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Efficient and long-time accurate partitioned method with multirate time steps for the nonstationary Stokes-Darcy model

Guangzhi Du * Liyun Zuo†

Abstract

This study investigates a multirate, decoupled scheme that adopts different time steps in different physical problem subdomains for the unsteady Stokes-Darcy system with the Beavers-Joseph interface condition. Under a modest time step constraint of the form $\Delta t \leq C(\text{phiysical parameters},r)$, we prove that the proposed algorithm is unconditional and long-time (over $0 \leq t < \infty$) stability. Based on this, we establish the optimal error estimation which is uniformly bounded in time over $0 \leq t < \infty$. Some numerical experiments are reported to demonstrate the theoretical results.

Keywords: Stokes-Darcy system; multiple-time-step; domain decomposition method; unconditional stability; long-time stability; uniform in time error estimate

MSC: 65M55, 65M12, 65M15, 65M60, 35Q35, 76D07, 76S05

1 Introduction

The mixed Stokes-Darcy system has many applications in various real life problems that involve but are not restricted to, the surface water and groundwater system, flow in vuggy porous media or around a horizontal wellbore, the flow of blood in the blood vessels, engineering filtration, and others. Thereby, there is not surprised that a large number of numerical algorithms have been developed and investigated for this coupled model, such as domain decomposition methods (DDMs) [4, 9, 16, 25], decoupled marching schemes [10, 11, 19, 24, 27, 28], coupled finite element methods [1, 7, 8], two-grid algorithms [3, 12, 13, 22, 23], and others [2, 20].

The nonstationary mixed Stokes-Darcy problem has different physical processes that are both time-dependent, thus the code for each sub-process needs to be optimized to solve the coupled system. In the surface water and groundwater system, the problems that evaluate the penetration of a plume of pollution from surface water to groundwater and remediation after such an infiltration are particularly important. For this, computations over long time interval are indispensable due to the fact that the spread and control of pollutants often exist a very long time scale. Consequently, it becomes particularly important to develop numerical algorithms that are long-time stability in the sense that approximations retain uniformly bounded in time. To establish the analysis of long-time stability, one should avoid using the Gronwall's inequality since the use of Gronwall's inequality results in a constant $C(T) \sim e^{aT}$ that increases exponentially with the final time T. To our knowledge, there were few studies on the analysis of long-time stability for numerical solutions to the mixed Stokes-Darcy problem. For instance, Layton et al. established the analysis of long time (over $0 \leq t < \infty$) stability of two first-order

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partitioned methods in [19], and derived the optimal error estimation. The long-time stability and errors of two second-order schemes and one third-order scheme were investigated by Chen et al. in [10, 11]. The long time stability of other four methods were studied in [21].

The fluid speed in the free flow region is much higher than in the porous media, hence it is natural to utilize a multirate-time-step scheme that allows use a small time step to solve the free flow system and a larger time step to solve the porous medium flow system. For the mixed unsteady Stokes-Darcy model, the first partitioned method with multirate time steps was proposed and investigated in [27] wherein the numerical analysis were established by using the Gronwall's inequality. A multiple-time-step scheme was investigated in [26] where the authors proved the long time stability under a time step restriction of the form $\Delta t \leq C(physical\ parameters, r)$.

In this study, we aim to consider a partitioned method with different subdomain time steps for the unsteady Stokes-Darcy with Beavers-Joseph (BJ) interface condition. Compared with the BJ interface condition, the simplified Beavers-Joseph-Saffman (BJS) interface condition neglects some contributions coming from the flow in the porous media flow to the coupling system. Note that the neglectful contributions might be significant in some applications such as karst aquifers. Thereby, the BJ interface condition is more accurate since it strictly reflects the contributions of the two flows in the coupling of the two systems. For the steady Stokes-Darcy model with BJ condition, in [5], Cao et. al. proved the well-posedness under an assumption that the parameter α is small enough. Then, Hou and Qin have perfectly proved the well-posedness for any physical parameters, especially for any $\alpha > 0$ in [18]. Numerical analysis indicates that our algorithm is unconditional and long-time (over $0 \leq t < \infty$) stability under a modest time step constraint of the form $\Delta t \leq C$(phiysical parameters,r) where r denotes the ratio between the coarse time step and the fine time step. The optimal error estimation that is uniformly bounded in time over $0 \leq t < \infty$ is also obtained.

The paper is organized as follows. In Section 2, we firstly show the time-dependent Stokes-Darcy model with BJ interface condition, then get the related weak formulation and describe the partitioned method with different subdomian time steps. In Section 3, we mainly establish the unconditional and long time stability of our algorithm. In Section 4, we give the optimal error estimation that is uniform in time over $0 \leq t < \infty$. In Section 5, we provide some numerical experiments to verify the theoretical findings.

2 Mixed Stokes-Darcy model

The mixed Stokes-Darcy system is defined on a bounded domain $\Omega \subset R^d (d = 2, 3)$ that involves a free-flow domain denoted by Ω_S and a porous medium domain denoted by Ω_D. The common boundary of Ω_S and Ω_D is $\Gamma = \overline{\Omega_S} \cap \overline{\Omega_D}$. Let $\boldsymbol{n}_S$ and $\boldsymbol{n}_D (= -\boldsymbol{n}_S)$ represent the unit outward normal vectors to the free-flow domain and porous media, respectively, and let $\boldsymbol{\tau}_i (i = 1, ..., d-1)$ denote the mutually orthogonal unit tangential vectors along the interface Γ. In this paper, we use t to denote the time variable.

The free flow in Ω_S satisfies the time-dependent Stokes system

$$\begin{cases} \frac{\partial \boldsymbol{u}}{\partial t} - \nabla \cdot \mathbb{T}(\boldsymbol{u}, p) = \boldsymbol{f}_S & \text{in } \Omega_S, \\ \nabla \cdot \boldsymbol{u} = 0 & \text{in } \Omega_S, \end{cases} \tag{2.1}$$

where $\mathbb{T}(\boldsymbol{u}, p) = -p\mathbb{I} + 2\nu\mathbb{D}(\boldsymbol{u})$ denotes the stress tensor, $\mathbb{D}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla^T \boldsymbol{u})$ represents the velocity deformation tensor, $\mathbb{I}$ is the identity matrix, ν is the kinematic viscosity, $\boldsymbol{u}$ is the fluid velocity, p is the kinematic pressure, and $\boldsymbol{f}_S$ is the external body force.

The flow in Ω_D satisfies the Darcy system

$$S_0 \frac{\partial \phi}{\partial t} - \nabla \cdot (\mathbb{K} \nabla \phi) = f_D, \tag{2.2}$$

where S_0 denotes the specific mass storativity coefficient, ϕ represents the piezometric head, $\mathbb{K} = \frac{g\Pi}{\nu}$ is the hydraulic conductivity tensor with g being the gravitational acceleration and Π the permeability of the porous media that satisfies $\mathbb{K} = \frac{\Pi g}{\nu}$. f_D denotes the body force. In this paper, we assume that $\mathbb{K}$ is a symmetric positive definite matrix such that it has the smallest eigenvalue $k_{min} > 0$. Let $\boldsymbol{u}_D$ denote the fluid discharge rate in the porous media, then it could be recovered from the formulation $\boldsymbol{u}_D = -\mathbb{K}\nabla\phi$.

To complete the mixed system, we consider the three interface conditions as follows

$$\begin{cases} \boldsymbol{u}\cdot\boldsymbol{n}_S = -\boldsymbol{u}_D\cdot\boldsymbol{n}_D, \\ -\boldsymbol{n}_S\cdot(\mathbb{T}(\boldsymbol{u},p)\cdot\boldsymbol{n}_S) = g\phi, \\ -\boldsymbol{\tau}_i\cdot(\mathbb{T}(\boldsymbol{u},p)\cdot\boldsymbol{n}_S) = \frac{\alpha\nu\sqrt{d}}{\sqrt{trace(\Pi)}}\boldsymbol{\tau}_i\cdot(\boldsymbol{u}+\mathbb{K}\nabla\phi), \quad i = 1,...d-1. \end{cases} \tag{2.3}$$

Note that the above three conditions denote the conservation of mass, the balance of forces normal to the interface and the BJ interface condition, respectively.

Note that boundary conditions are not essential to either the scheme or the theoretical analysis. For simplicity, we impose the following initial-boundary value conditions:

$$\boldsymbol{u}(x,0) = \boldsymbol{u}_0 \quad \text{in } \Omega_S, \quad \phi(x,0) = \phi_0 \quad \text{in } \Omega_D; \tag{2.4}$$

$$\boldsymbol{u}(x,t) = 0 \quad \text{on } \Gamma_S, \quad \phi(x,t) = 0 \quad \text{on } \Gamma_D. \tag{2.5}$$

For a region D ($D = \Omega_S$ or Ω_D), the L^2 inner product and $L^2(D)$ norm are denoted by $(\cdot,\cdot)_D$ and $\|\cdot\|_D$, respectively. The L^2 inner product on the interface Γ or the duality pairing between $(H_{00}^{1/2}(\Gamma))'$ and $H_{00}^{1/2}(\Gamma)$, and the $L^2(\Gamma)$ norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|_\Gamma$, respectively. Let us introduce the following functional spaces

$$H_S = \{\boldsymbol{v}\in\left(H^1(\Omega_S)\right)^d : \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_S\},\ Q = L^2(\Omega_S),$$
$$H_D = \{\psi\in H^1(\Omega_D) : \psi = 0 \text{ on } \Gamma_D\}.$$

In the rest of this article, we shall use C to denote a generic positive constant that does not rely on mesh size, time step, and final time. Note that the C that appears in different places represents different constants.

We introduce the following notations

$$a_S(\boldsymbol{u},\boldsymbol{v}) = 2\nu(\mathbb{D}(\boldsymbol{u}),\mathbb{D}(\boldsymbol{v}))_{\Omega_S} + \frac{\alpha\nu\sqrt{d}}{\sqrt{trace(\Pi)}}\langle P_{\boldsymbol{\tau}}\boldsymbol{u}, P_{\boldsymbol{\tau}}\boldsymbol{v}\rangle, \quad P_{\boldsymbol{\tau}}\boldsymbol{u} = \sum_{i=1}^{d-1}(\boldsymbol{u}\cdot\boldsymbol{\tau}_i)\boldsymbol{\tau}_i,$$
$$a_D(\phi,\psi) = (\mathbb{K}\nabla\phi,\nabla\psi)_{\Omega_D}, \quad b(\boldsymbol{v},p) = -(\nabla\cdot\boldsymbol{v},p)_{\Omega_S},$$
$$a_\Gamma(\boldsymbol{u},\psi) = g\langle\boldsymbol{u}\cdot\boldsymbol{n}_S,\psi\rangle \quad \widetilde{a}_\Gamma(\phi,\boldsymbol{v}) = \frac{\alpha\nu\sqrt{d}}{\sqrt{trace(\Pi)}}\langle P_{\boldsymbol{\tau}}(\mathbb{K}\nabla\phi), P_{\boldsymbol{\tau}}\boldsymbol{v}\rangle.$$

The variational formulation of the mixed Stokes-Darcy system reads as: Seek $(\boldsymbol{u},p,\phi) : [0,\infty)\to H_S\times Q\times H_D$ to satisfy the given initial conditions, and for $\forall(\boldsymbol{v},q,\psi)\in H_S\times Q\times H_D$ such that

$$(\frac{\partial\boldsymbol{u}}{\partial t},\boldsymbol{v})_{\Omega_S} + a_S(\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v},p) + a_\Gamma(\boldsymbol{v},\phi) + \widetilde{a}_\Gamma(\phi,\boldsymbol{v}) = (\boldsymbol{f}_S,\boldsymbol{v})_{\Omega_S}, \tag{2.6}$$

$$b(\boldsymbol{u},q) = 0, \tag{2.7}$$

$$g(S_0\frac{\partial\phi}{\partial t},\psi)_{\Omega_D} + ga_D(\phi,\psi) - a_\Gamma(\boldsymbol{u},\psi) = g(f_D,\psi)_{\Omega_D}. \tag{2.8}$$

For the steady, mixed Stokes-Darcy model with the BJ interface condition, its well-posedness has been investigated in [5, 18]. The finite element method for the unsteady case has been

studied in [6]. Note that in [18], Hou and Qin established the uniqueness result of the solution to the steady-state mixed Stokes/Darcy model with Beavers-Joseph interface condition for any physical parameters, especially for any $\alpha > 0$. In this paper, similar to [5], we always assume that the parameter α is small enough to establish our analysis results.

Let $\mathcal{T}_h$ be a regular and quasi-uniform triangulation of Ω with maximum triangle diameter being h. Let $\mathcal{T}_{Sh}$ and $\mathcal{T}_{Dh}$ be regular and quasi-uniform triangulations of Ω_S and Ω_D, respectively. Let $H_{S,h}, Q_h$ and $H_{D,h}$ be the responding finite element spaces. For simplicity, we choose the following specific finite element spaces

$$\begin{aligned} H_{S,h} &= \{\boldsymbol{v} \in (H^1(\Omega_S))^d : \boldsymbol{v}|_T \in (\mathbb{P}_2(T))^d,\ \forall\ T \in \mathcal{T}_{Sh},\ \boldsymbol{v}|_{\Gamma_S} = \boldsymbol{0}\}, \\ Q_h &= \{q \in L^2(\Omega_S) : q|_T \in \mathbb{P}_1(T),\ \forall\ T \in \mathcal{T}_{Sh}\}, \\ H_{D,h} &= \{\psi \in H^1(\Omega_D) : \psi|_T \in \mathbb{P}_2(T),\ \forall\ T \in \mathcal{T}_{Dh},\ \psi|_{\Gamma_D} = 0\}. \end{aligned}$$

Note that spaces $H_{S,h}$ and Q_h satisfy the discrete inf-sup condition, see [14, 15].

Note that the fluid velocity in the free flow region is in general much higher than in the porous medium, thereby, we could adopt different subdomain time steps. To do this, let ΔT and Δt denote the coarse time and fine grids, respectively. The ratio between these two time steps is denoted by $r := \frac{\Delta T}{\Delta t}$. Generally speaking, r is a positive integer. Let T be the final time, then define $N = \frac{T}{\Delta t}$ and $M = \frac{T}{\Delta T}$. In this study, the uniform time levels are applied, i.e.,

$$t^{n_k} = k\Delta T, k = 0, 1, ..., M, \quad t^n = n\Delta t, n = 0, 1, ..., N.$$

For simple, we write $\boldsymbol{f}_S^n = \boldsymbol{f}_S(t^n), f_D^{n_k} = f_D(t^{n_k})$ (applicable to other variables). Let $\boldsymbol{P}_h : H_S \to H_{S,h}$ and $P_h : H_D \to H_{D,h}$ be the regular orthogonal projections. With the above notations at hand, we are in the position to describe our algorithm.

Partitioned method with multirate time steps

Let $\boldsymbol{u}_h^0 = \boldsymbol{P}_h\boldsymbol{u}_0, \quad \phi_h^{n_0} = \phi_h^0 = P_h\phi_0$.
for $k = 0$ to $M - 1$ do
  for $n = n_k$ to $n_{k+1} - 1$ do
    Find $(\boldsymbol{u}_h^{n+1}, p_h^{n+1}) \in H_{S,h} \times Q_h$, for $\forall(\boldsymbol{v}_h, q_h) \in H_{S,h} \times Q_h$ such that

$$(\frac{\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n}{\Delta t}, \boldsymbol{v}_h)_{\Omega_S} + a_S(\boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, p_h^{n+1}) = (\boldsymbol{f}_S^{n+1}, \boldsymbol{v}_h)_{\Omega_S} - a_\Gamma(\boldsymbol{v}_h, \phi_h^{n_k}) - \widetilde{a}_\Gamma(\phi_h^{n_k}, \boldsymbol{v}_h), \tag{2.9}$$

$$b(\boldsymbol{u}_h^{n+1}, q_h) = 0. \tag{2.10}$$

  end for
  Find $\phi_h^{n_{k+1}} \in H_{D,h}$, for $\forall\psi_h \in H_{D,h}$ such that

$$(gS_0\frac{\phi_h^{n_{k+1}} - \phi_h^{n_k}}{\Delta T}, \psi_h)_{\Omega_D} + ga_D(\phi_h^{n_{k+1}}, \psi_h) = g(f_D^{n_{k+1}}, \psi_h)_{\Omega_D} + a_\Gamma(\boldsymbol{u}_h^{n_{k+1}}, \psi_h). \tag{2.11}$$

end for

---

3 Long-time stability analysis

In this section, we mainly show the long-time stability for our scheme over $0 \le t^{n_k} < \infty$. Numerical analysis implies that under the time restriction of the form $\Delta t \le C$(physical parameters,r) which will be subsequently obtained, it follows that

- the numerical solutions converge uniformly over $0 < t < \infty$;
- if $\boldsymbol{f}_S = f_D = 0$, $\boldsymbol{u}^{n_k}, \phi^{n_k} \to 0$ as $t^{n_k} \to \infty$, this means that if $(\boldsymbol{u}_i, \phi_i)(i = 1, 2)$ are two solutions derived by the same right-hand side and different initial conditions, then $\boldsymbol{u}_1^{n_k} - \boldsymbol{u}_2^{n_k}, \phi_1^{n_k} - \phi_2^{n_k} \to 0$ as $k \to \infty$;
- $\sup_{t^{n_k}}(\|\boldsymbol{u}^{n_k}\| + \|\phi^{n_k}\|) < \infty$ if $\|\boldsymbol{f}_S(t)\|, \|f_D(t)\|$ are uniformly bounded in time.

Now we present some basic inequalities in the following lemma for the subsequent analysis.

Lemma 3.1 *Let $C_K, C_{P,S}, C_{P,D}$ denote the positive constants in Korn's and Poincaré inequalities of the indicated regions, respectively. Let $\lambda_{max}(\mathbb{K}), \lambda_{min}(\Pi)$ respectively denote the largest and smallest eigenvalues of $\mathbb{K}$ and Π. For $\boldsymbol{u} \in H_S$ and $\phi \in H_D$, it follows that*

$$a_S(\boldsymbol{u}, \boldsymbol{u}) \geq C_K \nu \|\nabla \boldsymbol{u}\|_S^2, \tag{3.1}$$

$$\|\boldsymbol{u}\|_S \leq C_{P,S} \|\nabla \boldsymbol{u}\|_S, \quad \|\phi\|_D \leq C_{P,D} \|\nabla \phi\|_D, \tag{3.2}$$

$$\begin{aligned} \widetilde{a}_\Gamma(\phi, \boldsymbol{v}) &\leq \frac{\alpha \nu \sqrt{d}}{\sqrt{trace(\Pi)}} \|\mathbb{K} \nabla \phi\|_{(H_{00}^{1/2}(\Gamma))'} \|P_{\boldsymbol{\tau}} \boldsymbol{v}\|_{H_{00}^{1/2}(\Gamma)} \\ &\leq \frac{\alpha \nu \lambda_{max}(\mathbb{K})}{\sqrt{\lambda_{min}(\Pi)}} \|\phi\|_{H^1(\Omega_D)} \|\boldsymbol{v}\|_{H^1(\Omega_S)} \\ &\leq \frac{\sqrt{(1 + C_{P,D}^2)(1 + C_{P,S}^2)} \alpha \nu \lambda_{max}(\mathbb{K})}{\sqrt{\lambda_{min}(\Pi)}} \|\nabla \phi\|_D \|\nabla \boldsymbol{v}\|_S, \end{aligned} \tag{3.3}$$

$$\|\boldsymbol{u}\|_\Gamma \leq C(\Omega_S) \sqrt{\|\boldsymbol{u}\|_S \|\nabla \boldsymbol{u}\|_S}, \quad \|\phi\|_\Gamma \leq C(\Omega_D) \sqrt{\|\phi\|_D \|\nabla \phi\|_D}, \tag{3.4}$$

where $C(\Omega_{S/D}) = O(\sqrt{L_{S/D}})$ with $L_{S/D} = diameter(\Omega_{S/D})$.

Remark 3.1 *For the inequality (3.3), we refer the reads to [5] for more details.*

Denote

$$C_1 = \frac{r(r-1) g^2 C_{P,D}^2 C^4(\Omega_S) C^4(\Omega_D)}{4 k_{min}^2}, \quad C_2 = \frac{r g^3 C_{P,S}^2 C^4(\Omega_S) C^4(\Omega_D)}{2 S_0 C_K^2 \nu^2}.$$

and the fine time step restriction

$$\Delta t_{stable} := \min\{\frac{C_K \nu}{16 C_1 (r-1)}, \frac{g k_{min}}{32 C_2}\}. \tag{3.5}$$

With the above preparations completed, we show the unconditional, long-time stability of our algorithm in the following theorem.

Theorem 3.1 *Assume that the parameter α is sufficiently small to satisfy that*

$$\alpha \leq \min\{\frac{g k_{min} \sqrt{\lambda_{min}(\Pi)}}{8\sqrt{(1 + C_{P,D}^2)(1 + C_{P,S}^2)} \nu \lambda_{max}(\mathbb{K})}, \frac{C_K \sqrt{\lambda_{min}(\Pi)}}{4\sqrt{(1 + C_{P,D}^2)(1 + C_{P,S}^2)} \lambda_{max}(\mathbb{K})}\}. \tag{3.6}$$

Under the restriction of the fine time step

$$\Delta t \leq \Delta t_{stable}, \tag{3.7}$$

the partitioned method with multirate time steps is long-time and uniformly-in-time stable

$$\|\boldsymbol{u}_h^N\|_S^2 + g S_0 \|\phi_h^N\|_D^2 + \frac{C_K \nu \Delta t}{4} \|\nabla \boldsymbol{u}_h^N\|_S^2 + \frac{r g k_{min} \Delta t}{4} \|\nabla \phi_h^N\|_D^2$$

$$+\frac{C_K\nu\Delta t}{2}\sum_{k=0}^{M-1}\sum_{n=n_k}^{n_{k+1}-1}\|\nabla \boldsymbol{u}_h^{n+1}\|_S^2+\frac{rgk_{min}\Delta t}{2}\sum_{k=0}^{M-1}\|\nabla\phi_h^{n_{k+1}}\|_D^2 \tag{3.8}$$
$$\le \|\boldsymbol{u}_h^0\|_S^2+\|\phi_h^0\|_D^2+\frac{C_K\nu\Delta t}{4}\|\nabla \boldsymbol{u}_h^0\|_S^2+\frac{rgk_{min}\Delta t}{4}\|\nabla\phi_h^0\|_D^2$$
$$+\sum_{k=0}^{M-1}\sum_{n=n_k}^{n_{k+1}-1}\frac{4C_{P,S}^2\Delta t}{C_K\nu}\|\boldsymbol{f}_S^{n+1}\|_S^2+\sum_{k=0}^{M-1}\frac{2rgC_{P,D}^2\Delta t}{k_{min}}\|f_D^{n_{k+1}}\|_D^2.$$

Note that if $\boldsymbol{f}_S\in L^\infty((L^2(\Omega_S))^d), f_D\in L^\infty(L^2(\Omega_D))$, *there exists a positive constant* C *such that*

$$\sup_{0\le n_k\le\infty}\{\|\boldsymbol{u}_h^{n_k}\|_S^2+gS_0\|\phi_h^{n_k}\|_D^2\}\le C(\sup_{0\le n\le\infty}r\|\boldsymbol{f}_S^n\|_S^2+\sup_{0\le n_k\le\infty}\|f_D^{n_k}\|_D^2)$$
$$+\{\|\boldsymbol{u}_h^0\|_S^2+\frac{C_K\nu\Delta t}{4}\|\nabla\boldsymbol{u}_h^0\|_S^2+gS_0\|\phi_h^0\|_D^2+\frac{gk_{min}\Delta t}{4}\|\nabla\phi_h^0\|_D^2\}. \tag{3.9}$$

In particular, if $\boldsymbol{f}_S\equiv 0$ *and* $f_D\equiv 0$, *it follows*

$$\boldsymbol{u}_h^n\to 0,\quad \phi_h^{n_k}\to 0 \tag{3.10}$$

in $H^1(\Omega_S)$ *and* $H^1(\Omega_D)$, *respectively, as* $n,k\to\infty$.

Proof. Substituting $\boldsymbol{v}_h=\boldsymbol{u}_h^{n+1}$ in (2.9), $q_h=p_h^{n+1}$ in (2.10) and applying the identity that $2(a-b,a)=|a|^2-|b|^2+|a-b|^2$ yield

$$\frac{\|\boldsymbol{u}_h^{n+1}\|_S^2}{2\Delta t}-\frac{\|\boldsymbol{u}_h^n\|_S^2}{2\Delta t}+\frac{\|\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h^n\|_S^2}{2\Delta t}+a_S(\boldsymbol{u}_h^{n+1},\boldsymbol{u}_h^{n+1})$$
$$=(\boldsymbol{f}_S^{n+1},\boldsymbol{u}_h^{n+1})_{\Omega_S}-a_\Gamma(\boldsymbol{u}_h^{n+1},\phi_h^{n_k})-\widetilde{a}_\Gamma(\phi_h^{n_k},\boldsymbol{u}_h^{n+1}). \tag{3.11}$$

Sum of the above equation over the coarse time interval $[t^{n_k},t^{n_{k+1}-1}]$ leads to

$$\frac{\|\boldsymbol{u}_h^{n_{k+1}}\|_S^2}{2\Delta t}-\frac{\|\boldsymbol{u}_h^{n_k}\|_S^2}{2\Delta t}+\sum_{n=n_k}^{n_{k+1}-1}\frac{\|\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h^n\|_S^2}{2\Delta t}+\sum_{n=n_k}^{n_{k+1}-1}a_S(\boldsymbol{u}_h^{n+1},\boldsymbol{u}_h^{n+1})$$
$$=\sum_{n=n_k}^{n_{k+1}-1}(\boldsymbol{f}_S^{n+1},\boldsymbol{u}_h^{n+1})_{\Omega_S}-\sum_{n=n_k}^{n_{k+1}-1}a_\Gamma(\boldsymbol{u}_h^{n+1},\phi_h^{n_k})-\sum_{n=n_k}^{n_{k+1}-1}\widetilde{a}_\Gamma(\phi_h^{n_k},\boldsymbol{u}_h^{n+1}). \tag{3.12}$$

On the other hand, substituting $\psi_h=\phi_h^{n_{k+1}}$ in (2.11) gives

$$\frac{gS_0\|\phi_h^{n_{k+1}}\|_D^2}{2\Delta T}-\frac{gS_0\|\phi_h^{n_k}\|_D^2}{2\Delta T}+\frac{gS_0\|\phi_h^{n_{k+1}}-\phi_h^{n_k}\|_D^2}{2\Delta T}+ga_D(\phi_h^{n_{k+1}},\phi_h^{n_{k+1}})$$
$$=g(f_D^{n_{k+1}},\phi_h^{n_{k+1}})_{\Omega_D}+a_\Gamma(\boldsymbol{u}_h^{n_{k+1}},\phi_h^{n_{k+1}}). \tag{3.13}$$

Not that $\Delta T=r\Delta t$, we obtain

$$\frac{gS_0\|\phi_h^{n_{k+1}}\|_D^2}{2\Delta t}-\frac{gS_0\|\phi_h^{n_k}\|_D^2}{2\Delta t}+\frac{gS_0\|\phi_h^{n_{k+1}}-\phi_h^{n_k}\|_D^2}{2\Delta t}+rga_D(\phi_h^{n_{k+1}},\phi_h^{n_{k+1}})$$
$$=rg(f_D^{n_{k+1}},\phi_h^{n_{k+1}})_{\Omega_D}+ra_\Gamma(\boldsymbol{u}_h^{n_{k+1}},\phi_h^{n_{k+1}}). \tag{3.14}$$

The sum of (3.12) and (3.14) shows

$$\frac{\|\boldsymbol{u}_h^{n_{k+1}}\|_S^2}{2\Delta t}-\frac{\|\boldsymbol{u}_h^{n_k}\|_S^2}{2\Delta t}+\sum_{n=n_k}^{n_{k+1}-1}\frac{\|\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h^n\|_S^2}{2\Delta t}+\sum_{n=n_k}^{n_{k+1}-1}a_S(\boldsymbol{u}_h^{n+1},\boldsymbol{u}_h^{n+1})$$

$$
\begin{aligned}
&+\frac{gS_0\|\phi_h^{n_{k+1}}\|_D^2}{2\Delta t}-\frac{gS_0\|\phi_h^{n_k}\|_D^2}{2\Delta t}+\frac{gS_0\|\phi_h^{n_{k+1}}-\phi_h^{n_k}\|_D^2}{2\Delta t}+rga_D(\phi_h^{n_{k+1}},\phi_h^{n_{k+1}})\\
&=\sum_{n=n_k}^{n_{k+1}-1}(\boldsymbol{f}_S^{n+1},\boldsymbol{u}_h^{n+1})_{\Omega_S}+rg(f_D^{n_{k+1}},\phi_h^{n_{k+1}})_{\Omega_D} \qquad (3.15)\\
&-\sum_{n=n_k}^{n_{k+1}-1}a_\Gamma(\boldsymbol{u}_h^{n+1},\phi_h^{n_k})-\sum_{n=n_k}^{n_{k+1}-1}\widetilde{a}_\Gamma(\phi_h^{n_k},\boldsymbol{u}_h^{n+1})+ra_\Gamma(\boldsymbol{u}_h^{n_{k+1}},\phi_h^{n_{k+1}}).
\end{aligned}
$$

For the terms including $\boldsymbol{f}_S^{n+1}$ and $f_D^{n_{k+1}}$, there holds

$$
\begin{aligned}
&\sum_{n=n_k}^{n_{k+1}-1}(\boldsymbol{f}_S^{n+1},\boldsymbol{u}_h^{n+1})_{\Omega_S}+rg(f_D^{n_{k+1}},\phi_h^{n_{k+1}})_{\Omega_D} \qquad (3.16)\\
&\le C_{P,S}\sum_{n=n_k}^{n_{k+1}-1}\|\boldsymbol{f}_S^{n+1}\|_S\|\nabla\boldsymbol{u}_h^{n+1}\|_S+rgC_{P,D}\|f_D^{n_{k+1}}\|_D\|\nabla\phi_h^{n+1}\|_D\\
&\le\sum_{n=n_k}^{n_{k+1}-1}(\frac{2C_{P,S}^2}{C_K\nu}\|\boldsymbol{f}_S^{n+1}\|_S^2+\frac{C_K\nu}{8}\|\nabla\boldsymbol{u}_h^{n+1}\|_S^2)+\frac{rgC_{P,D}^2}{k_{min}}\|f_D^{n_{k+1}}\|_D^2+\frac{rgk_{min}}{4}\|\nabla\phi_h^{n+1}\|_D^2.
\end{aligned}
$$

Now we estimate the interface terms. Firstly, we have

$$
\begin{aligned}
&ra_\Gamma(\boldsymbol{u}_h^{n_{k+1}},\phi_h^{n_{k+1}})-\sum_{n=n_k}^{n_{k+1}-1}a_\Gamma(\boldsymbol{u}_h^{n+1},\phi_h^{n_k})\\
&=\sum_{n=n_k}^{n_{k+1}-1}a_\Gamma(\boldsymbol{u}_h^{n_{k+1}},\phi_h^{n_{k+1}})-\sum_{n=n_k}^{n_{k+1}-1}a_\Gamma(\boldsymbol{u}_h^{n+1},\phi_h^{n_k}) \qquad (3.17)\\
&=\sum_{n=n_k}^{n_{k+1}-1}a_\Gamma(\boldsymbol{u}_h^{n_{k+1}}-\boldsymbol{u}_h^{n+1},\phi_h^{n_{k+1}})+\sum_{n=n_k}^{n_{k+1}-1}a_\Gamma(\boldsymbol{u}_h^{n+1},\phi_h^{n_{k+1}}-\phi_h^{n_k}).
\end{aligned}
$$

Note that

$$
\begin{aligned}
&a_\Gamma(\boldsymbol{u}_h^{n_{k+1}}-\boldsymbol{u}_h^{n+1},\phi_h^{n_{k+1}})\le g\|\boldsymbol{u}_h^{n_{k+1}}-\boldsymbol{u}_h^{n+1}\|_\Gamma\|\phi_h^{n_{k+1}}\|_\Gamma\\
&\le gC(\Omega_S)C(\Omega_D)\|\boldsymbol{u}_h^{n_{k+1}}-\boldsymbol{u}_h^{n+1}\|_S^{1/2}\|\nabla(\boldsymbol{u}_h^{n_{k+1}}-\boldsymbol{u}_h^{n+1})\|_S^{1/2}\|\phi_h^{n_{k+1}}\|_D^{1/2}\|\nabla\phi_h^{n_{k+1}}\|_D^{1/2}\\
&\le gC(\Omega_S)C(\Omega_D)C_{P,D}^{1/2}\|\boldsymbol{u}_h^{n_{k+1}}-\boldsymbol{u}_h^{n+1}\|_S^{1/2}\|\nabla(\boldsymbol{u}_h^{n_{k+1}}-\boldsymbol{u}_h^{n+1})\|_S^{1/2}\|\nabla\phi_h^{n_{k+1}}\|_D\\
&\le\frac{\epsilon_1}{\Delta t}\|\boldsymbol{u}_h^{n_{k+1}}-\boldsymbol{u}_h^{n+1}\|_S^2+C_1\Delta t\|\nabla(\boldsymbol{u}_h^{n_{k+1}}-\boldsymbol{u}_h^{n+1})\|_S^2+\frac{gk_{min}}{4}\|\nabla\phi_h^{n_{k+1}}\|_D^2,
\end{aligned}
$$

and

$$
\begin{aligned}
&a_\Gamma(\boldsymbol{u}_h^{n+1},\phi_h^{n_{k+1}}-\phi_h^{n_k})\le g\|\boldsymbol{u}_h^{n+1}\|_\Gamma\|\phi_h^{n_{k+1}}-\phi_h^{n_k}\|_\Gamma\\
&\le gC(\Omega_S)C(\Omega_D)\|\boldsymbol{u}_h^{n+1}\|_S^{1/2}\|\nabla\boldsymbol{u}_h^{n+1}\|_S^{1/2}\|\phi_h^{n_{k+1}}-\phi_h^{n_k}\|_D^{1/2}\|\nabla(\phi_h^{n_{k+1}}-\phi_h^{n_k})\|_D^{1/2}\\
&\le gC(\Omega_S)C(\Omega_D)C_{P,S}^{1/2}\|\nabla\boldsymbol{u}_h^{n+1}\|_S\|\phi_h^{n_{k+1}}-\phi_h^{n_k}\|_D^{1/2}\|\nabla(\phi_h^{n_{k+1}}-\phi_h^{n_k})\|_D^{1/2}\\
&\le\frac{\epsilon_2}{\Delta t}\|\phi_h^{n_{k+1}}-\phi_h^{n_k}\|_D^2+C_2\Delta t\|\nabla(\phi_h^{n_{k+1}}-\phi_h^{n_k})\|_D^2+\frac{C_K\nu}{4}\|\nabla\boldsymbol{u}_h^{n+1}\|_S^2,
\end{aligned}
$$

where ϵ_1 and ϵ_2 are two positive constants, and C_1, C_2 satisfy that

$$
C_1=\frac{g^2C_{P,D}^2C^4(\Omega_S)C^4(\Omega_D)}{4\epsilon_1k_{min}^2},\quad C_2=\frac{g^4C_{P,S}^2C^4(\Omega_S)C^4(\Omega_D)}{4\epsilon_2C_K^2\nu^2}.
$$

Thereby, we derive

$$\sum_{n=n_k}^{n_{k+1}-1} a_\Gamma(\boldsymbol{u}_h^{n_{k+1}} - \boldsymbol{u}_h^{n+1}, \phi_h^{n_{k+1}}) = \sum_{n=n_k}^{n_{k+1}-2} a_\Gamma(\boldsymbol{u}_h^{n_{k+1}} - \boldsymbol{u}_h^{n+1}, \phi_h^{n_{k+1}})$$
$$\leq \sum_{n=n_k}^{n_{k+1}-2} (\frac{\epsilon_1}{\Delta t}\|\boldsymbol{u}_h^{n_{k+1}} - \boldsymbol{u}_h^{n+1}\|_S^2 + C_1\Delta t\|\nabla(\boldsymbol{u}_h^{n_{k+1}} - \boldsymbol{u}_h^{n+1})\|_S^2 + \frac{gk_{min}}{4}\|\nabla\phi_h^{n_{k+1}}\|_D^2)$$
$$\leq \sum_{n=n_k}^{n_{k+1}-1} \frac{\epsilon_1 r(r-1)}{2\Delta t}\|\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n\|_S^2 + 2C_1(r-1)\Delta t \sum_{n=n_k}^{n_{k+1}-1} (\|\nabla\boldsymbol{u}_h^{n+1}\|_S^2 + \|\nabla\boldsymbol{u}_h^n\|_S^2) + \frac{rgk_{min}}{4}\|\nabla\phi_h^{n_{k+1}}\|_D^2,$$

and

$$\sum_{n=n_k}^{n_{k+1}-1} a_\Gamma(\boldsymbol{u}_h^{n+1}, \phi_h^{n_{k+1}} - \phi_h^{n_k})$$
$$\leq \sum_{n=n_k}^{n_{k+1}-1} (\frac{\epsilon_2}{\Delta t}\|\phi_h^{n_{k+1}} - \phi_h^{n_k}\|_D^2 + C_2\Delta t\|\nabla(\phi_h^{n_{k+1}} - \phi_h^{n_k})\|_D^2 + \frac{C_K\nu}{4}\|\nabla\boldsymbol{u}_h^{n+1}\|_S^2)$$
$$\leq \frac{r\epsilon_2}{\Delta t}\|\phi_h^{n_{k+1}} - \phi_h^{n_k}\|_D^2 + 2C_2r\Delta t(\|\nabla\phi_h^{n_{k+1}}\|_D^2 + \|\nabla\phi_h^{n_k}\|_D^2) + \sum_{n=n_k}^{n_{k+1}-1} \frac{C_K\nu}{4}\|\nabla\boldsymbol{u}_h^{n+1}\|_S^2.$$

Consequently, we get

$$ra_\Gamma(\boldsymbol{u}_h^{n_{k+1}}, \phi_h^{n_{k+1}}) - \sum_{n=n_k}^{n_{k+1}-1} a_\Gamma(\boldsymbol{u}_h^{n+1}, \phi_h^{n_k}) \tag{3.18}$$
$$\leq \sum_{n=n_k}^{n_{k+1}-1} \frac{\epsilon_1 r(r-1)}{2\Delta t}\|\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n\|_S^2 + 2C_1(r-1)\Delta t \sum_{n=n_k}^{n_{k+1}-1} (\|\nabla\boldsymbol{u}_h^{n+1}\|_S^2 + \|\nabla\boldsymbol{u}_h^n\|_S^2) + \frac{rgk_{min}}{4}\|\nabla\phi_h^{n_{k+1}}\|_D^2$$
$$+ \frac{r\epsilon_2}{\Delta t}\|\phi_h^{n_{k+1}} - \phi_h^{n_k}\|_D^2 + 2C_2r\Delta t(\|\nabla\phi_h^{n_{k+1}}\|_D^2 + \|\nabla\phi_h^{n_k}\|_D^2) + \sum_{n=n_k}^{n_{k+1}-1} \frac{C_K\nu}{4}\|\nabla\boldsymbol{u}_h^{n+1}\|_S^2.$$

On the other hand, it follows that

$$\sum_{n=n_k}^{n_{k+1}-1} \widetilde{a}_\Gamma(\phi_h^{n_k}, \boldsymbol{u}_h^{n+1}) \tag{3.19}$$
$$\leq \sum_{n=n_k}^{n_{k+1}-1} \frac{\sqrt{(1+C_{P,D}^2)(1+C_{P,S}^2)}\alpha\nu\lambda_{max}(\mathbb{K})}{\sqrt{\lambda_{min}(\Pi)}}\|\nabla\phi_h^{n_k}\|_D\|\nabla\boldsymbol{u}_h^{n+1}\|_S$$
$$\leq \frac{\sqrt{(1+C_{P,D}^2)(1+C_{P,S}^2)}r\alpha\nu\lambda_{max}(\mathbb{K})}{2\sqrt{\lambda_{min}(\Pi)}}(\|\nabla\phi_h^{n_k}\|_D^2 + \|\nabla\phi_h^{n_{k+1}}\|_D^2)$$
$$+ \sum_{n=n_k}^{n_{k+1}-1} \frac{\sqrt{(1+C_{P,D}^2)(1+C_{P,S}^2)}\alpha\nu\lambda_{max}(\mathbb{K})}{2\sqrt{\lambda_{min}(\Pi)}}\|\nabla\boldsymbol{u}_h^{n+1}\|_S^2$$
$$\leq \frac{rgk_{min}}{16}(\|\nabla\phi_h^{n_k}\|_D^2 + \|\nabla\phi_h^{n_{k+1}}\|_D^2) + \sum_{n=n_k}^{n_{k+1}-1} \frac{C_K\nu}{8}\|\nabla\boldsymbol{u}_h^{n+1}\|_S^2,$$

under the assumption that α is sufficiently small. In particular, we demand that

$$\frac{\sqrt{(1+C_{P,D}^2)(1+C_{P,S}^2)}r\alpha\nu\lambda_{max}(\mathbb{K})}{2\sqrt{\lambda_{min}(\Pi)}} \leq \frac{rgk_{min}}{16}, \quad \frac{\sqrt{(1+C_{P,D}^2)(1+C_{P,S}^2)}\alpha\nu\lambda_{max}(\mathbb{K})}{2\sqrt{\lambda_{min}(\Pi)}} \leq \frac{C_K\nu}{8},$$

which implies that

$$\alpha \leq \min\{\frac{gk_{min}\sqrt{\lambda_{min}(\Pi)}}{8\sqrt{(1+C_{P,D}^2)(1+C_{P,S}^2)}\nu\lambda_{max}(\mathbb{K})}, \frac{C_K\sqrt{\lambda_{min}(\Pi)}}{4\sqrt{(1+C_{P,D}^2)(1+C_{P,S}^2)}\lambda_{max}(\mathbb{K})}\}.$$

By taking $\epsilon_1 = \frac{1}{r(r-1)}$ and $\epsilon_2 = \frac{gS_0}{2r}$ in , we derive

$$C_1 = \frac{r(r-1)g^2C_{P,D}^2C^4(\Omega_S)C^4(\Omega_D)}{4k_{min}^2}, \quad C_2 = \frac{rg^3C_{P,S}^2C^4(\Omega_S)C^4(\Omega_D)}{2S_0C_K^2\nu^2}$$

Under the restriction of the fine time step (3.7), plugging (3.16), (3.18) and (3.19) into (3.15) leads to

$$\begin{aligned}
&\frac{\|\boldsymbol{u}_h^{n_{k+1}}\|_S^2}{2\Delta t} - \frac{\|\boldsymbol{u}_h^{n_k}\|_S^2}{2\Delta t} + \frac{gS_0\|\phi_h^{n_{k+1}}\|_D^2}{2\Delta t} - \frac{gS_0\|\phi_h^{n_k}\|_D^2}{2\Delta t} + \frac{C_K\nu}{4}\sum_{n=n_k}^{n_{k+1}-1}\|\nabla\boldsymbol{u}_h^{n+1}\|_S^2 + \frac{rgk_{min}}{4}\|\nabla\phi_h^{n_{k+1}}\|_D^2 \\
&+ \frac{C_K\nu}{8}(\|\nabla\boldsymbol{u}_h^{n_{k+1}}\|_S^2 - \|\nabla\boldsymbol{u}_h^{n_k}\|_S^2) + \frac{rgk_{min}}{8}(\|\nabla\phi_h^{n_{k+1}}\|_D^2 - \|\nabla\phi_h^{n_k}\|_D^2) \\
&\leq \sum_{n=n_k}^{n_{k+1}-1}\frac{2C_{P,S}^2}{C_K\nu}\|\boldsymbol{f}_S^{n+1}\|_S^2 + \frac{rgC_{P,D}^2}{k_{min}}\|f_D^{n_{k+1}}\|_D^2. \qquad (3.20)
\end{aligned}$$

The summation of the above inequality form $k = 0$ to $M-1$ yields

$$\begin{aligned}
&\|\boldsymbol{u}_h^N\|_S^2 + gS_0\|\phi_h^N\|_D^2 + \frac{C_K\nu\Delta t}{4}\|\nabla\boldsymbol{u}_h^N\|_S^2 + \frac{rgk_{min}\Delta t}{4}\|\nabla\phi_h^N\|_D^2 \\
&+ \frac{C_K\nu\Delta t}{2}\sum_{k=0}^{M-1}\sum_{n=n_k}^{n_{k+1}-1}\|\nabla\boldsymbol{u}_h^{n+1}\|_S^2 + \frac{rgk_{min}\Delta t}{2}\sum_{k=0}^{M-1}\|\nabla\phi_h^{n_{k+1}}\|_D^2 \\
&\leq \|\boldsymbol{u}_h^0\|_S^2 + \|\phi_h^0\|_D^2 + \frac{C_K\nu\Delta t}{4}\|\nabla\boldsymbol{u}_h^0\|_S^2 + \frac{rgk_{min}\Delta t}{4}\|\nabla\phi_h^0\|_D^2 \\
&+ \sum_{k=0}^{M-1}\sum_{n=n_k}^{n_{k+1}-1}\frac{4C_{P,S}^2\Delta t}{C_K\nu}\|\boldsymbol{f}_S^{n+1}\|_S^2 + \sum_{k=0}^{M-1}\frac{2rgC_{P,D}^2\Delta t}{k_{min}}\|f_D^{n_{k+1}}\|_D^2.
\end{aligned}$$

This is (3.8).

Denote

$$Q(\Delta t) = \min\{\frac{2C_K\nu}{4C_{P,S}^2 + C_K\nu\Delta t}, \frac{2rk_{min}}{4S_0C_{P,D}^2 + rk_{min}\Delta t}\}\Delta t.$$

Then it follows that

$$\begin{aligned}
&Q(\Delta t)(\frac{1}{2\Delta t}\|\boldsymbol{u}_h^{n_{k+1}}\|_S^2 + \frac{C_K\nu}{8}\|\nabla\boldsymbol{u}_h^{n_{k+1}}\|_S^2) \\
\leq &Q(\Delta t)(\frac{C_{P,S}^2}{2\Delta t}\|\nabla\boldsymbol{u}_h^{n_{k+1}}\|_S^2 + \frac{C_K\nu}{8}\|\nabla\boldsymbol{u}_h^{n_{k+1}}\|_S^2)
\end{aligned}$$

$$\leq \frac{C_K\nu}{4}Q(\Delta t)\frac{4C_{P,S}^2+C_K\nu\Delta t}{2C_K\nu\Delta t}\|\nabla \boldsymbol{u}_h^{n_{k+1}}\|_S^2 \leq \frac{C_K\nu}{4}\|\nabla \boldsymbol{u}_h^{n_{k+1}}\|_S^2. \tag{3.21}$$

$$Q(\Delta t)(\frac{gS_0}{2\Delta t}\|\phi_h^{n_{k+1}}\|_D^2+\frac{rgk_{min}}{8}\|\nabla\phi_h^{n_{k+1}}\|_D^2) \leq \frac{rgk_{min}}{4}\|\nabla\phi_h^{n_{k+1}}\|_D^2. \tag{3.22}$$

Let

$$s_{n_{k+1}} = \frac{1}{2\Delta t}\|\boldsymbol{u}_h^{n_{k+1}}\|_S^2+\frac{C_K\nu}{8}\|\nabla\boldsymbol{u}_h^{n_{k+1}}\|_S^2+\frac{gS_0}{2\Delta t}\|\phi_h^{n_{k+1}}\|_D^2+\frac{gk_{min}}{8}\|\nabla\phi_h^{n_{k+1}}\|_D^2,$$
$$P = \frac{2rC_{P,S}^2}{C_K\nu}\sup_{0\leq N\leq\infty}\|\boldsymbol{f}_S^N\|_S^2+\frac{rgC_{P,D}^2}{k_{min}}\sup_{0\leq N\leq\infty}\|f_D^N\|_D^2.$$

Combining (3.21) and (3.22) with (3.20) results in

$$(1+Q(\Delta t))s_{n_{k+1}} - s_{n_k} \leq P,$$

which implies that

$$\begin{aligned}
s_{n_{k+1}} &\leq \frac{1}{1+Q(\Delta t)}(s_{n_k}+P)\\
&= \frac{1}{1+Q(\Delta t)}s_{n_k}+\frac{1}{1+Q(\Delta t)}P\\
&\leq .....\\
&\leq \frac{1}{(1+Q(\Delta t))^{k+1}}s_{n_0}+\frac{P}{Q(\Delta t)}.
\end{aligned}$$

Thereby, we have

$$\frac{1}{2\Delta t}\|\boldsymbol{u}_h^{n_{k+1}}\|_S^2+\frac{gS_0}{2\Delta t}\|\phi_h^{n_{k+1}}\|_D^2 \leq \frac{1}{(1+Q(\Delta t))^{k+1}}s_{n_0}+\frac{P}{Q(\Delta t)}.$$

Consequently, we get

$$\begin{aligned}
&\|\boldsymbol{u}_h^{n_{k+1}}\|_S^2+gS_0\|\phi_h^{n_{k+1}}\|_D^2 \leq 2\Delta t s_{n_0}+\frac{2\Delta t P}{Q(\Delta t)}\\
&\leq 2\Delta t s_{n_0}+\max\{\frac{4C_{P,S}^2}{C_K\nu}+\Delta t,\frac{4C_{P,D}^2S_0}{rk_{min}}+\Delta t\}P\\
&\leq \max\{\frac{4C_{P,S}^2}{C_K\nu}+\Delta t_{stable},\frac{4C_{P,D}^2S_0}{rk_{min}}+\Delta t_{stable}\}P\\
&+\{\|\boldsymbol{u}_h^0\|_S^2+\frac{C_K\nu\Delta t}{4}\|\nabla\boldsymbol{u}_h^0\|_S^2+gS_0\|\phi_h^0\|_D^2+\frac{gk_{min}\Delta t}{4}\|\nabla\phi_h^0\|_D^2\},
\end{aligned}$$

which implies (3.9).

Finally, if $\boldsymbol{f}_S \equiv 0$ and $f_D \equiv 0$, (3.8) implies that the series

$$\frac{C_K\nu}{2}\sum_{k=0}^{\infty}\sum_{n=n_k}^{n_{k+1}-1}\|\nabla\boldsymbol{u}_h^{n+1}\|_S^2+\frac{rgk_{min}}{2}\sum_{k=0}^{\infty}\|\nabla\phi_h^{n_{k+1}}\|_D^2$$

is convergent. Hence we get (3.10). □

4 Uniformly bounded error estimation

This section shows the uniformly bounded error estimation of the proposed partitioned method with multirate time steps over $0 \leq t < \infty$, which is based on the long-time stability analysis. We consider the following finite element space

$$V_h = H_{S,h} \cap \{\boldsymbol{v}_h : (q_h, \nabla \cdot \boldsymbol{v}_h)_{\Omega_S} = 0 \text{ for all } q_h \in Q_h\}.$$

During the following analysis, we assume that

$$\begin{aligned} &\boldsymbol{u} \in L^\infty(0,T;(H^3(\Omega_S))^d),\ \boldsymbol{u}_t \in L^\infty(0,T;(H^3(\Omega_S))^d),\ \boldsymbol{u}_{tt} \in L^\infty(0,T;(L^2(\Omega_S))^d),\\ &\phi \in L^\infty(0,T;H^3(\Omega_D)),\ \phi_t \in L^\infty(0,T;H^3(\Omega_D)),\ \phi_{tt} \in L^\infty(0,T;L^2(\Omega_D)),\\ &p \in L^\infty(0,T;H^2(\Omega_S)), \end{aligned} \tag{4.1}$$

where $T = N\Delta t = M\Delta T$. Note that if $T = \infty$, then $N = \infty$ and $M = \infty$. Let us introduce the following norms

$$\begin{aligned} &\|\boldsymbol{v}\|_{p,k,S} := \|\boldsymbol{v}\|_{L^p(0,T;(H^k(\Omega_S))^d)}, \quad |||\ \boldsymbol{v}\ |||_{\infty,k,S} := \sup_{0\leq n\leq N} \|\boldsymbol{v}^n\|_{(H^k(\Omega_S))^d},\\ &\|\psi\|_{p,k,D} := \|\psi\|_{L^p(0,T;H^k(\Omega_D))}, \quad |||\ \psi\ |||_{\infty,k,D} := \sup_{0\leq n\leq N} \|\psi^n\|_{H^k(\Omega_D)}. \end{aligned}$$

Firstly, we rewrite the weak formulation of the continuous problem as the discrete one that is forced by consistency errors as follows: for $\forall \boldsymbol{v}_h \in V_h, \psi_h \in H_{D,h}$ and $\forall \lambda_h^{n+1} \in Q_h$,

$$\begin{aligned} &(\frac{\boldsymbol{u}^{n+1}-\boldsymbol{u}^n}{\Delta t}, \boldsymbol{v}_h)_{\Omega_S} + a_S(\boldsymbol{u}^{n+1}, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, p^{n+1} - \lambda_h^{n+1})\\ &= (\boldsymbol{f}_S^{n+1}, \boldsymbol{v}_h)_{\Omega_S} - a_\Gamma(\boldsymbol{v}_h, \phi^{n+1}) - \widetilde{a}_\Gamma(\phi^{n+1}, \boldsymbol{v}_h) + \epsilon_S^{n+1}(\boldsymbol{v}_h) \end{aligned} \tag{4.2}$$

$$g(S_0\frac{\phi^{n_{k+1}}-\phi^{n_k}}{\Delta T}, \psi_h)_{\Omega_D} + g a_D(\phi^{n_{k+1}}, \psi) = g(f_D^{n_{k+1}}, \psi_h)_{\Omega_D} + \epsilon_D^{n_{k+1}}(\psi_h) + a_\Gamma(\boldsymbol{u}^{n_{k+1}}, \psi_h), \tag{4.3}$$

where the consistency errors are

$$\begin{aligned} \epsilon_S^{n+1}(\boldsymbol{v}_h) &= (\frac{\boldsymbol{u}^{n+1}-\boldsymbol{u}^n}{\Delta t} - \boldsymbol{u}_t^{n+1}, \boldsymbol{v}_h)_{\Omega_S},\\ \epsilon_D^{n_{k+1}}(\psi_h) &= g(S_0\frac{\phi^{n_{k+1}}-\phi^{n_k}}{\Delta T} - \phi_t^{n_{k+1}}, \psi_h)_{\Omega_D}. \end{aligned}$$

Denote the errors by $\boldsymbol{e}_{\boldsymbol{u}}^n = \boldsymbol{u}^n - \boldsymbol{u}_h^n, e_\phi^{n_k} = \phi^{n_k} - \phi_h^{n_k}$. Then we could get the errors system for our algorithm as follows: for $\forall \boldsymbol{v}_h \in V_h, \psi_h \in H_{D,h}$ and $\forall \lambda_h^{n+1} \in Q_h$,

$$\begin{aligned} &(\frac{\boldsymbol{e}_{\boldsymbol{u}}^{n+1}-\boldsymbol{e}_{\boldsymbol{u}}^n}{\Delta t}, \boldsymbol{v}_h)_{\Omega_S} + a_S(\boldsymbol{e}_{\boldsymbol{u}}^{n+1}, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, p^{n+1} - \lambda_h^{n+1})\\ &= a_\Gamma(\boldsymbol{v}_h, \phi_h^{n_k}) + \widetilde{a}_\Gamma(\phi_h^{n_k}, \boldsymbol{v}_h) - a_\Gamma(\boldsymbol{v}_h, \phi^{n+1}) - \widetilde{a}_\Gamma(\phi^{n+1}, \boldsymbol{v}_h) + \epsilon_S^{n+1}(\boldsymbol{v}_h), \end{aligned} \tag{4.4}$$

$$g(S_0\frac{e_\phi^{n_{k+1}}-e_\phi^{n_k}}{\Delta T}, \psi_h)_{\Omega_D} + g a_D(e_\phi^{n_{k+1}}, \psi_h) = \epsilon_D^{n_{k+1}}(\psi_h) + a_\Gamma(\boldsymbol{e}_{\boldsymbol{u}}^{n_{k+1}}, \psi_h). \tag{4.5}$$

Note that

$$\begin{aligned} &a_\Gamma(\boldsymbol{v}_h, \phi_h^{n_k}) + \widetilde{a}_\Gamma(\phi_h^{n_k}, \boldsymbol{v}_h) - a_\Gamma(\boldsymbol{v}_h, \phi^{n+1}) - \widetilde{a}_\Gamma(\phi^{n+1}, \boldsymbol{v}_h)\\ =&a_\Gamma(\boldsymbol{v}_h, \phi_h^{n_k} - \phi^{n_k}) + a_\Gamma(\boldsymbol{v}_h, \phi^{n_k} - \phi^{n+1}) + \widetilde{a}_\Gamma(\phi_h^{n_k} - \phi^{n_k}, \boldsymbol{v}_h) + \widetilde{a}_\Gamma(\phi^{n_k} - \phi^{n+1}, \boldsymbol{v}_h) \end{aligned}$$

$$=a_\Gamma(\boldsymbol{v}_h,\phi^{n_k}-\phi^{n+1})-a_\Gamma(\boldsymbol{v}_h,e_\phi^{n_k})+\widetilde{a}_\Gamma(\phi^{n_k}-\phi^{n+1},\boldsymbol{v}_h)-\widetilde{a}_\Gamma(e_\phi^{n_k},\boldsymbol{v}_h).$$

Hence, (4.4) could be rewritten into the following form

$$(\frac{\boldsymbol{e}_{\boldsymbol{u}}^{n+1}-\boldsymbol{e}_{\boldsymbol{u}}^{n}}{\Delta t},\boldsymbol{v}_h)_{\Omega_S}+a_S(\boldsymbol{e}_{\boldsymbol{u}}^{n+1},\boldsymbol{v}_h)+b(\boldsymbol{v}_h,p^{n+1}-\lambda_h^{n+1})=\widehat{\epsilon_S^{n+1}(\boldsymbol{v}_h)}-a_\Gamma(\boldsymbol{v}_h,e_\phi^{n_k})-\widetilde{a}_\Gamma(e_\phi^{n_k},\boldsymbol{v}_h),\tag{4.6}$$

where

$$\widehat{\epsilon_S^{n+1}(\boldsymbol{v}_h)}=\epsilon_S^{n+1}(\boldsymbol{v}_h)+a_\Gamma(\boldsymbol{v}_h,\phi^{n_k}-\phi^{n+1})+\widetilde{a}_\Gamma(\phi^{n_k}-\phi^{n+1},\boldsymbol{v}_h).$$

Theorem 4.1 *Under the assumption (3.6), the time step restriction (3.7) and the regularity condition (4.1), for our algorithm, for any $0\le t^{n_M}<\infty$, we get*

$$\begin{aligned}&\|\boldsymbol{e}_{\boldsymbol{u}}^{n_M}\|_S^2+gS_0\|e_\phi^{n_M}\|_D^2\\&\le Cr\Delta t^2(\|\boldsymbol{u}_{tt}\|_{\infty,0,S}^2+r^2\|\phi_t\|_{\infty,1,D}^2+r^2\|\phi_{tt}\|_{\infty,0,D}^2)\\&+Crh^6(\|\boldsymbol{u}_t\|_{\infty,3,S}^2+\|\phi_t\|_{\infty,3,D}^2)+Crh^4(|||\ \boldsymbol{u}\ |||_{\infty,3,S}^2+|||\ \phi\ |||_{\infty,3,D}^2+|||\ p\ |||_{\infty,2,S}^2)\\&+C(\Delta th^4+h^6).\end{aligned}$$

Proof. Decompose the errors in the following way

$$\begin{aligned}\boldsymbol{e}_{\boldsymbol{u}}^{n}&=(\boldsymbol{u}^n-\boldsymbol{U}^n)+(\boldsymbol{U}^n-\boldsymbol{u}_h^n):=\boldsymbol{\theta}_{\boldsymbol{u}}^{n}+\boldsymbol{\rho}_{\boldsymbol{u}}^{n},\\e_\phi^{n_k}&=(\phi^{n_k}-\Phi^{n_k})+(\Phi^{n_k}-\phi_h^{n_k}):=\theta_\phi^{n_k}+\rho_\phi^{n_k},\end{aligned}$$

where $\boldsymbol{U}^n,\Phi^{n_k}$ denote finite element interpolations of $\boldsymbol{u}^n,\phi^{n_k}$ in $V_h,H_{D,h}$, respectively. Note that they have the property of interpolation approximation. Then (4.6) and (4.5) could be rewritten as

$$\begin{aligned}&(\frac{\boldsymbol{\rho}_{\boldsymbol{u}}^{n+1}-\boldsymbol{\rho}_{\boldsymbol{u}}^{n}}{\Delta t},\boldsymbol{v}_h)_{\Omega_S}+a_S(\boldsymbol{\rho}_{\boldsymbol{u}}^{n+1},\boldsymbol{v}_h)\qquad(4.7)\\&=\widehat{\epsilon_S^{n+1}(\boldsymbol{v}_h)}-[(\frac{\boldsymbol{\theta}_{\boldsymbol{u}}^{n+1}-\boldsymbol{\theta}_{\boldsymbol{u}}^{n}}{\Delta t},\boldsymbol{v}_h)_{\Omega_S}+a_S(\boldsymbol{\theta}_{\boldsymbol{u}}^{n+1},\boldsymbol{v}_h)+b(\boldsymbol{v}_h,p^{n+1}-\lambda_h^{n+1})]\\&-a_\Gamma(\boldsymbol{v}_h,\rho_\phi^{n_k})-\widetilde{a}_\Gamma(\rho_\phi^{n_k},\boldsymbol{v}_h)-a_\Gamma(\boldsymbol{v}_h,\theta_\phi^{n_k})-\widetilde{a}_\Gamma(\theta_\phi^{n_k},\boldsymbol{v}_h),\\&g(S_0\frac{\rho_\phi^{n_{k+1}}-\rho_\phi^{n_k}}{\Delta T},\psi_h)_{\Omega_D}+ga_D(\rho_\phi^{n_{k+1}},\psi_h)\qquad(4.8)\\&=\epsilon_D^{n_{k+1}}(\psi_h)-[g(S_0\frac{\theta_\phi^{n_{k+1}}-\theta_\phi^{n_k}}{\Delta T},\psi_h)_{\Omega_D}+ga_D(\theta_\phi^{n_{k+1}},\psi_h)]+a_\Gamma(\boldsymbol{\rho}_{\boldsymbol{u}}^{n_{k+1}},\psi_h)+a_\Gamma(\boldsymbol{\theta}_{\boldsymbol{u}}^{n_{k+1}},\psi_h).\qquad(4.9)\end{aligned}$$

By taking $\boldsymbol{v}_h=\boldsymbol{\rho}_{\boldsymbol{u}}^{n+1}$ in (4.7), we get

$$\begin{aligned}&\frac{1}{2\Delta t}\|\boldsymbol{\rho}_{\boldsymbol{u}}^{n+1}\|_S^2-\frac{1}{2\Delta t}\|\boldsymbol{\rho}_{\boldsymbol{u}}^{n}\|_S^2+\frac{1}{2\Delta t}\|\boldsymbol{\rho}_{\boldsymbol{u}}^{n+1}-\boldsymbol{\rho}_{\boldsymbol{u}}^{n}\|_S^2+C_K\nu\|\nabla\boldsymbol{\rho}_{\boldsymbol{u}}^{n+1}\|_S^2\qquad(4.10)\\&\le\widehat{\epsilon_S^{n+1}(\boldsymbol{\rho}_{\boldsymbol{u}}^{n+1})}-[(\frac{\boldsymbol{\theta}_{\boldsymbol{u}}^{n+1}-\boldsymbol{\theta}_{\boldsymbol{u}}^{n}}{\Delta t},\boldsymbol{\rho}_{\boldsymbol{u}}^{n+1})_{\Omega_S}+a_S(\boldsymbol{\theta}_{\boldsymbol{u}}^{n+1},\boldsymbol{\rho}_{\boldsymbol{u}}^{n+1})+b(\boldsymbol{\rho}_{\boldsymbol{u}}^{n+1},p^{n+1}-\lambda_h^{n+1})]\\&-a_\Gamma(\boldsymbol{\rho}_{\boldsymbol{u}}^{n+1},\rho_\phi^{n_k})-\widetilde{a}_\Gamma(\rho_\phi^{n_k},\boldsymbol{\rho}_{\boldsymbol{u}}^{n+1})-a_\Gamma(\boldsymbol{\rho}_{\boldsymbol{u}}^{n+1},\theta_\phi^{n_k})-\widetilde{a}_\Gamma(\theta_\phi^{n_k},\boldsymbol{\rho}_{\boldsymbol{u}}^{n+1})\end{aligned}$$

By choosing $\psi_h=\rho_\phi^{n_{k+1}}$ in (4.8), we get

$$\frac{gS_0}{2\Delta T}\|\rho_\phi^{n_{k+1}}\|_D^2-\frac{gS_0}{2\Delta T}\|\rho_\phi^{n_k}\|_D^2+\frac{gS_0}{2\Delta T}\|\rho_\phi^{n_{k+1}}-\rho_\phi^{n_k}\|_D^2+gk_{min}\|\nabla\rho_\phi^{n_{k+1}}\|_D^2\tag{4.11}$$

$$\leq \epsilon_D^{n_{k+1}}(\rho_\phi^{n_{k+1}}) - [g(S_0\frac{\theta_\phi^{n_{k+1}}-\theta_\phi^{n_k}}{\Delta T},\rho_\phi^{n_{k+1}})_{\Omega_D} + ga_D(\theta_\phi^{n_{k+1}},\rho_\phi^{n_{k+1}})]$$
$$+ a_\Gamma(\boldsymbol{\rho}_{\boldsymbol{u}}^{n_{k+1}},\rho_\phi^{n_{k+1}}) + a_\Gamma(\boldsymbol{\theta}_{\boldsymbol{u}}^{n_{k+1}},\rho_\phi^{n_{k+1}}).$$

Summation of (4.10) over the coarse time interval $[n_k, n_{k+1}-1]$ and note that $\Delta T = r\Delta t$ in (4.11) imply that

$$\frac{1}{2\Delta t}\|\boldsymbol{\rho}_{\boldsymbol{u}}^{n_{k+1}}\|_S^2 - \frac{1}{2\Delta t}\|\boldsymbol{\rho}_{\boldsymbol{u}}^{n_k}\|_S^2 + \sum_{n=n_k}^{n_{k+1}-1}\frac{1}{2\Delta t}\|\boldsymbol{\rho}_{\boldsymbol{u}}^{n+1}-\boldsymbol{\rho}_{\boldsymbol{u}}^{n}\|_S^2 + \sum_{n=n_k}^{n_{k+1}} C_K\nu\|\nabla\boldsymbol{\rho}_{\boldsymbol{u}}^{n+1}\|_S^2$$
$$+ \frac{gS_0}{2\Delta t}\|\rho_\phi^{n_{k+1}}\|_D^2 - \frac{gS_0}{2\Delta t}\|\rho_\phi^{n_k}\|_D^2 + \frac{gS_0}{2\Delta t}\|\rho_\phi^{n_{k+1}}-\rho_\phi^{n_k}\|_D^2 + rgk_{min}\|\nabla\rho_\phi^{n_{k+1}}\|_D^2$$
$$\leq \underbrace{\sum_{n=n_k}^{n_{k+1}-1}\epsilon_S^{n+1}(\widehat{\boldsymbol{\rho}_{\boldsymbol{u}}^{n+1}}) + r\epsilon_D^{n_{k+1}}(\rho_\phi^{n_{k+1}})}_{T_1}$$
$$\underbrace{-\sum_{n=n_k}^{n_{k+1}-1}[(\frac{\boldsymbol{\theta}_{\boldsymbol{u}}^{n+1}-\boldsymbol{\theta}_{\boldsymbol{u}}^{n}}{\Delta t},\boldsymbol{\rho}_{\boldsymbol{u}}^{n+1})_{\Omega_S} + a_S(\boldsymbol{\theta}_{\boldsymbol{u}}^{n+1},\boldsymbol{\rho}_{\boldsymbol{u}}^{n+1}) + b(\boldsymbol{\rho}_{\boldsymbol{u}}^{n+1},p^{n+1}-\lambda_h^{n+1})]}_{T_2}$$
$$\underbrace{-[rg(S_0\frac{\theta_\phi^{n_{k+1}}-\theta_\phi^{n_k}}{\Delta T},\rho_\phi^{n_{k+1}})_{\Omega_D} + rga_D(\theta_\phi^{n_{k+1}},\rho_\phi^{n_{k+1}})]}_{T_3}$$
$$\underbrace{-\sum_{n=n_k}^{n_{k+1}-1}a_\Gamma(\boldsymbol{\rho}_{\boldsymbol{u}}^{n+1},\theta_\phi^{n_k}) - \sum_{n=n_k}^{n_{k+1}-1}\widetilde{a}_\Gamma(\theta_\phi^{n_k},\boldsymbol{\rho}_{\boldsymbol{u}}^{n+1}) + ra_\Gamma(\boldsymbol{\theta}_{\boldsymbol{u}}^{n_{k+1}},\rho_\phi^{n_{k+1}})}_{T_4}$$
$$\underbrace{-\sum_{n=n_k}^{n_{k+1}-1}a_\Gamma(\boldsymbol{\rho}_{\boldsymbol{u}}^{n+1},\rho_\phi^{n_k}) - \sum_{n=n_k}^{n_{k+1}-1}\widetilde{a}_\Gamma(\rho_\phi^{n_k},\boldsymbol{\rho}_{\boldsymbol{u}}^{n+1}) + ra_\Gamma(\boldsymbol{\rho}_{\boldsymbol{u}}^{n_{k+1}},\rho_\phi^{n_{k+1}})}_{\widetilde{I}}.$$

For the right-hand side terms, firstly, under the assumptions (3.6) and (3.7), we can readily derive that

$$\widetilde{I} \leq \sum_{n=n_k}^{n_{k+1}-1}\frac{1}{2\Delta t}\|\boldsymbol{\rho}_{\boldsymbol{u}}^{n+1}-\boldsymbol{\rho}_{\boldsymbol{u}}^{n}\|_S^2 + \sum_{n=n_k}^{n_{k+1}-1}\frac{C_K\nu}{8}(\|\nabla\boldsymbol{\rho}_{\boldsymbol{u}}^{n+1}\|_S^2 + \|\nabla\boldsymbol{\rho}_{\boldsymbol{u}}^{n}\|_S^2) + \sum_{n=n_k}^{n_{k+1}-1}\frac{3C_K\nu}{8}\|\nabla\boldsymbol{\rho}_{\boldsymbol{u}}^{n+1}\|_S^2$$
$$+ \frac{gS_0}{2\Delta t}\|\rho_\phi^{n_{k+1}}-\rho_\phi^{n_k}\|_D^2 + \frac{rgk_{min}}{8}(\|\nabla\rho_\phi^{n_{k+1}}\|_D^2 + \|\nabla\rho_\phi^{n_k}\|_D^2) + \frac{rgk_{min}}{4}\|\nabla\rho_\phi^{n_{k+1}}\|_D^2.$$

For the consistency errors, it follows that

$$T_1 = \sum_{n=n_k}^{n_{k+1}-1}\epsilon_S^{n+1}(\widehat{\boldsymbol{\rho}_{\boldsymbol{u}}^{n+1}}) + r\epsilon_D^{n_{k+1}}(\rho_\phi^{n_{k+1}})$$
$$= \sum_{n=n_k}^{n_{k+1}-1}((\frac{\boldsymbol{u}^{n+1}-\boldsymbol{u}^n}{\Delta t} - \boldsymbol{u}_t^{n+1},\boldsymbol{\rho}_{\boldsymbol{u}}^{n+1})_{\Omega_S} + a_\Gamma(\boldsymbol{\rho}_{\boldsymbol{u}}^{n+1},\phi^{n_k}-\phi^{n+1}) + \widetilde{a}_\Gamma(\phi^{n_k}-\phi^{n+1},\boldsymbol{\rho}_{\boldsymbol{u}}^{n+1}))$$
$$+ rg(S_0\frac{\phi^{n_{k+1}}-\phi^{n_k}}{\Delta T} - \phi_t^{n_{k+1}},\rho_\phi^{n_{k+1}})_{\Omega_D}$$

$$\leq C\sum_{n=n_k}^{n_{k+1}-1}(\|\frac{\boldsymbol{u}^{n+1}-\boldsymbol{u}^n}{\Delta t}-\boldsymbol{u}_t^{n+1}\|_S^2+\|\nabla(\phi^{n+1}-\phi^{n_k})\|_D^2)+Cr\|\frac{\phi^{n_{k+1}}-\phi^{n_k}}{\Delta T}-\phi_t^{n_{k+1}}\|_D^2$$
$$+\sum_{n=n_k}^{n_{k+1}-1}\frac{C_K\nu}{24}\|\nabla\boldsymbol{\rho}_{\boldsymbol{u}}^{n+1}\|_S^2+\frac{rgk_{min}}{12}\|\nabla\rho_\phi^{n_{k+1}}\|_D^2.$$

For T_2, it follows

$$T_2=-\sum_{n=n_k}^{n_{k+1}-1}[(\frac{\boldsymbol{\theta}_{\boldsymbol{u}}^{n+1}-\boldsymbol{\theta}_{\boldsymbol{u}}^n}{\Delta t},\boldsymbol{\rho}_{\boldsymbol{u}}^{n+1})_{\Omega_S}+a_S(\boldsymbol{\theta}_{\boldsymbol{u}}^{n+1},\boldsymbol{\rho}_{\boldsymbol{u}}^{n+1})+b(\boldsymbol{\rho}_{\boldsymbol{u}}^{n+1},p^{n+1}-\lambda_h^{n+1})]$$
$$\leq C\sum_{n=n_k}^{n_{k+1}-1}(\|\frac{\boldsymbol{\theta}_{\boldsymbol{u}}^{n+1}-\boldsymbol{\theta}_{\boldsymbol{u}}^n}{\Delta t}\|_S^2+\|\nabla\boldsymbol{\theta}_{\boldsymbol{u}}^{n+1}\|_S^2+\|p^{n+1}-\lambda_h^{n+1}\|_S^2)+\sum_{n=n_k}^{n_{k+1}-1}\frac{C_K\nu}{24}\|\nabla\boldsymbol{\rho}_{\boldsymbol{u}}^{n+1}\|_S^2.$$

For T_3, it follows

$$T_3=-[rg(S_0\frac{\theta_\phi^{n_{k+1}}-\theta_\phi^{n_k}}{\Delta T},\rho_\phi^{n_{k+1}})_{\Omega_D}+rga_D(\theta_\phi^{n_{k+1}},\rho_\phi^{n_{k+1}})]$$
$$\leq Cr(\|\frac{\theta_\phi^{n_{k+1}}-\theta_\phi^{n_k}}{\Delta T}\|_D^2+\|\nabla\theta_\phi^{n_{k+1}}\|_D^2)+\frac{rgk_{min}}{12}\|\nabla\rho_\phi^{n_{k+1}}\|_D^2.$$

For T_4, it follows

$$T_4=-\sum_{n=n_k}^{n_{k+1}-1}a_\Gamma(\boldsymbol{\rho}_{\boldsymbol{u}}^{n+1},\theta_\phi^{n_k})-\sum_{n=n_k}^{n_{k+1}-1}\widetilde{a}_\Gamma(\theta_\phi^{n_k},\boldsymbol{\rho}_{\boldsymbol{u}}^{n+1})+ra_\Gamma(\boldsymbol{\theta}_{\boldsymbol{u}}^{n_{k+1}},\rho_\phi^{n_{k+1}})$$
$$\leq C\sum_{n=n_k}^{n_{k+1}-1}\|\nabla\theta_\phi^{n_k}\|_D^2+Cr\|\nabla\boldsymbol{\theta}_{\boldsymbol{u}}^{n_{k+1}}\|_D^2+\sum_{n=n_k}^{n_{k+1}-1}\frac{C_K\nu}{24}\|\nabla\boldsymbol{\rho}_{\boldsymbol{u}}^{n+1}\|_S^2+\frac{rgk_{min}}{12}\|\nabla\rho_\phi^{n_{k+1}}\|_D^2.$$

Consequently, we get

$$\frac{1}{2\Delta t}\|\boldsymbol{\rho}_{\boldsymbol{u}}^{n_{k+1}}\|_S^2-\frac{1}{2\Delta t}\|\boldsymbol{\rho}_{\boldsymbol{u}}^{n_k}\|_S^2+\sum_{n=n_k}^{n_{k+1}}\frac{C_K\nu}{4}\|\nabla\boldsymbol{\rho}_{\boldsymbol{u}}^{n+1}\|_S^2+\sum_{n=n_k}^{n_{k+1}}\frac{C_K\nu}{8}(\|\nabla\boldsymbol{\rho}_{\boldsymbol{u}}^{n+1}\|_S^2-\|\nabla\boldsymbol{\rho}_{\boldsymbol{u}}^n\|_S^2)$$
$$+\frac{gS_0}{2\Delta t}\|\rho_\phi^{n_{k+1}}\|_D^2-\frac{gS_0}{2\Delta t}\|\rho_\phi^{n_k}\|_D^2+\frac{rgk_{min}}{4}\|\nabla\rho_\phi^{n_{k+1}}\|_D^2+\frac{rgk_{min}}{8}(\|\nabla\rho_\phi^{n_{k+1}}\|_D^2-\|\nabla\rho_\phi^{n_k}\|_D^2)$$
$$\leq\mathcal{R}, \tag{4.12}$$

where $\mathcal{R}$ is defined as

$$\mathcal{R}=\underbrace{C\sum_{n=n_k}^{n_{k+1}-1}(\|\frac{\boldsymbol{u}^{n+1}-\boldsymbol{u}^n}{\Delta t}-\boldsymbol{u}_t^{n+1}\|_S^2+\|\nabla(\phi^{n+1}-\phi^{n_k})\|_D^2)+Cr\|\frac{\phi^{n_{k+1}}-\phi^{n_k}}{\Delta T}-\phi_t^{n_{k+1}}\|_D^2}_{\mathcal{R}_1}$$
$$+\underbrace{C\sum_{n=n_k}^{n_{k+1}-1}(\|\frac{\boldsymbol{\theta}_{\boldsymbol{u}}^{n+1}-\boldsymbol{\theta}_{\boldsymbol{u}}^n}{\Delta t}\|_S^2+\|\nabla\boldsymbol{\theta}_{\boldsymbol{u}}^{n+1}\|_S^2+\|p^{n+1}-\lambda_h^{n+1}\|_S^2)}_{\mathcal{R}_2}$$
$$+\underbrace{Cr(\|\frac{\theta_\phi^{n_{k+1}}-\theta_\phi^{n_k}}{\Delta T}\|_D^2+\|\nabla\theta_\phi^{n_{k+1}}\|_D^2)}_{\mathcal{R}_3}+\underbrace{C\sum_{n=n_k}^{n_{k+1}-1}\|\nabla\theta_\phi^{n_k}\|_D^2+Cr\|\nabla\boldsymbol{\theta}_{\boldsymbol{u}}^{n_{k+1}}\|_D^2}_{\mathcal{R}_4}.$$

Firstly, by using Taylor's theorem, we get

$$C\sum_{n=n_k}^{n_{k+1}-1}\|\frac{\boldsymbol{u}^{n+1}-\boldsymbol{u}^n}{\Delta t}-\boldsymbol{u}_t^{n+1}\|_S^2\le C\Delta t^2\sum_{n=n_k}^{n_{k+1}-1}\sup_{t^n\le t\le t^{n+1}}\|\boldsymbol{u}_{tt}(t)\|_S^2\le Cr\Delta t^2\|\boldsymbol{u}_{tt}\|_{\infty,0,S}^2,$$
$$C\sum_{n=n_k}^{n_{k+1}-1}\|\nabla(\phi^{n+1}-\phi^{n_k})\|_D^2\le Cr^2\Delta t^2\sum_{n=n_k}^{n_{k+1}-1}\sup_{t^n\le t\le t^{n+1}}\|\nabla\phi_t(t)\|_D^2\le Cr^3\Delta t^2\|\phi_t\|_{\infty,1,D}^2,$$
$$Cr\|\frac{\phi^{n_{k+1}}-\phi^{n_k}}{\Delta T}-\phi_t^{n_{k+1}}\|_D^2\le Cr^3\Delta t^2\|\phi_{tt}\|_{\infty,0,D}^2.$$

Hence we get

$$\mathcal{R}_1\le Cr\Delta t^2(\|\boldsymbol{u}_{tt}\|_{\infty,0,S}^2+r^2\|\phi_t\|_{\infty,1,D}^2+r^2\|\phi_{tt}\|_{\infty,0,D}^2).$$

By applying the approximation property and choosing λ_h^{n+1} to be the interpolation of p^{n+1} in Q_h, we derive

$$\begin{aligned}\mathcal{R}_2&\le Cr(\|\boldsymbol{\theta}_{\boldsymbol{u},t}(t)\|_{\infty,0,S}^2+|||\,\boldsymbol{\theta}_{\boldsymbol{u}}\,|||_{\infty,1,S}^2)+Crh^4\,|||\,p\,|||_{\infty,2,S}^2\\&\le Crh^6\|\boldsymbol{u}_t\|_{\infty,3,S}^2+Crh^4\,|||\,\boldsymbol{u}\,|||_{\infty,3,S}^2+Crh^4\,|||\,p\,|||_{\infty,2,S}^2\,.\end{aligned}$$

For $\mathcal{R}_3$, there holds

$$\mathcal{R}_3\le Cr(\|\theta_{\phi,t}(t)\|_{\infty,0,D}^2+|||\,\theta_\phi\,|||_{\infty,1,D}^2)\le Crh^6\|\phi_t\|_{\infty,3,D}^2+Crh^4\,|||\,\phi\,|||_{\infty,3,D}^2\,.$$

For the last term, it follows

$$\mathcal{R}_4\le Crh^4(|||\,\boldsymbol{u}\,|||_{\infty,3,S}^2+|||\,\phi\,|||_{\infty,3,D}^2).$$

Combining the above estimates and plugging them into (4.12) lead to

$$\begin{aligned}&\frac{1}{2\Delta t}\|\boldsymbol{\rho}_{\boldsymbol{u}}^{n_{k+1}}\|_S^2-\frac{1}{2\Delta t}\|\boldsymbol{\rho}_{\boldsymbol{u}}^{n_k}\|_S^2+\sum_{n=n_k}^{n_{k+1}}\frac{C_K\nu}{4}\|\nabla\boldsymbol{\rho}_{\boldsymbol{u}}^{n+1}\|_S^2+\sum_{n=n_k}^{n_{k+1}}\frac{C_K\nu}{8}(\|\nabla\boldsymbol{\rho}_{\boldsymbol{u}}^{n+1}\|_S^2-\|\nabla\boldsymbol{\rho}_{\boldsymbol{u}}^{n}\|_S^2)\\&+\frac{gS_0}{2\Delta t}\|\rho_\phi^{n_{k+1}}\|_D^2-\frac{gS_0}{2\Delta t}\|\rho_\phi^{n_k}\|_D^2+\frac{rgk_{min}}{4}\|\nabla\rho_\phi^{n_{k+1}}\|_D^2+\frac{rgk_{min}}{8}(\|\nabla\rho_\phi^{n_{k+1}}\|_D^2-\|\nabla\rho_\phi^{n_k}\|_D^2)\\&\le Cr\Delta t^2(\|\boldsymbol{u}_{tt}\|_{\infty,0,S}^2+r^2\|\phi_t\|_{\infty,1,D}^2+r^2\|\phi_{tt}\|_{\infty,0,D}^2)\\&+Crh^6(\|\boldsymbol{u}_t\|_{\infty,3,S}^2+\|\phi_t\|_{\infty,3,D}^2)+Crh^4(|||\,\boldsymbol{u}\,|||_{\infty,3,S}^2+|||\,\phi\,|||_{\infty,3,D}^2+|||\,p\,|||_{\infty,2,S}^2).\end{aligned}\tag{4.13}$$

Utilizing the proof framework of (3.9) again, we could readily get

$$\begin{aligned}&\|\boldsymbol{\rho}_{\boldsymbol{u}}^{n_{k+1}}\|_S^2+gS_0\|\rho_\phi^{n_{k+1}}\|_D^2\\&\le Cr\Delta t^2(\|\boldsymbol{u}_{tt}\|_{\infty,0,S}^2+r^2\|\phi_t\|_{\infty,1,D}^2+r^2\|\phi_{tt}\|_{\infty,0,D}^2)\\&+Crh^6(\|\boldsymbol{u}_t\|_{\infty,3,S}^2+\|\phi_t\|_{\infty,3,D}^2)+Crh^4(|||\,\boldsymbol{u}\,|||_{\infty,3,S}^2+|||\,\phi\,|||_{\infty,3,D}^2+|||\,p\,|||_{\infty,2,S}^2)\\&+\|\boldsymbol{\rho}_{\boldsymbol{u}}^0\|_S^2+\frac{C_K\nu\Delta t}{4}\|\nabla\boldsymbol{\rho}_{\boldsymbol{u}}^0\|_S^2+gS_0\|\rho_\phi^0\|_D^2+\frac{gk_{min}\Delta t}{4}\|\nabla\rho_\phi^0\|_D^2.\end{aligned}$$

Note that

$$\|\boldsymbol{\rho}_{\boldsymbol{u}}^0\|_S\le\|\boldsymbol{u}^0-\boldsymbol{u}_h^0\|_S+\|\boldsymbol{u}^0-\boldsymbol{U}^0\|_S\le Ch^3,\quad\|\nabla\boldsymbol{\rho}_{\boldsymbol{u}}^0\|_S\le\|\nabla(\boldsymbol{u}^0-\boldsymbol{u}_h^0)\|_S+\|\nabla(\boldsymbol{u}^0-\boldsymbol{U}^0)\|_S\le Ch^2,$$

and

$$\|\rho_\phi^0\|_D\le Ch^3,\quad\|\nabla\rho_\phi^0\|_D\le Ch^2.$$

Finally, by using the triangle inequality, the conclusion could be derived. □

5 Numerical results

We provide two numerical experiments to examine the algorithm studied herein. In the first experiment, the theoretical results are verified. In the second one, a model test is given to demonstrate the robustness of our algorithm. All the numerical experiments are implemented by utilizing the software Freefem++ [17].

5.1 Test 1

The computational domain is $\Omega_S = (0,1)\times(0,1)$ and $\Omega_D = (0,1)\times(-1,0)$ with the interface $\Gamma = (0,1)\times\{0\}$. Let the exact solutions be

$$\begin{aligned}
&\boldsymbol{u} = (u_1, u_2) = \left(\frac{1}{\pi}\sin(2\pi y)\cos(x)e^t, (\frac{1}{\pi^2}\sin^2(\pi y) - 2k)\sin(x)e^t\right),\\
&p = (2 - \pi\sin(\pi x))\sin(\pi y)e^t,\\
&\phi = (e^y - e^{-y})\sin(x)e^t.
\end{aligned}$$

The boundary conditions, initial conditions and source terms are determined by the true solutions. The parameters ν, α, S_0, g are chosen to be 1 and $\mathbb{K} = \mathbb{I}$. The final time is set to 3.

Here we follow the way to compute the order of convergence provided in [24]. In particular, let w (w can be $\boldsymbol{u}, \nabla\boldsymbol{u}, p, \phi, \nabla\phi$) be the true solution, $w_{h,\Delta t}$ be the numerical solution derived with mesh size h and fine time step Δt. If there holds

$$w_{h,\Delta t}(\boldsymbol{x}, t^N) \approx w(\boldsymbol{x}, t^N) + c_1(\boldsymbol{x}, t^N)\Delta t^k + c_2(\boldsymbol{x}, t^N)h^\gamma,$$

then we have

$$w_{h,\Delta t}(\boldsymbol{x}, t^N) - w_{h/2,\Delta t}(\boldsymbol{x}, t^N) \approx c_2(\boldsymbol{x}, t^N)h^\gamma(1 - (\frac{1}{2})^\gamma),$$

which indicates that

$$\begin{aligned}
&\frac{w_{h,\Delta t}(\boldsymbol{x}, t^N) - w_{h/2,\Delta t}(\boldsymbol{x}, t^N)}{w_{h/2,\Delta t}(\boldsymbol{x}, t^N) - w_{h/4,\Delta t}(\boldsymbol{x}, t^N)} \approx \frac{c_2(\boldsymbol{x}, t^N)h^\gamma(1 - (\frac{1}{2})^\gamma)}{c_2(\boldsymbol{x}, t^N)h^\gamma((\frac{1}{2})^\gamma) - (\frac{1}{4})^\gamma)} = 2^\gamma,\\
&\frac{w_{h,\Delta t}(\boldsymbol{x}, t^N) - w_{h,\Delta t/2}(\boldsymbol{x}, t^N)}{w_{h,\Delta t/2}(\boldsymbol{x}, t^N) - w_{h,\Delta t/4}(\boldsymbol{x}, t^N)} \approx \frac{c_1(\boldsymbol{x}, t^N)\Delta t^k(1 - (\frac{1}{2})^k)}{c_1(\boldsymbol{x}, t^N)\Delta t^k((\frac{1}{2})^k) - (\frac{1}{4})^k)} = 2^k.
\end{aligned}$$

Firstly, we choose $r = 5$. Numerical results and convergence orders with respect to h and Δt are provided in Tabs.1 and 2, respectively. As seen from these two tables, we can conclude that our scheme derive the optimal convergence orders. Then we choose $r = 10$, and obtain the numerical results and convergence orders in Tabs.3 and 4. It is clear that our scheme also arrive at the optimal convergence orders with a larger r. By comparing Tab. 1 to Tab 3., we see that r has almost no effect on spatial errors. However, by comparing Tab. 2 to Tab. 4, it is easy to find that r directly affects the time errors.

Table 1: The convergence performance of our algorithm at time $T = 3$, with varying spacing h but fixed time step $\Delta t = 0.01$ and $r = 5$.

h	$\|\boldsymbol{u}^N_{h,\Delta t} - \boldsymbol{u}^N_{\frac{h}{2},\Delta t}\|_S$	order	$\|\nabla(\|\boldsymbol{u}^N_{h,\Delta t} - \boldsymbol{u}^N_{\frac{h}{2},\Delta t})\|_S$	order	$\|p^N_{h,\Delta t} - p^N_{\frac{h}{2},\Delta t}\|_S$	order
$\frac{1}{4}$	8.3398e-2	–	2.1995e-0	–	1.5871e-0	–
$\frac{1}{8}$	1.0834e-2	2.94	5.5989e-1	1.97	3.7149e-1	2.09
$\frac{1}{16}$	1.3713e-3	2.98	1.4060e-1	1.99	9.0717e-2	2.03
$\frac{1}{32}$	1.7208e-4	2.99	3.5196e-2	1.99	2.2554e-2	2.00
h	$\|\phi^N_{h,\Delta t} - \phi^N_{\frac{h}{2},\Delta t}\|_D$	order	$\|\nabla(\phi^N_{h,\Delta t} - \phi^N_{\frac{h}{2},\Delta t}\|_D$	order		
$\frac{1}{4}$	4.0587e-3	–	1.6387e-1	–		
$\frac{1}{8}$	5.0619e-4	3.00	4.1301e-2	1.98		
$\frac{1}{16}$	6.3327e-5	2.99	1.0358e-2	1.99		
$\frac{1}{32}$	7.9428e-6	2.99	2.5932e-3	1.99		

Table 2: The convergence performance of our algorithm at time $T = 3$, with varying time step Δt but fixed spacing $h = \frac{1}{64}$ and $r = 5$.

Δt	$\|\boldsymbol{u}^N_{h,\Delta t} - \boldsymbol{u}^N_{h,\frac{\Delta t}{2}}\|_S$	order	$\|\nabla(\|\boldsymbol{u}^N_{h,\Delta t} - \boldsymbol{u}^N_{h,\frac{\Delta t}{2}})\|_S$	order	$\|p^N_{h,\Delta t} - p^N_{h,\frac{\Delta t}{2}}\|_S$	order
$\frac{1}{50}$	1.5562e-1	–	2.9122e-1	–	2.7626e-1	–
$\frac{1}{100}$	7.6083e-2	1.03	1.4199e-1	1.03	1.4479e-1	0.93
$\frac{1}{200}$	3.7571e-2	1.01	6.9874e-2	1.02	7.4205e-2	0.96
$\frac{1}{400}$	1.8663e-2	1.00	3.4615e-2	1.01	3.7578e-2	0.98
Δt	$\|\phi^N_{h,\Delta t} - \phi^N_{h,\frac{\Delta t}{2}}\|_D$	order	$\|\nabla(\phi^N_{h,\Delta t} - \phi^N_{h,\frac{\Delta t}{2}}\|_D$	order		
$\frac{1}{50}$	1.1026e-1	–	1.7711e-1	–		
$\frac{1}{100}$	5.5964e-2	0.97	8.9928e-2	0.97		
$\frac{1}{200}$	2.8190e-2	0.98	4.5307e-2	0.98		
$\frac{1}{400}$	1.4147e-2	0.99	2.2739e-2	0.99		

Table 3: The convergence performance of our algorithm at time $T = 3$, with varying spacing h but fixed time step $\Delta t = 0.01$ and $r = 10$.

h	$\|\boldsymbol{u}^N_{h,\Delta t} - \boldsymbol{u}^N_{\frac{h}{2},\Delta t}\|_S$	order	$\|\nabla(\|\boldsymbol{u}^N_{h,\Delta t} - \boldsymbol{u}^N_{\frac{h}{2},\Delta t})\|_S$	order	$\|p^N_{h,\Delta t} - p^N_{\frac{h}{2},\Delta t}\|_S$	order
$\frac{1}{4}$	8.3401e-2	–	2.1996e-0	–	1.5876e-0	–
$\frac{1}{8}$	1.0835e-2	2.94	5.5992e-1	1.97	3.7171e-1	2.09
$\frac{1}{16}$	1.3717e-3	2.98	1.4061e-1	1.99	9.0819e-2	2.03
$\frac{1}{32}$	1.7236e-4	2.99	3.5217e-2	1.99	2.2630e-2	2.00
h	$\|\phi^N_{h,\Delta t} - \phi^N_{\frac{h}{2},\Delta t}\|_D$	order	$\|\nabla(\phi^N_{h,\Delta t} - \phi^N_{\frac{h}{2},\Delta t}\|_D$	order		
$\frac{1}{4}$	4.0365e-3	–	1.6320e-1	–		
$\frac{1}{8}$	5.0376e-4	3.00	4.1143e-2	1.98		
$\frac{1}{16}$	6.3121e-5	2.99	1.0323e-2	1.99		
$\frac{1}{32}$	7.9676e-6	2.98	2.5858e-3	1.99		

Table 4: The convergence performance of our algorithm at time $T = 3$, with varying time step Δt but fixed spacing $h = \frac{1}{64}$ and $r = 10$.

Δt	$\|\boldsymbol{u}^N_{h,\Delta t} - \boldsymbol{u}^N_{h,\frac{\Delta t}{2}}\|_S$	order	$\|\nabla(\|\boldsymbol{u}^N_{h,\Delta t} - \boldsymbol{u}^N_{h,\frac{\Delta t}{2}})\|_S$	order	$\|p^N_{h,\Delta t} - p^N_{h,\frac{\Delta t}{2}}\|_S$	order
$\frac{1}{50}$	2.8165e-1	–	5.3007e-1	–	4.8255e-1	–
$\frac{1}{100}$	1.3385e-1	1.07	2.5207e-1	1.07	2.6554e-1	0.86
$\frac{1}{200}$	6.4892e-2	1.04	1.2156e-1	1.04	1.3992e-1	0.92
$\frac{1}{400}$	3.1902e-2	1.02	5.9417e-2	1.03	7.1875e-2	0.96
Δt	$\|\phi^N_{h,\Delta t} - \phi^N_{h,\frac{\Delta t}{2}}\|_D$	order	$\|\nabla(\phi^N_{h,\Delta t} - \phi^N_{h,\frac{\Delta t}{2}}\|_D$	order		
$\frac{1}{50}$	2.1392e-1	–	3.4339e-1	–		
$\frac{1}{100}$	1.1026e-1	0.95	1.7711e-1	0.95		
$\frac{1}{200}$	5.5964e-2	0.97	8.9928e-2	0.97		
$\frac{1}{400}$	2.8190e-2	0.98	4.5307e-2	0.98		

Here we want to see the variation of the kinetic energy $E^{n_k} = \|\boldsymbol{u}_h^{n_k}\|_S^2 + \|\phi_h^{n_k}\|_D^2$ for each $k = 0, 1, 2, ..., M$. To this end, we choose $\boldsymbol{f}_S \equiv 0, f_D \equiv 0$ and let the initial conditions meet

$$\begin{aligned}
&\boldsymbol{u}(x, y, 0) = (u_1(x, y, 0), u_2(x, y, 0)) = \left(\frac{1}{\pi}\sin(2\pi y)\cos(x), (\frac{1}{\pi^2}\sin^2(\pi y) - 2)\sin(x)\right),\\
&p(x, y, 0) = (2 - \pi\sin(\pi x))\sin(\pi y),\\
&\phi(x, y, 0) = (e^y - e^{-y})\sin(x).
\end{aligned}$$

We calculate the kinetic energy $E^{n_k} = \|\boldsymbol{u}_h^{n_k}\|_S^2 + \|\phi_h^{n_k}\|_D^2$ for each $k = 0, 1, 2, ..., M$ with varied $r \in \{1, 2, 5, 10, 20\}$. We take $h = \Delta t = \frac{1}{20}$ and $T = 10$. For a comparison, we also implement the fully coupled implicit method to derive the numerical solution as a reference. The variation tendency of the kinetic energy is shown in Fig.1. As seen from this figure, we find that the smaller the r is, the closer the numerical solution is to the reference solution, which is in line with forecast.

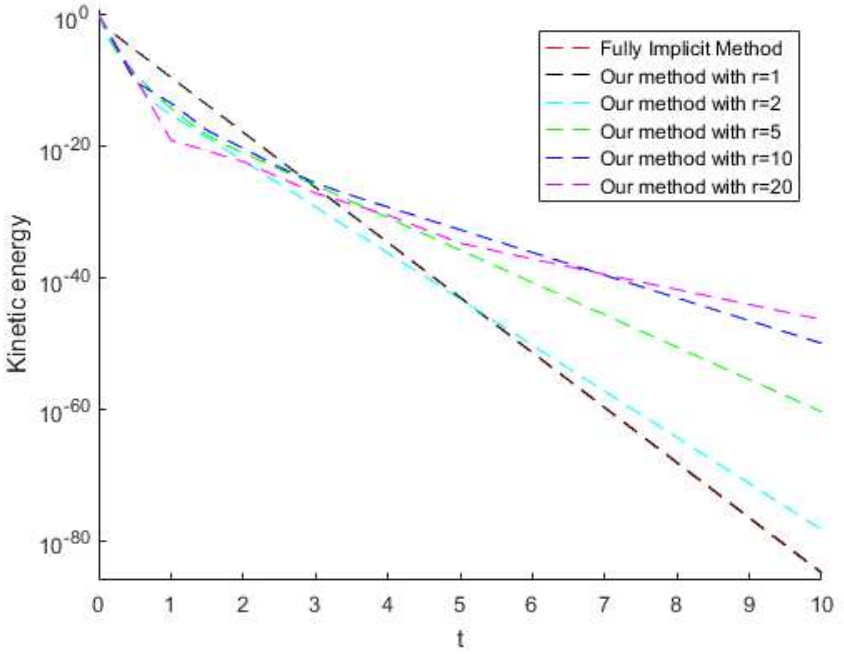


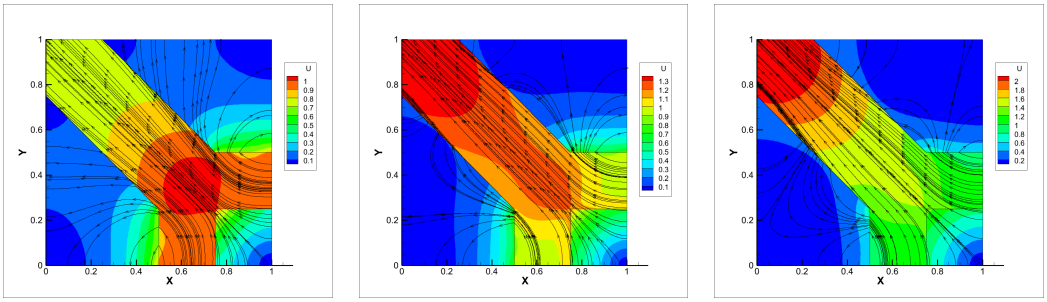
Figure 1: The decay of kinetic energy for different numerical schemes.

5.2 Test 2

In this test, we consider a model problem that models the fluid flow in karst groundwater aquifers. The problem region is $\Omega = [0,1] \times [0,1]$, where the free flow region Ω_S is a polygonal region $\overline{\text{ABCDEFGHIJ}}$. Here the points are $A = (0,1), B = (0,0.75), C = (0.5,0.25), D = (0.5,0), E = (0.75,0), F = (0.75,0.25), G = (1,0.25), H = (1,0.5), I = (0.75,0.5), J = (0.25,1)$. The porous medium region is $\Omega_D = \Omega \backslash \Omega_S$. We choose all parameters to 1 and let $\boldsymbol{f}_S = 0$ and $f_D = 0$. The final time is $T = 5$ and the ratio is $r = 10$. Let $\boldsymbol{u}^0 = \phi^0 = 0$. We consider the homogeneous Dirichlet boundary condition for the piezometric head on Γ_D, and the following boundary conditions for the velocity

$$\boldsymbol{u} = \begin{cases} (-s_1, 0)^T & \text{on } \overline{\text{GH}}, \\ (0, s_1)^T & \text{on } \overline{\text{DE}}, \\ (-s_2, s_2)^T & \text{on } \overline{\text{AB}} \text{ and } \overline{\text{JA}}, \end{cases}$$

in which s_1 and s_2 are two positive constants representing the inflow rate and the outflow rate, respectively. In the following, we fix $s_1 = 1$ and choose $s_2 \in \{0.5, 1, 1.5\}$. Note that if $s_2 = 0.5 < s_1$, the total inflow rate is larger than the total outflow rate; if $s_2 = 1 = s_1$, the total inflow rate is equal to the total outflow rate; if $s_2 = 1.5 > s_1$, the total inflow rate is less than the total outflow rate. We plot the speed and the streamlines in Fig.2. Note that the warmer color denotes higher speed of the flow, and the line with arrows represents the streamline. Fig.2 indicates that in the upper left corner of the calculation area, the flow rate increases successively. Moreover, the fluid flows from the fluid region to the porous medium region, but does not flow back if $s_1 > s_2$; the fluid that flows out of the fluid region will also return if $s_1 = s_2$; if $s_1 < s_2$, we see more fluid flows from a larger porous media domain into



(a) $s_1 = 1, s_2 = 0.5$ (b) $s_1 = 1, s_2 = 1$ (c) $s_1 = 1, s_2 = 1.5$

Figure 2: Plot of the speed and the streamlines of the velocity.

the free flow region than $s_1 = s_2$. In particular, we plot the variation of fluid flow over time in Fig.3 with $s_2 = 0.5$.

Declarations

Availability of supporting data

No data was used for the research described in the article.

Competing interests

The authors declare no competing interests.

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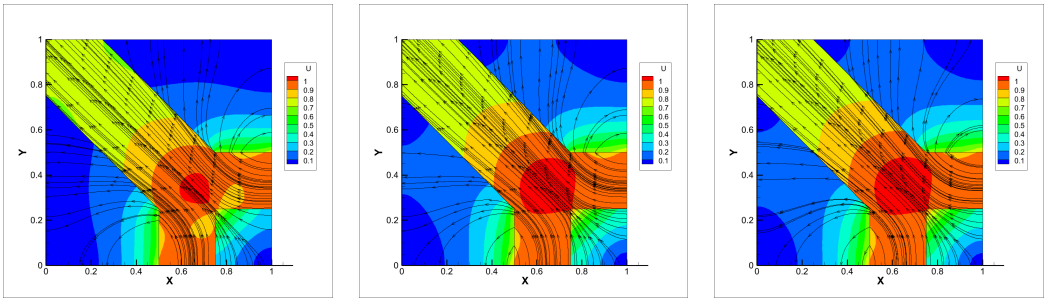
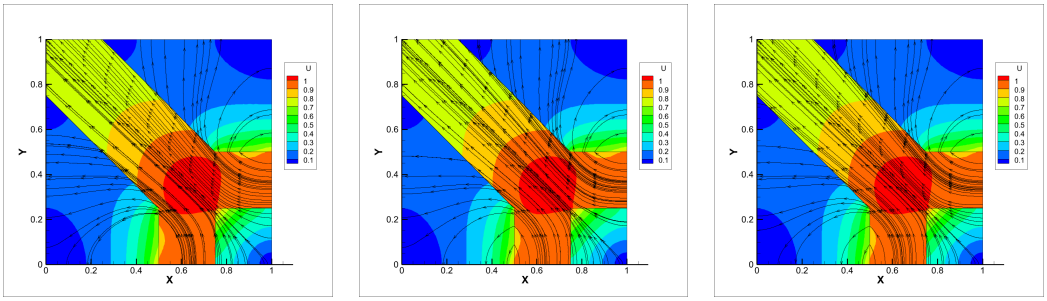


(a) $t = 0.1$ (b) $t = 0.2$ (c) $t = 0.3$



(d) $t = 0.4$ (e) $t = 0.5$ (f) $t = 5$

Figure 3: Plot of the speed and the streamlines of the velocity at different times.

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