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A TOPSIS Method For Multi-Attribute Group Decision Making Based on Hesitant Fuzzy Linguistic q-Rung Orthopair Fuzzy Sets and Its Application in Credit Evaluation

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A TOPSIS Method For Multi-Attribute Group Decision Making Based on Hesitant Fuzzy Linguistic *q*-Rung Orthopair Fuzzy Sets and Its Application in Credit Evaluation

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Abstract. Exploring the determination method of membership degree and non-membership degree with higher richness of linguistic elicitation is an important issue in the application research of *q*-rung orthopair fuzzy sets (*q*-ROFSs) in multi-attribute group decision-making (MAGDM). Firstly, the definition of hesitant fuzzy linguistic element normalized score function (HFLENSF) was proposed by using the linguistic scale function which can realize the transformation between qualitative information and quantitative data, and the mapping from hesitant fuzzy linguistic term set (HFLTS) to [0,1] interval was realized. Secondly, based on the HFLENSF, the definition of hesitant fuzzy linguistic *q*-rung orthopair fuzzy set (HFL*q*-ROFS) was then proposed, so that the HFLTS was reasonably introduced into the *q*-ROFS. Thirdly, the HFL*q*-ROFS was applied to MAGDM, and a TOPSIS method for MAGDM based on HFL*q*-ROFS was constructed. Finally, the proposed method was applied to the credit evaluation of the listed companies in strategic emerging industries in China, and taking q=3 as an example, an application example analysis was carried out. The application example analysis results show that the ranking of alternatives obtained by the proposed method is completely consistent with that of the TOPSIS method for MAGDM based on linguistic Fermatean fuzzy set, but the discrimination degree of the former for alternatives is 2.2257, which is higher than 2.0889 of the latter, which proves the feasibility and effectiveness of the proposed method. This study enriches the theoretical framework of *q*-ROFSs and expands the applicability and methodology of *q*-ROFSs in MAGDM.

Keywords: *q*-rung orthopair fuzzy sets; hesitant fuzzy linguistic term set; hesitant fuzzy linguistic element; linguistic scale function; multiple-attribute group decision-making; TOPSIS method; enterprise credit evaluation.

Declarations

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1 Introduction

Orthopair fuzzy sets [1] refer to a class of fuzzy sets in which the membership grades of element $x \in X$ are pairs of

values $\langle \mu(x), v(x) \rangle$ in the unit interval, where $\mu(x)$ denotes the support for membership in the fuzzy set, $\mu(x): X \to [0,1]$; $v(x)$ denotes the support against membership in the fuzzy set, $v(x): X \to [0,1]$. By introducing the boundary condition $0 \le (\mu(x))^q + (v(x))^q \le 1$ ($q \ge 1$) [1] in the orthopair fuzzy sets, then the orthopair fuzzy sets evolve into the *q*-rung orthopair fuzzy sets (*q*-ROFSs). When *q*=1, the *q*-rung orthopair fuzzy sets degenerate into the intuitionistic fuzzy sets [2]; when *q*=2, the *q*-rung orthopair fuzzy sets degenerate into the Pythagorean fuzzy sets [3]; when *q*=3, the *q*-rung orthopair fuzzy sets degenerate into the Fermatean fuzzy sets [4]. With the increase of *q*, the allowable orthopair space also increases, thus giving users more freedom to express their views on the membership grades [1]. Therefore, *q*-rung orthopair fuzzy sets (*q*-ROFSs) have been widely used in multi-attribute group decision-making (MAGDM) [5-22].

However, *q*-rung orthopair fuzzy sets face the challenge of determining membership degree and non-membership degree in practical applications. In recent years, existing literatures typically employed two types of methods—quantitative and qualitative—to characterize membership degree and non-membership degree. Among them, quantitative methods primarily included: expert assignment [1], hesitant fuzzy sets [5], interval values [6-8], incomplete interval values [9], interval 2-tuple [10], continuous function values [11,12], etc.; while qualitative methods mainly included: linguistic terms [13-15], linguistic interval values [16-18], 2-tuple linguistic information [19], probabilistic linguistic terms [20,21], and interval-valued probabilistic linguistic terms [22]. In general, compared with quantitative methods, qualitative methods have advantages in reflecting the fuzziness and hesitation of expert decision-making. However, the existing qualitative methods are still insufficient in the richness of linguistic elicitation. It is well known that the higher the richness of linguistic elicitation is, the more sufficient the opinions expressed by experts are. Therefore, it is an important issue to explore the method of determining membership degree and non-membership degree with high richness of linguistic elicitation in the application research of *q*-rung orthopair fuzzy sets in MAGDM.

In [23], the concept of a hesitant fuzzy linguistic term set (HFLTS) that was based on hesitant fuzzy sets [24] had been presented to manage situations in which experts hesitate between several values to assess an indicator, alternative, variable, etc. The hesitant fuzzy linguistic term set (HFLTS) provides a linguistic and computational basis to increase the richness of linguistic elicitation based on the fuzzy linguistic approach and the use of context-free grammars by using comparative terms [23], and thus provides a good solution for expressing uncertain information [25].

Based on this, some scholars have tried to introduce HFLTS into intuitionistic fuzzy set (IFS), using HFLTS to describe the membership degree and non-membership degree of the element $x \in X$ to the intuitionistic fuzzy set (IFS). Beg and Rashid [26] proposed the concept of hesitant intuitionistic fuzzy linguistic term set (HIFLTS) for the first time, which provides a linguistic and computational basis to manage the situations in which experts assess an alternative in possible linguistic interval and impossible linguistic interval. The follow-up studies were carried out under this theoretical framework. The main research results included: 1) Based on the linguistic scale function, the score function and accuracy function of hesitant intuitionistic fuzzy linguistic element were proposed [27], and the correlation measure and correlation coefficient were established [28]. 2) Several distance measure methods for hesitant intuitionistic fuzzy

linguistic term set were proposed [29-31]. 3) A new aggregation operator for hesitant intuitionistic fuzzy linguistic term set was proposed [32]. 4) The concept of probabilistic hesitant intuitionistic (fuzzy) linguistic term set was proposed by introducing probabilistic linguistic term set [33] and extended to a certain extent [34].

In summary, a series of valuable achievements have been made in existing studies. However, these studies also have certain limitations. For example, although the definition of hesitant intuitionistic fuzzy linguistic term set (HIFLTS) had been proposed, since the hesitant fuzzy linguistic element is a set composed of the elements in the linguistic term set, it does not have the property of real numbers. As a result, the existing studies either cannot define the ranges of membership degree and non-membership degree depicted by hesitant fuzzy linguistic elements, or the defined ranges of membership degree and non-membership degree are inconsistent with the ranges of membership degree and non-membership degree defined by orthopair fuzzy sets, which leads to bias in the defined boundary condition of the HIFLTS.

In view of this, inspired by reference [27], this paper proposes the definition of hesitant fuzzy linguistic element normalized score function (HFLENSF) by using the linguistic scale function that can realize the transformation between qualitative information and quantitative data, and realizes the mapping from hesitant fuzzy linguistic term set (HFLTS) to the [0,1] interval. Furthermore, based on the HFLENSF, the definition of hesitant fuzzy linguistic q-rung orthopair fuzzy set (HFLq-ROFS) is proposed, so that the hesitant fuzzy linguistic term set (HFLTS) is reasonably introduced into the q-rung orthopair fuzzy set (q-ROFS). Based on the HFLq-ROFS, a TOPSIS method for MAGDM is constructed and applied to the credit evaluation of the listed companies in strategic emerging industries in China.

The remaining part of this paper is structured as follows: Section 2 introduces the preparatory knowledge employed in this study. Section 3 presents the definition of hesitant fuzzy linguistic q-rung orthopair fuzzy set (HFLq-ROFS). Section 4 constructs a TOPSIS method for MAGDM based on HFLq-ROFS. Section 5 describes the application example analysis results. Section 6 discusses the results obtained and Section 7 concludes this paper.

2 Preparatory knowledge

This section introduced the related definitions of q-rung orthopair fuzzy set (q-ROFS), the related definitions of hesitant fuzzy linguistic term set (HFLTS) and the definition of linguistic scale function reported in the existing studies. This was the basis of Section 3 and Section 4.

2.1 *Related definitions of q-rung orthopair fuzzy set*

Definition 1 [1]: Let $X=\{x_1,x_2,\cdots,x_n\}$ be a universe of discourse, and a q-rung orthopair fuzzy set Z in X is an object with the following form

$$Z=\{\langle x,\mu_Z(x),\nu_Z(x)\rangle : x\in X\}$$

where $\mu_Z(x):X\to[0,1]$ and $\nu_Z(x):X\to[0,1]$, for all $x\in X$, the following conditions are satisfied

$$0\le(\mu_Z(x))^q+(\nu_Z(x))^q\le 1\ (q\ge 1)$$

the numbers $\mu_Z(x)$ and $\nu_Z(x)$ denote, respectively, the membership degree and the non-membership degree of the element *x* in the set *Z*.

For any *q*-rung orthopair fuzzy set Z and $x \in X$, $\pi_Z(x) = \sqrt[q]{1-(\mu_Z(x))^q-(\nu_Z(x))^q}$ is identified as the degree of indeterminacy of *x* to *Z*.

For simplicity, when there is only one element in *X*, we shall mention the symbol $Z=(\mu_Z,\nu_Z)$ for the *q*-rung orthopair fuzzy set, which is called the *q*-rung orthopair fuzzy number.

Definition 2 [1,4]: Let $Z=(\mu_Z,\nu_Z)$ be a *q*-rung orthopair fuzzy number, then the score function of *Z* can be expressed as

$$\hat{S}(Z)=(\mu_Z)^q-(\nu_Z)^q \tag{1}$$

where $\hat{S}(Z)\in[-1,1]$.

2.2 *Related definitions of hesitant fuzzy linguistic term set*

Definition 3 [23,35]: Let $X=\{x_1,x_2,\cdots,x_n\}$ be a universe of discourse, and $S=\{s_\beta \mid \beta=-\tau,\cdots,-1,0,1,\cdots,\tau\}$ be a linguistic term set. A hesitant fuzzy linguistic term set (HFLTS) in *X* is an object having the form

$$H_S=\{<x_i,h_S(x_i)>\mid x_i\in X\}$$

where $h_S(x_i)$ is the set of elements in *S*, called hesitant fuzzy linguistic element (HFLE), it can be expressed as $h_S(x_i)=\{s_{\eta_l}(x_i)\mid s_{\eta_l}(x_i)\in S, l=1,2,\cdots,L\}$, where $s_{\eta_l}(x_i)$ is the η_l-th element in $h_S(x_i)$, L denotes the number of elements in $h_S(x_i)$.

Definition 4 [35,36]: Let G_H be a context-free grammar, and $S=\{s_\beta \mid \beta=-\tau,\cdots,-1,0,1,\cdots,\tau\}$ be a linguistic term set. The elements of $G_H=(V_N,V_T,I,P)$ are defined as follows:

V_N = {primary term, composite term, unary relation, binary relation, conjunction}; V_T = {"less than", "more than", "at least", "at most", "between", "and", "$s_{-\tau}$", ⋯, "s_{-1}", "s_0", "s_1", ⋯, "s_τ"}; $I\in V_N$; P = { I refers to the primary term or composite term; the primary term refers to "$s_{-\tau}$", ⋯, "s_{-1}", "s_0", "s_1", ⋯, "s_τ"; the composite term refers to unary relation + primary term, or binary relation + primary term + conjunction + primary term; the unary relation refers to "less than" or "more than" or "at least" or "at most"; the binary relation refers to "between"; the conjunction refers to "and"}.

Definition 5 [35,36]: Let $S=\{s_\beta \mid \beta=-\tau,\cdots,-1,0,1,\cdots,\tau\}$ be a linguistic term set. Under the *S*, the linguistic expression generated by G_H is $ll\in S_{ll}$, where S_{ll} is the set of all linguistic expressions. Then the S_{ll} can be transformed into the HFLTS H_S by the transformation function $E_{G_H}: S_{ll}\to H_S$:

(1) $E_{G_H}(s_t)=\{s_t\mid s_t\in S\}$;

(2) $E_{G_H}(\text{at most } s_m)=\{s_t\mid s_t\in S, \text{ and } s_t\le s_m\}$;

(3) $E_{G_H}(\text{less than } s_m)=\{s_t\mid s_t\in S, \text{and } s_t< s_m\}$;

(4) $E_{G_H}(\text{at least } s_m) = \{s_t \mid s_t \in S, \text{and } s_t \geq s_m\}$;

(5) $E_{G_H}(\text{more than } s_m) = \{s_t \mid s_t \in S, \text{and } s_t > s_m\}$;

(6) $E_{G_H}(\text{between } s_m \text{ and } s_n) = \{s_t \mid s_t \in S, \text{and } s_m \leq s_t \leq s_n\}$.

2.3 *Definition of linguistic scale function*

For the element s_i in the linguistic term set $S = \{s_i \mid i = 0,1,2,\cdots,2t\}$, the relationship between s_i and its subscript *i* is strictly monotonically increasing [37]. Based on this, the definition of linguistic scale function is given in [38].

Definition 6 [38]: If $\theta_i \in R^+ (R^+ = \{r \mid r \geq 0, r \in R\})$ is a numerical value, then the linguistic scale function *f* that conducts the mapping from s_i to θ_i ($i = 0,1,2,\cdots,2t$) is defined as follows:

$$f : s_i \to \theta_i (i = 0,1,2,\cdots,2t)$$

where $0 \leq \theta_0 < \theta_1 < \cdots < \theta_{2t}$.

Clearly, function *f* is a strictly monotonically increasing function regarding subscript *i*. The symbols θ_i ($i = 0,1,2,\cdots,2t$) reflect the preference of the decision-makers when they are using the linguistic terms $s_i \in S$ ($i = 0,1,2,\cdots,2t$). Therefore, the function/value in fact denotes the semantics of the linguistic terms.

In [38], three possible choices of linguistic scale functions are listed:

(1) The definition of linguistic scale function based on the subscript function $I(s_i) = i$ is as follows:

$$f_1(s_i) = \theta_i = i / 2t \ (i = 0,1,\cdots,2t) \tag{2}$$

where $\theta_i \in [0,1]$.

The evaluation scale of the linguistic information given above is divided on average. Moreover, its form is simple and commonly used but lacks a reasonable theoretical basis [39].

(2) The definition of linguistic scale function based on [40] and [41] is as follows:

$$f_2(s_i) = \theta_i = \begin{cases} \dfrac{a^t - a^{t-i}}{2a^t - 2}, & i = 0,1,\cdots,t \\ \dfrac{a^t + a^{i-t} - 2}{2a^t - 2}, & i = t+1, t+2,\cdots,2t \end{cases} \tag{3}$$

In this case, with the extension from the middle of the linguistic term set to both ends, the absolute deviation between adjacent linguistic subscripts increases. The value of parameter *a* can be determined using a subjective method. It is generally believed that $a = \sqrt[k]{9}$, where *k* represents the scale level. If *k*=7, then $a = \sqrt[7]{9} \approx 1.37$ [40].

(3) Based on [39], the improved linguistic scale function can be defined as follows:

$$f_3(s_i) = \theta_i = \begin{cases} \dfrac{t^\alpha - (t-i)^\alpha}{2t^\alpha}, & i = 0,1,\cdots,t \\ \dfrac{t^\beta + (i-t)^\beta}{2t^\beta}, & i = t+1, t+2,\cdots,2t \end{cases} \tag{4}$$

where $\alpha, \beta \in [0,1]$.

In this case, with the extension from the middle of the linguistic term set to both ends, the absolute deviation between adjacent linguistic subscripts will decrease. When $2t = 6$, the parameters α and β usually take a value of 0.88 [31].

3 Hesitant fuzzy linguistic *q*-rung orthopair fuzzy set

This section proposed the definition of the hesitant fuzzy linguistic element normalized score function (HFLENSF), and then proposed the definition of hesitant fuzzy linguistic *q*-rung orthopair fuzzy set (HFL*q*-ROFS).

3.1 *Definition of HFLENSF*

Inspired by [27], according to Definitions 3 and 6, Definition 7 is proposed as follows:

Definition 7: Let $X = \{x_1, x_2, \cdots, x_n\}$ be a universe of discourse, $S = \{s_0, s_1, \cdots, s_{2t}\}$ be a linguistic term set, $H_S = \left\{< x_i, h_S(x_i) > \middle| x_i \in X, i = 1, 2, \cdots, n\right\}$ be a hesitant fuzzy linguistic term set on the domain X, where $h_S(x_i)$ is the hesitant fuzzy linguistic element, $h_S(x_i) = \left\{s_{\eta_l}(x_i) \middle| s_{\eta_l}(x_i) \in S, l = 1, 2, \cdots, L(h_S(x_i))\right\}$, $s_{\eta_l}(x_i)$ is the η_l-th element in $h_S(x_i)$, and $L(h_S(x_i))$ denotes the number of elements in $h_S(x_i)$. Then the normalized score function of $h_S(x_i)$ is $\hat{s}: H_S \to [0,1]$, and its expression is

$$\hat{s}(h_S(x_i)) = \sum_{s_{\eta l}(x_i) \in h_S(x_i)} \frac{f(s_{\eta_l}(x_i))}{L(h_S(x_i))} \tag{5}$$

where $\hat{s}(h_S(x_i))$ is the normalized score of $h_S(x_i)$, $\hat{s}(h_S(x_i)) \in [0,1]$; $f(\square)$ is the linguistic scale function [38].

Proof:

It is easy to prove that for the linguistic scale functions shown in Eq. (2), Eq. (3) and Eq. (4), the ranges of θ_i are all $[0,1]$, therefore, we have $f(s_{\eta_l}(x_i)) \in [0,1]$. When there is only one element s_0 in $h_S(x_i)$, $f(s_{\eta_l}(x_i)) = 0$. From Eq. (5), we have $\hat{s}(h_S(x_i)) = 0$. When there is only one element s_{2t} in $h_S(x_i)$, $f(s_{\eta_l}(x_i)) = 1$. From Eq. (5), we have $\hat{s}(h_S(x_i)) = 1$. Therefore, we have $\hat{s}(h_S(x_i)) \in [0,1]$. Q.E.D.

3.2 *Definition of HFLq-ROFS*

According to Definitions 1 and 7, Definition 8 is proposed as follows:

Definition 8: Let $X = \{x_1, x_2, \cdots, x_n\}$ be a universe of discourse, $S = \{s_0, s_1, \cdots, s_{2t}\}$ be a linguistic term set, and a hesitant fuzzy linguistic *q*-rung orthopair fuzzy set $Z_{\hat{s}}$ in X under S is an object with the following form

$$Z_{\hat{s}} = \left\{\left\langle x_i, \hat{s}(h_S^{\mu}(x_i)), \hat{s}(h_S^{\nu}(x_i))\right\rangle : x_i \in X\right\}$$

where $h_S^{\mu}(x_i) \subset S$ is the hesitant fuzzy linguistic element characterizing the membership degree of element x_i to $Z_{\hat{s}}$, it can be expressed as $h_S^{\mu}(x_i) = \left\{s_{\eta_l^{\mu}}(x_i) \middle| s_{\eta_l^{\mu}}(x_i) \in S, l = 1, 2, \cdots, L(h_S^{\mu}(x_i))\right\}$, $s_{\eta_l^{\mu}}(x_i)$ is the η_l^{μ}-th element in $h_S^{\mu}(x_i)$, and $L(h_S^{\mu}(x_i))$ represents the number of elements in $h_S^{\mu}(x_i)$; $\hat{s}(h_S^{\mu}(x_i))$ is the normalized score of the hesitant fuzzy linguistic element characterizing the membership degree of element x_i to $Z_{\hat{s}}$. According to Eq. (5), its expression is

$$\hat{s}(h_S^{\mu}(x_i)) = \sum_{s_{\eta_l^{\mu}}(x_i) \in h_S^{\mu}(x_i)} \frac{f(s_{\eta_l^{\mu}}(x_i))}{L(h_S^{\mu}(x_i))} \tag{6}$$

where $\hat{s}(h_S^{\mu}(x_i)) \in [0,1]$; $f(\square)$ is the linguistic scale function [38].

In addition, $h_S^{\nu}(x_i) \subset S$ is the hesitant fuzzy linguistic element characterizing the non-membership degree of element x_i to $Z_{\hat{s}}$, it can be expressed as $h_S^{\nu}(x_i) = \left\{ s_{\eta_l^{\nu}}(x_i) \middle| s_{\eta_l^{\nu}}(x_i) \in S, l = 1,2,\cdots,L(h_S^{\nu}(x_i)) \right\}$, $s_{\eta_l^{\nu}}(x_i)$ is the η_l^{ν}-th element in $h_S^{\nu}(x_i)$, and $L(h_S^{\nu}(x_i))$ represents the number of elements in $h_S^{\nu}(x_i)$; $\hat{s}(h_S^{\nu}(x_i))$ is the normalized score of the hesitant fuzzy linguistic element characterizing the non-membership degree of element x_i to $Z_{\hat{s}}$. According to Eq. (5), its expression is

$$\hat{s}(h_S^{\nu}(x_i)) = \sum_{s_{\eta_l^{\nu}}(x_i) \in h_S^{\nu}(x_i)} \frac{f(s_{\eta_l^{\nu}}(x_i))}{L(h_S^{\nu}(x_i))} \tag{7}$$

where $\hat{s}(h_S^{\nu}(x_i)) \in [0,1]$; $f(\square)$ is the linguistic scale function [38].

For all $x_i \in X$, $\hat{s}(h_S^{\mu}(x_i))$ and $\hat{s}(h_S^{\nu}(x_i))$ satisfy the boundary condition: $0 \le (\hat{s}(h_S^{\mu}(x_i)))^q + (\hat{s}(h_S^{\nu}(x_i)))^q \le 1$ ($q \ge 1$).

For any $Z_{\hat{s}}$ and $x_i \in X$, $\hat{s}(h_S^{\pi}(x_i)) = \sqrt[q]{1 - (\hat{s}(h_S^{\mu}(x_i)))^q - (\hat{s}(h_S^{\nu}(x_i)))^q}$ is identified as the normalized score of the hesitant fuzzy linguistic element characterizing the indeterminacy degree of x_i to $Z_{\hat{s}}$.

For simplicity, when there is only one element in X, the shorthand $Z_{\hat{s}} = \left\{ \hat{s}(h_S^{\mu}), \hat{s}(h_S^{\nu}) \right\}$ is called the hesitant fuzzy linguistic *q*-rung orthopair fuzzy number.

Obviously, when *q*=1, the HFL*q*-ROFS degenerate into the hesitant fuzzy linguistic intuitionistic fuzzy set (HFLIFS); when *q*=2, the HFL*q*-ROFS degenerate into the hesitant fuzzy linguistic Pythagorean fuzzy set (HFLPFS); when *q*=3, the HFL*q*-ROFS degenerate into the hesitant fuzzy linguistic Fermatean fuzzy set (HFLFFS).

4 A TOPSIS method for MAGDM based on HFL*q*-ROFS

In this section, the HFL*q*-ROFS proposed in Section 3 was used to construct a TOPSIS method for MAGDM.

4.1 *Problem description*

Let $A = (A_1, A_2, \cdots, A_m)$ be a finite set of alternatives A_i ($i = 1,2,\cdots,m$), and $C = (C_1, C_2, \cdots, C_n)$ be a set of attributes C_j ($j = 1,2,\cdots,n$) to compare the alternatives; Let $A^* = \{1,2,\cdots,m\}$ be a set of subscripts of alternatives, and $C^* = \{1,2,\cdots,n\}$ be a set of subscripts of attributes. The weight vector of attributes is $W_C = (w_1, w_2, \cdots, w_n)$, where $w_i \in [0,1]$, and $\sum_{i=1}^{n} w_i = 1$. Let $D = (D_1, D_2, \cdots, D_k)$ be a set of experts, and $D^* = \{1,2,\cdots,k\}$ be a set of subscripts of experts. The weight vector of experts is $W_D = (w_1, w_2, \cdots, w_k)$, where $w_p \in [0,1]$, and $\sum_{p=1}^{k} w_p = 1$.

4.2 *Decision-making steps*

The process of the TOPSIS method for MAGDM based on HFLq-ROFS is shown in Figure 1.

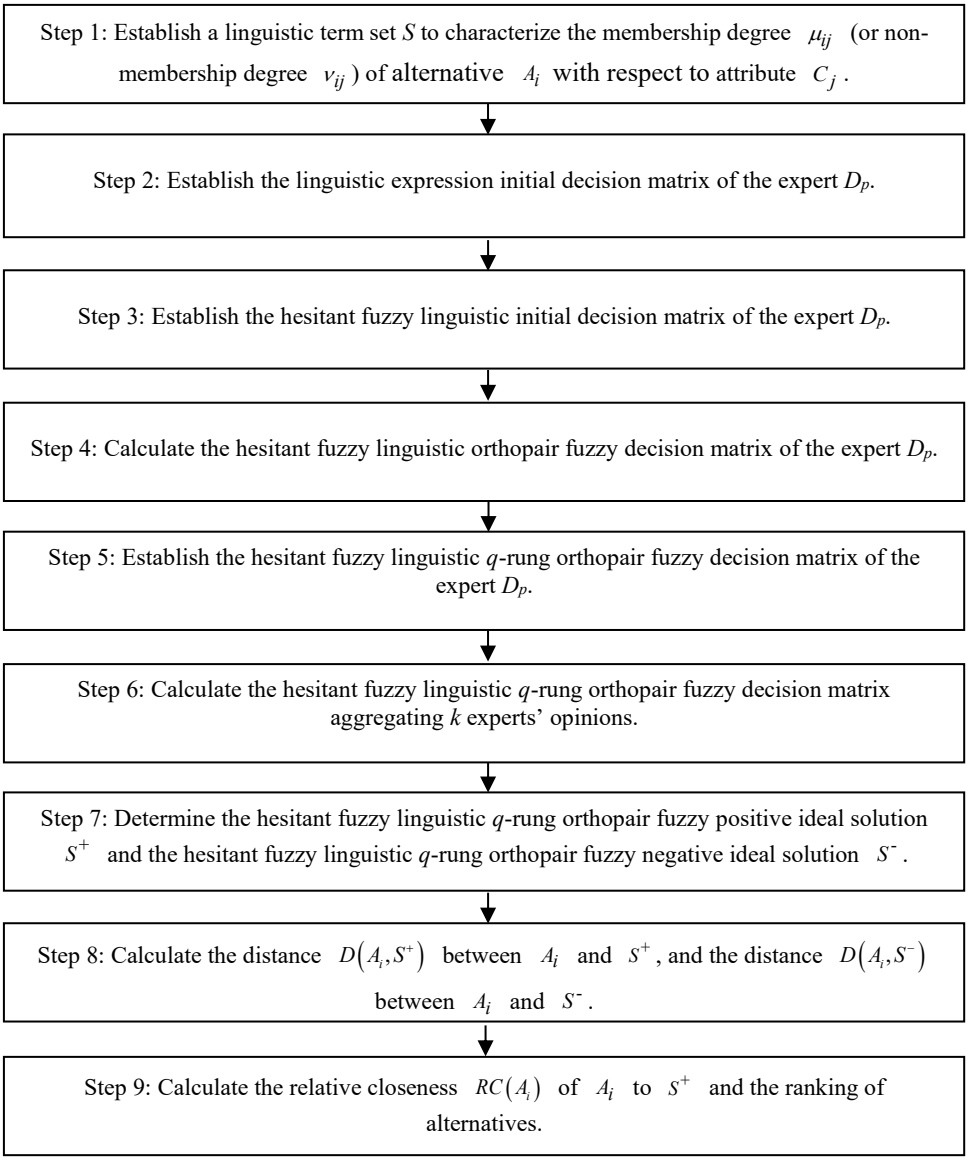


Figure 1. The process of the TOPSIS method for MAGDM based on HFLq-ROFS.

The specific steps of the TOPSIS method for MAGDM based on HFLq-ROFS are as follows:

Step 1: Establish a linguistic term set S to characterize the membership degree μ_{ij} (or non-membership degree ν_{ij}) of alternative A_i with respect to attribute C_j.

Let $S = \{s_\beta \mid \beta = 0,1,\cdots,\tau\}$ be the linguistic term set, where $s_0, s_1, \cdots, s_\tau$ are the linguistic terms that characterize the membership degree μ_{ij} (or non-membership degree ν_{ij}) of alternative A_i with respect to attribute C_j, the larger β is, the higher the membership degree μ_{ij} (or non-membership degree ν_{ij}) of alternative A_i with respect to attribute C_j is; the smaller β is, the lower the membership degree μ_{ij} (or non-membership degree ν_{ij}) of alternative A_i with respect to attribute C_j is. For example, when the linguistic term set S= { s_0 = extremely low, s_1 = very low, s_2 = low, s_3 = medium, s_4 = high, s_5 = very high, s_6 = extremely high}, where the linguistic term s_0 means

that the membership degree μ_{ij} (or non-membership degree ν_{ij}) of alternative A_i with respect to attribute C_j is extremely low, et cetera.

Step 2: Establish the linguistic expression initial decision matrix of the expert D_p.

According to Definition 4 and Definition 5, under the linguistic term set S, the expert D_p employs the linguistic expressions $ll \in S_{ll}$ generated by the context-free grammar G_H (see Definition 4) to evaluate the membership degree μ_{ij} and non-membership degree ν_{ij} of alternative A_i with respect to attribute C_j, and the linguistic expression initial decision matrix of the expert D_p ($p = 1, 2, \cdots, k$) can be obtained as follows:

$$\begin{bmatrix} (ll^{\mu_{11}(p)}, ll^{\nu_{11}(p)}) & (ll^{\mu_{12}(p)}, ll^{\nu_{12}(p)}) & \cdots & (ll^{\mu_{1n}(p)}, ll^{\nu_{1n}(p)}) \\ (ll^{\mu_{21}(p)}, ll^{\nu_{21}(p)}) & (ll^{\mu_{22}(p)}, ll^{\nu_{22}(p)}) & \cdots & (ll^{\mu_{2n}(p)}, ll^{\nu_{2n}(p)}) \\ \vdots & \vdots & \ddots & \vdots \\ (ll^{\mu_{m1}(p)}, ll^{\nu_{m1}(p)}) & (ll^{\mu_{m2}(p)}, ll^{\nu_{m2}(p)}) & \cdots & (ll^{\mu_{mn}(p)}, ll^{\nu_{mn}(p)}) \end{bmatrix}$$

where $ll^{\mu_{ij}(p)}$ is the linguistic expression used by expert D_p to evaluate the membership degree μ_{ij} of alternative A_i with respect to attribute C_j; $ll^{\nu_{ij}(p)}$ is the linguistic expression used by expert D_p to evaluate the non-membership degree ν_{ij} of alternative A_i with respect to attribute C_j.

Step 3: Establish the hesitant fuzzy linguistic initial decision matrix of the expert D_p.

Through the transformation function $E_{G_H} : S_{ll} \to H_S^{ij}$ (see Definition 5), the S_{ll} is further transformed into the hesitant fuzzy linguistic term set H_S^{ij}. Thus, the hesitant fuzzy linguistic initial decision matrix of the expert D_p ($p = 1, 2, \cdots, k$) can be obtained as follows:

$$\begin{bmatrix} (h_S^{\mu_{11}(p)}, h_S^{\nu_{11}(p)}) & (h_S^{\mu_{12}(p)}, h_S^{\nu_{12}(p)}) & \cdots & (h_S^{\mu_{1n}(p)}, h_S^{\nu_{1n}(p)}) \\ (h_S^{\mu_{21}(p)}, h_S^{\nu_{21}(p)}) & (h_S^{\mu_{22}(p)}, h_S^{\nu_{22}(p)}) & \cdots & (h_S^{\mu_{2n}(p)}, h_S^{\nu_{2n}(p)}) \\ \vdots & \vdots & \ddots & \vdots \\ (h_S^{\mu_{m1}(p)}, h_S^{\nu_{m1}(p)}) & (h_S^{\mu_{m2}(p)}, h_S^{\nu_{m2}(p)}) & \cdots & (h_S^{\mu_{mn}(p)}, h_S^{\nu_{mn}(p)}) \end{bmatrix}$$

where $h_S^{\mu_{ij}(p)} = \left\{ s_{\eta_l^{\mu_{ij}(p)}} \middle| s_{\eta_l^{\mu_{ij}(p)}} \in S, l = 1, 2, \cdots, L(h_S^{\mu_{ij}(p)}) \right\}$ is the hesitant fuzzy linguistic element used by expert D_p to evaluate the membership degree μ_{ij} of alternative A_i with respect to attribute C_j, $s_{\eta_l^{\mu_{ij}(p)}}$ is the $\eta_l^{\mu_{ij}(p)}$-th element in $h_S^{\mu_{ij}(p)}$, and $L(h_S^{\mu_{ij}(p)})$ represents the number of elements in $h_S^{\mu_{ij}(p)}$; $h_S^{\nu_{ij}(p)} = \left\{ s_{\eta_l^{\nu_{ij}(p)}} \middle| s_{\eta_l^{\nu_{ij}(p)}} \in S, l = 1, 2, \cdots, L(h_S^{\nu_{ij}(p)}) \right\}$ is the hesitant fuzzy linguistic element used by expert D_p to evaluate the non-membership degree ν_{ij} of alternative A_i with respect to attribute C_j, $s_{\eta_l^{\nu_{ij}(p)}}$ is the $\eta_l^{\nu_{ij}(p)}$-th element in $h_S^{\nu_{ij}(p)}$, and $L(h_S^{\nu_{ij}(p)})$ represents the number of elements in $h_S^{\nu_{ij}(p)}$.

Step 4: Calculate the hesitant fuzzy linguistic orthopair fuzzy decision matrix of the expert D_p.

According to Definition 6, we select Eq. (4) as the specific linguistic scale function. According to Definition 7, from Eq. (5), the normalized score of the hesitant fuzzy linguistic element $h_S^{\mu_{ij}(p)}$ can be calculated as

$$\hat{s}(h_S^{\mu_{ij}(p)}) = \sum_{s_{\eta_l^{\mu_{ij}(p)}} \in h_S^{\mu_{ij}(p)}} \frac{f_3(s_{\eta_l^{\mu_{ij}(p)}})}{L(h_S^{\mu_{ij}(p)})} \tag{8}$$

where $\hat{s}(h_S^{\mu_{ij}(p)}) \in [0,1]$.

From Eq. (5), the normalized score of the hesitant fuzzy linguistic element $h_S^{\nu_{ij}(p)}$ can be calculated as

$$\hat{s}(h_S^{\nu_{ij}(p)}) = \sum_{s_{\eta_l^{\nu_{ij}(p)}} \in h_S^{\nu_{ij}(p)}} \frac{f_3(s_{\eta_l^{\nu_{ij}(p)}})}{L(h_S^{\nu_{ij}(p)})} \tag{9}$$

where $\hat{s}(h_S^{\nu_{ij}(p)}) \in [0,1]$.

Thus, the hesitant fuzzy linguistic orthopair fuzzy decision matrix of the expert D_p ($p = 1,2,\cdots,k$) can be obtained as follows:

$$\begin{bmatrix} (\hat{s}(h_S^{\mu_{11}(p)}),\hat{s}(h_S^{\nu_{11}(p)})) & (\hat{s}(h_S^{\mu_{12}(p)}),\hat{s}(h_S^{\nu_{12}(p)})) & \cdots & (\hat{s}(h_S^{\mu_{1n}(p)}),\hat{s}(h_S^{\nu_{1n}(p)})) \\ (\hat{s}(h_S^{\mu_{21}(p)}),\hat{s}(h_S^{\nu_{21}(p)})) & (\hat{s}(h_S^{\mu_{22}(p)}),\hat{s}(h_S^{\nu_{22}(p)})) & \cdots & (\hat{s}(h_S^{\mu_{2n}(p)}),\hat{s}(h_S^{\nu_{2n}(p)})) \\ \vdots & \vdots & \ddots & \vdots \\ (\hat{s}(h_S^{\mu_{m1}(p)}),\hat{s}(h_S^{\nu_{m1}(p)})) & (\hat{s}(h_S^{\mu_{m2}(p)}),\hat{s}(h_S^{\nu_{m2}(p)})) & \cdots & (\hat{s}(h_S^{\mu_{mn}(p)}),\hat{s}(h_S^{\nu_{mn}(p)})) \end{bmatrix}$$

where $(\hat{s}(h_S^{\mu_{ij}(p)}),\hat{s}(h_S^{\nu_{ij}(p)}))$ is the hesitant fuzzy linguistic orthopair fuzzy number of the expert D_p.

Step 5: Establish the hesitant fuzzy linguistic q-rung orthopair fuzzy decision matrix of the expert D_p.

According to Definition 8, from the hesitant fuzzy linguistic orthopair fuzzy decision matrix of the expert D_p ($p = 1,2,\cdots,k$), we calculate the $(\hat{s}(h_S^{\mu_{ij}(p)}))^q + (\hat{s}(h_S^{\nu_{ij}(p)}))^q$, $i = 1,2,\cdots,m$, $j = 1,2,\cdots,n$.

If $0 \le (\hat{s}(h_S^{\mu_{ij}(p)}))^q + (\hat{s}(h_S^{\nu_{ij}(p)}))^q \le 1$ ($q \ge 1$), then $\hat{s}(h_S^{\mu_{ij}(p)})$ and $\hat{s}(h_S^{\nu_{ij}(p)})$ satisfy the boundary condition of q-rung orthopair fuzzy set, and the hesitant fuzzy linguistic orthopair fuzzy decision matrix of the expert D_p is also the hesitant fuzzy linguistic q-rung orthopair fuzzy decision matrix of the expert D_p.

If $(\hat{s}(h_S^{\mu_{ij}(p)}))^q + (\hat{s}(h_S^{\nu_{ij}(p)}))^q > 1$ ($q \ge 1$), then we repeat Steps 2-4 until $\hat{s}(h_S^{\mu_{ij}(p)})$ and $\hat{s}(h_S^{\nu_{ij}(p)})$ satisfy the boundary condition of q-rung orthopair fuzzy set.

Thus, the hesitant fuzzy linguistic q-rung orthopair fuzzy decision matrix of the expert D_p ($p = 1,2,\cdots,k$) can be obtained as follows:

$$\begin{bmatrix} (\hat{s}(h_S^{\mu_{Z,11}(p)}),\hat{s}(h_S^{\nu_{Z,11}(p)})) & (\hat{s}(h_S^{\mu_{Z,12}(p)}),\hat{s}(h_S^{\nu_{Z,12}(p)})) & \cdots & (\hat{s}(h_S^{\mu_{Z,1n}(p)}),\hat{s}(h_S^{\nu_{Z,1n}(p)})) \\ (\hat{s}(h_S^{\mu_{Z,21}(p)}),\hat{s}(h_S^{\nu_{Z,21}(p)})) & (\hat{s}(h_S^{\mu_{Z,22}(p)}),\hat{s}(h_S^{\nu_{Z,22}(p)})) & \cdots & (\hat{s}(h_S^{\mu_{Z,2n}(p)}),\hat{s}(h_S^{\nu_{Z,2n}(p)})) \\ \vdots & \vdots & \ddots & \vdots \\ (\hat{s}(h_S^{\mu_{Z,m1}(p)}),\hat{s}(h_S^{\nu_{Z,m1}(p)})) & (\hat{s}(h_S^{\mu_{Z,m2}(p)}),\hat{s}(h_S^{\nu_{Z,m2}(p)})) & \cdots & (\hat{s}(h_S^{\mu_{Z,mn}(p)}),\hat{s}(h_S^{\nu_{Z,mn}(p)})) \end{bmatrix}$$

where $(\hat{s}(h_S^{\mu_{Z,ij}(p)}),\hat{s}(h_S^{\nu_{Z,ij}(p)}))$ is the hesitant fuzzy linguistic q-rung orthopair fuzzy number of the expert D_p.

Step 6: Calculate the hesitant fuzzy linguistic q-rung orthopair fuzzy decision matrix aggregating k experts' opinions.

Referring to [1] and [42], we give the definition of hesitant fuzzy linguistic q-rung orthopair fuzzy numbers weighted geometric aggregation operator (HFLq-ROFNWG):

Definition 9: Let $Z_{\hat{s}}^{(p)} = \left\{\hat{s}(h_S^{\mu_Z(p)}),\hat{s}(h_S^{\nu_Z(p)})\right\}$ ($p = 1,2,\cdots,k$) be a group of hesitant fuzzy linguistic q-rung orthopair fuzzy numbers with the weight vector $(w_1,w_2,\cdots,w_k)$, where $w_p \in [0,1]$ and $\sum_{p=1}^{k} w_p = 1$. Let

HFLq-ROFNWG$:\Theta^k \to \Theta$, if

$$\text{HFL}q\text{-ROFNWG}(Z_{\hat{s}}^{(1)}, Z_{\hat{s}}^{(2)}, \cdots, Z_{\hat{s}}^{(k)}) = \left(\prod_{p=1}^{k} (\hat{s}(h_S^{\mu_Z(p)}))^{w_p}, \sqrt[q]{1 - \prod_{p=1}^{k} (1 - (\hat{s}(h_S^{v_Z(p)}))^q)^{w_p}} \right) \tag{10}$$

then HFLq-ROFNWG is called the hesitant fuzzy linguistic q-rung orthopair fuzzy numbers weighted geometric aggregation operator.

Using the HFLq-ROFNWG to aggregate k hesitant fuzzy linguistic q-rung orthopair fuzzy numbers $(\hat{s}(h_S^{\mu_{Z,ij}(p)}), \hat{s}(h_S^{v_{Z,ij}(p)}))$ ($p = 1, 2, \cdots, k$), the hesitant fuzzy linguistic q-rung orthopair fuzzy decision matrix aggregating k experts' opinions can be obtained as follows:

$$\begin{bmatrix} (\hat{s}(h_S^{\mu_{Z,11}}), \hat{s}(h_S^{v_{Z,11}})) & (\hat{s}(h_S^{\mu_{Z,12}}), \hat{s}(h_S^{v_{Z,12}})) & \cdots & (\hat{s}(h_S^{\mu_{Z,1n}}), \hat{s}(h_S^{v_{Z,1n}})) \\ (\hat{s}(h_S^{\mu_{Z,21}}), \hat{s}(h_S^{v_{Z,21}})) & (\hat{s}(h_S^{\mu_{Z,22}}), \hat{s}(h_S^{v_{Z,22}})) & \cdots & (\hat{s}(h_S^{\mu_{Z,2n}}), \hat{s}(h_S^{v_{Z,2n}})) \\ \vdots & \vdots & \ddots & \vdots \\ (\hat{s}(h_S^{\mu_{Z,m1}}), \hat{s}(h_S^{v_{Z,m1}})) & (\hat{s}(h_S^{\mu_{Z,m2}}), \hat{s}(h_S^{v_{Z,m2}})) & \cdots & (\hat{s}(h_S^{\mu_{Z,mn}}), \hat{s}(h_S^{v_{Z,mn}})) \end{bmatrix}$$

where $(\hat{s}(h_S^{\mu_{Z,ij}}), \hat{s}(h_S^{v_{Z,ij}}))$ is the hesitant fuzzy linguistic q-rung orthopair fuzzy number aggregating k experts' opinions.

Step 7: Determine the hesitant fuzzy linguistic q-rung orthopair fuzzy positive ideal solution S^+ and the hesitant fuzzy linguistic q-rung orthopair fuzzy negative ideal solution S^-.

Based on the hesitant fuzzy linguistic q-rung orthopair fuzzy decision matrix aggregating k experts' opinions, according to Definition 2, the score of $(\hat{s}(h_S^{\mu_{Z,ij}}), \hat{s}(h_S^{v_{Z,ij}}))$ is calculated as $\hat{S}(Z_{\hat{s}}^{ij}) = (\hat{s}(h_S^{\mu_{Z,ij}}))^q - (\hat{s}(h_S^{v_{Z,ij}}))^q$. Then the hesitant fuzzy linguistic q-rung orthopair fuzzy positive ideal solution S^+ is

$$\begin{aligned} S^+ &= \begin{cases} \max\limits_i \left\langle \hat{S}(Z_{\hat{s}}^{ij}) \right\rangle | j = 1, 2, \cdots, n, \text{ if } C_j \text{ is a benefit attribute} \\ \min\limits_i \left\langle \hat{S}(Z_{\hat{s}}^{ij}) \right\rangle | j = 1, 2, \cdots, n, \text{ if } C_j \text{ is a cost attribute} \end{cases} \\ &= \left\{ (\hat{s}(h_S^{\mu_Z})_1^+, \hat{s}(h_S^{v_Z})_1^+), (\hat{s}(h_S^{\mu_Z})_2^+, \hat{s}(h_S^{v_Z})_2^+), \cdots, (\hat{s}(h_S^{\mu_Z})_n^+, \hat{s}(h_S^{v_Z})_n^+) \right\} \end{aligned} \tag{11}$$

The hesitant fuzzy linguistic q-rung orthopair fuzzy negative ideal solution S^- is

$$\begin{aligned} S^- &= \begin{cases} \min\limits_i \left\langle \hat{S}(Z_{\hat{s}}^{ij}) \right\rangle | j = 1, 2, \cdots, n, \text{ if } C_j \text{ is a benefit attribute} \\ \max\limits_i \left\langle \hat{S}(Z_{\hat{s}}^{ij}) \right\rangle | j = 1, 2, \cdots, n, \text{ if } C_j \text{ is a cost attribute} \end{cases} \\ &= \left\{ (\hat{s}(h_S^{\mu_Z})_1^-, \hat{s}(h_S^{v_Z})_1^-), (\hat{s}(h_S^{\mu_Z})_2^-, \hat{s}(h_S^{v_Z})_2^-), \cdots, (\hat{s}(h_S^{\mu_Z})_n^-, \hat{s}(h_S^{v_Z})_n^-) \right\} \end{aligned} \tag{12}$$

Step 8: Calculate the distance $D(A_i, S^+)$ between the alternative A_i and the hesitant fuzzy linguistic q-rung orthopair fuzzy positive ideal solution S^+ and the distance $D(A_i, S^-)$ between the alternative A_i and the hesitant fuzzy linguistic q-rung orthopair fuzzy negative ideal solution S^-.

According to [1] and [4], the distance between the alternative A_i and the hesitant fuzzy linguistic q-rung orthopair fuzzy positive ideal solution S^+ is

$$
\begin{aligned}
&D\left(A_i,S^+\right)\\
&=\sum_{j=1}^{n} w_j d\left(C_j\left(A_i\right),C_j\left(S^+\right)\right)\\
&=\sum_{j=1}^{n} w_j \sqrt{\frac{1}{2}\left[\left(\left(\hat{s}(h_S^{\mu_{Z,ij}})\right)^q-\left(\hat{s}(h_S^{\mu_Z})_j^+\right)^q\right)^2+\left(\left(\hat{s}(h_S^{\nu_{Z,ij}})\right)^q-\left(\hat{s}(h_S^{\nu_Z})_j^+\right)^q\right)^2+\left(\left(\hat{s}(h_S^{\pi_{Z,ij}})\right)^q-\left(\hat{s}(h_S^{\pi_Z})_j^+\right)^q\right)^2\right]}\\
&i=1,2,\ldots,m
\end{aligned}
\tag{13}
$$

According to [1] and [4], the distance between the alternative A_i and the hesitant fuzzy linguistic q-rung orthopair fuzzy negative ideal solution S^- is

$$
\begin{aligned}
&D\left(A_i,S^-\right)\\
&=\sum_{j=1}^{n} w_j d\left(C_j\left(A_i\right),C_j\left(S^-\right)\right)\\
&=\sum_{j=1}^{n} w_j \sqrt{\frac{1}{2}\left[\left(\left(\hat{s}(h_S^{\mu_{Z,ij}})\right)^q-\left(\hat{s}(h_S^{\mu_Z})_j^-\right)^q\right)^2+\left(\left(\hat{s}(h_S^{\nu_{Z,ij}})\right)^q-\left(\hat{s}(h_S^{\nu_Z})_j^-\right)^q\right)^2+\left(\left(\hat{s}(h_S^{\pi_{Z,ij}})\right)^q-\left(\hat{s}(h_S^{\pi_Z})_j^-\right)^q\right)^2\right]}\\
&i=1,2,\ldots,m
\end{aligned}
\tag{14}
$$

Step 9: Calculate the relative closeness $RC\left(A_i\right)$ of the alternative A_i to the hesitant fuzzy linguistic q-rung orthopair fuzzy positive ideal solution S^+ and the ranking of alternatives.

According to [1] and [4], the relative closeness of the alternative A_i to the hesitant fuzzy linguistic q-rung orthopair fuzzy positive ideal solution S^+ is

$$
RC\left(A_i\right)=\frac{D\left(A_i,S^-\right)}{D\left(A_i,S^-\right)+D\left(A_i,S^+\right)} \tag{15}
$$

By ranking the relative closeness $RC\left(A_i\right)$ of the alternative A_i ($i=1,2,\cdots,m$) from large to small, the ranking of m alternatives A_i ($i=1,2,\cdots,m$) can be obtained. The larger the relative closeness $RC\left(A_i\right)$ is, the better the alternative A_i is. The smaller the relative closeness $RC\left(A_i\right)$ is, the worse the alternative A_i is.

5 Application Example

In this section, the method proposed in section 4 was applied to the credit evaluation of enterprises in strategic emerging industries in China. Taking q=3 as an example, the feasibility and effectiveness of the proposed method were illustrated, and the applicability of the HFLq-ROFSs in MAGDM was then proved.

5.1 Description of the problem

We randomly selected four sample enterprises of China's Strategic Emerging Industries Comprehensive Index (000891) to form an alternative set $A=\left\{A_1,A_2,A_3,A_4\right\}$, where A_1 was Espressif Systems (Shanghai) Co., Ltd. (688018), this enterprise belonged to the software and information technology services industry; A_2 was Beijing Wantai Biological Pharmacy Enterprise Co., Ltd. (603392), this enterprise belonged to the pharmaceutical manufacturing industry; A_3 was Fuzhou Rockchip Electronics Co., Ltd. (603893), this enterprise belonged to the computer, communication, and other

electronic equipment manufacturing industry; A_4 was Beijing United Information Technology Co., Ltd. (603613) , this enterprise belonged to the internet and related services.

This paper used five financial ratio indicators in the Z-Score model [43] to evaluate the credit status of sample enterprises. The five financial ratio indicators were: current capital / total assets (C_1), retained earnings / total assets (C_2), pre-tax earnings / total assets (C_3), preferred stock and common stock market value / total liabilities (C_4) and sales / total assets (C_5). The above five financial ratio indicators were all benefit-type indicators.

The sample period of this paper was set as 2024. The original data came from China Stock Market & Accounting Research Database (CSMAR). The five financial ratio indicators data were calculated based on the original data, and the calculation results were shown in Table 1.

Table 1. Financial ratio index data.

A_i	C_1	C_2	C_3	C_4	C_5
A_1	0.5577	0.3335	0.1239	51.0638	0.7692
A_2	0.4267	0.5280	0.0006	40.4545	0.1467
A_3	0.6837	0.3744	0.1277	63.8889	0.7209
A_4	0.4833	0.2701	0.1283	2.0879	3.0000

According to the data in Table 1, this paper used the entropy weight method [44] to determine the weights of five financial ratio indicators, and obtained the weight vector of the indicators: W_C = (0.1722, 0.1861, 0.2000, 0.2139, 0.2278). In addition, there were three experts: D_1, D_2 and D_3. This paper used the cyclic mutual evaluation method [45] to determine the weights of the experts, and obtained the weight vector of the experts: $W_D = (0.3976, 0.3012, 0.3012)$ (Note: The detailed calculation process see Appendix A in [46]).

5.2 Process and results of enterprise credit evaluation

Step 1: Established a linguistic term set S to characterize the membership degree μ_{ij} (or non-membership degree ν_{ij}) of alternative A_i with respect to attribute C_j.

According to step 1 in Section 4.2, a linguistic term set S was established to characterize the membership degree μ_{ij} (or non-membership degree ν_{ij}) of alternative A_i with respect to attribute C_j, S= { s_0 = extremely low, s_1 = very low, s_2 = low, s_3 = medium, s_4 = high, s_5 = very high, s_6 = extremely high}, where the linguistic term s_0 meant that the membership degree μ_{ij} (or non-membership degree ν_{ij}) of alternative A_i with respect to attribute C_j was extremely low, et cetera.

Step 2: Established the linguistic expression initial decision matrix of the expert D_p.

According to step 2 in Section 4.2, under the linguistic term set S, the expert D_p ($p = 1, 2, 3$) employed the linguistic expressions $ll \in S_{ll}$ generated by the context-free grammar G_H to evaluate the membership degree μ_{ij} and non-membership degree ν_{ij} of alternative A_i with respect to attribute C_j, and the linguistic expression initial decision matrix of the expert D_p ($p = 1, 2, 3$) was obtained (Note: Due to space limitations, the data is available upon request).

Step 3: Established the hesitant fuzzy linguistic initial decision matrix of the expert D_p.

According to step 3 in Section 4.2, through the transformation function $E_{G_H}: S_{ll} \to H_S^{ij}$ (see Definition 5), the S_{ll} was further transformed into the hesitant fuzzy linguistic term set H_S^{ij}. Thus, the hesitant fuzzy linguistic initial decision matrix of the expert D_p ($p = 1, 2, 3$) was obtained, as shown in Table 2, Table 3 and Table 4 respectively.

Table 2. Hesitant fuzzy linguistic initial decision matrix of the expert D_1.

A_i	C_1	C_2	C_3	C_4	C_5
A_1	$(\{s_4, s_5\}, \{s_1, s_2\})$	$(\{s_1, s_2, s_3\}, \{s_3, s_4\})$	$(\{s_4, s_5\}, \{s_0, s_1\})$	$(\{s_4\}, \{s_1, s_2\})$	$(\{s_3, s_4\}, \{s_2, s_3\})$
A_2	$(\{s_3, s_4\}, \{s_1, s_2, s_3\})$	$(\{s_3, s_4\}, \{s_0, s_1, s_2, s_3\})$	$(\{s_0\}, \{s_5, s_6\})$	$(\{s_3, s_4\}, \{s_2, s_3\})$	$(\{s_0, s_1\}, \{s_5, s_6\})$
A_3	$(\{s_5, s_6\}, \{s_0, s_1\})$	$(\{s_2, s_3\}, \{s_3, s_4\})$	$(\{s_5, s_6\}, \{s_0, s_1\})$	$(\{s_5, s_6\}, \{s_0, s_1\})$	$(\{s_3, s_4\}, \{s_2, s_3\})$
A_4	$(\{s_3, s_4\}, \{s_2\})$	$(\{s_1, s_2\}, \{s_4, s_5, s_6\})$	$(\{s_5, s_6\}, \{s_0, s_1\})$	$(\{s_0, s_1\}, \{s_5, s_6\})$	$(\{s_5, s_6\}, \{s_0\})$

Table 3. Hesitant fuzzy linguistic initial decision matrix of the expert D_2.

A_i	C_1	C_2	C_3	C_4	C_5
A_1	$(\{s_5\}, \{s_1, s_2\})$	$(\{s_1, s_2, s_3\}, \{s_4, s_5, s_6\})$	$(\{s_5\}, \{s_0, s_1, s_2\})$	$(\{s_4\}, \{s_0, s_1, s_2\})$	$(\{s_4\}, \{s_2, s_3\})$
A_2	$(\{s_3, s_4\}, \{s_2, s_3, s_4\})$	$(\{s_3, s_4\}, \{s_2, s_3\})$	$(\{s_0, s_1\}, \{s_5, s_6\})$	$(\{s_3\}, \{s_2, s_3\})$	$(\{s_1\}, \{s_5, s_6\})$
A_3	$(\{s_5, s_6\}, \{s_0, s_1, s_2\})$	$(\{s_2, s_3\}, \{s_4, s_5, s_6\})$	$(\{s_5, s_6\}, \{s_0, s_1\})$	$(\{s_5\}, \{s_0, s_1\})$	$(\{s_3, s_4\}, \{s_2, s_3\})$
A_4	$(\{s_4\}, \{s_2, s_3\})$	$(\{s_1, s_2\}, \{s_4, s_5\})$	$(\{s_5, s_6\}, \{s_0, s_1\})$	$(\{s_0, s_1\}, \{s_5, s_6\})$	$(\{s_5, s_6\}, \{s_0, s_1\})$

Table 4. Hesitant fuzzy linguistic initial decision matrix of the expert D_3.

A_i	C_1	C_2	C_3	C_4	C_5
A_1	$(\{s_4, s_5\}, \{s_1\})$	$(\{s_0, s_1, s_2\}, \{s_3, s_4\})$	$(\{s_4, s_5, s_6\}, \{s_1, s_2\})$	$(\{s_4, s_5\}, \{s_2\})$	$(\{s_3, s_4\}, \{s_2\})$
A_2	$(\{s_3, s_4\}, \{s_2, s_3, s_4\})$	$(\{s_3, s_4, s_5, s_6\}, \{s_2, s_3\})$	$(\{s_0, s_1\}, \{s_5, s_6\})$	$(\{s_3, s_4\}, \{s_3\})$	$(\{s_0, s_1\}, \{s_5\})$
A_3	$(\{s_4, s_5, s_6\}, \{s_0, s_1\})$	$(\{s_0, s_1, s_2, s_3\} \{s_3, s_4\})$	$(\{s_5, s_6\}, \{s_0, s_1\})$	$(\{s_5, s_6\}, \{s_1\})$	$(\{s_3, s_4\}, \{s_2, s_3\})$
A_4	$(\{s_3, s_4\}, \{s_2, s_3\})$	$(\{s_0, s_1\}, \{s_4, s_5\})$	$(\{s_5, s_6\}, \{s_1\})$	$(\{s_1\}, \{s_5, s_6\})$	$(\{s_5, s_6\}, \{s_0\})$

Step 4: Calculated the hesitant fuzzy linguistic orthopair fuzzy decision matrix of the expert D_p.

According to step 4 in Section 4.2, we selected Eq. (4) as the specific linguistic scale function. For the linguistic term set S, there was $2t = 6$, then the values of α and β in Eq. (4) were both 0.88. Thus, the linguistic scale values θ_i ($i = 0, 1, \cdots, 6$) of the seven linguistic terms in S were calculated, as shown in Table 5. Taking i=1 as an example, the linguistic scale value θ_1 was calculated as follows:

$$\theta_1 = \frac{3^{0.88} - (3-1)^{0.88}}{2 \times 3^{0.88}} = 0.15 .$$

Table 5. Linguistic scale values.

i	0	1	2	3	4	5	6
θ_i	0	0.15	0.3097	0.5	0.6901	0.8498	1

Based on Table 2, Table3, Table4 and Table5, the normalized score $\hat{s}(h_S^{\mu_{ij}(p)})$ of hesitant fuzzy linguistic element $h_S^{\mu_{ij}(p)}$ was calculated by using Eq. (8), and the normalized score $\hat{s}(h_S^{\nu_{ij}(p)})$ of hesitant fuzzy linguistic element $h_S^{\nu_{ij}(p)}$ was calculated by using Eq. (9). Thus, the hesitant fuzzy linguistic orthopair fuzzy decision matrix of the expert D_p ($p=1,2,3$) was obtained, as shown in Table 6, Table 7, and Table 8 respectively. Taking $(h_S^{\mu_{11}(1)}, h_S^{\nu_{11}(1)})$ as an example, $\hat{s}(h_S^{\mu_{11}(1)})$ and $\hat{s}(h_S^{\nu_{11}(1)})$ were calculated as follows:

$$\hat{s}(h_S^{\mu_{11}(1)}) = \frac{\theta_4+\theta_5}{2} = \frac{0.6901+0.8498}{2} = 0.7700;$$

$$\hat{s}(h_S^{\nu_{11}(1)}) = \frac{\theta_1+\theta_2}{2} = \frac{0.1500+0.3097}{2} = 0.2299.$$

Therefore, we had $(\hat{s}(h_S^{\mu_{11}(1)}), \hat{s}(h_S^{\nu_{11}(1)})) = (0.7700, 0.2299)$.

Table 6. Hesitant fuzzy linguistic orthopair fuzzy decision matrix of the expert D_1.

A_i	C_1	C_2	C_3	C_4	C_5
A_1	(0.7700, 0.2299)	(0.4049, 0.5951)	(0.7700, 0.0750)	(0.6901, 0.2299)	(0.5951, 0.4049)
A_2	(0.5951, 0.3199)	(0.5951, 0.2399)	(0.0000, 0.9249)	(0.5951, 0.4049)	(0.0750, 0.9249)
A_3	(0.9249, 0.0750)	(0.4049, 0.5951)	(0.9249, 0.0750)	(0.9249, 0.0750)	(0.5951, 0.4049)
A_4	(0.5951, 0.3097)	(0.2299, 0.9249)	(0.9249, 0.0750)	(0.0750, 0.9249)	(0.9249, 0.000)

Table 7. Hesitant fuzzy linguistic orthopair fuzzy decision matrix for expert D_2.

A_i	C_1	C_2	C_3	C_4	C_5
A_1	(0.8498, 0.2299)	(0.3199, 0.8466)	(0.8498, 0.1532)	(0.6901, 0.1532)	(0.6901, 0.4049)
A_2	(0.5951, 0.4999)	(0.5951,0.4049)	(0.0750, 0.9249)	(0.5000, 0.4049)	(0.1500, 0.9249)
A_3	(0.9249, 0.1532)	(0.4049, 0.8466)	(0.9249, 0.0750)	(0.8498, 0.0750)	(0.5951, 0.4049)
A_4	(0.6901, 0.4049)	(0.2299, 0.7700)	(0.9249, 0.0750)	(0.0750, 0.9249)	(0.9249, 0.0750)

Table 8. Hesitant fuzzy linguistic orthopair fuzzy decision matrix for expert D_3.

A_i	C_1	C_2	C_3	C_4	C_5
A_1	(0.7700, 0.1500)	(0.1532, 0.5951)	(0.8466, 0.2299)	(0.7700, 0.3097)	(0.5951, 0.3097)
A_2	(0.5951, 0.4999)	(0.7600, 0.4049)	(0.0750, 0.9249)	(0.5951, 0.5000)	(0.0750, 0.8498)
A_3	(0.8466, 0.0750)	(0.2399, 0.5951)	(0.9249, 0.0750)	(0.9249, 0.1500)	(0.5951, 0.4049)
A_4	(0.5951, 0.4049)	(0.0750, 0.7700)	(0.9249, 0.1500)	(0.1500, 0.9249)	(0.9249, 0.0000)

Step 5: Established the hesitant fuzzy linguistic Fermatean fuzzy decision matrix of the expert D_p.

According to step 5 in Section 4.2, based on Table 6, $(\hat{s}(h_S^{\mu_{ij}(1)}))^3 + (\hat{s}(h_S^{\nu_{ij}(1)}))^3$ ($i=1,2,3,4$; $j=1,2,\cdots,5$) can be calculated. The calculation results showed that:

$$\max_{ij}\left\langle (\hat{s}(h_S^{\mu_{ij}(1)}))^3 + (\hat{s}(h_S^{\nu_{ij}(1)}))^3 \right\rangle = 0.2299^3 + 0.9249^3 = 0.8033 < 1.$$

Thus, $\hat{s}(h_S^{\mu_{ij}(1)})$ and $\hat{s}(h_S^{\nu_{ij}(1)})$ satisfied the boundary condition of Fermatean fuzzy set, and it can be concluded that Table 6 was also the hesitant fuzzy linguistic Fermatean fuzzy decision matrix of the expert D_1.

Based on Table 7, $(\hat{s}(h_S^{\mu_{ij}(2)}))^3 + (\hat{s}(h_S^{\nu_{ij}(2)}))^3$ ($i=1,2,3,4$; $j=1,2,\cdots,5$) can be calculated. The calculation results

showed that:

$$\max_{ij}\left\langle (\hat{s}(h_S^{\mu_{ij}(2)}))^3 + (\hat{s}(h_S^{\nu_{ij}(2)}))^3 \right\rangle = 0.9249^3 + 0.1532^3 = 0.7948 < 1 .$$

Thus, $\hat{s}(h_S^{\mu_{ij}(2)})$ and $\hat{s}(h_S^{\nu_{ij}(2)})$ satisfied the boundary condition of Fermatean fuzzy set, and it can be concluded that Table 7 was also the hesitant fuzzy linguistic Fermatean fuzzy decision matrix of the D_2.

Based on Table 8, $(\hat{s}(h_S^{\mu_{ij}(3)}))^3 + (\hat{s}(h_S^{\nu_{ij}(3)}))^3$ ($i = 1,2,3,4$; $j = 1,2,\cdots,5$) can be calculated. The calculation results showed that:

$$\max_{ij}\left\langle (\hat{s}(h_S^{\mu_{ij}(3)}))^3 + (\hat{s}(h_S^{\nu_{ij}(3)}))^3 \right\rangle = 0.9249^3 + 0.1500^3 = 0.7946 < 1 .$$

Thus, $\hat{s}(h_S^{\mu_{ij}(3)})$ and $\hat{s}(h_S^{\nu_{ij}(3)})$ satisfy the boundary condition of Fermatean fuzzy sets, and it can be concluded that Table 8 is also the hesitant fuzzy linguistic Fermatean fuzzy decision matrix of the D_3.

At this point, the hesitant fuzzy linguistic orthopair fuzzy number $(\hat{s}(h_S^{\mu_{ij}(p)}), \hat{s}(h_S^{\nu_{ij}(p)}))$ was also the hesitant fuzzy linguistic Fermatean fuzzy number $(\hat{s}(h_S^{\mu_{F,ij}(p)}), \hat{s}(h_S^{\nu_{F,ij}(p)}))$.

Step 6: Calculated the hesitant fuzzy linguistic Fermatean fuzzy decision matrix aggregating three experts' opinions.

According to the data in Table 6, Table 7 and Table 8, three hesitant fuzzy linguistic Fermatean fuzzy numbers $(\hat{s}(h_S^{\mu_{F,ij}(p)}), \hat{s}(h_S^{\nu_{F,ij}(p)}))$ ($p = 1,2,3$) were aggregated by using Eq. (10). Thus, the hesitant fuzzy linguistic Fermatean fuzzy decision matrix aggregating three experts' opinions can be obtained, as shown in Table 9. Taking $(\hat{s}(h_S^{\mu_{F,11}(p)}), \hat{s}(h_S^{\nu_{F,11}(p)}))$ as an example, $\hat{s}(h_S^{\mu_{F,11}})$ and $\hat{s}(h_S^{\nu_{F,11}})$ were calculated as follows:

$$\hat{s}(h_S^{\mu_{F,11}}) = 0.7700^{0.3976} \times 0.8498^{0.3012} \times 0.7700^{0.3012} = 0.7932 ;$$

$$\hat{s}(h_S^{\nu_{F,11}}) = \sqrt[3]{1-(1-0.2299^3)^{0.3976} \times (1-0.2299^3)^{0.3012} \times (1-0.1500^3)^{0.3012}} = 0.2119 .$$

Therefore, we had $(\hat{s}(h_S^{\mu_{F,11}}), \hat{s}(h_S^{\nu_{F,11}})) = (0.7932, 0.2119)$.

Table 9. Hesitant fuzzy linguistic Fermatean fuzzy decision matrix aggregating three experts' opinions.

A_i	C_1	C_2	C_3	C_4	C_5
A_1	(0.7932, 0.2119)	(0.2814, 0.7115)	(0.8162, 0.1701)	(0.7133, 0.2462)	(0.6222, 0.3814)
A_2	(0.5951, 0.4471)	(0.6406, 0.3578)	(0.0000, 0.9249)	(0.5647, 0.4387)	(0.0000, 0.9080)
A_3	(0.9006, 0.1113)	(0.3458, 0.7115)	(0.9249, 0.0750)	(0.9016, 0.1095)	(0.5951, 0.4049)
A_4	(0.6222, 0.3731)	(0.1641, 0.8566)	(0.9249, 0.1095)	(0.0924, 0.9249)	(0.9249, 0.0503)

Step 7: Determined the hesitant fuzzy linguistic Fermatean fuzzy positive ideal solution S^+ and the hesitant fuzzy linguistic Fermatean fuzzy negative ideal solution S^-.

According to the data in Table 9, using Definition 2, the score $\hat{S}(F_{\hat{s}}^{ij}) = (\hat{s}(h_S^{\mu_{F,ij}}))^3 - (\hat{s}(h_S^{\nu_{F,ij}}))^3$ of $(\hat{s}(h_S^{\mu_{F,ij}}), \hat{s}(h_S^{\nu_{F,ij}}))$ was calculated. The results of the calculation were shown in Table 10. Taking $(\hat{s}(h_S^{\mu_{F,11}}), \hat{s}(h_S^{\nu_{F,11}}))$ as an example, the score of $(\hat{s}(h_S^{\mu_{F,11}}), \hat{s}(h_S^{\nu_{F,11}}))$ was calculated as follows:

$$\hat{S}(F_{\hat{s}}^{11}) = (\hat{s}(h_S^{\mu_{F,11}}))^3 - (\hat{s}(h_S^{\nu_{F,11}}))^3 = 0.7932^3 - 0.2119^3 = 0.4895 .$$

Table 10. The scores of $(\hat{s}(h_S^{\mu_{F,ij}}), \hat{s}(h_S^{\nu_{F,ij}}))$.

A_i	C_1	C_2	C_3	C_4	C_5
A_1	0.4895	-0.3379	0.5388	0.3480	0.1854
A_2	0.1214	0.2171	-0.7912	0.0956	-0.7486
A_3	0.7291	-0.3188	0.7908	0.7316	0.1444
A_4	0.1889	-0.6241	0.7899	-0.7904	0.7911

According to the data in Table 10, from Eq. (11), the hesitant fuzzy linguistic Fermatean fuzzy positive ideal solution was S^+ = {(0.9006, 0.1113), (0.6406, 0.3578), (0.9249, 0.0750), (0.9016, 0.1095), (0.9249, 0.0503)}; from Eq. (12), the hesitant fuzzy linguistic Fermatean fuzzy negative ideal solution was S^- = {(0.5951, 0.4471), (0.1641, 0.8566), (0.0000, 0.9249), (0.0924, 0.9249), (0.0000, 0.9080)}.

Step 8: Calculated the distance $D\left(A_i, S^+\right)$ between the alternative A_i and the hesitant fuzzy linguistic Fermatean fuzzy positive ideal solution S^+, and the distance $D\left(A_i, S^-\right)$ between the alternative A_i and the hesitant fuzzy linguistic Fermatean fuzzy negative ideal solution S^-.

From Eq. (13), the distance $D\left(A_i, S^+\right)$ between the alternative A_i and the hesitant fuzzy linguistic Fermatean fuzzy positive ideal solution S^+ was calculated. The results were shown in Table 11. From Eq. (14), the distance between the alternative A_i and the hesitant fuzzy linguistic Fermatean fuzzy negative ideal solution S^- was calculated. The results were shown in Table 11.

Step 9: Calculated the relative closeness $RC\left(A_i\right)$ of the alternative A_i to the hesitant fuzzy linguistic Fermatean fuzzy positive ideal solution S^+ and the ranking of alternatives.

From Eq. (15), the relative closeness $RC\left(A_i\right)$ of the alternative A_i to the hesitant fuzzy linguistic Fermatean fuzzy positive ideal solution S^+ was calculated. The results were shown in Table 11.

Table 11. Decision results.

A_i	$D\left(A_i, S^+\right)$	$D\left(A_i, S^-\right)$	$RC\left(A_i\right)$	**Ranking**
A_1	0.3385	0.5150	0.6034	2
A_2	0.5272	0.2302	0.3040	4
A_3	0.1774	0.5890	0.7685	1
A_4	0.3377	0.3396	0.5014	3

As shown in Table 11, according to the size of relative closeness $RC\left(A_i\right)$, the ranking of four alternatives is $A_3 \succ A_1 \succ A_4 \succ A_2$. This indicates that the credit status of alternative A_3 is the best and the credit status of alternative A_2 is the worst.

5.3 Comparative analysis

To illustrate the feasibility and effectiveness of the TOPSIS method for MAGDM based on hesitant fuzzy linguistic Fermatean fuzzy set (hereinafter referred to as Method 1) proposed in this paper, the decision results shown in Table 11 were compared with the decision results of the TOPSIS method for MAGDM based on linguistic Fermatean fuzzy set (hereinafter referred to as Method 2). When other conditions remain unchanged, the linguistic Fermatean fuzzy decision

matrix aggregating three experts' opinions obtained by Method 2 was shown in Table 12 (Note: In this paper, linguistic Fermatean fuzzy set is regarded as a special case of hesitant fuzzy linguistic Fermatean fuzzy set). The decision results obtained by Method 2 were shown in Table 13 (Note: Due to space limitations, the calculation process is available upon request).

Table 12. Linguistic Fermatean fuzzy decision matrix aggregating three experts' opinions.

A_i	C_1	C_2	C_3	C_4	C_5
A_1	(0.7469, 0.2501)	(0.2677, 0.7500)	(0.7900, 0.2003)	(0.6900, 0.3002)	(0.6000, 0.4100)
A_2	(0.6300, 0.4275)	(0.6720, 0.3420)	(0.0000, 0.8741)	(0.6089, 0.3992)	(0.0000, 0.8646)
A_3	(0.8730, 0.1134)	(0.3398, 0.7178)	(0.8924, 0.0824)	(0.8730, 0.1134)	(0.5870, 0.4202)
A_4	(0.6305, 0.3708)	(0.1648, 0.8343)	(0.8827, 0.1032)	(0.0927, 0.8925)	(0.8924, 0.0515)

Table 13. Decision results (Method 2).

A_i	$D(A_i, S^+)$	$D(A_i, S^-)$	$RC(A_i)$	**Ranking**
A_1	0.3236	0.4141	0.5613	2
A_2	0.4471	0.2091	0.3186	4
A_3	0.1647	0.5020	0.7530	1
A_4	0.3066	0.2952	0.4905	3

As shown in Table 13, according to the size of relative closeness $RC(A_i)$, the ranking of four alternatives is $A_3 \succ A_1 \succ A_4 \succ A_2$. It can be seen that the ranking result of alternatives obtained by Method 1 is completely consistent with the ranking result of alternatives obtained by Method 2, indicating the feasibility of Method 1.

We further investigated the discrimination of Method 1 for alternatives, as well as Method 2. Let the ranking of m alternatives A_i ($i = 1, 2, \cdots, m$) be $A_{(1)} \succ A_{(2)} \succ \cdots \succ A_{(m)}$, where (i) is the subscript of the alternatives ranked by $RC(A_{(1)}) > RC(A_{(2)}) > \cdots > RC(A_{(m)})$. According to the discrimination algorithm given in [47], the discrimination calculation formula of Method 1 (or Method 2) for alternatives is as follows:

$$\rho = \sum_{i=2}^{m} \frac{RC(A_{(1)}) - RC(A_{(i)})}{RC(A_{(1)})} + \sum_{i=3}^{m} \frac{RC(A_{(2)}) - RC(A_{(i)})}{RC(A_{(2)})} + \cdots + \frac{RC(A_{(m-1)}) - RC(A_{(i)})}{RC(A_{(m-1)})} \tag{16}$$

From Eq. (16), the discrimination of Method 1 for alternatives was calculated as $\rho_1 = 2.2257$. Similarly, the discrimination of Method 2 for alternatives was calculated as $\rho_2 = 2.0889$.

It can be seen that although the ranking results of the alternatives obtained by Method 1 and Method 2 are completely consistent, the discrimination of Method 1 for alternatives is higher than that of Method 2, which means that the decision results obtained by Method 1 are better than those by Method 2, indicating the effectiveness of Method 1.

6 Discussion

From the comparative analysis results in Section 5.3, it can be seen that in the case that the ranking results of the alternatives obtained by Method 1 and Method 2 were completely consistent, the discrimination of Method 1 for

alternatives was 2.2257, which was higher than 2.0889 of Method 2. This is because, from the calculation of the data in Table 11 and Table 13, the standard deviation of the relative closeness $RC\left(A_i\right)$ obtained by Method 1 is 0.1944, which is higher than 0.1798 of Method 2. According to Eq. (15), the specific reason is that the standard deviation of $D\left(A_i,S^+\right)/D\left(A_i,S^-\right)$ obtained by Method 1 is 0.8671, which is higher than 0.7696 of Method 2. This transmission mechanism is as follows:

$$\sigma_{D(A_i,S^+)/D(A_i,S^-)}\uparrow\Rightarrow\sigma_{RC(A_i)}\uparrow\Rightarrow\rho\uparrow$$

where $\sigma_{D(A_i,S^+)/D(A_i,S^-)}$ and $\sigma_{RC(A_i)}$ represents the standard deviation of $D\left(A_i,S^+\right)/D\left(A_i,S^-\right)$ and $RC\left(A_i\right)$, respectively, ρ represents the discrimination.

Obviously, the standard deviation of $D\left(A_i,S^+\right)/D\left(A_i,S^-\right)$ is closely related to the standard deviations of $D\left(A_i,S^+\right)$ and $D\left(A_i,S^-\right)$. Furthermore, according to Eq. (13) and Eq. (14), the standard deviations of $D\left(A_i,S^+\right)$ and $D\left(A_i,S^-\right)$ obtained by Method 1 are closely related to the elements in Table 9; similarly, the standard deviations of $D\left(A_i,S^+\right)$ and $D\left(A_i,S^-\right)$ obtained by Method 2 are closely related to the elements in Table 12.

In Method 1, the basis of Table 9 is the linguistic expression initial decision matrix of the expert D_p ($p=1,2,3$). This matrix is derived from, under the linguistic term set S, the expert D_p employs the linguistic expressions $ll\in S_{ll}$ generated by the context-free grammar G_H to evaluate the membership degree μ_{ij} and non-membership degree ν_{ij} of alternative A_i with respect to attribute C_j. However, in Method 2, the basis of Table 12 is the linguistic term initial decision matrix of the expert D_p ($p=1,2,3$). This matrix is derived from the expert D_p employs the linguistic term in S to evaluate the membership degree μ_{ij} and non-membership degree ν_{ij} of alternative A_i with respect to attribute C_j.

Method 2 has 7 linguistic terms (see Step 1 in Section 5.2), and Method 1 has 12 linguistic expressions (see Definition 5). It can be seen that compared with Method 2, Method 1 has a higher richness of linguistic elicitation. This may be reflected in: compared with the elements in Table 12, the elements in Table 9 are generally different in the standard deviations of membership degree, non-membership degree and indeterminacy degree. According to the calculation of the data in Table 9 and Table 12, the standard deviations of membership, non-membership and indeterminacy degree of all elements in Table 9 are 0.3176, 0.3075 and 0.1286, respectively, which are slightly higher than 0.3068, 0.2933 and 0.0932 in Table 12.

7 Conclusions

The marginal contributions of this work may be as follows: 1) By using the linguistic scale function which can realize the transformation between qualitative information and quantitative data, the definition of hesitant fuzzy linguistic element normalized score function (HFLENSF) was proposed and the mapping from hesitant fuzzy linguistic term set (HFLTS) to [0,1] interval was realized. Based on the HFLENSF, the definition of hesitant fuzzy linguistic q-rung orthopair fuzzy set (HFLq-ROFS) was then proposed. As a result, the hesitant fuzzy linguistic term set (HFLTS) was

reasonably introduced into the q-rung orthopair fuzzy set (q-ROFS). 2) The HFLq-ROFS was applied to MAGDM, and a TOPSIS method for MAGDM based on HFLq-ROFS was constructed. 3) The TOPSIS method for MAGDM based on HFLq-ROFS was applied to the credit evaluation of the listed companies in strategic emerging industries in China. Taking q=3 as an example, the application example analysis results showed the feasibility and effectiveness of the proposed method, and proved the applicability of HFLq-ROFSs in MAGDM.

The advantages of this work may be as follows: 1) Compared with [1,5-22], this paper introduced hesitant fuzzy linguistic term set (HFLTS) into q-rung orthopair fuzzy set (q-ROFS), using hesitant fuzzy linguistic term set to describe the membership degree and non-membership degree of the element $x_i \in X$ to q-rung orthopair fuzzy set, which improved the richness of linguistic elicitation, so it was more reasonable. 2) Compared with [26], this paper theoretically defined the ranges of membership degree and non-membership degree characterized by hesitant fuzzy linguistic elements, so it was more scientific. 3) Compared with [27], the ranges of membership degree and non-membership degree depicted by hesitant fuzzy linguistic elements defined by this paper were consistent with the ranges of membership degree and non-membership degree defined by orthopair fuzzy set, so it was more reasonable. 4) Compared with [26-34], this paper extended the application field of hesitant fuzzy linguistic term set from intuitionistic fuzzy sets to q-rung orthopair fuzzy sets, so it was more universal.

The academic value of this work may be reflected as follows: 1) The definition of hesitant fuzzy linguistic q-rung orthopair fuzzy set (HFLq-ROFS) proposed in this paper provided a new method and idea for exploring the determination method of membership degree and non-membership degree with higher richness of linguistic elicitation, it had high theoretical value. 2) The TOPSIS method for MAGDM based on HFLq-ROFS proposed in this paper can be widely applied to MAGDM problems, such as credit evaluation, ESG evaluation, digital inclusive finance evaluation, ecological environment quality evaluation and so on, it had high practical application value and broad application prospects. In summary, this work enriched the theoretical framework of q-rung orthopair fuzzy sets and expanded the applicability and methodology of q-rung orthopair fuzzy sets in MAGDM.

In this paper, the linguistic term set established in Section 5.2 to characterize the membership degree (or non-membership degree) of alternative A_i with respect to attribute C_j contained only seven linguistic terms, and there was still room for improvement in the richness of linguistic elicitation. In the future, the application example analysis will consider adding elements in the linguistic term set to improve the richness of linguistic elicitation. Furthermore, only three experts were set in the application example analysis, and the number was small. In the future, the number of experts will be increased to improve the stability of MAGDM and the persuasiveness of the application example analysis results. At the same time, this paper used the cyclic mutual evaluation method to determine the weights of experts in the application example analysis, which had certain limitations. The future application example analysis will consider introducing more advanced methods [48,49] to determine the weights of experts to improve the rationality of the weights of experts. In addition, Method 1 should be compared with other methods except Method 2, such as linguistic interval

values [16-18], 2-tuple linguistic information [19], probabilistic linguistic terms [20,21], and interval-valued probabilistic linguistic terms [22], etc. This is one of the shortcomings of this work and needs to be improved in the future.

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