A thermodynamic bound on Hadley cell expansion: Supplementary Information

1 Thermodynamic parameters

For convenience of calculation we define q to be the mass mixing ratio of water vapor in air, equal to $q^\star$ at saturation, and the following dimensionless parameters: $\beta = L_v/\left(c_pT\right)$ where L_v is the latent heat of vaporatization and c_p is the specific heat of air at constant pressure p, $\kappa = R_d/c_p$ where R_d is the gas constant for air, $\epsilon = R_d/R_v$ where R_v is the gas constant for water vapor, and for relative variations of the latent heat to the specific heat [1]

$$\gamma = \beta\left(\frac{\partial q^\star\left(T\right)}{\partial \ln T}\right)_p \tag{1}$$

Noting the Clausius-Clapeyron relationship for the near exponential dependence of the saturation :

$$\left(\frac{\partial q^\star\left(T\right)}{\partial \ln T}\right)_p = \frac{\epsilon\beta}{\kappa}q^\star \tag{2}$$

we obtain

$$\gamma = \frac{\epsilon\beta^2}{\kappa}q^\star \tag{3}$$

which is, very roughly, $\gamma \simeq 200q^\star$. We also note the pressure scale height

$$H_p = R_dT/g \tag{4}$$

where g is the gravitational acceleration.

2 The available potential energy of buoyant oscillations in a stratified atmosphere

As described in the main text, the buoyant acceleration around some neutral mid-level $z_0 \simeq H/2$ is given by

$$\frac{d^2 z'}{dt^2} + N^2 z' = 0 \tag{5}$$

where $z' = z - z_0$ and N is the frequency of buoyancy oscillations for a parcel of air displaced vertical from its position of neutral buoyancy in a stably stratified atmosphere. If $\Gamma_d = g/c_p$ is the dry adiabatic lapse rate, and $\Gamma = -dT/dz$ is the actual mean vertical gradient for the layer-averaged temperature T in the Tropics, and neglecting the second-order influence of water vapor vertical gradients on buoyancy, then [2]

$$N^2 = \frac{g}{T}\left(\Gamma_d - \Gamma\right) \tag{6}$$

Integrating Eq 5, the potential energy of the oscillation becomes

$$\mu_z = \alpha N^2 H^2 \tag{7}$$

where α is a constant coefficient.

Assuming the atmosphere approximates a moist adiabat $\Gamma = \Gamma^\star$ where [2]

$$\Gamma^\star = \frac{g}{c_p}\frac{1 + \gamma/\left(\epsilon\beta\right)}{1+\gamma} \tag{8}$$

Substituting in Eq. 6 yields the square of the buoyancy frequency $N^\star$ for a moist adiabat

$$\begin{aligned} N^{\star 2} &= \frac{g}{T}\left(\Gamma_d - \Gamma^\star\right) && (9)\\ &= \kappa\frac{g}{H}\left(\frac{\epsilon\beta - 1}{\epsilon\beta}\right)\frac{\gamma}{1+\gamma} && (10) \end{aligned}$$

This then leads to an expression for the available potential energy of a moist adiabatic atmosphere, available in the sense that it can be converted to kinetic energy. From Eq. 7

$$\mu_z = \alpha N^{\star 2} H^2 \tag{11}$$

$$= \alpha\kappa \left(\frac{\epsilon\beta - 1}{\epsilon\beta}\right) \frac{\gamma}{1+\gamma} gH \tag{12}$$

In a very low-temperature, dry atmosphere with $\gamma \simeq 0$, the potential energy is zero. But in a higher temperature moist atmosphere approximating a saturated adiabat, μ_z is positive because rising parcels cool and condense to release latent heat aloft, thereby stabilizing the atmosphere with respect to a dry adiabat. From inspection of Eq. 12, this available potential energy is determined by its geopotential parameter gH, as modified by the parameter $\gamma/\left(1+\gamma\right)$ where γ represents the ratio of changes in the column averaged latent heat of saturated air to the specific heat of dry air. Both parameters are a function of temperature.

For the sake of argument, a candidate vertical root-mean-square displacement $\sqrt{2}H/4$ from a mean height $H/2$, $\alpha = 1/16$. Then, from ERA-5 reanalysis datasets (Methods), and using Eq. 10 with $\alpha = 1/16$ and the pressure scale height for H, the value of μ_z in Eq. 11 is $718\,\mathrm{J\,kg^{-1}}$. From the density weighted value of β and γ over the depth H then from Eq. 12 $\mu_z = 722\,\mathrm{J\,kg^{-1}}$. A natural comparison is to the Convective Available Potential Energy (CAPE), as determined by the difference between the lapse rate and the saturated adiabatic lapse rate $\Gamma - \Gamma^\star$ integrated over the atmospheric depth above the level of free convection (as opposed to $\Gamma_d - \Gamma^\star$ and $\sqrt{2}H/4$ as was done here). While strictly CAPE and μ_z are different quantities (CAPE is determined from parcel theory by following a moist adiabat from the lifting condensation level to the level of neutral buoyancy), both have calculated magnitudes that are of comparable value [3].

3 Scaling results for the Hadley cell width

In a nearly inviscid, axisymmetric model of the atmosphere that yielded qualitatively realistic Hadley and Ferrel cells, the Hadley cell width was shown to be controlled by a balance between convective heating in the deep tropics and radiation in the upper, poleward-flowing branch of the cell[4]. The width of the Hadley cell scales as (from Eq. 21 of their study)

$$y_{hc}\left(S77\right) = 5^{\frac{1}{4}}\left(R_L a\right)^{\frac{1}{2}},$$

where R_L is the polar Rossby deformation radius for a stratified atmosphere, expressed as

$$R_L = \frac{NH}{2\Omega}, \tag{13}$$

and H is the depth of the tropical atmosphere in their model. Eq. 13 implies that

$$y_{hc}\left(S77\right) = \left(\frac{5}{4}\right)^{\frac{1}{4}}\left(\frac{aNH}{\Omega}\right)^{\frac{1}{2}}, \tag{14}$$

In a similar but nearly-inviscid axisymmetric formulation for the tropical circulation [5], the latitude of the Hadley cell edge scales as (from Eq. 16 of their study)

$$\theta_H(H80) = \left(\frac{5}{3}R\right)^{\frac{1}{2}}, \tag{15}$$

where

$$R = \left(gH\Delta_H\right)/\left(\Omega^2 a^2\right)$$

is a thermal Rossby number, H is the height of the rigid lid in their model, and Δ_H is the fractional latitudinal potential temperature contrast over a hemisphere. Eq. 15 implies

$$\theta_H(H80) = \left(\frac{5gH\Delta_H}{3\Omega^2 a^2}\right)^{\frac{1}{2}}.$$

Denoting fractional vertical potential temperature contrast throughout the troposphere as Δ_V, defining $\zeta \equiv \Delta_H/\Delta_V$, and letting

$$N^2 \simeq \frac{g}{\theta}\frac{\partial\theta}{\partial z} \simeq \frac{g\Delta_V}{H},$$

we see that

$$\Delta_V = \frac{N^2 H}{g} \tag{16}$$

and

$$\Delta_H = \frac{\zeta N^2 H}{g}.$$

Substituting this expression in for Δ_H in the right-hand side of equation 15 and noting that $\phi_H = y_{hc}/a$ gives

$$y_{hc}(H80) = \left(\frac{5\zeta N^2 H^2}{3\Omega^2}\right)^{\frac{1}{2}}. \tag{17}$$

Eq. 17 and Eq. 14 are equivalent if

$$\zeta = \frac{2}{3}\frac{a\Omega}{NH},$$

which (substituting representative values for the global mean tropopause height, potential temperature, tropospheric potential temperature gradients, and Brunt-Väisälä frequency) is true to within about 10%. This equivalency is weak, in that N itself scales with mean temperature via Clausius-Clapeyron, but it suggests a sort of global equipartition in Held and Hou's model.

Yet another analysis noted that, as air in the upper branch of the Hadley cell moves poleward, the Coriolis force increases vertical wind shear, leading progressively towards a baroclinically unstable profile, thereby governing the growth rate of baroclinic eddies on the Hadley cell's poleward flanks ^{6}. The scaling for this limit of the Hadley cell latitudinal width was argued to be given (on page 36) by

$$\phi'_H(H00) \approx (R^*\Delta_V)^{\frac{1}{4}}$$

where Δ_V is as defined above, and $R^\star$ is another thermal Rossby number, this time given by

$$R^* \equiv \frac{gH}{a^2\Omega^2},$$

and H is the mean thickness of a shallow water model. Substituting Eq. 16 for Δ_V and again noting that $\phi_H = y_{hc}/a$, this scaling reduces to:

$$y_H\,(H00) \approx \left(\frac{NHa}{\Omega}\right)^{\frac{1}{2}}. \tag{18}$$

Eq. 18 and equation 14 differ only by a constant, and by the use of a mean thickness (in Eq. 18 rather than a tropical tropopause height (in Eq. 14).

As described in the main text, Eq. 18 can also be obtained quite directly by treating the Tropics as a system of two coupled oscillators, where the potential energy of a vertical mode μ_z associated with moist convection (Eq. 11) balances the rotational energy of a horizontal mode μ_y that is related to meridional displacement against the Coriolis force. Here, μ_y is obtained analogously to the derivation of μ_z above (Eq. 7). The average Coriolis frequency of a Hadley cell of width ϕ_H with a mean distance from the equator $y_{hc}/2$ is $f = 2\Omega \sin(\phi_H/2)$. Assuming the horizontal oscillatory motions of Hadley cell circulations have a root-mean-squared displacement of $\sqrt{2}y_{hc}/4$, and applying the small angle approximation $\sin\left(\phi_H/2\right) \approx y_{hc}/\left(2a\right)$, then the average Coriolis frequency is $f = \Omega y_{hc}/a$ and

$$\mu_y = \alpha \frac{\Omega^2 y_{hc}^4}{a^2} \tag{19}$$

where applying similar reasoning to the vertical mode for the candidate displacement $\alpha = 1/16$. Equating μ_y with μ_z given by Eq. 7 leads to the expression given by Eq. 18 without requiring any treatment of deterministic atmospheric dynamics. Note that the coefficient α cancels, so a different choice of average displacement than $1/16$ would not change the result.

4 The Hadley cell depth and buoyancy frequency

From the perspective of climate change on Earth, the only free parameters in Eq. 18 are N and H, whose product has units of speed. More generally, allowing for variability in the planetary radius a and rotation rate Ω, Eq. 18 reproduces well the observed Hadley cell width on a wide range of planetary bodies in the solar system, although allowance must be made for any atmospheric super-rotation, which augments Ω relative to the planetary rotation rate and is important on Venus and Jupiter [?].

However, there is some ambiguity, about the appropriate choice for H in Eq. 18. Most obviously this would be either the pressure scale height or the tropospheric depth as applied to Earth by [6]. However, there might also be an effective scale height that is suitable that accounts for deepening of the scale height due to internal atmospheric heating [7]. For example, for the gas giants of Jupiter, Saturn, and Neptune, deepening owes to the fractional excess of outgoing longwave radiation relative to absorbed solar radiation that arises from internal heating from Kelvin-Helmholtz contraction. Such deepening is particularly notable on Neptune where the Hadley cell height reaches well into the stratosphere. There is no Kelvin-Helmholtz contraction on Earth. However, Earth is unique among solar system planetary bodies for the importance of internal latent heating from condensation of water vapor. In the Earth's tropics the relative magnitude of latent heat flux to the outgoing longwave radiative flux is 0.37 [8]. Such internal heating was argued to imply an effective scale height of $H_{\text{eff}} = 10.3\,\text{km}$, which is higher than the density scale height of about 8.5 km but lower than the radiatively determined tropopause height of about 14 km [9]. Substituting H_{eff} for H in Eq. 18, and assuming a column integrated value for N based on ERA-5 reanalysis data sets of $\simeq 0.012\ \text{s}^{-1}$, yields a value for y_{hc} of 3250 km, reproducing well the observed value of 3274 kilometers or 29.5 degrees latitude (Methods).

Here, however, we account for the role of moisture condensation and latent heat release by maintaining the pressure scale height of 8544 m for H and by accounting for moisture by adjusting the buoyancy frequency N to its value given by Eq. 10, assuming the tropical atmosphere follows a moist adiabat. In this case, based on ERA-5 reanalysis data sets, $N^\star = 0.0126\,\text{s}^{-1}$, so from Eq. 18 the value for y_{hc} is 3073 km, which is 5% lower than 3250 km. Thus, either approach, when used to calculate the Hadley cell width in Eq. 18, yields values that match well with observations.

5 The surface temperature dependence of Hadley cell width

The advantage of the second approach mentioned above is that it provides a pathway for an analytical formulation for the Hadley Cell width that includes moist thermodynamics a calculation of its sensitivity to surface temperature changes. Both the scale height (Eq. 4) and the buoyancy frequency assuming a moist adiabat (Eq. 10) are functions of temperature. Substituting the scale height and moist buoyancy frequency into Eq. 18 yields a new expression for Hadley cell width:

$$y_{hc} = \left(\kappa \frac{a^2}{\Omega^2} \left(\frac{\epsilon\beta - 1}{\epsilon\beta} \right) \frac{\gamma}{1+\gamma} gH \right)^{1/4} \tag{20}$$

Moving forward, the sensitivity of y_{hc} to changes in surface temperature T_S can be expressed, in dimensionless terms for ease of calculation, as:

$$S_{hc} = \frac{d \ln y_{hc}}{d \ln T_S} \simeq \frac{1}{4} \left(\frac{d \ln H}{d \ln T} + \frac{1}{1+\gamma} \frac{d \ln \gamma}{d \ln T} \right) \frac{d \ln T}{d \ln T_s} \tag{21}$$

Using reanalysis data for the tropics, $d \ln T / d \ln T_S = 1.1$. The sensitivity is determined by a factor of one quarter owing to a quarter-root scaling of y_{hc} on μ_y and hence μ_z (Eq. 19, and in parentheses, a thermal expansion term and a latent heating term. Since from Eq. 4, the pressure scale height scales with temperature, it follows that $d \ln H / d \ln T = 1$. From Eq. 3, the sensitivity of γ to changes in temperature at constant pressure is $d \ln \gamma / d \ln T = d \ln q^\star / d \ln T + d \ln \beta / d \ln T = d \ln e^\star / d \ln T - 2$ where $e^\star$ is the saturation vapor pressure and through the Clausius-Clapeyron relation and $d \ln e^\star / d \ln T = \epsilon\beta/\kappa$. and density-weighted mean values are $\gamma = 1.6$, and $\varepsilon\beta/\kappa \simeq 19.4$. Thus the latent heating term dominates in the parentheses with a value of 7.7. Thus, we obtain

$$S_{hc} = \frac{d \ln y_{hc}}{d \ln T_S} \simeq 2.1 \tag{22}$$

Relative to the current state y_{hc0}, it follows that $y_{hc} = y_{hc0}\left(T_S/T_{S0}\right)^{2.1}$, or 23 km ($0.21°$ latitude or 0.7%) per degree Kelvin of surface warming. Since γ increases exponentially with temperature through the Clausius-Clapeyron relation, or at a rate of approximately 7% per Kelvin, the coefficient $1/\left(1+\gamma\right)$ in Eq. 21 acts as a negative feedback on the sensitivity associated with any further warming. However, the effect is small. For example, if the column were to warm by 4 K, then $S_{hc} \simeq 1.8$ and the expansion rate decreases slightly to 20 km ($0.18°$ latitude) per degree Kelvin.

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