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A By-Production Approach to Deriving the Production Possibility Frontier and the Shadow Prices of CO₂ Reduction

Alexandre Repkine¹

Abstract

This paper develops a novel method to derive an explicit Production Possibility Frontier (PPF) for desirable and undesirable outputs within a by-production framework. We formulate a constrained optimization model that, under proven regularity conditions, ensures a unique PPF where technology operates on its efficient boundary for both types of outputs. Marginal abatement costs (MACs) are obtained directly as PPF slopes, consistent with the marginal rate of transformation, and without requiring the arbitrary directional vectors used in directional distance function approaches.

We apply the method to panel data for sixteen Asia-Pacific economies from 2010 to 2019, estimating an explicit PPF and corresponding MACs. These estimates capture the short-run opportunity costs of emissions abatement along a fixed technological frontier, rather than engineering costs of fuel switching or technological change. The framework offers policymakers a comprehensive view of tradeoffs between output and emissions at scales ranging from marginal reductions to economy-wide transitions.

Keywords: marginal abatement costs, production possibilities frontier, constrained optimization

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1. Introduction

Modeling polluting technologies provides policymakers with tools to quantify the negative externalities of production, design mechanisms to internalize them, and evaluate the associated economic costs. At a macroeconomic level, these costs are often framed as the sacrifice in GDP required to achieve a given emissions target. Analogous to standard production theory, this tradeoff can be conceptualized through a marginal rate of transformation (MRT) between the good output y (GDP) and the bad output b (pollution).

However, in contrast to standard production theory where the MRT between two good outputs reflects a reallocation of resources, the trade-off between y and b is fundamentally different. Pollution b is a by-product of producing the desirable output y , meaning the same inputs are used simultaneously for producing both desirable and undesirable outputs. The tradeoff between y and b arises because reducing pollution necessitates curtailing the use of polluting input (e.g., coal), which simultaneously constrains the production of good output. This fundamental characteristic is well captured by the by-production technology framework developed by Murty et al.'s (2012), which explicitly models the joint production of goods and pollutants through separate sub-technologies linked by the use of polluting inputs.

Given the theoretical and policy-related importance of estimating the marginal abatement costs of harmful emissions, a variety of methods have been developed in the existing literature. A significant number of these estimation methods are based on both parametric and non-parametric estimation of the radial output distance functions and their directional analogues, i.e. directional output distance functions (DODF.) These methods have been widely applied in environmental policy modeling, for instance in constructing MAC curves (Fare and Grosskopf, 1998; Lee et al., 2014).

Despite their theoretical appeal, such methods face notable challenges. For instance, DODF-based estimates can be highly sensitive to input data and the choice of directional output distance vectors (Atkinson and Tsionas, 2016), while MACs may become undefined at certain points on the efficiency frontier in case of the non-parametric estimation based on the data envelopment analysis (DEA), as discussed e.g. in Kuosmanen and Zhou (2021.) Furthermore, because these methods are

inherently marginal, they may poorly approximate the economic costs of large-scale emission reductions (Ward, 2014; Kesicki and Ekins, 2012).

Since marginal abatement cost of harmful emissions is essentially the marginal rate of transformation between good and bad output, it is natural to measure it as a slope of production possibilities frontier (PPF) linking emissions b to the maximum amount of good output y that can

be produced given a specific level of emissions, i.e. $MAC(b) := \frac{dy^*(b)}{db}$ where $y^*(b)$ is the PPF

linking y and b . However, to our knowledge not many (if any) studies address the problem of evaluating an explicit functional form of this kind of PPF, instead measuring MACs indirectly as e.g. ratios of the distance functions' derivatives at the projection point on the PPF, as popularized by Färe et al. (2005). It is important to stress that under this approach the PPF itself is not explicitly estimated as from the mathematical perspective the estimation of MACs is based on the application of a theorem on differentiating an implicit PPF function defined by the generic relationship $D(\square) = 0$ where $D(\square)$ is the (directional) distance function linking inputs and outputs.

To address these challenges, this study proposes a novel framework for computing MACs that derives an explicit production possibility frontier (PPF) directly from a constrained optimization problem within Murty et al.'s (2012) by-production model of a polluting technology. In this framework, the marginal abatement cost is given by the slope of the PPF with respect to an undesirable output, representing the economic cost of reducing one unit of pollution and corresponding to the shadow price of the associated emissions constraint, i.e., its Lagrange multiplier. This approach ensures that MACs are well-defined under certain regularity conditions imposed on the polluting technology and provides a clear economic interpretation aligned with the classical definition of MACs stemming from the work of Baumol and Oates (1971) and Pigou (1920).

Compared to the standard DODF-based estimation approaches, our framework offers several advantages. First, by explicitly generating the PPF, it allows policymakers to evaluate the economic cost of non-marginal emission reductions, for example, determining how much desirable output must be sacrificed to reduce emissions by 100 tons of CO₂ rather than by an infinitesimal amount. Second, our method avoids the problem of choosing directional output vectors that can

influence DODF-based MAC estimates (Atkinson and Tsionas, 2016; also Ma et al., 2019) by providing a policy maker with an overall view of the production possibilities set, enabling the analyst to choose in what direction to move towards the efficient frontier.

Finally, while important properties of the production possibility frontier (PPF)—such as its existence, uniqueness, and concavity—have often been assumed rather than formally established, our framework derives them explicitly by imposing additional regularity conditions on the by-production polluting technology beyond those in Murty et al. (2012), which ensures both the PPF and the associated marginal abatement costs are well defined.

We illustrate the practical applicability of this approach by estimating the parameters of a polluting technology using a cross-country dataset covering sixteen Asia-Pacific economies for the period between 2010 and 2019, deriving the associated PPF and marginal abatement costs.

This paper is organized as follows. Section 2 presents literature overview. Section 3 describes our modeling framework that allows us to obtain PPF in the space of good and bad outputs as a solution to a constrained optimization problem. Section 4 presents the results of applying our framework to a set of Asia-Pacific economies. Section 5 discusses and concludes.

2. Literature Overview

2.1 Historical origins of the marginal abatement costs concept

The concept of marginal abatement costs (MACs) originates in classical welfare economics. Pigou (1920) introduced the notion of externalities and corrective taxation, laying the groundwork for analyzing pollution control costs. Dales (1968) extended this analysis by proposing tradable pollution rights, relying implicitly on heterogeneous abatement costs across firms. Baumol and Oates (1971) and Baumol and Oates (1975) formally integrated MACs into optimization frameworks, showing that efficient pollution control requires the equalization of MACs across firms. In their formulation, MACs emerge as the Lagrange multipliers of an aggregate emissions constraint, reflecting the shadow cost of reducing pollution by one unit. Equivalently, MAC

represents the slope of the production possibility frontier (PPF) in the space of desirable and undesirable outputs, or the marginal rate of transformation between them.

This theoretical literature provided the foundation for empirical estimation of MACs using output distance functions (ODFs), including their directional variants (directional output distance functions, or DODFs), estimated parametrically or via nonparametric approaches such as DEA.

2.2 Parametric directional output distance functions approach to the estimation of MACs

Shepard's (1970) concept of the radial distance function laid the groundwork for modeling multi-output production technologies by allowing one to treat harmful emissions as another output. Chung et al. (1997) and Färe et al. (2005) generalized this concept with the directional output distance function (DODF), which allows for the simultaneous expansion of desirable outputs and contraction of undesirable ones along a pre-specified directional output distance vector.

Parametric approaches to DODF estimation, such as Färe et al. (2005) and Coelli et al. (2007), estimate functional forms of a DODF characterizing the polluting technology. The marginal abatement cost (MAC) is defined as the shadow price of reducing the bad output in terms of forgone good output. Formally, for a good output y , bad output b , and inputs $\vec{x}$, let $D(\vec{x}, y, b)$ denote the DODF. The MAC at the efficient frontier projection ($D(\vec{x}, y, b) = 1$) is

$$MAC = - \frac{\partial D(\vec{x}, y, b)}{\partial b} \bigg/ \frac{\partial D(\vec{x}, y, b)}{\partial y}.$$

This ratio is the marginal rate of transformation between b and y , showing the good output sacrifice needed to abate one extra unit of bad output. Mathematically, the negative of the ratio of two partial

derivatives $-\frac{D_b(\square)}{D_y(\square)}$ results from an application of the theorem of differentiating an implicit PPF

function $y^*(b)$ defined by the generic relationship $D(\square) = 0$ that represents an efficient boundary of the production possibilities set of a polluting technology. Implicit differentiation happens at the efficient projection point of the observed input-output mix along the directional output distance vector.

While the DODF approach to the estimation of MACs has a solid foundation in economic theory, Lauwers (2009) has shown it to be in conflict with the law of physical conservation of mass and energy as the directional output distance function approach is based on the assumption that good outputs can be expanded, and bad outputs contracted without adjusting the mix of inputs. In addition, Zhou et al. (2021) notice that for observations already on the efficient frontier such output changes will not be feasible, making MAC estimation via the ratio of the DODF's partial derivatives challenging.

Computationally, when the partial derivative of the directional output distance function with respect to the good output $\frac{\partial D(\square)}{\partial y}$ is small, the computed MACs can become unrealistically high.

In the case of a single good and a single bad output, this would happen when a nearly vertical PPF $y^*(b)$ occurs for a firm that is already operating at low emissions levels for its technology. In such cases, even small changes in the denominator—due, for instance, to measurement errors—can lead to large variations in the estimated marginal abatement costs. These limitations motivate the development of a method that does not rely on partial derivative ratios and ensures computability even for units on the efficient frontier.

Another persistent challenge with the DODF approach is its reliance on the choice of a directional output distance vector. A common but arbitrary choice is the 45-degree vector, where emissions and good output change by an equal number of units. This lack of economic rationale means a variety of MACs can be obtained for the same observed input-output mix by choosing alternative directional vectors. To address this issue, studies like Atkinson and Tsionas (2016) propose a method to endogenously identify optimal, firm-specific directions for each input and output based on profit-maximizing and cost-minimizing behavior. Similarly, Färe et al. (2017) acknowledge the traditional "ad hoc nature" of the directional vector choice and suggest an endogenous alternative. Wang et al. (2017) provide a comprehensive review of the challenges in selecting directional output distance vectors.

2.3 DEA-based approach to the estimation of DODF parameters and MACs

Data envelopment analysis (DEA), pioneered by Charnes et al. (1978), is a widely used approach for estimating directional output distance functions (DODFs) in the context of a polluting technology and the associated marginal abatement costs (MACs) computation. Its primary advantage over parametric methods is that it requires no pre-specified functional form for a polluting technology. This application of DEA for estimating pollution shadow prices is well-established, as extensively reviewed by Lee et al. (2014).

DEA constructs a piecewise-linear efficiency frontier whose vertices are the observed, fully efficient DMUs. For any inefficient DMU, the model identifies a virtual benchmark projection on this frontier, defined as a linear combination of these efficient peers, which provides a target for improvement by using less input to produce an equal or greater amount of output. Within the DODF framework, the efficient counterpart on the DEA frontier for an observed inefficient input-output mix is found as a projection on the frontier along the direction output distance vector.

Mathematically, DEA solves a constrained optimization problem for each producer. For a single good output y and bad output b , the marginal abatement cost is given by $MAC = \frac{\lambda_b}{\lambda_y}$, where λ_b and λ_y are the shadow prices for each output. This ratio quantifies the trade-off between pollution reduction and the loss of desirable output. Geometrically, the calculated MAC represents the slope of the piecewise-linear production possibility frontier at the projection point. Equivalently, DEA-based MACs are the ratios of the partial derivatives of efficiency scores with respect to bad and good outputs, reflecting their marginal trade-off. These ratios are conceptually the same with the ones used for MAC computation within the parametric DODF framework.

Despite its strengths, however, the standard DEA approach presents two major challenges for MAC estimation. First, MACs are only well-defined on the linear facets of the frontier. At the vertices, the slope—and thus the MAC—is undefined, complicating the evaluation of efficient DMUs. To resolve this, smoothing approaches like the convex nonparametric least squares method by Kuosmanen and Zhou (2021) construct a smooth frontier to yield unique and stable shadow prices.

Second, even when defined, the MAC estimates can be highly unstable since a minor change in the input-output mix can shift this projection to an adjacent facet with a different slope, causing a large, discontinuous jump in the MAC. This instability makes results highly sensitive to data noise

or measurement error. As Lee et al. (2002) emphasize, the slope of the production possibility frontier—and thus the MAC—can jump dramatically due to arbitrarily small data perturbations.

2.4 Marginal Abatement Costs as Shadow Prices of Pollution Constraints

The approaches to measuring marginal abatement costs of harmful emissions outlined above are well-grounded in economic theory, but they are not directly based on the classical Baumol and Oates (1971) definition of MAC as the shadow price of an emissions constraint, or equivalently, as slopes of the explicit production possibilities frontiers in the space of good and bad outputs. To resolve this disconnect, the following section develops a novel framework that allows for an explicit derivation of PPF within Murty et al. (2012)’s by-production framework that models a polluting technology via two distance functions representing the production of a good and a bad output.

We derive an explicit PPF by maximizing desirable output subject to the distance-function constraints that ensure feasibility of the solution, and an exogenous emissions cap. Our approach allows one to derive a well-behaved production possibilities frontier in the space of good and bad outputs, which provides a stable, economically intuitive measure of the marginal abatement costs of emissions that is fully aligned with the classical MAC definition.

3. Theoretical Framework

3.1 Representation of a by-production polluting technology by two distance functions

Our modeling approach follows Murty et al’s (2012) by-production framework, in which a polluting technology is represented as the intersection of two sub-technologies: one producing the good output y and the other generating bad output (emissions) b . Both are produced with an input

vector $\bar{x} = (k, l, f)$, where k is the stock of physical capital, l is labor input, and f is fuel, the only input whose use results in environmentally harmful emissions b .

Each sub-technology is represented by a distance function: $D^y(y, k, l, f)$ for the desirable, or good, output y , and $D^b(b, k, l, f)$ for the undesirable emissions b . Murty et al. (2012) show that in order for these two distance functions to represent a by-production polluting technology, they must satisfy the properties of monotonicity and homogeneity. Specifically, $D^y(y, k, l, f) \leq 1$ must be non-decreasing in y , non-increasing in k , l , and f , and homogeneous of degree 1 in y . Similarly, $D^b(b, k, l, f) \leq 1$ must be non-increasing in b , non-decreasing in k , l , and f , and homogeneous of degree -1 in b . The equalities $D^y(y, k, l, f) = 1$ and $D^b(b, k, l, f) = 1$ then define the efficient boundaries of the production possibilities sets for the good and bad outputs, respectively.

While this framework allows one to model a polluting by-production technology as an intersection of two sub-technologies, it does not in itself provide a direct method for deriving the production possibilities frontier (PPF) linking desirable and undesirable outputs. In what follows, we formulate a constrained maximization problem that yields the PPF in the (b, y) -space. To ensure this frontier is well-behaved, we impose additional regularity conditions on the distance functions $D^y(\square)$ and $D^b(\square)$. As we demonstrate below, this formulation not only ensures the existence and uniqueness of the PPF but also allows for the computation of the marginal abatement costs of emissions according to their classical definition as the marginal rates of transformation, which are given by the slopes of the PPF in (b, y) -space.

3.2 Constrained maximization problem

For any cap on the emissions level b_0 we define a production possibilities frontier (PPF) in the (b, y) -space as the maximum level of good output $y^*(b_0)$ that can be produced as a solution to the following constrained maximization problem:

$$\begin{cases} \max_{k,l,f,b} y \\ s.t. \\ D^y(y,k,l,f) \leq 1 \\ D^b(b,k,l,f) \leq 1 \\ b \leq b_0 \end{cases} \quad (1)$$

The first two constraints in (1) require that an input-output combination (y,k,l,f,b) is technologically feasible, while the emissions constraint $b \leq b_0$ imposes an upper bound on the level of emissions.

To ensure the PPF derived from (1) is well-behaved, we impose the following additional regularity conditions on the distance functions $D^y(\square)$ and $D^b(\square)$:

1. $D^y(y,k,l,f)$ is strictly increasing in y , non-increasing in (k,l,f) , homogeneous of degree 1 in y , and continuously differentiable in all arguments.
2. $D^b(b,k,l,f)$ is strictly decreasing in b , non-decreasing in (k,l,f) , homogeneous of degree minus 1 in b , and continuously differentiable in all arguments.
3. The feasible set $F(b_0) = \{(y,k,l,f,b) \in \mathfrak{R}_+^5 : D^y(y,k,l,f) \leq 1, D^b(b,k,l,f) \leq 1, b \leq b_0\}$ is non-empty and compact for each $b_0 > 0$.
4. Non-degeneracy: $\nabla_{(k,l,f)} D^y(y,k,l,f) \neq 0$ for all $(y,k,l,f,b) \in \text{int}(F(b_0))$, i.e. for all input-output combinations in the interior of $F(b_0)$.

Conditions 1 and 2 extend the original constraints by Murty et al (2012) by requiring $D^y(\square)$ to be strictly increasing in y , $D^b(\square)$ to be strictly decreasing in b , and both distance functions to be continuously differentiable. The strict monotonicity of $D^y(\square)$ with respect to y ensures the good-

output distance function constraint binds at the optimum. Strict monotonicity of $D^b(\square)$ with respect to b that the Lagrange multiplier on the emissions constraint is positive, as shown in Theorem 1 below. Conditions 3 and 4 are new: they guarantee that the optimization problem (1) is well-defined (via compactness of the feasible set) and that the bad-output and emissions-cap constraints also bind at the optimum, preventing degenerate solutions.

Under the regularity Conditions 1-4 the solution $y^*(b_0)$ to (1) defines a unique production possibility frontier in the (b, y) -space for a given emissions cap b_0 . The marginal abatement cost is then given by the Lagrange multiplier on the emissions constraint $b \leq b_0$, or equivalently the slope of the PPF $y^*(b_0)$ at $b = b_0$, i.e. $\frac{dy^*(b_0)}{db_0}$.

3.3 Derivation of the production possibilities frontier $y^*(b_0)$ and the marginal abatement costs

Theorem 1 is based on the definition of production possibilities frontier in the (b, y) -space as the maximal amount of good output y that can be produced given an emissions cap $b \leq b_0$. In our further exposition we will refer to Conditions 1 through 4 from Section 3.2 without citing the Section number.

Theorem 1. (Existence and Uniqueness of the Production Possibilities Frontier)

Assume that Conditions 1 through 4 hold. Then, for any emissions cap $b_0 > 0$:

- (a) **Existence:** The solution $y^*(b_0)$ to the maximization problem (1) exists.
- (b) **Efficiency:** For any input mix (k^*, l^*, f^*) that can produce $y^*(b_0)$ a solution $(y^*(b_0), k^*, l^*, f^*)$ satisfies $D^y(y^*(b_0), k^*, l^*, f^*) = 1$, $D^b(b_0, k^*, l^*, f^*) = 1$, and $b^* = b_0$.

Proof of Theorem 1:

(a) Existence

The objective function y is continuous. By Condition 3 in Section 3.2, the feasible set $F(b_0)$ is non-empty and compact. Therefore, by the Weierstrass extreme value theorem, a maximum $y^*(b_0)$ exists. It is important to note that, while $y^*(b_0)$ is unique, the input mix (k^*, l^*, f^*) may not be unique.

(b) Efficiency

Consider the Lagrangian

$$L(y, b, k, l, f; \mu, \nu, \lambda) = y + \mu(1 - D^y(y, k, l, f)) + \nu(1 - D^b(b, k, l, f)) + \lambda(b_0 - b) \quad (2)$$

with $\mu, \nu, \lambda \geq 0$. The first-order conditions include:

$$\left\{ \begin{array}{ll} \frac{\partial L}{\partial y} = 1 - \mu \frac{\partial D^y}{\partial y} = 0 & \text{(ia)} \\ \frac{\partial L}{\partial b} = -\nu \frac{\partial D^b}{\partial b} - \lambda = 0 & \text{(ib)} \\ \frac{\partial L}{\partial x} = -\mu \frac{\partial D^y}{\partial x} - \nu \frac{\partial D^b}{\partial x} = 0 & \text{(ic)} \\ x = k, l, f & \\ \mu(1 - D^y) = 0 & \text{(ii)} \\ \nu(1 - D^b) = 0 & \text{(iii)} \\ \lambda(b_0 - b) = 0 & \text{(iv)} \\ \mu, \lambda, \nu \geq 0 & \end{array} \right. \quad (3)$$

Conditions (ia), (ib) and (ic) in (3) ensure stationarity of the Lagrangian, while conditions (ii), (iii) and (iv) represent complementary slackness.

From (ia) in (3), $\mu \frac{\partial D^y}{\partial y} = 1$. Since $D^y(\square)$ is strictly increasing in y (Condition 1), $\frac{\partial D^y}{\partial y} > 0$, which implies $\mu > 0$. From complementary slackness in part (ii) of (3), this gives $D^y(y^*, k^*, l^*, f^*) = 1$ at the optimal solution to (1).

Now, suppose for contradiction that $\nu = 0$. The first-order conditions (ic) for the stationarity of the Lagrangian in (2) with respect to inputs are:

$$\mu \frac{\partial D^y}{\partial \vec{x}} + \nu \frac{\partial D^b}{\partial \vec{x}} = 0, \quad \vec{x} = (k, l, f) \quad (4)$$

If $\nu = 0$, this implies $\mu \frac{\partial D^y}{\partial \vec{x}} = 0$. Since $\mu > 0$, this means $\nabla_{(k,l,f)} D^y = 0$, which contradicts the non-degeneracy assumption in Condition 4. Therefore, $\nu > 0$, and from part (iii) in the FOC (3), $D^b(b^*, k^*, l^*, f^*) = 1$.

Finally, we prove $b^* = b_0$. Suppose $b^* < b_0$. By homogeneity of degree -1 of $D^b(\square)$ (Condition 2):

$$D^b(b_0, k^*, l^*, f^*) = D^b\left(b^* \cdot \frac{b_0}{b^*}, k^*, l^*, f^*\right) = \frac{b^*}{b_0} D^b(b^*, k^*, l^*, f^*) = \frac{b^*}{b_0} < 1. \quad \text{Thus, at the optimal}$$

input bundle (k^*, l^*, f^*) the constraint $D^b(b, k^*, l^*, f^*) \leq 1$ is slack at $b = b_0$.

Since $D^b(b_0, k^*, l^*, f^*) < 1$ and $D^b(\square)$ is continuously differentiable, there exists a neighborhood around (k^*, l^*, f^*) where $D^b(b_0, k, l, f) < 1$. By the non-degeneracy condition (Condition 4), $\nabla_{(k,l,f)} D^y(y^*, k^*, l^*, f^*) \neq 0$, so there exists a direction (η_k, η_l, η_f) such that for sufficiently small $\varepsilon > 0$, at $(k_\varepsilon, l_\varepsilon, f_\varepsilon) = (k^* + \varepsilon \eta_k, l^* + \varepsilon \eta_l, f^* + \varepsilon \eta_f)$, we have:

1. $D^b(b_0, k_\varepsilon, l_\varepsilon, f_\varepsilon) \leq 1$ (by continuity and the fact that the constraint was slack at the optimal inputs)
2. $D^y(y^*, k_\varepsilon, l_\varepsilon, f_\varepsilon) < 1$ (since we're moving in a direction that decreases $D^y(\square)$)

Since $D^y(\square)$ is strictly increasing in y and continuous, there exists $y' > y^*$ such that $D^y(y', k_\varepsilon, l_\varepsilon, f_\varepsilon) \leq 1$.

Thus, the point $(y', k_\varepsilon, l_\varepsilon, f_\varepsilon, b_0)$ is feasible and yields a higher level of good output than y^* , contradicting the optimality of y^* . Therefore, $b^* = b_0$.

Finally, from the first-order condition (ib) in (3), we have $\lambda = -\nu \frac{\partial D^b}{\partial b} > 0$ since, as shown above, $\nu > 0$ and $D^b(\square)$ is strictly decreasing in b by the regularity Condition 2.

End of proof of Theorem 1.

From the Lagrangian (2) in Theorem 1, the envelope theorem gives $\frac{dy^*(b_0)}{db_0} = \frac{\partial L}{\partial b_0} \Big|_{opt} = \lambda$. By definition, the marginal abatement cost is the rate at which the maximal good output changes with the emissions cap, i.e. $MAC(b_0) = \frac{dy^*(b_0)}{db_0} = \lambda > 0$.

Since in our setting the value function of (1) coincides with the PPF, i.e. $V(b_0) \equiv y^*(b_0)$. If $V(b_0)$ is concave, then its derivative $V'(b_0) \equiv MAC(b_0)$ is non-increasing, and the PPF $y^*(b_0)$ is concave as well, satisfying the properties of the non-increasing marginal costs of abating pollution with increased emission levels.

In Theorem 1 we impose additional conditions on the polluting technology compared to Murty et al (2012) in order to guarantee the efficiency properties at the optimum, i.e. $D^y(\square) = D^b(\square) = 1$ and $b = b_0$ at an optimal solution $(y^*(b_0), k^*, l^*, f^*, b_0)$. These additional constraints are the strict monotonicity of $D^y(\square)$ in y , and the non-degeneracy condition $\nabla_{(k,l,f)} D^y(y, k, l, f) \neq 0$.

Note.

Theorem 1 establishes the conditions under which a unique production possibility frontier $y^*(b_0)$ exists, which is compatible with the situation when the optimal input mix (k^*, l^*, f^*) that achieves this maximum may not be unique. It is worth noting that while strict convexity of the feasible set $F(b_0)$ would be sufficient to guarantee a unique input vector (the proof is available immediately upon request), it is not a necessary condition for the core results of this theorem. More importantly, for the widely-used Cobb-Douglas specifications of the distance functions in (5) that we introduce in the next Section, imposing global strict convexity on $F(b_0)$ would necessitate parameter restrictions such as e.g. decreasing returns to scale in the production of the good output y that are often empirically untenable.

By demonstrating that the existence and efficiency of the PPF depend only on the milder regularity Conditions 1-4 in Theorem 1, our framework remains both theoretically rigorous and empirically relevant, accommodating a wider range of realistic technological configurations. The potential for non-unique input mixes at the frontier is not a limitation but a reflection of the fact that there may be multiple technologically efficient ways to combine inputs to achieve the same maximal output under a given emissions constraint.

4. Empirical Exercise

In this section we provide specific functional forms for the two distance functions $D^y(y, k, l, f)$ and $D^b(b, k, l, f)$ to use it as a basis for the derivation of an analytical PPF $y^*(b_0)$. We then estimate the parameters of this polluting technology along with the associated marginal abatement costs using the data for a set of OECD economies for the period between 2010 and 2019.

4.1 Functional specifications for a polluting technology and parameter constraints to ensure regularity conditions

We employ the following functional specifications for the two distance functions defining our polluting technology:

$$D^y(y, k, l, f) = \frac{y}{Ck^\alpha l^\beta f^\gamma} \quad \text{and} \quad D^b(b, k, l, f) = \frac{Tk^{\theta_1} l^{\theta_2} f^{\theta_3}}{b} \quad (5)$$

While the existing literature provides little guidance on the functional form for $D^b(\square)$, we opted for a multiplicative specification as it embodies the fundamental property of null-jointness specified in Shepard and Färe (1974). This property, which states that positive output of the good is impossible without generating some of the bad, i.e. $b = 0 \Rightarrow y = 0$, is guaranteed by our choice of the functional forms in (5). As Theorem 1 demonstrates, at the optimum $b = Tk^{\theta_1} l^{\theta_2} f^{\theta_3}$ and $y = Ck^\alpha l^\beta f^\gamma$. Consequently, if $b = 0$, at least one input must be zero, forcing $y = 0$, thus ensuring the property of null-jointness.

It is straightforward to verify that distance function $D^y(\square)$ is strictly increasing in good output y and homogeneous of degree 1 in y . The coefficient $C > 0$ represents a Hicks-neutral efficiency parameter for the production of good output. To ensure that $D^y(\square)$ is non-increasing in inputs k, l , and f , we require $\alpha, \beta, \gamma > 0$. Similarly, the bad-output distance function $D^b(\square)$ is strictly decreasing in bad output b and homogeneous of degree -1 in b . The coefficient T can be thought of as a pollution intensity parameter: a higher T corresponds to a dirtier technology, meaning that for any given input vector the a producer can produce a larger amount of bad output b while still satisfying the distance function constraint. To ensure that $D^b(\square)$ is non-decreasing in k, l , and f we require $\theta_1, \theta_2, \theta_3 > 0$. Both distance functions are continuously differentiable in all arguments, so that the regularity conditions 1 and 2 from Section 3.2 are satisfied if $\alpha, \beta, \gamma > 0$ and $\theta_1, \theta_2, \theta_3 > 0$.

Having established that our functional forms satisfy the core properties of the by-production framework (Conditions 1 and 2), we now verify that they also meet the additional regularity conditions required for Theorem 1.

First, we consider Condition 3, which requires the feasible set $F(b_0)$ to be non-empty and compact for any $b_0 > 0$. The set is non-empty, as a point with zero inputs and outputs is trivially feasible. To ensure compactness, we observe that for a given emissions cap b_0 , the input and output quantities are naturally bounded from above. The constraints $D^y(\square) \leq 1$ and $D^b(\square) \leq 1$ imply that the good output y is bounded by a function of the inputs, and the inputs themselves are constrained by $b = Tk^{\theta_1}l^{\theta_2}f^{\theta_3} \leq b_0$. Given that all exponents $\theta_i, i=1..3$ are positive, this relationship imposes an upper bound on feasible input combinations, thus ensuring the feasible set is bounded. As all constraint functions are continuous, the set $F(b_0)$ is also closed, confirming its compactness.

Finally, we verify the non-degeneracy Condition 4. The gradient of $D^y(\square)$ with respect to the inputs is $\nabla_{(k,l,f)} D^y(\square) = \left(-\frac{\alpha y}{Ck^{\alpha+1}l^{\beta}f^{\gamma}}, -\frac{\beta y}{Ck^{\alpha}l^{\beta+1}f^{\gamma}}, -\frac{\gamma y}{Ck^{\alpha}l^{\beta}f^{\gamma+1}} \right)$. Given that $C, \alpha, \beta, \gamma > 0$, and all inputs and the output y are positive for any non-trivial production plan within $F(b_0)$, this gradient vector is non-zero for all feasible points involving positive production. This satisfies Condition 4 and guarantees that the two sub-technologies are never locally decoupled, ensuring that the efficiency conditions $D^y(\square) = 1$ and $D^b(\square) = 1$ bind simultaneously at the optimum.

Thus, under the parameter restrictions $C, T, \alpha, \beta, \gamma, \theta_1, \theta_2, \theta_3 > 0$, the functional forms in (5) satisfy all regularity Conditions 1 through 4, guaranteeing the existence and efficiency properties of the production possibility frontier as established in Theorem 1 so that the PPF $y^*(b_0)$ is uniquely defined. The PPF concavity under the distance functions in (5) will be established in the next Section 4.2 where we derive an explicit-form PPF.

4.2 Constrained maximization problem for the derivation of PPF and MACs

Given the distance functions $D^y(\square)$ and $D^b(\square)$ in (5), the Lagrangian in (2) solving constrained maximization problem (1) becomes

$$L = y + \mu \left(1 - \frac{y}{Ck^\alpha l^\beta f^\gamma} \right) + \nu \left(1 - T \frac{k^{\theta_1} l^{\theta_2} f^{\theta_3}}{b} \right) + \lambda (b_0 - b) \quad (6)$$

with multipliers $\mu, \nu, \lambda \geq 0$.

The first-order conditions for a stationary point of (6) are:

$$\begin{cases} \frac{\partial L}{\partial y} = 1 - \frac{\mu}{Ck^\alpha l^\beta f^\gamma} = 0 \\ \frac{\partial L}{\partial b} = \nu \frac{Tk^{\theta_1} l^{\theta_2} f^{\theta_3}}{b^2} - \lambda = 0 \\ \frac{\partial L}{\partial k} = \frac{\mu\alpha}{k} - \nu \frac{T\theta_1 k^{\theta_1-1} l^{\theta_2} f^{\theta_3}}{b} = 0 \\ \frac{\partial L}{\partial l} = \frac{\mu\beta}{l} - \nu \frac{T\theta_2 k^{\theta_1} l^{\theta_2-1} f^{\theta_3}}{b} = 0 \\ \frac{\partial L}{\partial f} = \frac{\mu\gamma}{f} - \nu \frac{T\theta_3 k^{\theta_1} l^{\theta_2} f^{\theta_3-1}}{b} = 0 \end{cases} \quad (7)$$

By Theorem 1 each constraint in the constrained optimization problem (1) binds at the optimum, so complementary slackness implies $\lambda, \mu, \nu > 0$. Denote y^* the optimal solution to (1).

From the first equation in (7) and the fact that constraint $D^y(\square) \leq 1$ is binding, $\mu = y^*$. The second equation in (7) implies that $\lambda = \frac{\nu}{b}$ since by Theorem 1 the constraint $D^b(\square) \leq 1$ is binding at the optimum so $b = Tk^{\theta_1} l^{\theta_2} f^{\theta_3}$. Since constraint $D^y(\square) \leq 1$ binds at the optimum, i.e. $y^* = Ck^\alpha l^\beta f^\gamma$, the third equation in (7) implies $\mu \frac{\alpha}{k} - \nu \frac{\theta_1}{k} = 0$ or $\mu\alpha = \nu\theta_1$. Since $\mu = y^*$ we have $\nu = \frac{y^* \alpha}{\theta_1}$.

Similarly, from the FOC on l , $\mu\beta = \nu\theta_2$ and $\nu = \frac{y^* \beta}{\theta_2}$. Finally, from the FOC for f , $\nu = \frac{y^* \gamma}{\theta_3}$.

Equating the above expressions for ν we obtain $\nu = \frac{y^* \alpha}{\theta_1} = \frac{y^* \beta}{\theta_2} = \frac{y^* \gamma}{\theta_3}$, which results in the

following proportionality constraints on the polluting technology's parameters:

$$\frac{\alpha}{\theta_1} = \frac{\beta}{\theta_2} = \frac{\gamma}{\theta_3} = m \quad (8)$$

where $m = \frac{\nu}{y^*}$, so $\nu = my^*$.

The first-order conditions reveal a critical necessary condition that a solution to the set of first-order conditions (7) must satisfy: the ratios of the good output elasticities to the bad output elasticities must be equal across all inputs. The proportionality constraint in (8) emerges endogenously from finding the stationary points of Lagrangian in (6) and defines the trade-off between the two sub-technologies.

From (8) and the fact that constraint $D^y(\square) \leq 1$ binds at the optimum it follows that $y^* = Ck^\alpha l^\beta f^\gamma = Ck^{m\theta_1} l^{m\theta_2} f^{m\theta_3} = C(k^{\theta_1} l^{\theta_2} f^{\theta_3})^m$. Since constraints $D^b(\square) \leq 1$ and $b \leq b_0$ bind at the optimum as well, $b_0 = Tk^{\theta_1} l^{\theta_2} f^{\theta_3}$ and we obtain an explicit form PPF as follows:

$$y^*(b_0) = C \left(\frac{b_0}{T} \right)^m \quad (9)$$

where m is given by the proportionality requirement in (8). If $m < 1$, the production possibilities frontier $y^*(b_0)$ is concave, which is consistent with the property of decreasing marginal costs of reducing emissions with more pollution. MACs are then trivially computed as

$$MAC(b_0) = \frac{mC}{T^m} b_0^{m-1} \quad (10)$$

4.3 Empirical evaluation of the polluting technology parameters

We estimate the parameters of the two distance functions in (5) by maximizing a constrained likelihood function derived from a stochastic frontier analysis (SFA) framework. This approach explicitly accounts for the fact that observed combinations of inputs and outputs may deviate from the efficient frontier *both ways* due to exogenous random shocks and technical inefficiency.

Formally, for each producer $i = 1..N$ and year $t = 1..T$, the deterministic efficient boundary of the production possibilities set is given by $D^y(y, k, l, f) = 1$ and $D^b(b, k, l, f) = 1$. In case of the inefficient production practices such as e.g. the choice of an inefficient input mix (k', l', f') for a particular level of pollution b that is different from the efficient (k^*, l^*, f^*) derived in Theorem 1,

the observed values of $\frac{y}{Ck^\alpha l^\beta f^\gamma}$ and $T \frac{k^{\theta_1} l^{\theta_2} f^{\theta_3}}{b}$ defining $D^y(\square)$ and $D^b(\square)$, respectively, will be strictly less than unity. If in addition production practices are affected by exogenous shocks outside of control of the polluting producers, these observed values may be greater or less than unity.

Denoting the observed values of the two ratios $\frac{y}{Ck^\alpha l^\beta f^\gamma}$ and $T \frac{k^{\theta_1} l^{\theta_2} f^{\theta_3}}{b}$ as $\tilde{D}^y(y, k, l, f)$ and $\tilde{D}^b(b, k, l, f)$, respectively, we formulate the following stochastic frontier model:

$$\left\{ \begin{array}{l} \frac{\tilde{D}^y(y_{it}, k_{it}, l_{it}, f_{it})}{D^y(y_{it}^*, k_{it}^*, l_{it}^*, f_{it}^*)} = e^{v_{it}^y - u_{it}^y} \\ \frac{\tilde{D}^b(b_{it}^0, k_{it}, l_{it}, f_{it})}{D^b(b_{it}^0, k_{it}^*, l_{it}^*, f_{it}^*)} = e^{v_{it}^b - u_{it}^b} \\ u_{it}^y \sim f_u(u_{it}^y) = \mu e^{-\mu u_{it}^y} \\ u_{it}^b \sim f_u(u_{it}^b) = \lambda e^{-\lambda u_{it}^b} \\ v_{it}^y \sim i.i.d.N(0, \sigma_y^2) \\ v_{it}^b \sim i.i.d.N(0, \sigma_b^2) \end{array} \right. \quad (11)$$

where b_{it}^0 is the level of emissions b in country i in year t , and y_{it}^* is the theoretical maximum value of the good output y from Theorem 1. Vector $(k_{it}^*, l_{it}^*, f_{it}^*)$ is an optimal input mix for b_{it}^0 defined in Theorem 1. As established in the proof of Theorem 1, this efficient input mix is not necessarily unique. The left-hand side of each equation in (11) then represents deviations of the actually observed $\tilde{D}^y(y, k, l, f)$ and $\tilde{D}^b(b, k, l, f)$ from their optimal counterparts $D^y(y_{it}^*, k_{it}^*, l_{it}^*, f_{it}^*)$ and $D^b(b_{it}^0, k_{it}^*, l_{it}^*, f_{it}^*)$.

Since by construction, $D^y(y_{it}^*, k_{it}^*, l_{it}^*, f_{it}^*) = D^b(b_{it}^0, k_{it}^*, l_{it}^*, f_{it}^*) = 1$, model (11) reduces to

$$\begin{cases} \tilde{D}^y(y_{it}, k_{it}, l_{it}, f_{it}) = e^{v_{it}^y - u_{it}^y} \\ \tilde{D}^b(b_{it}^0, k_{it}, l_{it}, f_{it}) = e^{v_{it}^b - u_{it}^b} \end{cases} \quad (12)$$

where u -s and v -s are distributed as specified in (11).

Following the standard result from the stochastic frontier analysis in Aigner et al. (1977) and Meeusen and van den Broeck (1977), the composite error term $\varepsilon_{it}^y = v_{it}^y - u_{it}^y$ is distributed according to the density $f_\varepsilon(\varepsilon_{it}) = \mu \exp\left(\mu\varepsilon + \frac{\mu^2\sigma_v^2}{2}\right) \Phi\left(-\frac{\varepsilon + \mu\sigma_v^2}{\sigma_v}\right)$ where $\Phi(\cdot)$ is the standard normal cumulative density and σ_v^2 is the variance of the symmetric component of the composite error term ε_{it} . Density for the composite error term $\varepsilon_{it}^b = v_{it}^b - u_{it}^b$ is determined analogously.

The log-likelihood from (12) is

$$\begin{aligned} l_{NT}(\vec{\Psi}) &= \\ &= \sum_{i=1}^N \sum_{t=1}^T \left[\ln(\mu) + \mu\varepsilon_{it}^y + \frac{\mu^2\sigma_y^2}{2} + \ln\Phi\left(-\frac{\varepsilon_{it}^y + \mu\sigma_y^2}{\sigma_y}\right) + \ln(\lambda) + \lambda\varepsilon_{it}^b + \frac{\lambda^2\sigma_b^2}{2} + \ln\Phi\left(-\frac{\varepsilon_{it}^b + \lambda\sigma_b^2}{\sigma_b}\right) \right] \end{aligned} \quad (13)$$

where $\varepsilon_{it}^y = \ln y_{it} - \ln C - \alpha \ln k_{it} - \beta \ln l_{it} - \gamma \ln f_{it}$ and $\varepsilon_{it}^b = \ln T + \theta_1 \ln k_{it} + \theta_2 \ln l_{it} + \theta_3 \ln f_{it} - \ln b_{it}^0$ and $\vec{\Psi} = (\alpha, \beta, \gamma, \theta_1, \theta_2, \theta_3, C, T, \mu, \lambda, \sigma_y, \sigma_b)$ is the parameter vector.

The constrained maximization problem is as follows:

$$\left\{ \begin{array}{l} \max_{\vec{\Psi}} l_{NT}(\vec{\Psi}) \\ s.t. \\ \vec{\Psi} > 0 \\ m < 1 \\ \frac{\alpha}{\theta_1} = \frac{\beta}{\theta_2} = \frac{\gamma}{\theta_3} = m < 1 \\ \alpha, \beta, \gamma, \theta_1, \theta_2, \theta_3 > 0 \end{array} \right. \quad (14)$$

We solve (14) with the maxLik package developed by Henningsen and Toomet (2011) for the R environment.

4.4 Data and estimation results

We apply our methodology to sixteen Asia-Pacific economies over 2010–2019. The region is characterized by economic and environmental heterogeneity: it includes four advanced economies (Korea, Japan, New Zealand, and Australia), several smaller developing countries (e.g., Malaysia and Thailand), and two major carbon emitters (China and India), capturing a wide spectrum of development stages and emission profiles.

Our approach assumes a common production technology across countries that remains approximately constant over time. Gross domestic product (GDP, constant 2017 US dollars), net physical capital stock (constant 2017 US dollars), and population data are from the Penn World Tables (PWT), while carbon dioxide (CO₂) emissions (million tonnes) and energy consumption (tons of oil equivalent) are obtained from the World Bank’s World Development Indicators (WDI).

The study period ends before 2020 to avoid the unprecedented economic disruption caused by the COVID-19 pandemic, which led to temporary, non-technological collapses in output and emissions, creating severe outliers and potentially distorting the estimation of production technology parameters. A ten-year period is short enough for the assumption of roughly constant technology to remain reasonable. The final balanced panel consists of sixteen countries and 160 country-year observations.

In the distance functions (5), we use energy consumption per capita as the polluting input instead of total energy. This choice is motivated by a high correlation (95%) between total energy use (in tons of oil equivalent) and physical capital, which raises multicollinearity concerns. Using the per-capita measure reduces this correlation to 18.7%, thereby mitigating potential estimation bias. Moreover, the correlation with labor shifts from +80% (total energy) to −48% (per capita), reflecting a sample composition effect: highly developed, energy-intensive economies (e.g., South Korea, Australia, Japan) tend to have smaller populations, whereas populous nations (e.g., China, India) exhibit lower per-capita energy use.

Table 1 presents summary statistics for sixteen Asia-Pacific economies over 2010–2019.

Table 1: Summary Statistics

	Mean	Median	Standard Deviation	Min	Max
CO₂, gigaton emissions	1.061	0.227	2.569	0.014	11.851
GDP, trillion USD, constant 2017 prices	2.553	0.814	4.356	0.149	20.573
Physical Capital, trillion USD, constant 2017 prices	10.156	2.390	17.388	0.438	99.609
Population, million people	240.413	79.00	428.034	4.00	14343.00
Fuel Consumption, TOE per capita	2.269	1.924	1.887	0.196	6.313

Note: TOE stands for “tons of oil equivalent.” GDP, physical capital and population data come from the Penn World Tables, while CO₂ emissions and per-capita fuel consumption are from the World Development Indicators (WDI).

All variables are positively skewed with the mean exceeding the median due to large economies such as China, India, Japan, and Australia. GDP and physical capital exhibit particularly wide variation with maximum values far above the median, reflecting the contrast between advanced and smaller developing economies. Population is highly concentrated in the largest countries, i.e. China and India, while CO₂ emissions and per capita fuel consumption vary substantially, illustrating the region’s diversity in economic scale and energy use.

4.4.1 Parameters of the polluting technology

Table 2 presents the estimated parameters of two distance functions in (5).

Table 2: Estimated Parameters of $D^y(\cdot)$ and $D^b(\cdot)$

Parameters	$D^y(y, k, l, f) = \frac{y}{Ck^\alpha l^\beta f^\gamma}$	Parameters	$D^b(b, k, l, f) = \frac{Tk^{\theta_1} l^{\theta_2} f^{\theta_3}}{b}$
Capital α	0.217 (0.027)***	Capital θ_1	0.303 (0.036)***
Labor β	0.605 (0.016)***	Labor θ_2	0.846 (0.028)***
TOE per capita γ	0.565 (0.016)***	TOE per capita θ_3	0.789 (0.027)***
Rate parameter μ	1.525 (0.140)***	Rate parameter λ	3.148 (0.306)***
C	0.070 (0.002)***	T	0.003 (0.001)***
μ	1.525 (0.140)***	λ	3.148 (0.306)***
σ_y	0.002 (0.001)**	σ_b	0.096 (0.020)***
m	0.715 (0.008)***		

Note: Standard errors are in parentheses. *** indicates significance at a 1% level, ** is a 5% level. Rate parameters refer to the exponential distributions of the distance functions. Parameter m links the technology coefficients in both

distance functions as $m = \frac{\alpha}{\theta_1} = \frac{\beta}{\theta_2} = \frac{\gamma}{\theta_3}$. Only three decimal places are reported for each coefficient so

$\frac{C}{T^m} = 4.456$ computed from the values reported in Table 2 is marginally different from the value of 4.756 reported below in the PPF formula, Section 4.4.2.

All estimated parameters are statistically significant and have the expected signs. Standard errors were computed using the asymptotic properties of maximum likelihood estimators. The covariance matrix was estimated by inverting the observed Fisher information matrix (the negative Hessian of the log-likelihood), with the robust Penrose pseudo-inverse employed to ensure numerical stability.

Our results imply that, in addition to physical capital and labor, fuel input plays an important role in the production of both desirable and undesirable outputs. While labor does not directly produce

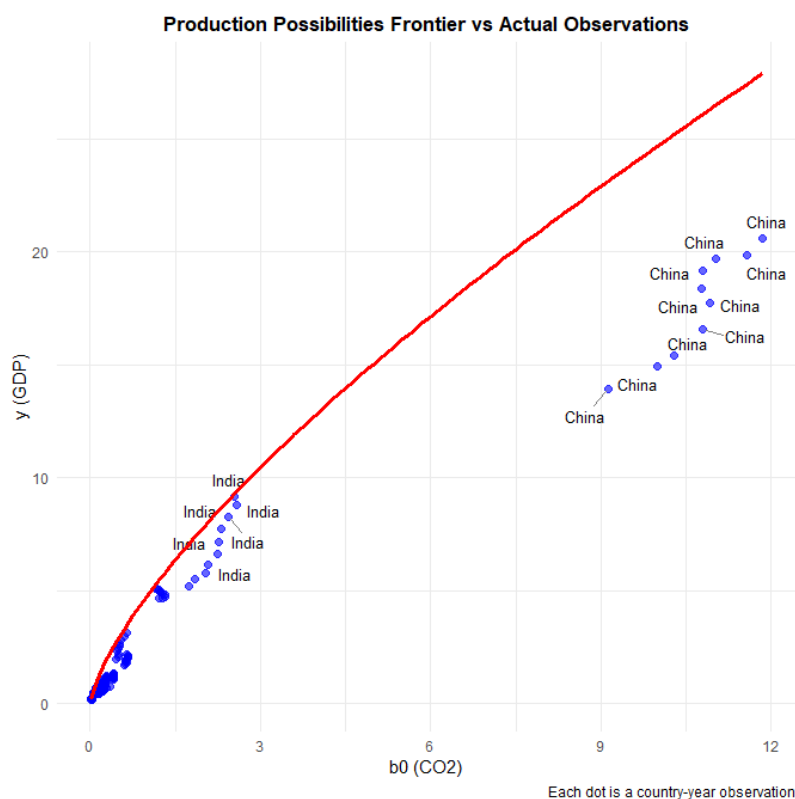
pollution, it contributes to harmful emissions by operating capital equipment that consumes polluting inputs, such as a blast furnace at a steel plant using coke and injected coal. Coke chemically strips oxygen from iron ore (Fe_2O_3) to produce pure molten iron (Fe), releasing CO_2 into the atmosphere. The injected coal acts as an auxiliary fuel, combusting to sustain the extreme temperatures required for the process and generating further emissions. Neither the chemical reduction by coke nor the intensive combustion of coal—the direct sources of pollution—would be possible without the labor required to operate and maintain the furnace. Thus, labor’s environmental effect arises indirectly through its interaction with energy-intensive capital.

4.4.2 *Explicit-form PPF and the marginal abatement costs*

Our estimates imply the following power-function PPF from (9): $y^*(b_0) = C \left(\frac{b_0}{T} \right)^m = 4.756 \times b_0^{0.715}$

with the marginal abatement costs $MAC(b_0) = \frac{mC}{T^m} b_0^{m-1} = 3.023 \times b_0^{-0.285}$ from (10). Figure 1 below shows the theoretical PPF superimposed on a scatterplot of the actually observed levels of good output y and pollution levels b :

Figure 1: Theoretical PPF versus Observed Output and Pollution Levels



Note: y is in trillion USD, b_0 is in gigaton CO_2 . Red line is the estimated PPF, blue points are the actually observed combinations of pollution and good output.

Consistent with our theoretical framework, all of the observed combinations of polluting output b and good output y lie below the estimated PPF represented by a concave line in Figure 1 despite the fact that we allow for the possibility of exogenous shocks in (11) that could in principle push the observed values of good output y above the estimated PPF. China is a clear outlier in our sample

both in terms of the size of its GDP and pollution, and in terms of the distance from the PPF, suggesting a substantial room for efficiency improvements in this economy. The same, although to a smaller extent, refers to India. It is important to keep in mind, however, that the vertical distance between the estimated PPF and the actually observed level of good output y is not entirely due to the inefficiency as part of this distance is captured by the exogenous shocks whose magnitude we are not evaluating in this study.

Our approach to estimating the marginal abatement costs of emissions in (10) consists of evaluating them as slopes of the PPF at the vertical projection of the observed output combination (b, y) . We report our estimates of these shadow prices in Table 3.

Table 3: Marginal Abatement Costs of Reducing CO₂, thousand USD / ton CO₂: 2010-2019

Country	Median	Standard Deviation	Min	Max	Average GDP, tr USD	Average CO ₂ , Gigaton
China	1.727	0.037	1.682	1.811	17.617	10.717
India	2.699	0.102	2.598	2.903	7.042	2.206
Japan	3.203	0.043	3.138	3.277	4.886	1.241
Korea	3.867	0.039	3.811	3.937	1.943	0.640
Indonesia	4.129	0.121	3.867	4.287	2.509	0.522
Australia	4.388	0.018	4.368	4.427	1.209	0.408
Thailand	4.911	0.072	4.866	5.070	1.070	0.270
Malaysia	5.091	0.099	5.018	5.266	0.667	0.239
Viet Nam	5.464	0.407	4.629	5.772	0.566	0.209
Pakistan	5.600	0.195	5.282	5.782	0.883	0.179
Philippines	6.404	0.383	5.862	6.874	0.717	0.112
Bangladesh	6.962	0.412	6.410	7.553	0.570	0.084
Singapore	7.835	0.108	7.700	8.036	0.423	0.053
Hong Kong	8.257	0.076	8.100	8.378	0.340	0.045
New Zealand	8.841	0.097	8.680	9.016	0.171	0.035
Sri Lanka	10.555	0.623	9.845	11.590	0.240	0.020

Note: Authors' calculations. GDP is in trillions of 2017 dollars. Average values are computed for the period between 2010 and 2019.

As expected, the median MACs tend to decrease with a larger average amount of pollution for a ten-year period, reflecting the fact that reducing pollution is associated with higher opportunity costs in terms of the forgone GDP in case of a smaller economy. Thus, the highest MACs are estimated for Sri Lanka at the level of \$10,555 per ton of CO₂ that is also characterized by the smallest amount of total pollution at the level of 0.02 Gigatons of carbon dioxide. China, on the other hand, is by far the largest polluter of all economies in our sample emitting 10.7 Gigatons of CO₂ on average and having to pay \$1,727 for reducing one ton of carbon dioxide emissions.

Our estimates of the CO₂ marginal abatement costs (MACs) are substantially higher than those reported in several recent country-level studies. For instance, Jiang et al. (2022) find MACs below \$60 per ton for China and India in a general equilibrium framework, while Chu (2023) estimates a range of \$40–\$300 per ton for Vietnam's rice sector, and Tang et al. (2021) report a mean shadow price of \$17.60 per ton for Australian dryland agriculture.

This orders-of-magnitude discrepancy stems from a fundamental difference in the underlying abatement concept. The cited studies primarily capture the cost of shifting the production possibilities frontier (PPF) inward through sector-specific adjustments or, crucially, by incorporating technological change and input substitution (e.g., switching to cleaner fuels or more efficient practices). The MAC estimates based on these technological changes thus reflect the cost of *avoiding* emissions by altering the production process. In contrast, our approach estimates the pure opportunity cost of reducing emissions while holding the technology constant—that is, the direct economic output that must be sacrificed by curtailing polluting production activities.

In terms of Figure 1, while the cited studies measure the cost of moving the PPF itself, we measure the steepness of the existing frontier. Consequently, our higher estimates reveal the significant economic cost of achieving emission reductions without technological progress, highlighting the critical value of the investment and innovation considered in other models.

5. Discussion and Conclusion

This paper introduces a novel methodology for deriving the production possibility frontier and marginal abatement costs within the by-production framework. Our primary contribution is threefold. First, we provide an explicit-form derivation of the PPF from the underlying technology given by the two distance functions representing the subtechnologies of producing a good and a bad output. Second, we establish the precise and relatively mild regularity conditions required for the PPF existence and for the key efficiency properties, i.e. $D^y(\square) = D^b(\square) = 1$ and $b = b_0$ to hold at the optimum, which are essential for the existence of an explicit-form PPF. Third, we compute the MACs directly as slopes of this PPF, which obviates the need for an arbitrary choice of a directional output vector used in the models based on the estimation of directional output distance functions, and ensures that MACs are consistent with their classical definition as a marginal rate of transformation, rather than a ratio of the inefficiency scores from a dual formulation.

Despite its theoretical coherence, our empirical framework entails several assumptions that point to the necessity of future research beyond the obvious extension to the case of multiple good and bad outputs.

A primary consideration is the potential endogeneity of inefficiency. Our model assumes independent inefficiency terms in the production of good and bad outputs. Yet, it is plausible that a country's technical inefficiency in GDP production is correlated with its inefficiency in pollution management. While beyond the present scope, modeling this dependency—for instance, through copula-based methods as explored in Repkine (2014)—would yield a more robust representation of the polluting technology.

Furthermore, the specification of the polluting technology warrants further investigation. While the Cobb-Douglas form is a natural choice for the good-output distance function $D^y(\square)$, the

theoretical basis for its application to the bad-output technology $D^b(\square)$ is less firm. Future work could employ alternative flexible functional forms or incorporate physically grounded models to better capture the properties of a sub-technology generating pollution.

Our analysis also aggregates countries under a single global technological frontier. A logical extension is to account for technological heterogeneity using a meta-frontier framework. Segmenting countries into groups (e.g., by development stage or region) would allow for a decomposition of inefficiency into a managerial component (the distance from a group's frontier) and a technology-gap component (the distance between the group frontier and the meta-frontier), providing deeper insights into the sources of performance differentials.

The case of China, which our results identify as a significant outlier, illustrates another refinement opportunity. The estimated distance to the frontier conflates persistent technical inefficiency with transient exogenous shocks. Applying a conditional efficiency approach developed in Jondrow et al. (1982) would allow one to isolate the pure inefficiency components of the composite error terms in (11), making it possible to calculate the extent to which a particular combination of pollution levels and GDP falls short of the estimated PPF due to the inefficient production practices alone.

Our derivation benefits from the explicit specifications of the distance functions $D^y(\square)$ and $D^b(\square)$, which yields an explicit-form PPF. However, our estimation framework is applicable even in the absence of these distance functions' explicit-form specifications. For any well-defined distance functions, the PPF can be numerically reconstructed by solving the constrained maximization problem over a grid of emissions caps, with MACs computed via numerical differentiation, ensuring the general applicability of our approach in those cases where an explicit-form derivation of the PPF is impossible.

Deriving more general necessary and sufficient conditions for the PPF concavity remains an important task. Global strict convexity of the feasible set $F(b_0)$ is sufficient for the PPF concavity in our framework, but appears to be too restrictive as it requires, for instance, decreasing returns to scale in the production of good output in case of the distance functions' specifications in (5). In other words, while strict convexity of $F(b_0)$ guarantees strict concavity of the PPF, it is not a necessary condition, as demonstrated by our empirical finding of a concave PPF in the absence of

strict convexity of the feasible set. Identifying milder conditions that guarantee non-increasing MACs with pollution would definitely be desirable.

Finally, our framework carries significant implications for policymakers as it enables the analysis of large-scale emission reductions as opposed to the marginal abatement costs associated with e.g. one ton of pollutant. For instance, our model can quantify the total economic cost of achieving emissions reductions on the order of million tons of CO₂, a calculation that is elusive when considering only the cost of the last ton abated. This shift from a marginal to a perspective based on the entire production possibilities set in the space of good and bad outputs appears to be essential for designing coherent policies aimed at a realistic reduction of harmful emissions.

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