Supplementary Material for "Depinning of KPZ Interfaces in Fractional Brownian Landscapes"

A : Correlated Random Media via Fractional Brownian Motion

In this study, we investigate the dynamics of a driven interface propagating through a random host medium. Unlike previous studies that typically assume uncorrelated (white) disorder, we consider a *correlated* quenched random landscape, modeled using two-dimensional fractional Brownian motion (2D FBM). Before introducing the dynamical model, we first provide a brief overview of FBM and explain how it is used to construct the correlated disorder field.

Our model considers interface motion in a $1+1$ dimensional geometry (i.e., scalar displacement over a two-dimensional substrate) where the underlying disorder exhibits long-range spatial correlations. These correlations are characterized by the Hurst exponent H and generated via FBM. The dynamical evolution of the interface is then modeled by the QKPZ_H equation, which will be detailed in the following section.

The Hurst exponent H controls the nature of the correlations: for $H = 0.5$, the increments of FBM are uncorrelated (standard Brownian motion), while $H > 0.5$ ($H < 0.5$) corresponds to positively (negatively) correlated disorder, resulting in smoother (rougher) surfaces,

FBM is a Gaussian stochastic process with stationary, power-law correlated increments. It is widely used as a prototypical model for anomalous diffusion and correlated random media.

The 2D FBM field $B_H(\mathbf{r})$, where $\mathbf{r} = (x, y)$, is defined such that

$$\langle B_H(\mathbf{r}) - B_H(\mathbf{r}_0)\rangle = 0, \qquad \langle [B_H(\mathbf{r}) - B_H(\mathbf{r}_0)]^2\rangle = |\mathbf{r} - \mathbf{r}_0|^{2H}, \tag{1}$$

where H is the Hurst exponent controlling the degree of correlation.

To numerically generate 2D FBM fields on an $L_x \times L_y$ lattice, we use the fast Fourier transform method. The approach begins by generating a Gaussian white noise field $\zeta(\mathbf{r})$ with the following statistical properties:

$$\langle \zeta(\mathbf{r})\rangle = 0, \quad \langle \zeta(\mathbf{r})\zeta(\mathbf{r}')\rangle = D\,\delta^2(\mathbf{r} - \mathbf{r}'), \tag{2}$$

where D controls the disorder strength.

We then define the Fourier-transformed FBM field as:

$$\tilde{B}_H(\mathbf{k}) = \left(k_x^2 + k_y^2\right)^{-(H+1)/2} \tilde{\zeta}(\mathbf{k}), \tag{3}$$

with

$$\tilde{\zeta}(\mathbf{k}) = \frac{1}{\sqrt{L_x L_y}} \sum_{\mathbf{r}} \zeta(\mathbf{r}) e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{4}$$

The real-space FBM field is then obtained via the inverse FFT:

$$B_H(\mathbf{r}) = \frac{1}{\sqrt{L_x L_y}} \sum_{\mathbf{k}} \tilde{B}_H(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{r}}. \tag{5}$$

Importantly, this construction satisfies the self-affine scaling property:

$$B_H(\alpha\mathbf{r}) \overset{d}{=} \alpha^H B_H(\mathbf{r}), \tag{6}$$

where $\overset{d}{=}$ denotes equality in distribution. The scaling behavior can be derived from the transformation properties of the white noise field $\zeta(\mathbf{r})$ under spatial rescaling

$$\zeta(\alpha\mathbf{r}) \overset{d}{=} \alpha^{-1}\zeta(\mathbf{r}), \quad \tilde{\zeta}(\alpha^{-1}\mathbf{k}) \overset{d}{=} \tilde{\zeta}(\mathbf{k}), \tag{7}$$

which, together with Eq. 3, implies:

$$\tilde{B}_H(\alpha^{-1}\mathbf{k}) \overset{d}{=} \alpha^{H+1}\tilde{B}_H(\mathbf{k}), \tag{8}$$

leading to the scaling form in Eq. 6.

The power spectral density of the resulting FBM landscape in 2D is given by:

$$S(\mathbf{k}) = \frac{a}{\left(k_x^2 + k_y^2\right)^{(H+1)/2}}, \tag{9}$$

where a is a normalization constant.

B: Scaling Arguments

This appendix details the scaling properties of Fractional Brownian Motion (FBM). The process is defined by its correlation function (Eq. 1):

$$\langle [B_H(\mathbf{r}) - B_H(\mathbf{r}')]^2 \rangle \propto |\mathbf{r} - \mathbf{r}'|^{2H}. \tag{10}$$

To analyze its scaling behavior, we apply a transformation $\mathbf{r} \to \alpha \mathbf{r}$, where $\alpha > 0$ is a scaling factor. For Eq. 10 to remain consistent under this transformation, the FBM field must satisfy the following scaling relation:

$$B_H(\alpha \mathbf{r}) \stackrel{d}{=} \alpha^H B_H(\mathbf{r}), \tag{11}$$

where $\stackrel{d}{=}$ denotes equality in probability distributions.

Our objective is to demonstrate that the FBM generation method described in the main text produces a field that adheres to this property. The method begins by generating an uncorrelated Gaussian white noise field $\zeta(\mathbf{r})$ across a lattice, characterized by:

$$\langle \zeta(\mathbf{r}) \rangle = 0, \quad \langle \zeta(\mathbf{r}) \zeta(\mathbf{r}') \rangle = \delta^{(2)}(\mathbf{r} - \mathbf{r}'). \tag{12}$$

The scaling property of the Dirac delta function,

$$\langle \zeta(\alpha \mathbf{r}) \zeta(\alpha \mathbf{r}') \rangle = \alpha^{-2} \delta^{(2)}(\mathbf{r} - \mathbf{r}'), \tag{13}$$

implies that the noise itself scales as:

$$\zeta(\alpha \mathbf{r}) \stackrel{d}{=} \alpha^{-1} \zeta(\mathbf{r}). \tag{14}$$

We now examine the scaling behavior in Fourier space. The Fourier transform of the noise is defined as:

$$\tilde{\zeta}(\mathbf{k}) \equiv \frac{1}{L} \sum_{\mathbf{r}} \zeta(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}}, \tag{15}$$

where $L = \sqrt{L_x L_y}$ for a square lattice ($L_x = L_y$). Applying the scaling relation from Eq. 14 yields:

$$\frac{1}{L} \sum_{\mathbf{k}} \tilde{\zeta}(\mathbf{k}) e^{i\alpha\mathbf{k}\cdot\mathbf{r}} = \alpha^{-1} \frac{1}{L} \sum_{\mathbf{k}} \tilde{\zeta}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{16}$$

Transitioning to the thermodynamic limit, where $\sum_{\mathbf{k}} \to \left(\frac{L}{2\pi}\right)^2 \int \mathrm{d}^2\mathbf{k}$, this equation becomes:

$$\frac{1}{\tilde{L}} \left(\frac{\tilde{L}}{2\pi}\right)^2 \int \mathrm{d}^2\mathbf{Q}\, \tilde{\zeta}\left(\frac{\mathbf{Q}}{\alpha}\right) e^{i\mathbf{Q}\cdot\mathbf{r}} = \frac{1}{L} \left(\frac{L}{2\pi}\right)^2 \int \mathrm{d}^2\mathbf{k}\, \tilde{\zeta}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{17}$$

where we have made the change of variables $\mathbf{Q} \equiv \alpha \mathbf{k}$ and defined a scaled system size $\tilde{L} \equiv \alpha^{-1} L$. This leads to the key scaling relation for the Fourier-transformed noise:

$$\tilde{\zeta}(\alpha^{-1}\mathbf{k}) \stackrel{d}{=} \tilde{\zeta}(\mathbf{k}). \tag{18}$$

(Note: While the specific pre-factors in the Fourier transform definition can affect the exact form of this relation, the final result for B_H remains unchanged.)

The FBM field in Fourier space is constructed by filtering the noise:

$$\tilde{B}_H(\mathbf{k}) = \mathbf{k}^{-H-1} \tilde{\zeta}(\mathbf{k}), \tag{19}$$

as given in Eq. 3. Using the scaling relation for the noise (Eq. 18), we find the corresponding scaling for $\tilde{B}_H$:

$$\begin{aligned} \tilde{B}_H(\alpha^{-1}\mathbf{Q}) &= (\alpha^{-1}\mathbf{Q})^{-H-1} \tilde{\zeta}(\alpha^{-1}\mathbf{Q}) \\ &= \alpha^{H+1} \mathbf{Q}^{-H-1} \tilde{\zeta}(\mathbf{Q}) \\ &= \alpha^{H+1} \tilde{B}_H(\mathbf{Q}). \end{aligned} \tag{20}$$

Finally, we confirm that this leads to the desired real-space scaling by performing the inverse Fourier transform:

$$
\begin{aligned}
B_H(\alpha \mathbf{r}) &= \frac{1}{L}\left(\frac{L}{2\pi}\right)^2 \int \mathrm{d}^2\mathbf{k}\, \tilde{B}_H(\mathbf{k}) e^{i\alpha \mathbf{k}\cdot\mathbf{r}} \\
&= \frac{\alpha^{-1}}{\tilde{L}}\left(\frac{\tilde{L}}{2\pi}\right)^2 \int \mathrm{d}^2\mathbf{Q}\, \tilde{B}_H(\alpha^{-1}\mathbf{Q}) e^{i\mathbf{Q}\cdot\mathbf{r}} && \text{(substitute } \mathbf{Q} = \alpha\mathbf{k}\text{)} \\
&= \frac{\alpha^{-1}}{\tilde{L}}\left(\frac{\tilde{L}}{2\pi}\right)^2 \int \mathrm{d}^2\mathbf{Q}\, \left[\alpha^{H+1}\tilde{B}_H(\mathbf{Q})\right] e^{i\mathbf{Q}\cdot\mathbf{r}} && \text{(apply Eq. 21)} \\
&= \alpha^H \left[\frac{1}{\tilde{L}}\left(\frac{\tilde{L}}{2\pi}\right)^2 \int \mathrm{d}^2\mathbf{Q}\, \tilde{B}_H(\mathbf{Q}) e^{i\mathbf{Q}\cdot\mathbf{r}}\right] \\
&= \alpha^H B_H(\mathbf{r}).
\end{aligned}
\tag{21}
$$

This result confirms that the generated FBM field indeed obeys the scaling law stated in Eq. 11.