only. Figure (4n) shows simulation of the effect of treatment rate (α_3) on cumulative new cases of monkey pox only. It can be observed that increasing the treatment rate (α_3) leads to the reduction in the cumulative new cases of monkey pox only. Figure (4o) shows simulation of the effect of treatment rate (α_3) on exposed human population to monkey pox only. It can be observed that increasing the treatment rate (α_3) leads to the reduction in the exposed human population to monkey pox only. (4p) shows simulation of the

effect of treatment rate (α_3) on quarantined population of monkey pox only. It can be observed that increasing the treatment rate (α_3) leads to the reduction in the human population quarantined of monkey pox only.

Figure (4q) shows simulation of the effect of treatment rate (α_3) on human population infected with monkey pox only. It can be observed that increasing the treatment rate (α_3) leads to the reduction in the population of human infected with monkey pox only. Figure (4r) shows simulation of the effect of treatment rate (α_3) on vaccinated human population against monkey pox only. It can be observed that increasing the treatment rate (α_3) leads to increase in the vaccinated human population against monkey pox only. Figure (4s) Shows simulation of the effect of treatment rate (α_3) on recovered human population from monkey pox only. It can be observed that increasing the treatment rate (α_3) leads to decrease in the recovered human population from monkey pox only. Figure (4t) shows simulation of the effect of treatment rate (α_3) on human population on monkey pox only treatment. It can be observed that increasing the treatment rate (α_3) leads to decrease in the population of human on monkey pox only treatment.

Figure (4u) shows simulation of the effect of treatment rate (α_3) on exposed human population to monkey pox only. It can be observed that increasing the treatment rate (α_3) leads to decrease in the population of exposed human to monkey pox only. (4v) shows simulation of the effect of treatment rate (α_3) on quarantined human population of monkey pox only. It can be observed that increasing the treatment rate (α_3) leads to decrease in the population of quarantined human of monkey pox only. Figure (4w) shows simulation of the effect of treatment rate (α_3) on infected human population with monkey pox only. It can be observed that increasing the treatment rate (α_3) leads to decrease in the population of infected human with monkey pox only. Figure (4x) shows simulation of the effect of treatment rate (α_3) on human population on monkey pox only treatment. It can be observed that increasing the treatment rate (α_3) leads to increase in the population of human on monkey pox only treatment. Figure (5a) it can be observed that the basic reproduction number R_0^M reaches a peak below one (1) as the values of α_3 and γ_2 increases. This indicates that increasing these parameters will ultimately alleviate the impact of monkey pox only in the population. Conversely, if appropriate measures are implemented α_3 and γ_2 can reduce the prevalence of monkey pox only in the human population. This is evident from their effect on R_0^M (5b) Illustrates the contour plot of concerning R_0^M . Upon examination of the numerical streams within the graph, it is evident that the maximum value of R_0^M attained by varying α_3 and γ_2 is 0.6, indicating a value below unity (1). This observation suggests that increasing these parameters would reduce a significant outbreak of monkey pox only in the human population. (5c) it can be observed that the basic reproduction number R_0^M reaches a peak below one (1) as the values of β_1 and β_2 decreases. This indicates that decreasing these parameters will ultimately alleviate the

impact of monkey pox only in the population. Conversely, if appropriate measures are not implemented β_1 and β_2 can trigger the prevalence of monkey pox in the human population. This is evident from their effect on R_0^L . (5d) Illustrates the contour plot of concerning R_0^M . Upon examination of the numerical streams within the graph, it is evident that the maximum value of R_0^L attained by varying β_1 and β_2 is 0.6, indicating a value below unity (1). This observation suggests that reducing β_1 and β_2 would reduce a significant outbreak of monkey pox in the human population. (5e) it can be observed that the basic reproduction number R_0^T reaches a peak below one (1) as the values of σ_1 and ψ_1 decreases. This indicates that decreasing these parameters will ultimately alleviate the impact of typhoid fever only in the population. Conversely, if appropriate measures are not implemented σ_1 and ψ_1 can trigger the prevalence of typhoid fever in the human population. This is evident from their effect on R_0^T (5f) Illustrates the contour plot of concerning R_0^T . Upon examination of the numerical streams within the graph, it is evident that the maximum value of R_0^T attained by varying σ_1 and ψ_1 is 0.6, indicating a value below unity (1). This observation suggests that reducing σ_1 and ψ_1 would reduce a significant outbreak of typhoid fever in the human population. (5g) it can be observed that the basic reproduction number R_0^T reaches a peak below one (1) as the values of α_1 and γ_2 increases. This indicates that increasing these parameters will ultimately alleviate the impact of typhoid fever only in the population. Conversely, if appropriate measures are implemented α_1 and γ_2 can reduce the prevalence of typhoid fever only in the human population. This is evident from their effect on R_0^T (5h) Illustrates the contour plot of concerning R_0^T . Upon examination of the numerical streams within the graph, it is evident that the maximum value of R_0^T attained by varying α_1 and γ_2 is 0.6, indicating a value below unity (1). This observation suggests that increasing these parameters would reduce a significant outbreak of typhoid fever only in the human population.

5.0 Conclusion

The fractal-fractional order model developed in this thesis provides a comprehensive and robust framework for analyzing monkeypox and typhoid fever co-infection transmission dynamics, significantly advancing beyond traditional integer-order models through the incorporation of fractal-fractional derivatives. These derivatives synergistically combine fractional calculus, which captures memory effects and long-term dependencies in dual pathogen progression, with fractal geometry, which models self-similar, heterogeneous transmission patterns across diverse population scales and zoonotic-environmental co-infection pathways. This integrated approach proves particularly effective for monkey pox-typhoid co-infection systems, as it accounts for the non-local nature of dual infection dynamics, zoonotic transmission patterns, and complex contact structures that characterize real-world epidemic behavior in regions with high typhoid endemicity and emerging monkeypox outbreaks. Analytical proofs confirm mathematical well-posedness through solution positivity and boundedness, while establishing global asymptotic stability at endemic equilibria when basic reproduction numbers exceed unity, providing theoretical foundations for epidemic persistence and control in zoonotic and environmentally transmitted dual

pathogen systems. The derivation of monkeypox-free and typhoid-free equilibria offers insights into single-pathogen elimination strategies and their feasibility under varying transmission intensities.

Numerical simulations and robust analytical results reveal that the fractal-fractional order α and fractal dimension σ significantly influence population dynamics, with lower fractional orders intensifying oscillatory behaviors in co-infected compartments and creating more pronounced memory effects that persist throughout zoonotic transmission cycles and beyond acute infection episodes. This finding emphasizes that monkeypox-typhoid co-infection exhibits strong historical dependencies where current health status, immune competence, and transmission potential are influenced by complete infection trajectories, previous typhoid exposure history, and cumulative immunological damage rather than just immediate clinical presentations. The oscillatory patterns observed across all population compartments, particularly those reflecting zoonotic monkeypox transmission peaks, suggest that co-infection management requires sustained, integrated intervention strategies that account for both persistent typhoid transmission through environmental reservoirs and periodic monkeypox outbreaks. Compared to integer-order models, the fractal-fractional framework demonstrates superior capability in capturing multi-scale epidemic dynamics that span from individual immune response patterns to population-level zoonotic and environmental fluctuations, revealing temporal complexities that classical approaches cannot adequately represent. The model's ability to generate realistic oscillatory behaviors mimicking observed monkeypox zoonotic patterns combined with memory-dependent typhoid progression dynamics provides enhanced insights into epidemic patterns and offers more accurate predictions for intervention planning, resource allocation, and timing of zoonotic control and sanitation campaigns. The convergence of all population compartments toward stable equilibrium values, regardless of the fractal-fractional order, indicates robust long-term epidemic behavior while highlighting that the path to equilibrium varies substantially depending on memory effects and zoonotic forcing. This finding has important implications for public health planning in regions with high typhoid prevalence and emerging monkeypox, suggesting that early intervention timing, intensity of zoonotic control measures, and coordination between typhoid and monkeypox programs can dramatically influence epidemic trajectories and long-term co-infection burden.

Future extensions of this work could incorporate spatial heterogeneity to model geographic variations in typhoid transmission intensity and monkeypox prevalence, integrate antibiotic resistance mechanisms for typhoid treatment, and expand the framework to include additional zoonotic reservoirs and environmental contamination dynamics. The incorporation of age-structured dynamics, particularly for typhoid immunity development in children, and waterborne transmission pathways could further enhance the model's realism and policy relevance. Additionally, investigating the effects of different fractal-fractional operators, exploring optimal control strategies for integrated zoonotic and sanitation management within the fractal-fractional framework, and incorporating behavioral changes in response to zoonotic monkeypox risk represent promising directions for advancing monkeypox-typhoid co-infection modeling and developing evidence-based control strategies for dual epidemic management in resource-limited settings.

Clinical trial number

Not Applicable

Declarations

Ethics approval and consent to participate in clinical trial

Not applicable

Consent for publication

Not applicable

Availability of data.

The data used in this study are presented and referenced in table 4 of this manuscript. The source code data used for simulation is available from the corresponding author on reasonable request.

Competing Interests

The authors declare that they have no competing interests.

Author Contribution

Benedict Celestine Agbata: Writing – original draft, Introduction, formal analysis, Simulations. **Hawah Oyiza Rabi:** Writing – original draft, Writing literature review & editing, Formal analysis, Simulations. **Jeremiah Amos** Introduction, formal analysis, Simulations. **Aseel Smerat** Writing , review, editing, simulations, supervision. **Hambeer Singh** Writing literature review Review & editing and supervision, editing. **A.K.Awasthi** Analysis, Simulations, Review & editing and supervision, editing. **Godwin Onuche Acheneje** Analysis, Simulations, Review & editing and supervision, editin. **Bolarinwa Bolaji** Writing , review, editing, simulations, supervision.

Acknowledgements

Not applicable.

Funding Statement

This research work received no funding. As a corresponding author from low income country Nigeria, I hereby request full APC waiver

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