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Novel Bias Drift Mechanism of Whole-Angle Mode Gyroscopes: Oscillatory Rectification Errors

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Abstract This work originally identifies a novel error mechanism, Oscillatory Rectification Errors(ORE), of the whole-angle mode gyroscope (WAMG) under oscillatory rotation and conducts important insights for WAMG based inertial navigation applications. Conventionally, WAMG is widely recognized as an ideal solution for high-dynamic measurement, but this study reveals significant inherent measurement risks in such scenarios with periodic motions.

The dominant dynamic equation is firstly solved by using equivalent transformation and superposition principle. The conducted analytical solution clearly proves that the output of WAMG will drift to 3 attraction regions under oscillatory angular rates. There is a complex relation between the final bias state with its initial states, asymmetry parameters, and input signals, but the overall insight is that larger amplitude and lower frequency of oscillatory rotation, larger frequency difference, higher damping anisotropy degree, and less vibration energy will accelerate the drift process.

This ORE is fatal for long term inertial navigation applications: the measurement outputs of WAMG will drift to certain bias attraction points under periodic motions. To intuitively demonstrate this phenomenon, a typical high precision hemispherical resonator gyro can drift 5.8156 deg in 60 minutes, which is a severe hazard in navigation missions.

Keywords Asymmetry Dynamics · Inertial Sensing · Resonant Gyroscopes · Whole-Angle Mode

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1 Introduction

1.1 Motivation

Gyroscopes play an essential role in attitude control and navigation applications, since they can accurately measure the rate/angle status of their carriers. Among the emerging gyroscopic technologies, the Whole-Angle Mode Gyroscope (WAMG) stands out for its superior high-dynamic and wide band measurement characteristics, which have been widely acknowledged in the field[1]. Compared to its conventional Force-to-Rebalance (FTR) mode, WAMG overcomes the limitation restricted angular rate range ($\leq 10^\circ/\text{s}$) and low closed-loop bandwidth (no more than 20 Hz)[2,3], which cannot satisfy the demands of high-frequency dynamic applications. The WAMG is theoretically endowed with an "unlimited" measurement range because its internal dynamics is running as a free stand wave. The experimental results also proved a linear working range larger than $\geq \pm 5000^\circ/\text{s}$ while maintaining a bandwidth over 100 Hz[4,5]. This enables the whole-angle mode gyroscope to exhibit superior performance in terms of dynamic response range and anti-interference ability[6]. Therefore, these advantages also make it particularly suitable for fields with significant high-frequency dynamic and complex scenarios, such as aerospace navigation and robotics [7].

However, the comprehensive performance of the WAMG system is not fully evaluated yet. The prior works are focused on constant rate input conditions to test scale factor linearity, static output stability and constant measurement range, etc. Meanwhile, the dynamics of WAMG under oscillatory operating conditions remain as an open issue. In practice, the motions of gyro's carrier can be very complex which is combined with constant and oscillatory patterns. The barrier is that the WAGMs have complex nonlinear terms in their dynamics equations. Scientists either analyze the major WAGM equation in an indirect way [8] [9], or utilize numerical methods to achieve quantified results[10]. A hypothesis is that oscillatory rate inputs will lead to harmonics, just like its conventional FTR mode [11] [12] [13]. None of these efforts give a consistent conclusion about the actual performance of WAGM with complex dynamic inputs.

It is no doubt that the most straight forward way would be solving the major WAGM equation. This work firstly conducts a set of analytical solution for nonlinear WAGM dynamics and overturn the hypothesis before: there are no harmonics but another bias drift in the oscillatory motions. This induced drift is defined as oscillatory rectification errors (ORE). It can be considered as a dual problem to the vibration rectification errors in accelerometers, whom are also important components in inertial navigation systems [14]. Nevertheless, it is the first time to report this type of error in WAMG sensors.

1.2 Related Work

It is necessary to review the related works to support the novelty of this work. The ultimate research goal is improving the WAGM's measurement precision to support high-end navigation systems. Thus the relevant literature is distributed from manufacturing, design, dynamics to interface systems.

The first clue is investigating the asymmetry dynamics WAMGs to gain fundamental knowledge of WAMGs' error mechanism. Zega et al. used the implicit static condensation method to build a nonlinear reduced-order model, establishing the first analytical link between resonator frequency splitting and angle random walk (ARW) and suppressing 4th-order harmonic errors by 99.2% [15]. Gill et al. developed a comprehensive motion equation for ring gyro dynamic behavior under vibration and achieved $0.05^\circ/\text{s}$ -level bias error prediction within $\pm 50^\circ/\text{s}$ dynamic range [16]. However, these quasi-static (dynamic frequency $\ll 10$ Hz) models neglect the nonlinear interference coupling that occurs at hundreds of hertz of dynamic frequency. Park et al. proposed a composite control algorithm for MEMS angle-measuring gyroscopes by integrating dynamic analysis and error compensation, but lacked high-dynamic condition analysis and compensation validation [17]. Yu et al. enhanced electrostatic drive gain nonlinearity, suppressing the 4th-component angle-dependent bias by over 95% to $\ll 0.0001^\circ/\text{s}$, with their error model lacking long-term stability verification [18]. Though existing works made impressive progresses about WAMG dynamics, these are limited to constant rate situations while the motion is mixed with constant and oscillatory signals in real navigation applications.

Since the device asymmetry is the major error contributor, dynamic system level calibration/compensation is an essential topic other than the precision manufacturing approach. David Horsley et al. employed anisotropic damping compensation to lower the measurement threshold to $1^\circ/\text{s}$ [19], and stabilized the Allan deviation at $2^\circ/\text{h}$ using continuous-time mode reversal [20]. Li et al. further proposed a gain mismatch identification method based on mode reversal, compressing angular bias error to 1.124° through drive/detection phase alternation [21]. Huang et al. proposed the amplitude-phase compensation mode matching method, which achieves a tuning accuracy of 0.06 Hz but suffers from identification divergence under 500 Hz sinusoidal dithering [22]. Wu et al. presented a real-time capacitive displacement calibration method, reducing the scale factor nonlinearity to 0.79 ppm [23]. However, harmonic distortion above 1 kHz vibration causes compensation failure, with the nonlinearity increasing to 12 ppm [24]. Thus, the core bottleneck in this direction is that existing control laws ignore phase lag and amplitude attenuation from high dynamic rotation, resulting in closed-loop system instability under dynamic conditions.

Another notable research topic is the virtual electrical rotation technology for WAMGs. This method utilizes an artificial electrostatic force to generate equivalent rotation signals, which could be used to eliminate the dead-zone problem or as an excitation source for system identification. Luo et al. pro-

posed an asynchronous virtual rotation scheme that adopted a dual-gyro configuration for online rate estimation, boosting precession angle measurement SNR by 12.28 times [25]. Wang et al. presented a sinusoidal-injection self-calibration method that shortened the temperature-dependent stabilization time to 10 minutes [26]. However, these technologies optimized for static calibration are ill-suited for high-frequency multi-interference conditions. Li et al. presented a virtual Coriolis force mode matching method, achieving a room-temperature bias instability of $2.8331^\circ/\text{h}$ [27] yet failing above 1.5 kHz owing to virtual force instability and control breakdown. Meng et al. proposed an online estimation method for asymmetry parameters based on high-frequency injection, attaining a tracking accuracy of the 10^{-3} order but lacking practical experimental validation [10]. The most famous work in this area would be canceling the measurement dead-zone by adding a virtual constant rate [9]. It claims that WAMGs have dead-zones for small rate signals but are ideal for large rotation motions. This concept has been widely accepted in the navigation community, which is over-turned by this work. The reminder of this paper will prove that there are unique error mechanism for WAMGs under more complex situations.

In summary, current research are concentrated on constant rate conditions and control methods, whereas systematic studies on periodic measurements remain open. These unknowns limits the understanding of WAMG error mechanisms in vibration-intensive real dynamic environments, weakens model predictive performance, and raises significant risks for high performance navigation applications.

1.3 Contributions and Novelty

To address the limitations of existing research, this paper originally solved the nonlinear WAMG dynamic equation to make solid conclusions about the characteristics under oscillatory rates. The specific research methodology is outlined as follows: Firstly, a mathematical model of WAMG incorporating high-dynamic rotational signals is established, and a reduced-order expression for the measurement output of WAMG is derived. Secondly, parameterized simulation experiments are designed and implemented to systematically analyze the influence of different inherent parameter configurations and dynamic conditions on the response trajectory of the gyroscope's measurement output. Finally, based on the aforementioned research, a novel physical phenomenon exhibited by WAMG under high-frequency dynamic scenarios is identified and experimentally verified. The primary contributions of this paper are summarized as follows:

- (1) An analytical solution is first developed for the WAMG dynamics of asymmetry, which accurately characterizes the measurement output under the inputs of the oscillatory rate.

(2) The novel ORE mechanism is originally claimed, which turned over the previous harmonic based hypothesis. The factors of ultimate bias are also developed in a consistent way.

(3) A comprehensive case study is performed to quantify the ORE and asymmetry parameters. It is an important prior knowledge for navigation engineers who decide to utilize WAMG sensors.

2 The definition of Oscillatory Rectification Error of WAMGs

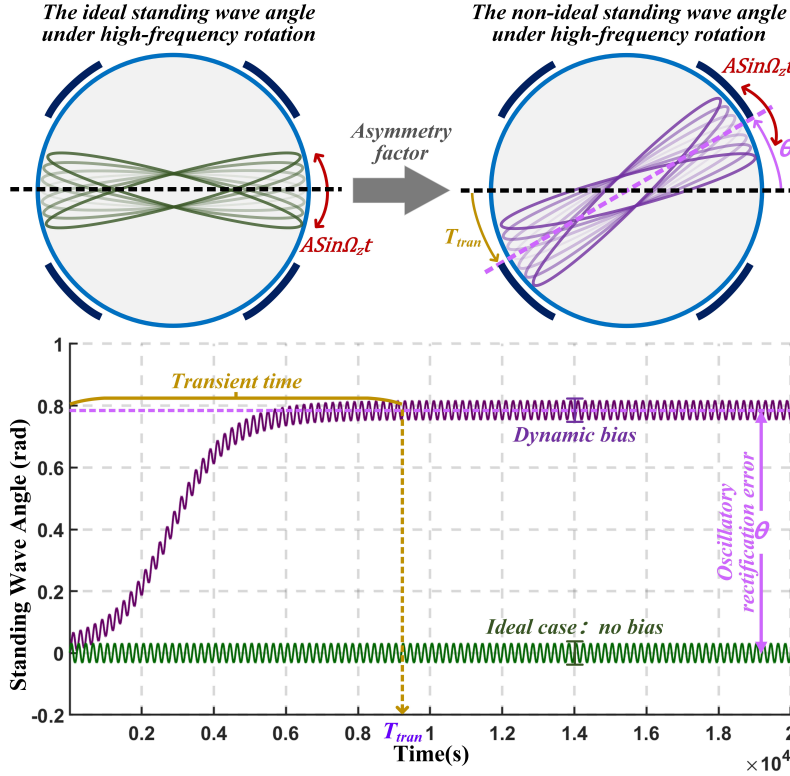


Fig. 1: The ORE of an asymmetry WAMG under oscillatory rotation inputs.

The ORE in whole-angle mode gyroscopes is defined as the oscillatory/periodic rotation induced bias drift. Drawing on the inherent characteristic of WAMG to directly output rotational angles, the measurement output is taken as the core research object. Ideally, the measurement output of a WAMG is expected to oscillate around its initial position with a constant central amplitude when subjected to alternating rotation. However, mathematical derivations and experimental results presented in this paper demonstrate that non-ideal con-

ditions—including damping asymmetry, stiffness asymmetry, and quadrature errors—induce additional dynamic behaviors. Specifically, the measurement output not only oscillates but also exhibits a time-dependent central amplitude drift until dynamic equilibrium is achieved. To quantitatively characterize this phenomenon, three key parameters are defined: transition time (duration for the central amplitude to reach dynamic stability), ORE (oscillatory rectification error, gyro bias drift induced by rectification of oscillatory/periodic rotation), and dynamic bias range (fluctuation interval of the measurement output relative to ORE). A comparative illustration of the measurement output under ideal and non-ideal conditions is provided in Fig.1.

3 Dynamics of Whole-Angle Mode Gyroscope

This section first establishes the second-order motion equation of the gyroscope's harmonic oscillator, thereby reconstructing the motion trajectory of the gyroscope resonator in the whole-angle mode. Based on this dynamic model, the first-order differential equations governing the standing wave angle, energy, quadrature component, and phase are systematically derived. It should be noted that the standing wave angle serves as the measurement output of the WAMG. This work lays a theoretical foundation for investigating the novel error phenomena of WAMG under oscillatory rotational scenarios.

Distinguished from the single-mode drive of the FTR, the WAMG maintains constant total vibration energy via its energy control loop. It then adjusts the drive amplitudes distributed to the two modes according to the measurement output, thereby enabling full-circle rotational angle measurement. The WAMG is essentially a two-degree-of-freedom vibration system. By means of the averaging solution method [28], the motion trajectory of its resonator can be derived and is illustrated in Figure 2. In the figure, a denotes the major axis of the gyroscope's trajectory (i.e., the amplitude of the main wave), representing the vibration amplitude of the gyroscope resonator; q denotes the minor axis of the trajectory (i.e., the amplitude of the quadrature wave), which approaches 0 under ideal conditions; θ denotes the angle between the standing wave axis and the 0° electrode, corresponding to the precession angle of the vibration mode and serving as the rotational angle measured by the gyroscope; ϕ_0 denotes the initial phase of the main wave; ω denotes the resonant frequency of the hemispherical resonator; Ω_z denotes the externally input angular velocity; and γ denotes the scaling factor of the gyroscope resonator.

The general mathematical model for a whole-angle mode gyroscope is presented as follows:

$$\begin{aligned} m\ddot{x} + c_x\dot{x} + k_x x &= F_x + 4\gamma\Omega_z m\dot{y} \\ m\ddot{y} + c_y\dot{y} + k_y y &= F_y - 4\gamma\Omega_z m\dot{x} \end{aligned} \quad (1)$$

where m is the mass of the gyroscope, x and y are the displacements of the two modes of the gyroscope mode X and mode Y , c_x and c_y are the damping

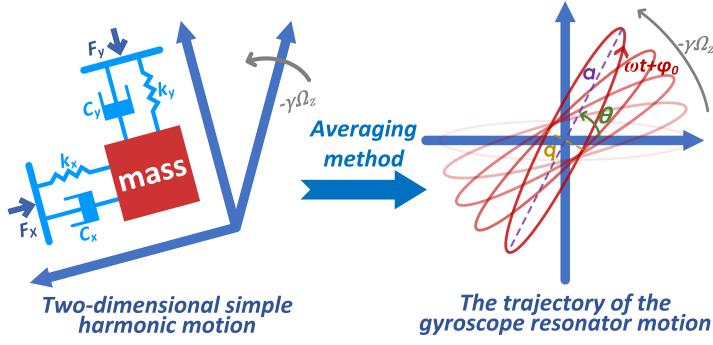


Fig. 2: The operational principle of a WAMG.

coefficients along the mode X and mode Y , respectively. k_x and k_y are the stiffness coefficients along the X mode and the Y mode, respectively. F_x is the driving forces on the X mode, F_y is the driving forces on the Y mode. In the absence of applied control forces and external rotational speed inputs, by solving Eq.1, the vibration displacement expressions of the gyroscopic resonator on the two mutually orthogonal coordinate axes in the Cartesian coordinate system are as follows:

$$\begin{aligned}
 x &= a \cos 2\theta \cos \phi - q \sin 2\theta \sin \phi \\
 y &= a \sin 2\theta \cos \phi + q \cos 2\theta \sin \phi \\
 \dot{x} &= \omega (-a \cos 2\theta \sin \phi - q \sin 2\theta \cos \phi) \\
 \dot{y} &= \omega (-a \sin 2\theta \sin \phi + q \cos 2\theta \cos \phi) \\
 \phi &= \omega t + \phi_0
 \end{aligned} \tag{2}$$

where ϕ represents the phase of the primary wave.

When an external rotational input is applied, the resonant frequency of the WAMG is significantly higher than the external input angular velocity, i.e., $\omega \gg \Omega_z$. Under this condition, the system equations can be regarded as a mathematical model describing the slow precession process. Relative to the input angular velocity, a , q , θ , and ϕ exhibit rapid variations and are classified as fast-changing variables, while Ω_z changes slowly and is classified as a slow-changing variable. As a result, the dynamic behavior of the system can be equivalently described using four elliptical characteristic parameters, which are presented as follows:

$$\begin{aligned}
\dot{a} &= -\gamma\Omega_z q \sin 2\phi \\
\dot{q} &= \gamma\Omega_z a \sin 2\phi \\
\dot{\theta} &= \frac{2\gamma\Omega_z(q^2 \cos^2 \phi - a^2 \sin^2 \phi)}{a^2 - q^2} \\
\dot{\phi}_0 &= -\frac{2\gamma\Omega_z a q \cos 2\phi}{a^2 - q^2}
\end{aligned} \tag{3}$$

Nevertheless, under non-ideal conditions, the damping coefficients and stiffness coefficients of the gyroscopic resonator exhibit inconsistencies ($c_x \neq c_y$, $k_x \neq k_y$). To address this issue, this paper defines the damping decay time constant τ and the resonant frequency ω , and their corresponding expressions are given as follows:

$$\begin{aligned}
\tau_x &= \frac{2m}{c_x}, \quad \tau_y = \frac{2m}{c_y} \\
\omega_x &= \sqrt{\frac{k_x}{m}}, \quad \omega_y = \sqrt{\frac{k_y}{m}} \\
\frac{1}{\tau} &= \frac{1}{2} \left(\frac{1}{\tau_x} + \frac{1}{\tau_y} \right), \quad \Delta \frac{1}{\tau} = \frac{1}{\tau_x} - \frac{1}{\tau_y} \\
\omega^2 &= \frac{\omega_x^2 + \omega_y^2}{2}, \quad \omega \Delta \omega = \frac{\omega_x^2 - \omega_y^2}{2}
\end{aligned} \tag{4}$$

where τ_x represents the damping decay time constant size of the X mode, τ_y represents the damping decay time constant size of the Y mode, ω_x represents the resonant frequency of the X mode, ω_y represents the resonant frequency of the Y mode, $\Delta 1/\tau$ represents the degree of damping anisotropy, and $\Delta \omega$ represents the degree of frequency splitting.

The total vibrational energy of the gyroscopic system can be expressed as $E = a^2 + q^2$, while the orthogonal term can be represented as $Q = 2aq$. From this, the following relationships can be derived:

$$\begin{aligned}
\dot{E} &= 2a\dot{a} + 2q\dot{q} \\
\dot{Q} &= 2\dot{a}q + 2a\dot{q}
\end{aligned} \tag{5}$$

Based on the integration averaging method proposed by D.D Lynch[28], the gyro parameters of the gyroscope resonator's motion trajectory under non-ideal conditions are calculated. The results are as follows:

$$\begin{aligned}
\dot{E} &= -\frac{2}{\tau}E - \Delta\frac{1}{\tau}\sqrt{E^2 - Q^2}\cos 4(\theta - \theta_\tau) \\
\dot{Q} &= -\frac{2}{\tau}Q - \Delta\omega\sqrt{E^2 - Q^2}\sin 4(\theta - \theta_\omega) \\
\dot{\theta} &= \frac{E}{2\sqrt{E^2 - Q^2}}\Delta\frac{1}{\tau}\sin 4(\theta - \theta_\tau) + \frac{Q}{2\sqrt{E^2 - Q^2}}\Delta\omega\cos 4(\theta - \theta_\omega) - \gamma\Omega_z \\
\dot{\phi}_0 &= -\frac{Q}{2\sqrt{E^2 - Q^2}}\Delta\frac{1}{\tau}\sin 4(\theta - \theta_\tau) - \frac{E}{2\sqrt{E^2 - Q^2}}\Delta\omega\cos 4(\theta - \theta_\omega)
\end{aligned} \tag{6}$$

where θ_τ represents the angle between the damping axis with the smallest damping attenuation time constant and the 0° detection electrode, which is referred to as the damping deviation angle, and θ_ω represents the angle between the stiffness axis with the lower resonant frequency and the 0° detection electrode, which is called the frequency deviation angle.

In the real practice, Eq.6 can be further concentrated. Because the total vibrational energy E can be maintained in a stable state by implementing energy closed-loop control, the orthogonal term Q can be suppressed through orthogonal rejection control, and ϕ_0 can be tracked by a phase lock loop. Thus the major dynamics to be analyzed is described as:

$$\dot{\theta} = \frac{E}{2\sqrt{E^2 - Q^2}}\Delta\frac{1}{\tau}\sin 4(\theta - \theta_\tau) + \frac{Q}{2\sqrt{E^2 - Q^2}}\Delta\omega\cos 4(\theta - \theta_\omega) - \gamma\Omega_z \tag{7}$$

Recalling to the ideal case that $\dot{\theta} = -\gamma\Omega_z$, which has a simple relation between the input angular rate Ω_z and output variable θ . This non-ideal case suffers from these nonlinear sinusoidal terms and solving this dynamic equation is difficult.

4 Reduced-order Analytical Solution for the measurement output of the Whole-angle Gyroscope

Based on the aforementioned Eq. 7, the differential equation for measurement output is solved to derive its analytical solution. First, both sides of the equation are multiplied by 4 and the substitution $\tilde{\theta}(t) = 4\theta(t)$ is introduced. Consequently, the differential equation for measurement output can be rearranged as follows:

$$\frac{d\tilde{\theta}(t)}{dt} = \frac{2E}{\sqrt{E^2 - Q^2}}\Delta\frac{1}{\tau}\sin \tilde{\theta}(t) + \frac{2Q}{\sqrt{E^2 - Q^2}}\Delta\omega\cos \tilde{\theta}(t) - 4\gamma\Omega_z \tag{8}$$

The aforementioned equation is decoupled into three distinct components, such that its derivative form can be expressed as the superposition of three individual differential terms:

$$\frac{d\tilde{\theta}(t)}{dt} = \frac{d\tilde{\theta}_1(t)}{dt} + \frac{d\tilde{\theta}_2(t)}{dt} + \frac{d\tilde{\theta}_3(t)}{dt} \quad (9)$$

where the three individual differential terms are respectively defined as:

$$\frac{d\tilde{\theta}_1(t)}{dt} = \frac{2E}{\sqrt{E^2 - Q^2}} \Delta \frac{1}{\tau} \sin \tilde{\theta}_1(t) \quad (10)$$

$$\frac{d\tilde{\theta}_2(t)}{dt} = \frac{2Q}{\sqrt{E^2 - Q^2}} \Delta \omega \cos \tilde{\theta}_2(t) \quad (11)$$

$$\frac{d\tilde{\theta}_3(t)}{dt} = -4\gamma\Omega_z \quad (12)$$

By rearranging Eq. 10 and computing its indefinite integral, the resulting expression is obtained as follows:

$$\begin{aligned} \int \frac{d\tilde{\theta}_1(t)}{\sin \tilde{\theta}_1(t)} &= \int \csc \tilde{\theta}_1(t) = \ln |\csc \tilde{\theta}_1(t) - \cot \tilde{\theta}_1(t)| \\ &= \frac{2E}{\sqrt{E^2 - Q^2}} \Delta \frac{1}{\tau} t + C_1 \end{aligned} \quad (13)$$

where C_1 represents the constant term generated from the computation of the indefinite integral.

Further derivation of Eq. 13 yields the following relationship:

$$\begin{aligned} \left| \csc \tilde{\theta}_1(t) - \cot \tilde{\theta}_1(t) \right| &= \exp \left(\frac{2E}{\sqrt{E^2 - Q^2}} \cdot \Delta \frac{1}{\tau} \cdot t + C_1 \right) \\ &= A_1 \exp \left(\frac{2E}{\sqrt{E^2 - Q^2}} \cdot \Delta \frac{1}{\tau} \cdot t \right) \end{aligned} \quad (14)$$

where $A_1 = e^{C_1}$. Define $B_1 = \pm A_1$ to eliminate the absolute value. The sign of B_1 is determined by $\tilde{\theta}_1(0)$, and thus the Eq. 14 is transformed into the following form:

$$\csc \tilde{\theta}_1(t) - \cot \tilde{\theta}_1(t) = B_1 \exp \left(\frac{2E}{\sqrt{E^2 - Q^2}} \cdot \Delta \frac{1}{\tau} \cdot t \right) \quad (15)$$

On this basis, from the Pythagorean identity for sine and cosine, $\sin^2 \tilde{\theta}_1(t) + \cos^2 \tilde{\theta}_1(t) = 1$, the following relationship can be further derived:

$$\sin \tilde{\theta}_1(t) = \frac{2B_1 \exp\left(\frac{2E}{\sqrt{E^2-Q^2}} \cdot \Delta \frac{1}{\tau} \cdot t\right)}{\left(B_1 \exp\left(\frac{2E}{\sqrt{E^2-Q^2}} \cdot \Delta \frac{1}{\tau} \cdot t\right)\right)^2 + 1} \quad (16)$$

Accordingly, substituting Eq. 16 into Eq. 10, the analytical expression for $\tilde{\theta}_1(t)$ is obtained as follows:

$$\begin{aligned} \tilde{\theta}_1(t) &= 2 \int \frac{d\left(B_1 \exp\left(\frac{2E}{\sqrt{E^2-Q^2}} \cdot \Delta \frac{1}{\tau} \cdot t\right)\right)}{\left(B_1 \exp\left(\frac{2E}{\sqrt{E^2-Q^2}} \cdot \Delta \frac{1}{\tau} \cdot t\right)\right)^2 + 1} \\ &= 2 \arctan\left(B_1 \exp\left(\frac{2E}{\sqrt{E^2-Q^2}} \cdot \Delta \frac{1}{\tau} \cdot t\right)\right) + D_1 \end{aligned} \quad (17)$$

where D_1 denotes the integration constant resulting from the indefinite integral.

A similar methodology is adopted to solve for $\tilde{\theta}_2(t)$ and $\tilde{\theta}_3(t)$ separately. For the sake of concise expression, the derived results are summarized as follows:

$$\int \frac{d\tilde{\theta}_2(t)}{\cos \tilde{\theta}_2(t)} = \frac{2Q}{\sqrt{E^2-Q^2}} \Delta \omega t + C_2 \quad (18)$$

$$\begin{aligned} \tilde{\theta}_2 &= 2 \int \frac{d\left(B_2 \exp\left(\frac{2Q}{\sqrt{E^2-Q^2}} \Delta \omega t\right)\right)}{\left(B_2 \exp\left(\frac{2Q}{\sqrt{E^2-Q^2}} \Delta \omega t\right)\right)^2 + 1} \\ &= 2 \arctan\left(B_2 \exp\left(\frac{2Q}{\sqrt{E^2-Q^2}} \Delta \omega t\right)\right) + D_2 \end{aligned} \quad (19)$$

$$\tilde{\theta}_3 = -4\gamma \int \Omega_z dt + D_3 \quad (20)$$

where C_2 , D_2 and D_3 represent the constant terms generated from the computation of the indefinite integral, A_2 represents e^{C_2} , and this paper defines $B_2 = \pm A_2$, its sign determined by $\tilde{\theta}_2(0)$.

Theorem 1 *In accordance with the superposition principle for solutions of first-order differential equations[29], the analytical expression for measurement output of WAMG is derived as follows:*

$$\begin{aligned}\tilde{\theta}(t) &= \tilde{\theta}_1(t) + \tilde{\theta}_2(t) + \tilde{\theta}_3(t) \\ &= 2 \arctan \left(B_1 \exp \left(\frac{2E}{\sqrt{E^2 - Q^2}} \Delta \frac{1}{\tau} t \right) \right) + 2 \arctan \left(B_2 \exp \left(\frac{2Q}{\sqrt{E^2 - Q^2}} \Delta \omega t \right) \right) \\ &\quad - 4\gamma \int \Omega_z dt + D_4 \quad (21)\end{aligned}$$

where D_4 denotes the sum of the integral constants D_1 , D_2 and D_3 .

Proposition 1 (The measurement output of WAMG under oscillatory rotation) *For the case where the input rotational angular velocity Ω_z is an oscillatory rotational signal (expressed as $\Omega_z = A \sin \Omega t$), the time-varying measurement output equation (Eq. 21) from Theorem 1 can be further rearranged into a closed-form expression as follows:*

$$\begin{aligned}\tilde{\theta}(t) &= \tilde{\theta}_1(t) + \tilde{\theta}_2(t) + \tilde{\theta}_3(t) \\ &= 2 \arctan \left(B_1 \exp \left(\frac{2E}{\sqrt{E^2 - Q^2}} \Delta \frac{1}{\tau} t \right) \right) + 2 \arctan \left(B_2 \exp \left(\frac{2Q}{\sqrt{E^2 - Q^2}} \Delta \omega t \right) \right) \\ &\quad + \frac{4A\gamma}{\Omega} \cos(\Omega t) + D_4. \quad (22)\end{aligned}$$

From the above analytical solutions (i.e., the general form in Theorem 1 and the closed-form for oscillatory rotational input in Proposition 1), it can be concluded that the presence of damping anisotropy and frequency splitting in the WAMG induces a time-dependent cumulative deviation in its measured output angle of rotation. The magnitude of this deviation is correlated with the sign and magnitude of the integral constants, and this deviation not only directly degrades the gyroscope's measurement accuracy but also exerts a persistent detrimental effect on its zero-bias performance. Ultimately, such time-dependent cumulative deviation leads to a significant decline in the overall measurement accuracy of the whole-angle mode gyroscope.

5 The Measurement Output Bias in Whole-Angle Mode Gyroscope under Oscillatory Rotation

5.1 The ORE for Whole-Angle Mode Gyroscope under Oscillatory Rotational Scenarios

This section focuses on exploring the final bias result of the measurement output in WAMG under oscillatory rotational scenarios, which is important to navigation systems. Observing and analyzing this measurement output can

reveal the system's stability and dynamic characteristics under varying internal and external conditions—serving as a key basis for system optimization and control, and laying a solid theoretical foundation for improving the gyroscope's reliability and overall performance.

From the definition of the arctangent function $\arctan(x)$, it is well known that $\arctan(x) \rightarrow \pi/2$ as $x \rightarrow +\infty$ and $\arctan(x) \rightarrow -\pi/2$ as $x \rightarrow -\infty$. Based on Eq. 22, for the case of oscillatory rotational motion, as time $t \rightarrow +\infty$, the term $\tilde{\theta}_1(t)$ converges to a specific limit whose magnitude is determined by the sign of B_1 . By the same token, the limit of $\tilde{\theta}_2(t)$ is governed by the sign of B_2 . In addition, the term $\tilde{\theta}_3(t)$ is a trigonometric function bounded above by $4A\gamma/\Omega$ and below by $-4A\gamma/\Omega$ [30]. Finally, the constant term D_4 corresponds to the sum of all integral-induced constant terms.

At the electrical activation initial stage of the WAMG, the standing wave angle is initialized to 0° . Under this initial condition, the following relationship is obtained:

$$\tilde{\theta}(0) = 2 \arctan(B_1) + 2 \arctan(B_2) + \frac{4A\gamma}{\Omega} + D_4 = 0 \quad (23)$$

In addition, by differentiating Eq. 22 and substituting $t = 0$ into this derived equation and Eq. 8, the following relationship can be derived:

$$\frac{2E \cdot \Delta \frac{1}{\tau} \cdot B_1}{1 + B_1^2} + \frac{2Q \cdot \Delta\omega \cdot B_2}{1 + B_2^2} = Q \cdot \Delta\omega \quad (24)$$

As $t \rightarrow +\infty$, the limit of $\tilde{\theta}(t)$ is determined by the signs of B_1 and B_2 , which are in turn determined by the initial standing wave angle $\tilde{\theta}(0)$. In addition, the parameters $\Delta \frac{1}{\tau}$, $\Delta\omega$, γ , and Ω are all positive constants. Accordingly, the following cases are discussed:

Assumption 1 $B_1 > 0$ and $B_2 > 0$.

Proposition 2 Under Assumption 1, the limit of $\tilde{\theta}(t)$ as $t \rightarrow +\infty$ is given by:

$$\begin{aligned} \lim_{t \rightarrow +\infty} \tilde{\theta}(t) &= \lim_{t \rightarrow +\infty} \left\{ 2 \arctan \left(B_1 \exp \left(\frac{2E}{\sqrt{E^2 - Q^2}} \Delta \frac{1}{\tau} t \right) \right) + \right. \\ &\quad \left. 2 \arctan \left(B_2 \exp \left(\frac{2Q}{\sqrt{E^2 - Q^2}} \Delta\omega t \right) \right) + \frac{4A\gamma}{\Omega} \cos(\Omega t) + D_4 \right\} \\ &= D_4 + 2\pi + \frac{4A\gamma}{\Omega} \lim_{t \rightarrow +\infty} \cos \Omega t \quad (25) \end{aligned}$$

Assumption 2 B_1 and B_2 have opposite signs (i.e., $B_1 B_2 < 0$).

Proposition 3 Under Assumption 2, the limit of $\tilde{\theta}(t)$ as $t \rightarrow +\infty$ is given by:

$$\begin{aligned} \lim_{t \rightarrow +\infty} \tilde{\theta}(t) &= \lim_{t \rightarrow +\infty} \left\{ 2 \arctan \left(B_1 \exp \left(\frac{2E}{\sqrt{E^2 - Q^2}} \Delta \frac{1}{\tau} t \right) \right) + \right. \\ &\quad \left. 2 \arctan \left(B_2 \exp \left(\frac{2Q}{\sqrt{E^2 - Q^2}} \Delta \omega t \right) \right) + \frac{4A\gamma}{\Omega} \cos(\Omega t) + D_4 \right\} \\ &= D_4 + \frac{4A\gamma}{\Omega} \lim_{t \rightarrow +\infty} \cos(\Omega t) \quad (26) \end{aligned}$$

Assumption 3 $B_1 < 0$ and $B_2 < 0$.

Proposition 4 Under Assumption 3, the limit of $\tilde{\theta}(t)$ as $t \rightarrow +\infty$ is given by:

$$\begin{aligned} \lim_{t \rightarrow +\infty} \tilde{\theta}(t) &= \lim_{t \rightarrow +\infty} \left\{ 2 \arctan \left(B_1 \exp \left(\frac{2E}{\sqrt{E^2 - Q^2}} \Delta \frac{1}{\tau} t \right) \right) + \right. \\ &\quad \left. 2 \arctan \left(B_2 \exp \left(\frac{2Q}{\sqrt{E^2 - Q^2}} \Delta \omega t \right) \right) + \frac{4A\gamma}{\Omega} \cos(\Omega t) + D_4 \right\} \\ &= D_4 - 2\pi + \frac{4A\gamma}{\Omega} \lim_{t \rightarrow +\infty} \cos \Omega t \quad (27) \end{aligned}$$

5.2 The Transition Time for Whole-Angle Mode Gyroscope under Oscillatory Rotational Scenarios

As a key parameter determining the dynamic response of the whole-angle mode gyroscope, the transition time under oscillatory rotation directly affects its angular measurement accuracy under sustained high-dynamic vibration. Imposing a continuous alternating signal on the WAMG induces a time-varying bias in its measurement output, which stabilizes after a specific transition period. Therefore, this study quantifies the time required for the WAMG's measurement output to stabilize at the ORE, which is a crucial step in improving the measurement accuracy under oscillatory rotational scenarios.

Definition 1 (Convergence Time Threshold) Under the conditions of Assumption 1, let T_0 denote a sufficiently large positive constant such that the measurement output $\tilde{\theta}(t)$ of the WAMG is bounded as $\tilde{\theta}(t)_{\inf} \leq \tilde{\theta}(t) \leq \tilde{\theta}(t)_{\sup}$ for all $t > T_0$. Two critical time thresholds are defined as follows: $T_{0,1}$: the time threshold at which $\tilde{\theta}(t)$ attains its upper bound $\tilde{\theta}(T_{0,1})_{\sup}$ (within the convergence range) for all $t > T_{0,1}$; $T_{0,2}$: the time threshold at which $\tilde{\theta}(t)$ attains its lower bound $\tilde{\theta}(T_{0,2})_{\inf}$ (within the convergence range) for all $t > T_{0,2}$.

For $t \geq \max\{T_{0,1}, T_{0,2}\}$, $\tilde{\theta}(t)$ converges to its bounded range with a constant bias. The convergence time threshold for the WAMG is thereby defined as:

$$T_0 = \max\{T_{0,1}, T_{0,2}\}.$$

Based on the above definition and Assumption 1, the explicit expressions for the upper and lower bounds of the WAMG's measurement output at the critical time thresholds $T_{0,1}$ and $T_{0,2}$ are derived as:

$$\tilde{\theta}(T_{0,1})_{\text{sup}} = 2\pi + D_4 + \frac{4A\gamma}{\Omega} \quad (28)$$

$$\tilde{\theta}(T_{0,2})_{\text{inf}} = 2\pi + D_4 - \frac{4A\gamma}{\Omega} \quad (29)$$

Proposition 5 (Derivatives of Bounded Measurement Output) *By substituting Eqs. 28 and 29 into the first-order differential equation governing the WAMG's measurement output (Eq. 6), the time derivatives of the upper and lower bounds of $\tilde{\theta}(t)$ are derived as follows:*

$$\begin{aligned} \frac{d\tilde{\theta}(T_{0,1})_{\text{sup}}}{dt} &= \frac{2E}{\sqrt{E^2 - Q^2}} \Delta \frac{1}{\tau} \sin\left(D_4 + \frac{4A\gamma}{\Omega}\right) \\ &\quad + \frac{2Q}{\sqrt{E^2 - Q^2}} \Delta \omega \cos\left(D_4 + \frac{4A\gamma}{\Omega}\right) - 4A\gamma \sin(\Omega T_{0,1}) \end{aligned} \quad (30)$$

$$\begin{aligned} \frac{d\tilde{\theta}(T_{0,2})_{\text{inf}}}{dt} &= \frac{2E}{\sqrt{E^2 - Q^2}} \Delta \frac{1}{\tau} \sin\left(D_4 - \frac{4A\gamma}{\Omega}\right) \\ &\quad + \frac{2Q}{\sqrt{E^2 - Q^2}} \Delta \omega \cos\left(D_4 - \frac{4A\gamma}{\Omega}\right) - 4A\gamma \sin(\Omega T_{0,2}) \end{aligned} \quad (31)$$

Proposition 6 (Simplified Derivatives and Transition Time Constraint)

When the WAMG's measurement output attains its upper and lower bounds at the critical time thresholds, its first-order time derivative vanishes, i.e., $\frac{d\tilde{\theta}_{\text{sup}}(T_{0,1})}{dt} = \frac{d\tilde{\theta}_{\text{inf}}(T_{0,2})}{dt} = 0$. Substituting this critical condition into Eqs. 30 and 31, the simplified analytical expressions are derived as follows:

$$\begin{aligned} \sin(\Omega T_{0,1}) &= \frac{1}{2A\gamma\sqrt{E^2 - Q^2}} \left[E\Delta \frac{1}{\tau} \sin\left(D_4 + \frac{4A\gamma}{\Omega}\right) \right. \\ &\quad \left. + Q\Delta \omega \cos\left(D_4 + \frac{4A\gamma}{\Omega}\right) \right] \end{aligned} \quad (32)$$

$$\begin{aligned} \sin(\Omega T_{0,2}) &= \frac{1}{2A\gamma\sqrt{E^2 - Q^2}} \left[E\Delta \frac{1}{\tau} \sin\left(D_4 - \frac{4A\gamma}{\Omega}\right) \right. \\ &\quad \left. + Q\Delta \omega \cos\left(D_4 - \frac{4A\gamma}{\Omega}\right) \right] \end{aligned} \quad (33)$$

For the inverse trigonometric solution of Eqs. 32 and 33, the range constraint of the sine function and the periodicity of the argument must be considered. As indicated by Eq. 28 and the definition of convergence time thresholds, $T_{0,1}$ and $T_{0,2}$ correspond to the time instants at which the ORE is attained within one period after the transient stage. Accordingly, the time difference between $T_{0,1}$ and $T_{0,2}$ satisfies the constraint:

$$\pi - \xi \leq \Omega|T_{0,1} - T_{0,2}| \leq \pi + \xi, \quad (34)$$

where ξ denotes a sufficiently small positive constant.

Proposition 7 (Inverse Trigonometric Forms of $\Omega T_{0,1}$ and $\Omega T_{0,2}$) To characterize the transition time T_0 , the inverse trigonometric solutions for $\Omega T_{0,1}$ and $\Omega T_{0,2}$ are categorized according to their value intervals, i.e., $[0, \pi/2]$ and $[\pi/2, +\infty)$. The key scenarios and their corresponding expressions are detailed as follows:

Case 1: $\Omega T_{0,1}, \Omega T_{0,2} \in [0, \pi/2]$ The closed-form expressions for $\Omega T_{0,1}$ and $\Omega T_{0,2}$ are given by:

$$\Omega T_{0,1} = \arcsin \left(\frac{E \Delta \frac{1}{\tau} \sin \left(D_4 + \frac{4A\gamma}{\Omega} \right) + Q \Delta \omega \cos \left(D_4 + \frac{4A\gamma}{\Omega} \right)}{2A\gamma \sqrt{E^2 - Q^2}} \right) \quad (35)$$

$$\Omega T_{0,2} = \arcsin \left(\frac{E \Delta \frac{1}{\tau} \sin \left(D_4 - \frac{4A\gamma}{\Omega} \right) + Q \Delta \omega \cos \left(D_4 - \frac{4A\gamma}{\Omega} \right)}{2A\gamma \sqrt{E^2 - Q^2}} \right) \quad (36)$$

Case 2: $\Omega T_{0,1}, \Omega T_{0,2} \in [\pi/2, +\infty)$ In this scenario, $\Omega T_{0,1}$ has two equivalent forms for integers k , as follows:

$$\Omega T_{0,1} = 2k\pi + \arcsin \left(\frac{E \Delta \frac{1}{\tau} \sin \left(D_4 + \frac{4A\gamma}{\Omega} \right) + Q \Delta \omega \cos \left(D_4 + \frac{4A\gamma}{\Omega} \right)}{2A\gamma \sqrt{E^2 - Q^2}} \right), \quad k \in \mathbb{N}^+ \quad (37)$$

or

$$\Omega T_{0,1} = (2k+1)\pi - \arcsin \left(\frac{E \Delta \frac{1}{\tau} \sin \left(D_4 + \frac{4A\gamma}{\Omega} \right) + Q \Delta \omega \cos \left(D_4 + \frac{4A\gamma}{\Omega} \right)}{2A\gamma \sqrt{E^2 - Q^2}} \right), \quad k \in \mathbb{N} \quad (38)$$

Similarly, $\Omega T_{0,2}$ can be expressed in the same form, i.e.,:

$$\Omega T_{0,2} = 2k\pi + \arcsin\left(\frac{E\Delta\frac{1}{\tau}\sin\left(D_4 - \frac{4A\gamma}{\Omega}\right) + Q\Delta\omega\cos\left(D_4 - \frac{4A\gamma}{\Omega}\right)}{2A\gamma\sqrt{E^2 - Q^2}}\right),$$

$$k \in \mathbb{N}^+ \quad (39)$$

or

$$\Omega T_{0,2} = (2k+1)\pi - \arcsin\left(\frac{E\Delta\frac{1}{\tau}\sin\left(D_4 - \frac{4A\gamma}{\Omega}\right) + Q\Delta\omega\cos\left(D_4 - \frac{4A\gamma}{\Omega}\right)}{2A\gamma\sqrt{E^2 - Q^2}}\right),$$

$$k \in \mathbb{N} \quad (40)$$

5.2.1 Case 1: $\Omega T_{0,1} \in [0, \frac{\pi}{2}]$, $\Omega T_{0,2} \in [0, \frac{\pi}{2}]$

When both $\Omega T_{0,1}$ and $\Omega T_{0,2}$ lie in the interval $[0, \frac{\pi}{2}]$, their closed-form analytical expressions can be explicitly obtained from Eqs. 35 and 36 in Proposition 7.

A fundamental property of the arcsine function is that if two arguments x and y lie within $[0, \frac{\pi}{2}]$, their absolute difference also falls within this interval. By substituting $x = \Omega T_{0,1}$ and $y = \Omega T_{0,2}$ into this property, it can be immediately derived that

$$0 \leq \Omega|T_{0,1} - T_{0,2}| \leq \frac{\pi}{2}.$$

This inequality is in direct contradiction with the transition time constraint specified in Eq. 34. Given this contradiction, the invalidity of this case is thus proven.

5.2.2 Case 2: $\Omega T_{0,1} \in [0, \frac{\pi}{2}]$, $\Omega T_{0,2} \in (\frac{\pi}{2}, +\infty)$

When $\Omega T_{0,1} \in [0, \frac{\pi}{2}]$ and $\Omega T_{0,2} \in (\frac{\pi}{2}, +\infty)$, the expression for $\Omega T_{0,1}$ is given by Eq. 35 in Proposition 7, while the expressions for $\Omega T_{0,2}$ are provided in Eqs. 39 and 40 of Proposition 7. Therefore, it is necessary to examine the sign of $\sin(\Omega T_{0,2})$ (as defined in Eq. 33) for different cases to determine the specific form of the transition time T_0 .

$0 \leq \sin(\Omega T_{0,2}) \leq 1$:

For the condition $0 \leq \sin(\Omega T_{0,2}) \leq 1$, the form of T_0 is analyzed as follows. If Eq. 39 is satisfied, the prerequisite condition in Eq. 34 is not met, hence this case is deemed invalid. If Eq. 40 is satisfied, the prerequisite condition in Eq. 34 holds. In this scenario, $\Omega T_{0,2} > \Omega T_{0,1}$, and the expression for the transition time is derived as follows:

$$T_0 = \max\{T_{0,1}, T_{0,2}\} = T_{0,2} = \pi - \arcsin \left(\frac{E \Delta \left(\frac{1}{\tau} \right) \sin \left(D_4 - \frac{4A\gamma}{\Omega} \right) + Q \Delta \omega \cos \left(D_4 - \frac{4A\gamma}{\Omega} \right)}{2A\gamma \sqrt{E^2 - Q^2}} \right) \quad (41)$$

$-1 \leq \sin(\Omega T_{0,2}) \leq 0$:

For the condition $-1 \leq \sin(\Omega T_{0,2}) \leq 0$, the form of T_0 is analyzed as follows. If Eq. 39 is satisfied, the prerequisite condition in Eq. 34 is met. In this scenario, $\Omega T_{0,2} > \Omega T_{0,1}$, and the expression for the transition time is derived as follows:

$$T_0 = \max\{T_{0,1}, T_{0,2}\} = T_{0,2} = 2\pi + \arcsin \left(\frac{E \Delta \left(\frac{1}{\tau} \right) \sin \left(D_4 - \frac{4A\gamma}{\Omega} \right) + Q \Delta \omega \cos \left(D_4 - \frac{4A\gamma}{\Omega} \right)}{2A\gamma \sqrt{E^2 - Q^2}} \right) \quad (42)$$

If Eq. 40 is satisfied, the prerequisite condition in Eq. 34 is met. In this scenario, $\Omega T_{0,2} > \Omega T_{0,1}$, and the expression for the transition time is consistent with that given in Eq. 41.

5.2.3 Case 3: $\Omega T_{0,1} \in (\frac{\pi}{2}, +\infty)$, $\Omega T_{0,2} \in [0, \frac{\pi}{2}]$

When $\Omega T_{0,1} \in (\frac{\pi}{2}, +\infty)$ and $\Omega T_{0,2} \in [0, \frac{\pi}{2}]$, the expressions for $\Omega T_{0,1}$ are provided in Eqs. 37 and 38 of Proposition 7, while the expression for $\Omega T_{0,2}$ is given by Eq. 36 in Proposition 7. Therefore, it is necessary to examine the sign of $\sin(\Omega T_{0,1})$ (as defined in Eq. 32) for different cases to determine the specific form of the transition time T_0 .

$0 \leq \sin(\Omega T_{0,1}) \leq 1$:

For the condition $0 \leq \sin(\Omega T_{0,1}) \leq 1$, the form of T_0 is analyzed as follows. If Eq. 37 is satisfied, the prerequisite condition in Eq. 34 is not met, hence this case is deemed invalid. If Eq. 38 is satisfied, the prerequisite condition in Eq. 34 is met. In this scenario, $\Omega T_{0,1} > \Omega T_{0,2}$, and the expression for the transition time is derived as follows:

$$T_0 = \max\{T_{0,1}, T_{0,2}\} = T_{0,1} = \pi - \arcsin\left(\frac{E \Delta\left(\frac{1}{\tau}\right) \sin\left(D_4 + \frac{4A\gamma}{\Omega}\right) + Q \Delta\omega \cos\left(D_4 + \frac{4A\gamma}{\Omega}\right)}{2A\gamma\sqrt{E^2 - Q^2}}\right) \quad (43)$$

$-1 \leq \sin(\Omega T_{0,1}) \leq 0$:

For the condition $-1 \leq \sin(\Omega T_{0,1}) \leq 0$, the form of T_0 is analyzed as follows. If Eq. 37 is satisfied, the prerequisite condition in Eq. 34 is met. In this scenario, $\Omega T_{0,1} > \Omega T_{0,2}$, and the expression for the transition time is derived as follows:

$$T_0 = \max\{T_{0,1}, T_{0,2}\} = T_{0,1} = 2\pi + \arcsin\left(\frac{E \Delta\left(\frac{1}{\tau}\right) \sin\left(D_4 + \frac{4A\gamma}{\Omega}\right) + Q \Delta\omega \cos\left(D_4 + \frac{4A\gamma}{\Omega}\right)}{2A\gamma\sqrt{E^2 - Q^2}}\right) \quad (44)$$

If Eq. 38 is satisfied, the prerequisite condition in Eq. 34 is met. In this scenario, $\Omega T_{0,1} > \Omega T_{0,2}$, and the expression for the transition time is consistent with that given in Eq. 43.

5.2.4 Case 4: $\Omega T_{0,1} \in (\frac{\pi}{2}, +\infty)$, $\Omega T_{0,2} \in (\frac{\pi}{2}, +\infty)$

When $\Omega T_{0,1} \in (\frac{\pi}{2}, +\infty)$ and $\Omega T_{0,2} \in (\frac{\pi}{2}, +\infty)$, the expressions for $\Omega T_{0,1}$ are provided in Eqs. 37 and 38 of Proposition 7, while the expressions for $\Omega T_{0,2}$ are provided in Eqs. 39 and 40 of Proposition 7. Therefore, it is necessary to examine the sign of $\sin(\Omega T_{0,1})$ (as defined in Eq. 32) and $\sin(\Omega T_{0,2})$ (as defined in Eq. 33) for different cases to determine the specific form of the transition time T_0 .

$0 \leq \sin(\Omega T_{0,1}) \leq 1$, $0 \leq \sin(\Omega T_{0,2}) \leq 1$:

For the conditions that $0 \leq \sin(\Omega T_{0,1}) \leq 1$ and $0 \leq \sin(\Omega T_{0,2}) \leq 1$ are both satisfied, $\Omega T_{0,1}$ and $\Omega T_{0,2}$ each have two distinct forms.

If the form of $\Omega T_{0,1}$ is given by Eq. 37 and the form of $\Omega T_{0,2}$ is given by Eq. 39, the prerequisite condition stated in Eq. 34 is not met, hence this case is deemed invalid. If the form of $\Omega T_{0,1}$ is given by Eq. 37 and the form of $\Omega T_{0,2}$ is given by Eq. 40, the prerequisite condition stated in Eq. 34 holds if and only if the following equations are satisfied:

$$\begin{aligned}
& \arcsin \left(\frac{E \Delta \left(\frac{1}{\tau} \right) \sin \left(D_4 + \frac{4A\gamma}{\Omega} \right) + Q \Delta \omega \cos \left(D_4 + \frac{4A\gamma}{\Omega} \right)}{2A\gamma \sqrt{E^2 - Q^2}} \right) = 0, \\
& \arcsin \left(\frac{E \Delta \left(\frac{1}{\tau} \right) \sin \left(D_4 - \frac{4A\gamma}{\Omega} \right) + Q \Delta \omega \cos \left(D_4 - \frac{4A\gamma}{\Omega} \right)}{2A\gamma \sqrt{E^2 - Q^2}} \right) = 0.
\end{aligned} \tag{45}$$

If the form of $\Omega T_{0,1}$ is given by Eq. 38 and the form of $\Omega T_{0,2}$ is given by Eq. 39, the prerequisite condition stated in Eq. 34 holds if and only if the conditions of Eq. 45 as previously stated are satisfied. If the form of $\Omega T_{0,1}$ is given by Eq. 38 and the form of $\Omega T_{0,2}$ is given by Eq. 40, the prerequisite condition stated in Eq. 34 is not met, hence this case is deemed invalid.

$0 \leq \sin(\Omega T_{0,1}) \leq 1, -1 \leq \sin(\Omega T_{0,2}) \leq 0$:

If the form of $\Omega T_{0,1}$ is given by Eq. 37 and the form of $\Omega T_{0,2}$ is given by Eq. 39, the prerequisite condition stated in Eq. 34 holds if and only if the following equations are satisfied:

$$\begin{aligned}
& \arcsin \left(\frac{E \Delta \left(\frac{1}{\tau} \right) \sin \left(D_4 + \frac{4A\gamma}{\Omega} \right) + Q \Delta \omega \cos \left(D_4 + \frac{4A\gamma}{\Omega} \right)}{2A\gamma \sqrt{E^2 - Q^2}} \right) = \frac{\pi}{2}, \\
& \arcsin \left(\frac{E \Delta \left(\frac{1}{\tau} \right) \sin \left(D_4 - \frac{4A\gamma}{\Omega} \right) + Q \Delta \omega \cos \left(D_4 - \frac{4A\gamma}{\Omega} \right)}{2A\gamma \sqrt{E^2 - Q^2}} \right) = -\frac{\pi}{2}.
\end{aligned} \tag{46}$$

If the form of $\Omega T_{0,1}$ is given by Eq. 37 and the form of $\Omega T_{0,2}$ is given by Eq. 40, the prerequisite condition stated in Eq. 34 holds if and only if the following equations are satisfied:

$$\begin{aligned}
& \arcsin \left(\frac{E \Delta \left(\frac{1}{\tau} \right) \sin \left(D_4 + \frac{4A\gamma}{\Omega} \right) + Q \Delta \omega \cos \left(D_4 + \frac{4A\gamma}{\Omega} \right)}{2A\gamma \sqrt{E^2 - Q^2}} \right) + \\
& \arcsin \left(\frac{E \Delta \left(\frac{1}{\tau} \right) \sin \left(D_4 - \frac{4A\gamma}{\Omega} \right) + Q \Delta \omega \cos \left(D_4 - \frac{4A\gamma}{\Omega} \right)}{2A\gamma \sqrt{E^2 - Q^2}} \right) = 0.
\end{aligned} \tag{47}$$

If the form of $\Omega T_{0,1}$ is given by Eq. 38 and the form of $\Omega T_{0,2}$ is given by Eq. 39, the prerequisite condition stated in Eq. 34 holds if and only if the conditions of Eq. 47 as previously stated are satisfied. If the form of $\Omega T_{0,1}$ is given by Eq. 38 and the form of $\Omega T_{0,2}$ is given by Eq. 40, the prerequisite condition stated in Eq. 34 holds if and only if the conditions of Eq. 46 as previously stated are satisfied.

$-1 \leq \sin(\Omega T_{0,1}) \leq 0$, $0 \leq \sin(\Omega T_{0,2}) \leq 1$:

If the form of $\Omega T_{0,1}$ is given by Eq. 37 and the form of $\Omega T_{0,2}$ is given by Eq. 39, the prerequisite condition stated in Eq. 34 holds if and only if the following equations are satisfied:

$$\begin{aligned} \arcsin \left(\frac{E \Delta \left(\frac{1}{\tau}\right) \sin \left(D_4 + \frac{4A\gamma}{\Omega}\right) + Q \Delta \omega \cos \left(D_4 + \frac{4A\gamma}{\Omega}\right)}{2A\gamma \sqrt{E^2 - Q^2}} \right) &= -\frac{\pi}{2}, \\ \arcsin \left(\frac{E \Delta \left(\frac{1}{\tau}\right) \sin \left(D_4 - \frac{4A\gamma}{\Omega}\right) + Q \Delta \omega \cos \left(D_4 - \frac{4A\gamma}{\Omega}\right)}{2A\gamma \sqrt{E^2 - Q^2}} \right) &= \frac{\pi}{2}. \end{aligned} \quad (48)$$

If the form of $\Omega T_{0,1}$ is given by either Eq. 37 or Eq. 38 and the corresponding form of $\Omega T_{0,2}$ is given by Eq. 40 or Eq. 39, respectively, the prerequisite condition stated in Eq. 34 holds if and only if the conditions of Eq. 45 as previously stated are satisfied. If the form of $\Omega T_{0,1}$ is given by Eq. 38 and the form of $\Omega T_{0,2}$ is given by Eq. 40, the prerequisite condition stated in Eq. 34 holds if and only if the conditions of Eq. 48 as previously stated are satisfied.

$-1 \leq \sin(\Omega T_{0,1}) \leq 0$, $-1 \leq \sin(\Omega T_{0,2}) \leq 0$:

For the conditions that $-1 \leq \sin(\Omega T_{0,1}) \leq 0$ and $-1 \leq \sin(\Omega T_{0,2}) \leq 0$ are both satisfied, $\Omega T_{0,1}$ and $\Omega T_{0,2}$ each have two distinct forms.

If $\Omega T_{0,1}$ takes the form specified in Eq. 37 and $\Omega T_{0,2}$ takes the form specified in Eq. 39, or if $\Omega T_{0,1}$ follows Eq. 38 and $\Omega T_{0,2}$ follows Eq. 40, the prerequisite condition in Eq. 34 is not met, hence these two cases are deemed invalid.

In contrast, if $\Omega T_{0,1}$ takes the form specified in Eq. 37 and $\Omega T_{0,2}$ takes the form specified in Eq. 40, or if $\Omega T_{0,1}$ follows Eq. 38 and $\Omega T_{0,2}$ follows Eq. 39, the prerequisite condition in Eq. 34 holds if and only if the conditions outlined in Eq. 45 are satisfied.

5.3 A Brief Summary of Claimed ORE

This derivation and analysis clearly prove the existence of ORE. For a WAMG under oscillatory rotation input, the specific form of its measurement output is given by Eq. 22. ORE is a cumulative bias in the gyroscope's measurement output induced by oscillatory rotation input, whose magnitude depends on the initial value of θ and the gyroscope's inherent parameters. The final ORE value has three attraction regions based on the initial standing wave angle, which are given by: $D_4 + 2\pi \pm \frac{4A\gamma}{\Omega}$, $D_4 \pm \frac{4A\gamma}{\Omega}$ and $D_4 - 2\pi \pm \frac{4A\gamma}{\Omega}$, where D_4 is determined by the gyroscope's initial state (the initial standing wave angle and inherent parameters). Differences within each attraction region are governed by the gyroscope's inherent parameters: vibrational energy E , orthogonal term Q , frequency splitting degree $\Delta\omega$, and damping anisotropy degree $\Delta 1/\tau$.

To quantify the time required to reach the ORE attraction region, a transition time is introduced to describe this cumulative process. Its general form is presented in Proposition 7, and all forms indicate that the transition time correlates with vibrational energy, frequency splitting, damping anisotropy, as well as the amplitude and frequency of the oscillatory rotation signal, with the applicable conditions for the specific forms analyzed in detail. It is noteworthy that the transition time is invalid in the short time interval $[0, \frac{\pi}{2}]$, which confirms that the transition process cannot be completed within a short time after the oscillatory rotation signal is injected into the gyroscope. This also demonstrates the practical application value of researching WAMGs under continuous oscillatory rotation.

6 Validation

6.1 Parameter Configurations

To intuitively evaluate the influence of ORE, this work did a case study with real parameters of the authors' own hemispherical resonant gyroscopes. It is an ultra precision gyroscope whose bias stability can reach 0.001 deg per hour in conventional FTR mode. And its specific characteristic values are presented in Table 1. Then, a comprehensive numerical simulation was performed. It is noted that the orders of magnitude of the gyroscope's inherent characteristic parameters are relatively small, and those of the error parameters are even lower. To amplify the differences in experimental results, this study assigns error parameter values with distinctly different orders of magnitude to the same type of gyroscope in each control experiment, aiming to investigate the specific effects of error parameters on the performance of the WAMG.

Table 1: The Gyroscopic Characteristic Parameters Set for Simulation

Characteristic parameter	Value	Unit
c_x	1.57×10^{-4}	$N \cdot s/m$
c_y	1.56×10^{-4}	$N \cdot s/m$
$\Delta \frac{1}{\tau}$	5×10^{-6}	$N \cdot s/m \cdot kg$
ω_x	8235.187	Hz
ω_y	8235.186	Hz
$\Delta \omega$	0.001	Hz
E	1.00×10^6	Dimensionless
Q	9.00×10^4	Dimensionless
γ	0.285	Dimensionless

To investigate the effects of three types of error parameters (frequency difference, damping difference, and energy difference) on the performance of the WAMG, this study designs five groups of univariate control experiments. In each group, only the target variable is varied, while all other irrelevant vari-

ables are held constant: initial standing wave angle = 0° , oscillatory rotational signal amplitude = 1, and oscillatory rotational signal frequency = 50 Hz. The specific parameter configurations for these five groups of experiments involving the three types of error parameters are presented in Table 2.

Table 2: Comparison Experimental Parameter Setting Table for $\Delta\omega$, $\Delta(1/\tau)$, and E

Group No.	1	2	3	4	5
$\Delta\omega(\text{Hz})$	0.0005	0.0075	0.001	0.005	0.01
$\Delta(1/\tau)(N \cdot s/m \cdot kg)$	5×10^{-8}	5×10^{-7}	5×10^{-6}	2.5×10^{-5}	5×10^{-5}
E	100000	500000	750000	1000000	2000000

Furthermore, this study conducts additional experiments to investigate the characteristics of the measurement output of the WAMG under different initial standing wave angles. The initial standing wave angles are categorized into two groups: positive and negative angles, with eight experimental sets configured for each group. All other irrelevant variables are kept consistent with the parameters specified in Table 1. The specific values of the eight experimental sets for the two categories of initial standing wave angles are presented in Table 3.

Table 3: Comparison Experimental Parameter Setting Table for Initial Standing Wave Angles

Group No.	1	2	3	4	5	6	7	8
Positive Angle (rad)	+0.1745	+0.3491	+0.5236	+0.6981	+0.8727	+1.0472	+1.2217	+1.3963
Negative Angle (rad)	-0.1745	-0.3491	-0.5236	-0.6981	-0.8727	-1.0472	-1.2217	-1.3963

Finally, this paper conducts experiments to investigate the measurement output characteristics of the WAMG under different amplitudes and frequencies of the oscillatory rotational signal. For the experiments on varying oscillatory rotational amplitudes, a total of eight experimental groups with distinct oscillatory rotational amplitudes are configured. Except for the target variable, all other irrelevant variables are kept consistent with the parameters specified in Table 1, and the specific values for these eight groups are presented in Table 4. For the experiments on varying oscillatory rotational frequencies, thirteen experimental groups with distinct oscillatory rotational frequencies are designed. Among these groups, the irrelevant variables for five groups are consistent with the parameters in Table 1, while those for the remaining eight groups are set to scaled-up values; the specific parameters for all groups are provided in Table 5.

Table 4: Comparison Experimental Parameter Setting Table for Varying Amplitudes of Oscillatory Rotation

Group No.	1	2	3	4	5	6	7	8
Amplitude(rad)	1	10	50	100	200	300	400	500

Table 5: Comparison Experimental Parameter Settings for Varying Frequencies of Oscillatory Rotation

(a) Parameter Group 1 (5 Sets)								
Group No.	1	2	3	4	5			
Frequency(Hz)	5	10	15	25	500			
(b) Parameter Group 2 (8 Sets)								
Group No.	1	2	3	4	5	6	7	8
Frequency(Hz)	1	5	10	15	20	25	30	35

6.2 Results and Discussions

The simulation experiment first performed this work: the measurement output of the hemispherical resonant gyroscope in whole-angle mode is presented under a typical oscillatory rotational input with 1 rad amplitude and 10 Hz frequency. The gyroscope parameters used are fully consistent with those in Table 2. As shown in Fig. 3, the transition time for the gyroscope to reach a ORE is approximately 3.6×10^4 s (about 60 minutes), with a bias magnitude of around 0.406 rad ($\tilde{\theta} \approx 23.2623^\circ$, $\theta \approx 5.8156^\circ$).

To investigate the influence of frequency difference, damping difference, and energy difference on the measurement output in the whole-angle mode gyroscope, a set of controlled experiments is conducted based on the parameters in Table 2. The experimental results are shown in Fig.4. The response curve in Fig.4 indicates that after oscillatory rotational signals are injected, the measurement output of the gyroscope does not stay at the initial zero position. Instead, a dynamic drift occurs first, followed by a gradual convergence to ORE. Additionally, three key factors—discrepancies in resonant frequencies, variations in damping, and differences in energy among the gyroscope modes—are identified as influencing the transition time and final ORE values. The difference in the final ORE values can be clearly observed through the Fig.5.

To examine the effect of the initial standing wave angle on the response of the whole-angle mode gyroscope, another series of controlled experiments is conducted based on the parameter settings in Table 3. The experimental results are visualized in Fig. 6. It shows that the gyroscope's standing wave angle drifts from its initial set position after an oscillatory rotation signal is injected, and the transition time is dependent on the initial position. Moreover, the final ORE value converges to a consistent value within a specific range.

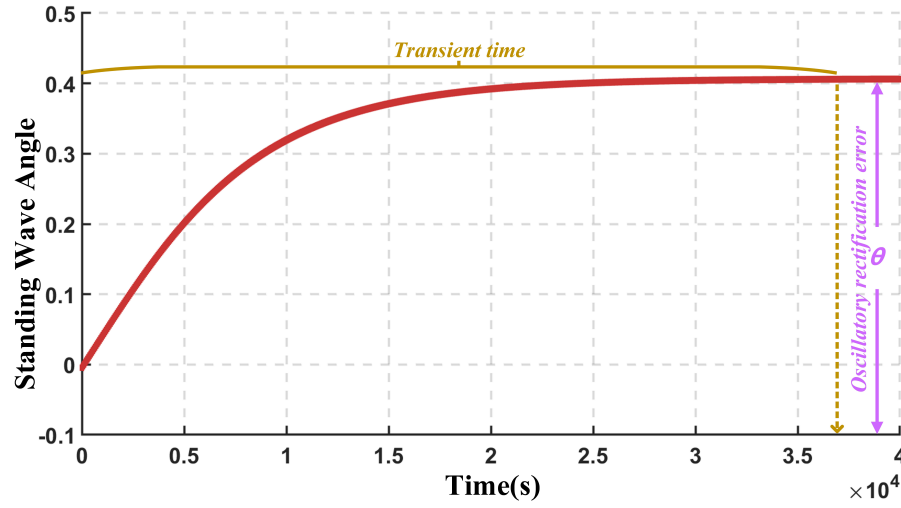


Fig. 3: The measurement output of the HRG under typical operating conditions

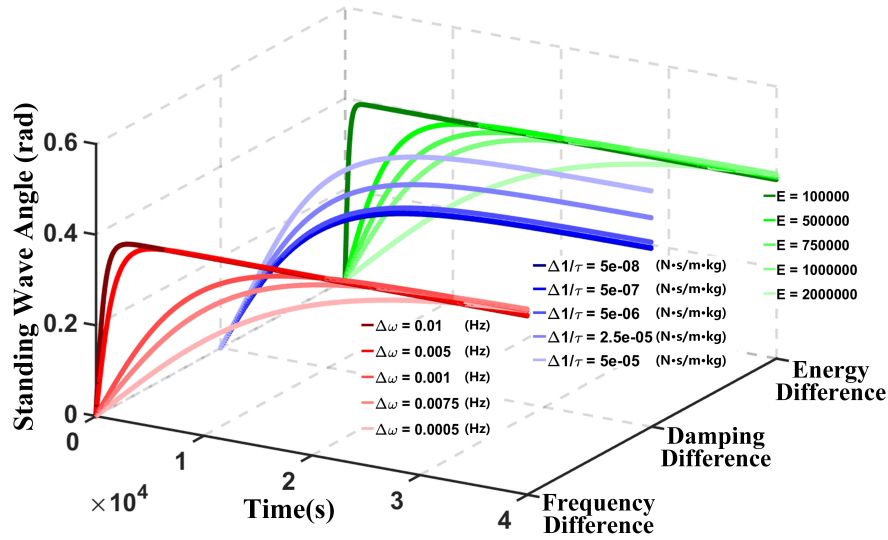


Fig. 4: The Measurement Output Results Under Frequency Differences, Damping Differences, and Energy Differences

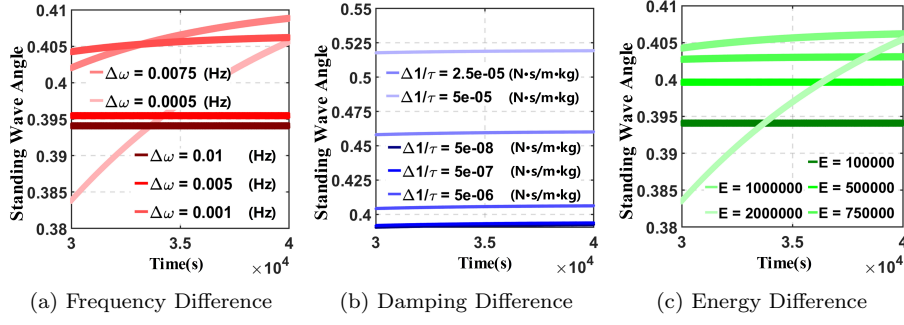
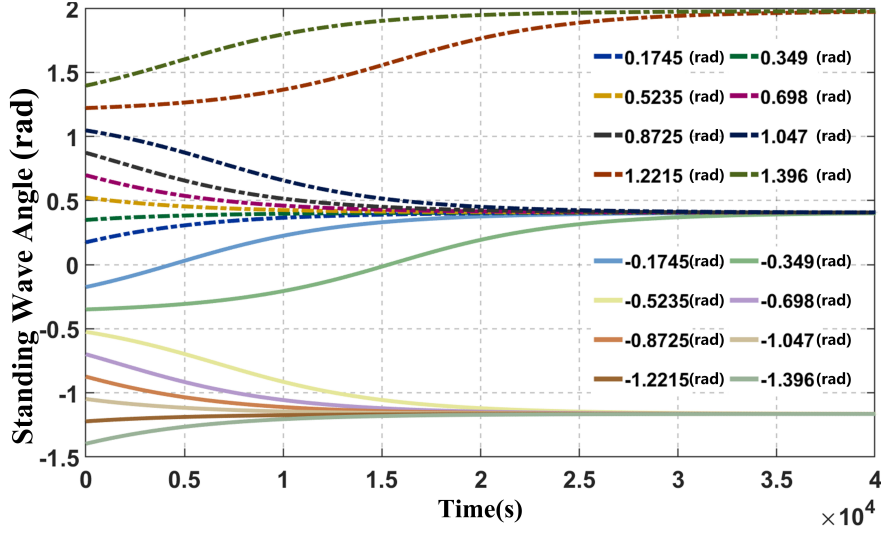
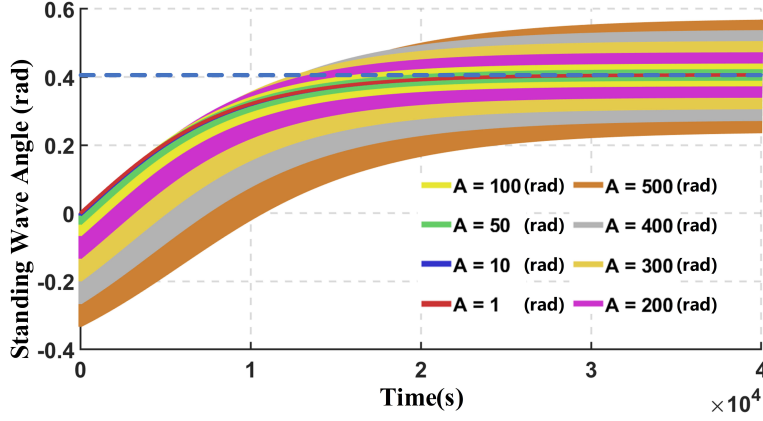
Fig. 5: The Final ORE Values During $3 \times 10^4 \text{s} \sim 4 \times 10^4 \text{s}$ 

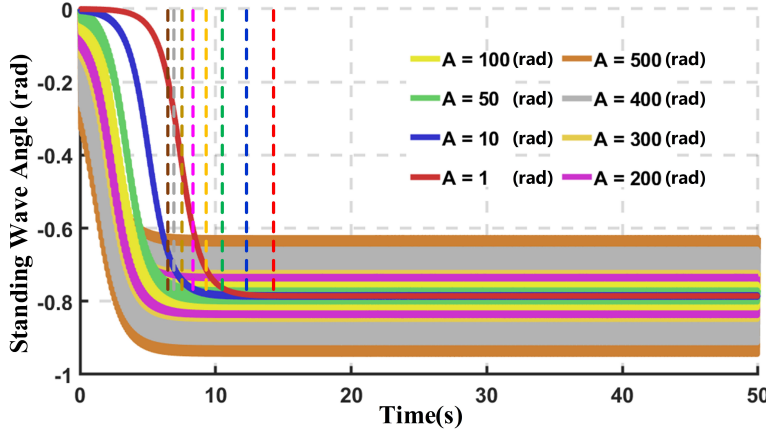
Fig. 6: The Measurement Output Results under Different Initial Standing Wave Angles

To investigate the influence of the amplitude of oscillatory rotational input signals on the measurement output of WAMG, this study designs a control experiment with fixed frequency and differentiated amplitudes based on the parameters in Table 4. The experimental results are shown in Fig.7. To more clearly characterize the variation trend of measurement output during oscillatory rotational signal measurement, two sets of experiments are conducted: one adopts the gyroscope's actual parameter values (results in Fig.7a), and the other uses scaled-up values of the gyroscope's actual parameters (results in Fig.7b). The response curves in Fig.7a indicate the following conclusions: the constant term of the steady-state offset remains consistent, and the amplitude of the dynamic offset is positively correlated with that of the input signal, i.e., a larger input signal amplitude corresponds to a higher dynamic

offset amplitude. From the response curves in Fig.7b, it can be concluded that the transition time is negatively correlated with the input signal amplitude, i.e., a larger input signal amplitude leads to a shorter transition time.



(a) The Measurement Output under Actual Gyroscope Parameters



(b) The Measurement Output under Alternative Actual Gyroscope Parameters

Fig. 7: The Measurement Output Results under Different Amplitudes of Oscillatory Rotation

To investigate the influence of the frequency of oscillatory rotational input signals on the measurement output of WAMG, a controlled experiment with fixed amplitude and differentiated frequencies is designed on the basis of the parameters listed in Table 4. The output data of the gyroscope's measurement output after oscillatory rotational signal injection are collected, and the experimental results are presented in Fig.8. Similarly, to characterize the variation trend of the measurement output of WAG more clearly during the

measurement of oscillatory rotational signals, two sets of experiments are conducted: one group adopts the actual parameter values of the gyroscope, with the results shown in Fig.8a; the other group uses the scaled-up values of the gyroscope's actual parameters, with the results shown in Fig.8b. The following conclusions can be drawn from the response curves in Fig.8a: the value of the constant term of the steady-state offset remains consistent, and the amplitude of the dynamic offset is negatively correlated with the frequency of the input signal; i.e., the higher the frequency of the input signal, the smaller the amplitude of the dynamic offset. The following conclusions can be drawn from the response curves in Fig.8b: the transition time is positively correlated with the frequency of the input signal; i.e., the higher the frequency of the input signal, the longer the transition time.

Based on the experimental results presented above, this study provides clear validation of the effectiveness of Eq.22. Specifically, when a oscillatory rotational signal is input into a whole-angle mode gyroscope, its measurement output evolves over time and eventually enters ORE. In this state, the measurement output oscillates near a fixed angle, with the oscillation being a cosine signal whose frequency equals that of the oscillatory rotation signal. The amplitude of this cosine signal is positively correlated with the amplitude of the oscillatory rotation signal and negatively correlated with its frequency.

1. In terms of the factors influencing the ORE:

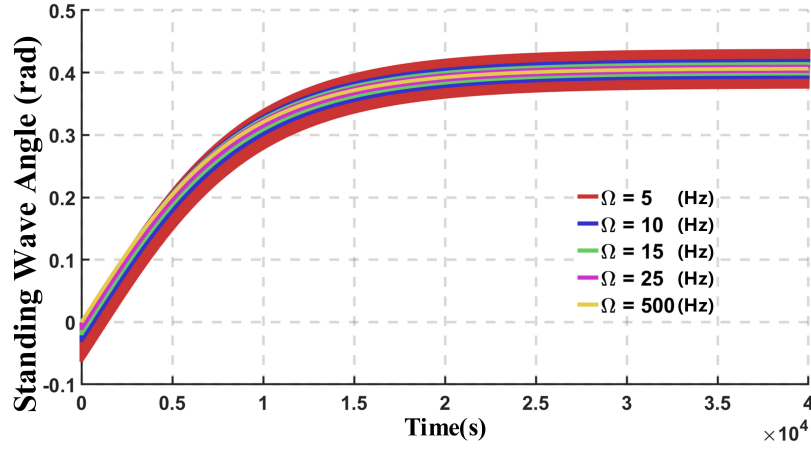
When the initial standing wave angle is set to 0, the ORE is negatively correlated with the frequency splitting and damping anisotropy degree (i.e., a larger frequency splitting or damping anisotropy degree leads to a smaller bias and vice versa), while it is positively correlated with the vibration energy (i.e., a higher vibration energy results in a larger bias and vice versa).

A significant correlation exists between the range of the initial standing wave angle and the convergence characteristics of the final ORE. These characteristics fall into three categories: the final ORE values of the measurement output tend to be consistent for the initial standing wave angle in the range of -0.349rad to $+1.047\text{rad}$; the final ORE values converges to another set of consistent values for that in the range of -0.5235rad to -1.396rad ; and the final ORE values also show consistency for the initial standing wave angle at $+1.2215\text{rad}$ and $+1.396\text{rad}$.

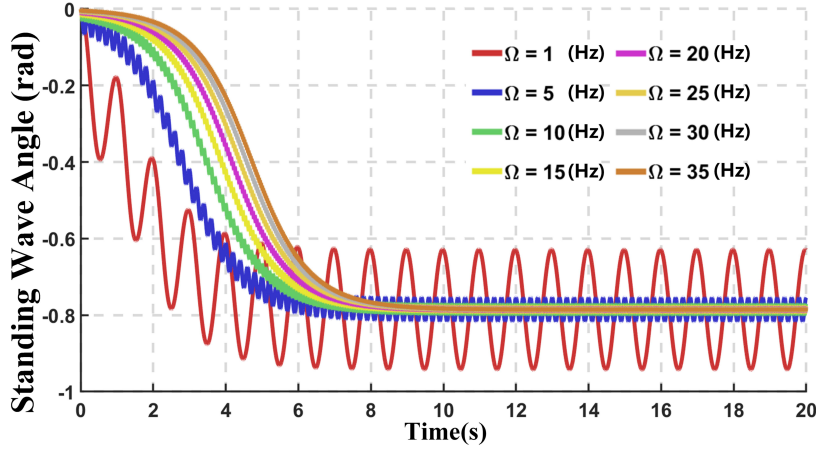
The influences of the amplitude and frequency of the oscillatory rotational signal on the ORE can offset each other. Consequently, when the amplitude and frequency of the input signal are different, the ORE value measured by the gyroscope can remain consistent.

2. In terms of the factors influencing the transition time:

A smaller frequency splitting, a smaller damping anisotropy degree or a larger vibration energy all prolong the time required for the gyroscope to reach the ORE; conversely, a larger frequency splitting, a larger damping anisotropy degree or a smaller vibration energy result in a shorter transition time. It is worth noting that the initial standing wave angle determines the



(a) The Measurement Output under Actual Gyroscope Parameters



(b) The Measurement Output under Alternative Actual Gyroscope Parameters

Fig. 8: The Measurement Output Results under Different Frequency of Oscillatory Rotation

dynamic characteristics during the transition process: the closer the initial angle is to the final ORE value, the shorter the transition time, and vice versa. For the oscillatory rotational signal, its amplitude and frequency are correlated with the transition time such that a larger amplitude or a lower frequency leads to a shorter transition time, and vice versa.

Based on the above results, the existence of ORE is well proved with the theoretical studies. The case study conducted a shocking conclusion that even an ultra precision quartz based WAMG sensor has the potential to drift 5 deg within 60 minutes in a dynamic environment. It is a remainder that high

performance navigation requires a bias drift less than 0.1 deg per hour as a bottom line, and a 0.001 deg per hour level performance is preferred.

This mechanism has not been discovered for more than 15 years since the first WAMG sensor was reported in 2011 [31]. The reason is that analyzing the WAMG dominant equation is difficult because of its nonlinearity. Only over simplified or indirect works were published with limited insights. Further more, the test standard only covers the static metrics in the industry.

7 Conclusion

This study claims a novel error mechanism for precision WAMG angular sensors, which is defined as Oscillatory Rectification Errors. It refers to the fact that there is a bias drift of WAMG under the input of periodic/oscillatory rates.

It is developed with the practical WAMG model and its firstly derived analytical solution. It has multiple converged final bias values, but has a certain dependency of WAMG's initial θ . Other factors such as frequency splitting, damping anisotropy, and vibration energy determine not only the transition time but also the converged value. A validation demonstrated that even an ultra high precision WAMG can drift a level of 5 deg on a hourly basis. This is definitely an alarm for navigation engineers, despite the promising metrics of WAMG under constant rate conditions.

Future work suggested would be developing adequate compensation methods to eliminate the influence of ORE for high-end inertial navigation applications.

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