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Global convergence for systems combining time-delayed negative feedback with time-periodic driving and applications

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Global convergence for systems combining time-delayed negative feedback with time-periodic driving and applications

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Abstract

For the infinite dimensional system combining time-delayed negative feedback and time-periodic driving, we investigate the discrete-time dynamical system generated by the Poincaré map T of this system and present an amenable condition which enables us to obtain the global convergence of the orbits. This yields the global convergence to the harmonic periodic solutions of the corresponding time-periodic delayed systems with negative feedback. Our approach is motivated by embedding the negative feedback system into a larger time-periodic monotone delayed dynamical systems. We further utilize the theoretical results to obtain the global convergence to periodic solutions for the time periodically-forced gene regulatory models with distribution delay. Numerical simulations are exhibited to illustrate the feasibility of our theoretical results for these models.

Keywords: global convergence, time-delayed negative feedback, time-periodic driving, Poincaré map, gene regulatory models

1. Introduction

Delay equations reveal systems whose future states depend on past values, vital for capturing time-lagged phenomena in biology, engineering, and epidemiology to predict dynamics accurately [3, 7]. Building on this foundation, feedback control systems dynamically adjust outputs to meet targets, ensuring stability from automation to biological regulation [1, 11, 14, 15, 16]. Among these, negative feedback systems are particularly prevalent in biological processes [8, 17, 21, 22]. Meanwhile, periodic phenomena arise naturally in population biology when accounting for external cyclic drivers like day-night rhythms or seasonal parameter variations [17, 20, 24]. Consequently, investigating the dynamics of systems combining time-delayed negative feedback and periodic driving presents both significant challenges and profound fascination. In the present paper, we consider the following time-periodic delayed cyclic system with negative feedback:

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$$\begin{cases} \dot{x}_1 = F_1(t, x_t) \equiv G_1\left(t, x_1, x_n(t-r)\right), \\ \dot{x}_i = F_i(t, x_t) \equiv G_i\left(t, x_i, x_{i-1}(t-r)\right), \qquad 2 \leqslant i \leqslant n-1, \\ \dot{x}_n = F_n(t, x_t) \equiv G_n\left(t, x_n, x_{n-1}(t-r)\right), \end{cases} \tag{1.1}$$

where $r > 0$ is the given delay and the nonlinearity $G = (G_1, G_2, \cdots, G_n)$ is continuous, together with their partial derivatives with respect to x, for any $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$. Here $\mathbb{R}_+ = [0, \infty)$. Moreover, there exists $\delta_i \in \{-1, +1\}$ such that

$$\delta_i \frac{\partial G_i(t, \xi, \eta)}{\partial \eta} > 0, \quad \text{for all } (t, \xi, \eta) \in \mathbb{R}_+ \times \mathbb{R}^2 \text{ and } 1 \leqslant i \leqslant n.$$

Let $\mathcal{X} \subset \mathcal{C} = C\left([-r, 0], \mathbb{R}^n\right)$ be the nonempty state space. It is also assumed that there exists $\tau > 0$ such that

$$F(t, x_t) = F\left(t + \tau, x_t\right), \ (t, x_t) \in \mathbb{R}_+ \times \mathcal{X}, \tag{1.2}$$

where $x_t(s) = x(t+s), \ -r \leqslant s \leqslant 0$.

For the autonomous case, that is, F does not depend on t in (1.1), Mallet-Paret and Sell [15] have established the remarkable Poincaré-Bendixson theory with delay, which yields that the ω-limit set of any bounded orbit must be a nontrival periodic orbit if it contains no equilibrium. Following [16], we call system (1.1)-(1.2) a time-periodic delayed *monotone cyclic feedback system* (MCFS). Let $\Delta = \delta_1 \delta_2 \cdots \delta_n = -1$. Then system (1.1)-(1.2) turns out to be a time-periodic delayed MCFS with negative feedback. It is worth pointing out that such system is not monotone in the delayed sense of Hirsch [9, 10, 19] with respect to certain usual convex cones.

Due to the complexity introduced by external periodic driving, very few studies have concentrated on dynamics of the time-periodic delayed MCFS with negative feedback. In the present paper, we will focus on the global asymptotic behavior of the time-periodic delayed negative feedback system (1.1)-(1.2) that can be viewed as a closed-loop system

$$\dot{x} = F(t, x_t) \equiv f\left(t, x_t, \mathbf{h}(x_t)\right), \tag{1.3}$$

which satisfies the following assumptions (see more details in Section 2):

- $(\mathbf{A1})'$ For each $\varphi \in \mathcal{U}$, $\dot{x} = f(t, x_t, \varphi)$ can generate a time-periodic monotone delayed system on $\mathcal{X}$ in the usual order relation sense.
- $(\mathbf{A2})'$ For each $(t, \xi) \in \mathbb{R}_+ \times \mathcal{X}$, $f(t, x_t, \varphi)$ is increasing in φ relative to an order relation.
- $(\mathbf{A3})'$ The output function $\mathbf{h} : \mathcal{X} \to \mathcal{U}$ is decreasing.

Here, the set $\mathcal{U} \subset C\left([-r, 0], \mathbb{R}^m\right)$ is called the input space and the assumption $(\mathbf{A3})'$ exhibits characteristic of the delayed negative feedback.

A prototypical example of system (1.3) is the following gene regulatory models with discrete

delay (see [18] for more details):

$$\begin{cases} \dot{x}_1 = g(x_n(t-r)) - \alpha_1(t)x_1, \\ \dot{x}_i = x_{i-1}(t-r) - \alpha_i(t)x_i, \qquad 2 \leqslant i \leqslant n-1, \\ \dot{x}_n = x_{n-1}(t-r) - \alpha_n(t)x_n. \end{cases} \tag{1.4}$$

Here $\alpha_i(t) = \alpha_i(t+\tau) > 0$ are continuous for any $i = 1, \cdots, n$, and $g : \mathbb{R}_+ \to \mathbb{R}_+$ is continuously differentiable and satisfies $g(0) > 0$ and its derivative $g' < 0$.

Enciso et al. [4] investigated the asymptotic behavior of the autonomous delayed system (1.3) with negative feedback. Under the assumptions $(\mathbf{A1})'$-$(\mathbf{A3})'$ with f being independent of time t, they constructed some positively invariant subset $B \subset X = \{x \in \mathbb{R}^n : \tilde{x} \in \mathcal{X} \subset \mathcal{C},\ \tilde{x}|_{[-r,0]} \equiv x\}$; and moreover, under the additional condition

$$a, b \in B \ \text{ with } \ a \leqslant b \ \text{ and } \ f(\tilde{a}, \mathbf{h}(\tilde{b})) = 0 = f(\tilde{b}, \mathbf{h}(\tilde{a})) \Longrightarrow a = b,$$

they have obtained the global convergence of any solution for such autonomous system.

In this paper, we will investigate the dynamics of the periodic map (i.e., Poincaré map), devoted by T, for the time-periodic closed-loop delayed negative feedback system (1.3). Motivated by the work in [4], we will embed the discrete-time dynamical system $\{T^n\}_{n\geqslant 0}$ on $\mathcal{X}$ into an extended discrete-time dynamical system $\{\tilde{T}^n\}_{n\geqslant 0}$, which is defined on the Cartesian Square $\mathcal{X} \times \mathcal{X}$, generated by the time-periodic symmetric system

$$\begin{cases} \dot{x} = f(t, x_t, \mathbf{h}(y_t)), \\ \dot{y} = f(t, y_t, \mathbf{h}(x_t)), \end{cases} \tag{1.5}$$

such that $\tilde{T}(\xi, \xi) = (T\xi, T\xi)$ on the invariant diagonal of $\mathcal{X} \times \mathcal{X}$. Furthermore, we provide an amenable condition:

$(\mathbf{A4})'$ There exist $x_0, y_0 \in X$ satisfying $x_0 \leqslant y_0$ with $[\tilde{x}_0, \tilde{y}_0] \subset \mathcal{X}$ and

$$\begin{aligned} &\bullet \ \int_0^\theta f(t, y_t^0, \mathbf{h}(x_t^0))\,\mathrm{d}t \leqslant 0 \leqslant \int_0^\theta f(t, x_t^0, \mathbf{h}(y_t^0))\,\mathrm{d}t, \ \text{for } \theta \in [\tau - r, \tau], \quad \text{if } \tau \geqslant r, \\ &\bullet \ \int_0^\theta f(t, y_t^0, \mathbf{h}(x_t^0))\,\mathrm{d}t \leqslant 0 \leqslant \int_0^\theta f(t, x_t^0, \mathbf{h}(y_t^0))\,\mathrm{d}t, \ \text{for } \theta \in [0, \tau], \qquad \text{if } \tau < r, \end{aligned} \tag{1.6}$$

where $\big(x^0(t), y^0(t)\big) = \Phi(t, 0, (\tilde{x}_0, \tilde{y}_0))$ is the solution of system (1.5). Moreover, for any two τ-periodic continuous functions $a(\cdot), b(\cdot) \in C([0, \tau], X)$ satisfying $a(\cdot) \leqslant b(\cdot)$ and $a_\tau, b_\tau \in [\tilde{x}_0, \tilde{y}_0]$, there holds

$$\int_0^\tau f(t, a_t, \mathbf{h}(b_t))\,\mathrm{d}t = 0 = \int_0^\tau f(t, b_t, \mathbf{h}(a_t))\,\mathrm{d}t \Longrightarrow a(\cdot) = b(\cdot). \tag{1.7}$$

Here (1.6) enables us to find a positively invariant region $\mathcal{B} \subset \mathcal{X}$ for the Poincaré map T associated with system (1.3) and (1.7) ensures the uniqueness of periodic solutions for system (1.3). It is worth noting that the global convergence of the time-periodic systems with negative

feedback has been obtained by $(\mathbf{A4})'$ without delay (i.e., $r = 0$)(see [21]). For the given delay $r > 0$, we will show that the global convergence of the orbits for the discrete-time dynamical system $\{T^n\}_{n\geqslant 0}$ holds whether the period $\tau \leqslant r$ or $\tau > r$ (see Theorem 2.1). In the terminology of differential equations, the global convergence to a unique τ-periodic solution is thus obtained for system (1.3) (see Corollary 2.2).

By applying our theoretical results to the time periodically-forced delayed gene regulatory system (1.4) with $\alpha_i(t) = \alpha_i, 1 \leqslant i \leqslant n-1$, we obtain that any solution will approach to a unique τ-periodic solution provided that the following condition holds:

$$(\mathbf{H1}) \qquad \max\left\{-g'(u) : 0 \leqslant u \leqslant \alpha^{-1}g(0)\right\} < \alpha, \quad \text{where} \quad \alpha = \prod_{i=1}^{n} \alpha_i \quad \text{and} \quad \alpha_n = \min_{0\leqslant t\leqslant \tau} \alpha_n(t).$$

Another prototypical application of system (1.3) is the following negative feedback systems with distribution delay:

$$\begin{cases} \dot{x}_1 = g(L_n x_n^t) - \alpha_1(t)x_1, \\ \dot{x}_i = L_{i-1}x_{i-1}^t - \alpha_i(t)x_i, \qquad 2 \leqslant i \leqslant n-1, \\ \dot{x}_n = L_{n-1}x_{n-1}^t - \alpha_n(t)x_n, \end{cases} \tag{1.8}$$

where $\alpha_i(t) = \alpha_i(t+\tau) > 0$ is continuous for any $i = 1, \cdots, n$, and $g : \mathbb{R}_+ \to \mathbb{R}_+$ is continuously differentiable and satisfies $g(0) > 0$ and its derivative $g' < 0$, and

$$L_i x_i^t = \frac{1}{r}\int_{-r}^{0} x_i(t+s)\,\mathrm{d}s, \ 1 \leqslant i \leqslant n.$$

According to our theoretical results for the time periodically-forced delayed negative feedback system (1.8) with $\alpha_i(t) = \alpha_i, 1 \leqslant i \leqslant n-1$, we obtain the existence and global convergence of periodic solution provided that the following condition holds:

$$(\mathbf{H2}) \quad \max\left\{-g'(u) : 0 \leqslant u \leqslant \alpha^{-1}g(0)\right\} < \frac{\alpha r}{N\tau}, \quad \text{where} \quad \alpha = \prod_{i=1}^{n} \alpha_i \quad \text{and} \quad \alpha_n = \min_{0\leqslant t\leqslant \tau} \alpha_n(t),$$

where N is the minimal positive integer such that $N\tau \geqslant r$. Moreover, numerical simulations for the dynamics of the time-periodic delayed gene regulatory model are illustrated to show the obtained theoretical results for periodic systems (1.4) and (1.8).

This paper is organized as follows. In Section 2, we introduce some notations and preliminaries, and state main results for the time-periodic delayed closed-loop system (1.3) with negative feedback. In Section 3, we give the proof of our main results. Finally, we explore the dynamics of time periodically-forced gene regulatory models with delay in Section 4.

2. Notations and main results

Let $\mathbb{R}^n$ be ordered by "$\leqslant_K$" generated by a cone K with nonempty interior and denote by K^* the dual cone to K. This cone K can induce a cone $\mathcal{C}_K$ in the Banach space $\mathcal{C} = C\left([-r, 0], \mathbb{R}^n\right)$,

where the delay $r > 0$, defined by

$$\mathcal{C}_K = \{\xi \in \mathcal{C} : \xi(s) \geqslant 0,\ -r \leqslant s \leqslant 0\}.$$

It has nonempty interior in $\mathcal{C}$ given by $\mathrm{Int}(\mathcal{C}_K) = \{\xi \in \mathcal{C} : \xi(s) \gg 0,\ -r \leqslant s \leqslant 0\}$, where the usual notion "$\leqslant, <, \ll$" will be used for the various order relations on $\mathcal{C}$ generated by $\mathcal{C}_K$. In particular,

$$\xi \leqslant \eta \ \text{ in } \ \mathcal{C} \iff \ \text{ for any } \ s \in [-r, 0],\ \xi(s) \leqslant \eta(s) \ \text{ in } \ \mathbb{R}^n.$$

We will use the same notation "$\leqslant$" for the order relation in $\mathbb{R}^n$ and $\mathcal{C}$ without explanation in the future. Let $\mathbb{R}^m$ be ordered by "$\leqslant_U$" generated by a cone K_U with nonempty interior. Similarly, the cone K_U induces a cone $\mathcal{C}_{K_U}$ in the Banach space $\mathcal{C}_U = C([-r, 0], \mathbb{R}^m)$, defined by

$$\mathcal{C}_{K_U} = \{\varphi \in \mathcal{C}_U : \varphi(s) \geqslant 0,\ -r \leqslant s \leqslant 0\}.$$

There exist two natural imbeddings which are $\mathbb{R}^n$ into $\mathcal{C}$ and $\mathbb{R}^m$ into $\mathcal{C}_U$ respectively. Let

$$X = \{x \in \mathbb{R}^n : \tilde{x} \in \mathcal{X} \subset \mathcal{C},\ \tilde{x}|_{[-r,0]} \equiv x\}.$$

Denote $\mathcal{U} \subset \mathcal{C}_U$ and $U = \{u \in \mathbb{R}^m : \tilde{u} \in \mathcal{U}\}$ in the same way. Let $x_t(s) = x(t + s),\ -r \leqslant s \leqslant 0$.

We focus on the asymptotic behavior of the time-periodic closed-loop system

$$\dot{x} = F(t, x_t) \equiv f\left(t, x_t, \mathbf{h}(x_t)\right),\ (t, x_t) \in \mathbb{R}_+ \times \mathcal{X} \tag{2.1}$$

obtained from the open-loop system

$$\begin{cases} \dot{x} = f(t, x_t, \varphi),\ \varphi \in \mathcal{U}, \\ y = \mathbf{h}(x_t) \end{cases}$$

by identifying input and output: $y = \varphi$ (see [1] for more details). Here $f(t, \xi, \varphi) = f(t + \tau, \xi, \varphi)$ is a τ-period vector field.

Assume that $f : \mathbb{R}_+ \times \mathcal{X} \times \mathcal{U} \to \mathbb{R}^n$ and $\mathbf{h} : \mathcal{X} \to \mathcal{U}$ are continuous and satisfy

($\mathbf{A1}$)$'$ For each $(t, \varphi) \in \mathbb{R}_+ \times \mathcal{U}$, $\xi \mapsto f(t, \xi, \varphi)$ is quasimonotone in the sense of (**QMD**): for any $\xi, \eta \in \mathcal{X}$, $\lambda \in K^*$ with $\xi \leqslant \eta$ and $\lambda(\xi(0)) = \lambda(\eta(0))$ implies $\lambda\left(f(t, \xi, \varphi)\right) \leqslant \lambda\left(f(t, \eta, \varphi)\right)$.

($\mathbf{A2}$)$'$ For each $(t, \xi) \in \mathbb{R}_+ \times \mathcal{X}$, $\varphi_1 \leqslant \varphi_2 \Longrightarrow f(t, \xi, \varphi_1) \leqslant f(t, \xi, \varphi_2)$.

($\mathbf{A3}$)$'$ For any $x \in X$, there exists $u = h(x) \in U$ such that $\mathbf{h}(\tilde{x}) = \tilde{u};\ \xi_1 \leqslant \xi_2 \Longrightarrow \mathbf{h}(\xi_2) \leqslant \mathbf{h}(\xi_1)$.

Assume that solutions of initial value problems in $\mathcal{X}$ for (2.1) are unique and write $\Psi(t, t_0, \xi)$ for the maximally extended solution satisfying $\Psi_{t_0}(t_0, \xi) = \xi$, i.e., $\Psi(t_0 + s, t_0, \xi) = \xi(s)$ for any $s \in [-r, 0]$. We will assume without further mentioning that the domain of $\Psi(t, t_0, \xi)$ includes $[t_0 - r, \infty)$ in case $\xi \in \mathcal{X}$. The closed-loop system (2.1) can be imbedded into a larger symmetric

system on the Cartesian Square $\mathcal{X}\times\mathcal{X}$ as

$$\begin{cases}\dot{x}=f(t,x_t,\mathbf{h}(y_t)),\\ \dot{y}=f(t,y_t,\mathbf{h}(x_t)).\end{cases}\tag{2.2}$$

For $\sigma\in\mathcal{X}\times\mathcal{X}$, denote $\Phi(t,t_0,\sigma)$ be the unique solution of system (2.2) satisfying $\Phi_{t_0}(t_0,\sigma)=\sigma$. Assume also that $G(t,(\xi,\eta))=(f(t,\xi,\mathbf{h}(\eta)),f(t,\eta,\mathbf{h}(\xi)))$, the right-hand side of (2.2), is completely continuous (i.e., mapping bounded sets to precompact sets). Due to the uniqueness of solutions, the diagonal

$$D=\{(\xi,\xi):\xi\in\mathcal{X}\}$$

is positively invariant for (2.2). More precisely, $\Phi(t,t_0,\sigma)\in D$ whenever $\sigma=(\xi,\xi)\in D$, for all $t_0\geqslant 0$. In particular, $\Phi(t,t_0,\sigma)=(\Psi(t,t_0,\xi),\Psi(t,t_0,\xi))$.

It is convenient to define the fundamental object of study in this section, i.e., the Poincaré map for the τ-periodic system (2.1):

$$T\xi=\Psi_\tau(0,\xi),\ \text{ for any } \xi\in\mathcal{X}.$$

Here $(T\xi)(s)=\Psi_\tau(0,\xi)(s)=\Psi(\tau+s,0,\xi)$, for any $s\in[-r,0]$.
According to the existence and uniqueness of the solution of the initial value problem and the periodicity of the vector field, one has

$$T^n\xi=\Psi_{n\tau}(0,\xi),\ \text{ for any } n\geqslant 1,$$

where $T^n=\underbrace{T\circ T\circ\cdots\circ T}_{n}$ is an n-order composite mapping.
Furthermore, we define the Poincaré map $\tilde{T}$ for the τ-periodic system (2.2) as

$$\tilde{T}\sigma=\Phi_\tau(0,\sigma),\ \text{ for any } \sigma\in\mathcal{X}\times\mathcal{X}.$$

In particular, if $\sigma=(\xi,\xi)\in D$, then $\tilde{T}(\xi,\xi)=(T\xi,T\xi)$.

By virtue of $(\mathbf{A1})'$-$(\mathbf{A3})'$, the Poincaré map $\tilde{T}$ for the extended system (2.2) can generate a discrete-time monotone system on $\mathcal{X}\times\mathcal{X}\subset\mathcal{C}\times\mathcal{C}$ with regard to the cone $\mathcal{P}=\mathcal{C}_K\times(-\mathcal{C}_K)$ as below, where $\mathcal{P}$ gives rise to the order relation

$$(\xi,\eta)\leqslant_{\mathcal{P}}(\bar{\xi},\bar{\eta})\iff\xi\leqslant\bar{\xi}\ \text{ and }\ \bar{\eta}\leqslant\eta.$$

Now we introduce the following amenable condition and state our main results as follows.

$(\mathbf{A4})'$ There exist $x_0,y_0\in X$ satisfying $x_0\leqslant y_0$ with $[\tilde{x}_0,\tilde{y}_0]\subset\mathcal{X}$ and

$$\begin{aligned}&\bullet\ \int_0^\theta f(t,y_t^0,\mathbf{h}(x_t^0))\,\mathrm{d}t\leqslant 0\leqslant\int_0^\theta f(t,x_t^0,\mathbf{h}(y_t^0))\,\mathrm{d}t,\ \text{for } \theta\in[\tau-r,\tau],\quad \text{if } \tau\geqslant r,\\ &\bullet\ \int_0^\theta f(t,y_t^0,\mathbf{h}(x_t^0))\,\mathrm{d}t\leqslant 0\leqslant\int_0^\theta f(t,x_t^0,\mathbf{h}(y_t^0))\,\mathrm{d}t,\ \text{for } \theta\in[0,\tau],\qquad \text{if } \tau<r,\end{aligned}\tag{2.3}$$

where $\left(x^0(t), y^0(t)\right) = \Phi(t, 0, (\tilde{x}_0, \tilde{y}_0))$ is the solution of system (2.2). Moreover, for any two τ-periodic continuous functions $a(\cdot)$, $b(\cdot) \in C([0,\tau], X)$ satisfying $a(\cdot) \leqslant b(\cdot)$ and $a_\tau, b_\tau \in [\tilde{x}_0, \tilde{y}_0]$, there holds

$$\int_0^\tau f(t, a_t, \mathbf{h}(b_t))\,\mathrm{d}t = 0 = \int_0^\tau f(t, b_t, \mathbf{h}(a_t))\,\mathrm{d}t \Longrightarrow a(\cdot) = b(\cdot). \tag{2.4}$$

Theorem 2.1. *Assume* $(\mathbf{A1})'$-$(\mathbf{A4})'$ *hold. Then the Poincaré map* T *for system* (2.1) *possesses a unique fixed point* $\bar{r} \in [\tilde{x}_0, \tilde{y}_0]$ *such that*

$$\omega_T([\tilde{x}_0, \tilde{y}_0]) = \{\bar{r}\}.$$

In the terminology of differential equations, we have the following result for system (2.1).

Corollary 2.2. *Assume* $(\mathbf{A1})'$-$(\mathbf{A4})'$ *hold. Then system* (2.1) *has a unique* τ*-periodic solution* $r(t)$*. Moreover, for any* $\bar{\xi} \in [\tilde{x}_0, \tilde{y}_0]$,

$$\|\Psi(t, 0, \bar{\xi}) - r(t)\| \to 0, \quad \text{as } t \to \infty.$$

3. Proof of the main results

Throughout this section, we always assume that $(\mathbf{A1})'$-$(\mathbf{A4})'$ hold. In order to prove our main results, we need the following Lemmas.

Lemma 3.1. *The Poincaré map* $\tilde{T}$ *for* (2.2) *generates a discrete-time monotone system on* $\mathcal{X} \times \mathcal{X}$ *with respect to* $\leqslant_{\mathcal{P}}$.

Proof. Let $G(t, (\xi, \eta)) = (f(t, \xi, \mathbf{h}(\eta)), f(t, \eta, \mathbf{h}(\xi)))$. To prove the monotonicity of the Poincaré map $\tilde{T}$, one only needs to show that $G(t, (\xi, \eta))$ satisfies (**QMD**) condition relative to the cone $\mathcal{P}$, (see [10, Section 4, Theorem 4.1]). To this purpose, for any $(\xi, \eta) \leqslant_{\mathcal{P}} (\bar{\xi}, \bar{\eta})$ and $(\lambda, -\mu) \in C^*$ with $(\lambda, -\mu)(\xi(0), \eta(0)) = (\lambda, -\mu)\left(\bar{\xi}(0), \bar{\eta}(0)\right)$, we must verify that

$$(\lambda, -\mu) G(t, (\xi, \eta)) \leqslant (\lambda, -\mu) G\left(t, (\bar{\xi}, \bar{\eta})\right), \text{ for any } t \geqslant 0.$$

In fact, $(\lambda, -\mu)(\xi(0), \eta(0)) = (\lambda, -\mu)\left(\bar{\xi}(0), \bar{\eta}(0)\right)$ and $\lambda, \mu \in K^*$ imply that

$$0 \leqslant \lambda\left(\bar{\xi}(0) - \xi(0)\right) = \mu\left(\bar{\eta}(0) - \eta(0)\right) \leqslant 0.$$

Thus, we obtain

$$\lambda(\xi(0)) = \lambda(\bar{\xi}(0)) \text{ and } \mu(\eta(0)) = \mu(\bar{\eta}(0)).$$

Note that $\xi \leqslant \bar{\xi}$ and $\bar{\eta} \leqslant \eta$. Then, one has

$$\begin{aligned}
(\lambda, -\mu) G(t, (\xi, \eta)) &= \lambda(f(t, \xi, \mathbf{h}(\eta))) - \mu(f(t, \eta, \mathbf{h}(\xi))) \\
&\leqslant \lambda(f(t, \xi, \mathbf{h}(\bar{\eta}))) - \mu\left(f(t, \eta, \mathbf{h}(\bar{\xi}))\right) \\
&\leqslant \lambda\left(f(t, \bar{\xi}, \mathbf{h}(\bar{\eta}))\right) - \mu\left(f(t, \bar{\eta}, \mathbf{h}(\bar{\xi}))\right)
\end{aligned}$$

$$= (\lambda, -\mu) G\left(t, (\bar{\xi}, \bar{\eta})\right),$$

where the first inequality is due to the monotonicity of f and h with respect to $(\mathbf{A2})'$-$(\mathbf{A3})'$, and the second inequality follows from $(\mathbf{A1})'$. □

Lemma 3.2. *If (2.3) holds, then the order interval*

$$I = [(\tilde{x}_0, \tilde{y}_0), (\tilde{y}_0, \tilde{x}_0)]_{\mathcal{P}} = \{(\xi, \eta) : (\tilde{x}_0, \tilde{y}_0) \leqslant_{\mathcal{P}} (\xi, \eta) \leqslant_{\mathcal{P}} (\tilde{y}_0, \tilde{x}_0)\}.$$

is positively invariant with respect to $\tilde{T}$ for system (2.2). In particular, $[\tilde{x}_0, \tilde{y}_0]$ is positively invariant with respect to T for system (2.1).

Proof. Note that $x_0 \leqslant y_0$ implies $(\tilde{x}_0, \tilde{y}_0) \leqslant_{\mathcal{P}} (\tilde{y}_0, \tilde{x}_0)$. Denote $\tilde{a}_0 = (\tilde{x}_0, \tilde{y}_0)$, $\tilde{b}_0 = (\tilde{y}_0, \tilde{x}_0)$.
Case 1. When $\tau \geqslant r$, for any $s \in [-r, 0]$, by (2.3), we obtain

$$\begin{aligned}
(\tilde{T}\tilde{a}_0)(s) - \tilde{a}_0(s) &= \Phi(\tau + s, 0, \tilde{a}_0) - \tilde{a}_0(s) \\
&= \Phi(\tau + s, 0, \tilde{a}_0) - \tilde{a}_0(0) \\
&= \int_0^{\tau+s} G\left(t, \Phi_t(0, \tilde{a}_0)\right) \mathrm{d}t \\
&= \left(\int_0^{\tau+s} f(t, x_t^0, \mathbf{h}(y_t^0)) \, \mathrm{d}t, \int_0^{\tau+s} f(t, y_t^0, \mathbf{h}(x_t^0)) \, \mathrm{d}t \right) \geqslant (0, 0).
\end{aligned}$$

Case 2. When $\tau \leqslant r$, for any $s \in [-r, 0]$, we discuss it separately as follows.
If $-r \leqslant s \leqslant -\tau$, by the definition of the Poincaré map $\tilde{T}$, one has

$$\begin{aligned}
(\tilde{T}\tilde{a}_0)(s) - \tilde{a}_0(s) &= \Phi(\tau + s, 0, \tilde{a}_0) - \tilde{a}_0(s) \\
&= \tilde{a}_0(\tau + s) - \tilde{a}_0(s) = (0, 0);
\end{aligned}$$

If $-\tau \leqslant s \leqslant 0$, by (2.3), one has

$$\begin{aligned}
(\tilde{T}\tilde{a}_0)(s) - \tilde{a}_0(s) &= \int_0^{\tau+s} G\left(t, \Phi_t(0, \tilde{a}_0)\right) \mathrm{d}t \\
&= \left(\int_0^{\tau+s} f(t, x_t^0, \mathbf{h}(y_t^0)) \, \mathrm{d}t, \int_0^{\tau+s} f(t, y_t^0, \mathbf{h}(x_t^0)) \, \mathrm{d}t \right) \geqslant (0, 0).
\end{aligned}$$

In a word, we conclude

$$\tilde{T}\tilde{a}_0 \geqslant_{\mathcal{P}} \tilde{a}_0.$$

Observe also that system (2.2) is symmetric. Then it yields that

$$\Phi(t, 0, \tilde{b}_0) = (y^0(t), x^0(t)), \quad \text{for any } \ t \geqslant 0.$$

Thus, by repeating the same arguments above, one can obtain $\tilde{T}\tilde{b}_0 \leqslant_{\mathcal{P}} \tilde{b}_0$. Consequently, for any $\sigma \in I$, it follows from Lemma 3.1 that

$$\tilde{a}_0 \leqslant_{\mathcal{P}} \tilde{T}\tilde{a}_0 \leqslant_{\mathcal{P}} \tilde{T}\sigma \leqslant_{\mathcal{P}} \tilde{T}\tilde{b}_0 \leqslant_{\mathcal{P}} \tilde{b}_0,$$

which entails that I is positively invariant for $\tilde{T}$. Note also that the diagonal

$$D = \{(\xi, \xi) : \xi \in \mathcal{X}\}$$

is positively invariant for system (2.2). Then $I \cap D$ is positively invariant for $\tilde{T}$. Since $\{\tilde{T}^n\}_{n\geqslant 0}$ reduces to $\{T^n\}_{n\geqslant 0}$ on D, we obtain that $[\tilde{x}_0, \tilde{y}_0]$ is positively invariant with respect to T. □

Lemma 3.3. *The Poincaré map $\tilde{T}$ has two fixed points $\bar{p}, \bar{q} \in I$ with $\bar{p} \leqslant_{\mathcal{P}} \bar{q}$.*

Proof. Given any $\sigma \in I$, one has $\tilde{a}_0 \leqslant_{\mathcal{P}} \sigma \leqslant_{\mathcal{P}} \tilde{b}_0$. By virtue of Lemmas 3.1-3.2, we obtain

$$\tilde{a}_0 \leqslant_{\mathcal{P}} \tilde{T}\tilde{a}_0 \leqslant_{\mathcal{P}} \tilde{T}\sigma \leqslant_{\mathcal{P}} \tilde{T}\tilde{b}_0 \leqslant_{\mathcal{P}} \tilde{b}_0.$$

Observe that the vector field $G(t, (\xi, \eta))$ is completely continuous and the order interval I is unifomly bounded, then $O_{\tilde{T}}(\tilde{a}_0)$, $O_{\tilde{T}}(\tilde{b}_0) \subset I$ are equicontinuous. By Arzelà-Ascoli Theorem, they have compact closure. Then there exist $\bar{p}, \bar{q} \in I$ such that

$$\omega_{\tilde{T}}(\tilde{a}_0) = \{\bar{p}\} \quad \text{and} \quad \omega_{\tilde{T}}(\tilde{b}_0) = \{\bar{q}\}$$

by using monotone convergence criterion (see [10, Section 5, Lemma 5.3]). Moreover,

$$\tilde{T}^n\tilde{a}_0 \to \bar{p}, \ \tilde{T}^n\tilde{b}_0 \to \bar{q}, \quad \text{as } n \to \infty.$$

Clearly, $\bar{p}$ and $\bar{q}$ are two fixed points of $\tilde{T}$ with $\bar{p} \leqslant_{\mathcal{P}} \bar{q}$. □

An immediate consequence of Lemma 3.3 is the following result.

Lemma 3.4. *System (2.2) possesses two τ-periodic solutions $p(t) = (p^1(t), p^2(t))$, $q(t) = (p^2(t), p^1(t))$ satisfying $p(\tau + s) = \bar{p}(s)$, $q(\tau + s) = \bar{q}(s)$, for any $s \in [-r, 0]$ and $p(t) \leqslant_{\mathcal{P}} q(t)$, for any $t \geqslant 0$. Moreover,*

$$\|\Phi(t, 0, \tilde{a}_0) - p(t)\| \to 0 \ \text{ and } \ \|\Phi(t, 0, \tilde{b}_0) - q(t)\| \to 0, \quad \text{as } t \to \infty.$$

Proof. By Lemma 3.3, $\bar{p}$ and $\bar{q}$ are two fixed points of $\tilde{T}$ satisfying $\bar{p} \leqslant_{\mathcal{P}} \bar{q}$. Then

$$p(t) = \Phi(t, 0, \bar{p}) \ \text{ and } \ q(t) = \Phi(t, 0, \bar{q})$$

are two τ-periodic solutions of the system (2.2). Moreover, we also obtain

$$p(\tau + s) = \Phi(\tau + s, 0, \bar{p}) = (\tilde{T}\bar{p})(s) = \bar{p}(s), \ \text{ for any } \ s \in [-r, 0].$$

Similarly, $q(\tau + s) = \bar{q}(s)$, for any $s \in [-r, 0]$. Next, we will show

$$\|\Phi(t, 0, \tilde{a}_0) - p(t)\| \to 0, \ \text{ as } t \to \infty.$$

The proof for $q(t)$ is analogous. Since $\tilde{T}^n\tilde{a}_0 \to \bar{p}$, one has

$$\begin{aligned}\|\Phi(t,0,\tilde{T}^n\tilde{a}_0) - p(t)\| &= \|\Phi\left(t,0,\Phi_{n\tau}(0,\tilde{a}_0)\right) - p(t)\| \\ &= \|\Phi\left(t+n\tau,n\tau,\Phi_{n\tau}(0,\tilde{a}_0)\right) - p(t)\| \\ &= \|\Phi(t+n\tau,0,\tilde{a}_0) - p(t)\| \to 0,\end{aligned} \tag{3.1}$$

as $n \to \infty$ uniformly for $t \in [0,\tau]$ (see [7, Chapter 2, Theorem 2.2]). For any sequence $t_n \to \infty$, we write $t_n = m_n\tau + s_n$, where m_n are nonnegative integers and $s_n \in [0,\tau)$. Then

$$\begin{aligned}\|\Phi(t_n,0,\tilde{a}_0) - \Phi(t_n,0,\bar{p})\| &= \|\Phi(m_n\tau+s_n,0,\tilde{a}_0) - \Phi(m_n\tau+s_n,0,\bar{p})\| \\ &= \|\Phi(m_n\tau+s_n,0,\tilde{a}_0) - \Phi(s_n,0,\bar{p})\|.\end{aligned}$$

Together with (3.1), we obtain that

$$\|\Phi(t,0,\tilde{a}_0) - p(t)\| = \|\Phi(t,0,\tilde{a}_0) - \Phi(t,0,\bar{p})\| \to 0,$$

as $n \to \infty$. Similarly,

$$\|\Phi(t,0,\tilde{b}_0) - q(t)\| \to 0, \quad \text{as } t \to \infty.$$

Finally, donote

$$p(t) = (p^1(t), p^2(t)).$$

Then the symmetry of system (2.2) implies that

$$q(t) = (p^2(t), p^1(t)).$$

Again, by Lemma 3.1, one has

$$p(t) = \Phi(t,0,\bar{p}) \leqslant_{\mathcal{P}} \Phi(t,0,\bar{q}) = q(t), \ \text{ for any } \ t \geqslant 0.$$

Thus, we have completed the proof. □

Now, we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. By Lemma 3.4, $p(t) = (p^1(t), p^2(t))$, $q(t) = (p^2(t), p^1(t))$, defined on $[0,\infty)$, are two τ-periodic solutions of system (2.2) satisfying

$$p(\tau+s) = \bar{p}(s), \ \ q(\tau+s) = \bar{q}(s), \ \text{ for any } \ s \in [-r,0],$$

and

$$p^1(t) \leqslant p^2(t), \ \text{for any } t \geqslant 0.$$

Then $p^1(t)$, $p^2(t)$ are two τ-periodic continuous functions and satisfy

$$\begin{cases}\dot{p}^1(t) = f(t,p^1_t,\mathbf{h}(p^2_t)),\\ \dot{p}^2(t) = f(t,p^2_t,\mathbf{h}(p^2_t)),\end{cases}$$

and

$$\left(p^1(\tau+s),p^2(\tau+s)\right)=p(\tau+s)=\bar{p}(s)=\left(\bar{p}^1(s),\bar{p}^2(s)\right),\ \text{ for any } s\in[-r,0].$$

Therefore, we obtain

$$0=p^1(\tau)-p^1(0)=\int_0^\tau f(t,p_t^1,\mathbf{h}(p_t^2))\,\mathrm{d}t,$$

and

$$0=p^2(\tau)-p^2(0)=\int_0^\tau f(t,p_t^2,\mathbf{h}(p_t^1))\,\mathrm{d}t.$$

By virtue of (2.4) in $(\mathbf{A4})'$, there holds

$$p^1(t)=p^2(t),\ \text{ for any } t\geqslant 0.$$

Hence, one has

$$\Phi(t,0,\bar{p})=p(t)=q(t)=\Phi(t,0,\bar{q}),\ \text{ for any }\ t\geqslant 0.$$

This yields that

$$\bar{p}(s)=p(\tau+s)=q(\tau+s)=\bar{q}(s)\triangleq(\bar{r}(s),\bar{r}(s)),\ \text{ for any } s\in[-r,0].$$

Due to Lemma 3.3, it is clear that $\bar{p}(=\bar{q})$ is the unique fixed point of $\tilde{T}$ in I. Therefore, we can obtain

$$\omega_{\tilde{T}}(I)=\omega_{\tilde{T}}\left([\tilde{a}_0,\tilde{b}_0]_{\mathcal{P}}\right)=\{\bar{p}\}\,(=\{\bar{q}\}).$$

This implies that $\bar{r}$ is the unique fixed point of T in $[\tilde{x}_0,\tilde{y}_0]$ and

$$\omega_T([\tilde{x}_0,\tilde{y}_0])=\{\bar{r}\}.$$

Thus, we have proved Theorem 2.1. □

4. Applications to time-periodic delayed gene regulatory models

In this section, we explore the existence of periodic solutions and global convergence for time periodically-forced delayed gene regulatory models. Numerical simulations are also presented to validate the theoretical results.

4.1. Time-periodic gene regulatory models with discrete delay

Firstly, we explore the dynamics of the time-periodic gene regulatory model with discrete delay, modeled by the equations

$$\begin{cases}\dot{x}_1=g(x_n(t-r))-\alpha_1x_1,\\ \dot{x}_i=x_{i-1}(t-r)-\alpha_ix_i, \qquad 2\leqslant i\leqslant n-1,\\ \dot{x}_n=x_{n-1}(t-r)-\alpha_n(t)x_n.\end{cases}\tag{4.1}$$

Here $\alpha_i>0$ for any $i=1,\cdots,n-1$, and the τ-periodic function $\alpha_n(t)=\alpha_n(t+\tau)>0$ is continuous, and $g:\mathbb{R}_+\to\mathbb{R}_+$ is continuously differentiable and satisfies $g(0)>0$ and $g'<0$.

Let $\mathcal{C} = C\left([-r, 0], \mathbb{R}^n_+\right)$ and $\mathcal{C}_U = C\left([-r, 0], \mathbb{R}_+\right)$. The map $\mathbf{h} : \mathcal{C} \to \mathcal{C}_U$ is defined by $\mathbf{h}(\xi) = \tilde{g}(\xi_n(-r))$, that is, $\mathbf{h}(\xi)$ is a constant function. With these hypotheses, the time-dependent vector field of system (4.1) naturally satisfies assumptions $(\mathbf{A1})'$-$(\mathbf{A3})'$. Take the cone $K = \mathbb{R}^n_+$. Let

$$X = [x_0, y_0] \triangleq \left[0, g(0)\left(\alpha_1^{-1}, \alpha_1^{-1}\alpha_2^{-1}, \cdots, \alpha_1^{-1}\alpha_2^{-1}\cdots\alpha_n^{-1}\right)^\top\right], \ \mathcal{X} = [\tilde{x}_0, \tilde{y}_0] \tag{4.2}$$

and introduce the assumption

$$(\mathbf{H1}) \qquad \max\left\{-g'(u) : 0 \leqslant u \leqslant \alpha^{-1}g(0)\right\} < \alpha, \quad \text{where } \ \alpha = \prod_{i=1}^{n}\alpha_i \ \text{ and } \ \alpha_n = \min_{0\leqslant t\leqslant\tau}\alpha_n(t).$$

By applying Corollary 2.2 to the time-periodic gene regulatory model (4.1) with discrete delay, we have the following

Theorem 4.1. *Assume* $(\mathbf{H1})$ *holds. Then system* (4.1) *has a unique* τ*-periodic solution* $r(t)$ *in* X *such that for any* $\bar{\xi} \in \mathcal{X}$,

$$\|\Psi(t, 0, \bar{\xi}) - r(t)\| \to 0, \quad \text{as } t \to \infty,$$

where $\Psi(t, 0, \bar{\xi})$ *is the solution of system* (4.1) *satisfying* $\Psi_0(0, \bar{\xi}) = \bar{\xi}$.

Proof. We first show that the solution $\Psi(t, 0, \bar{\xi})$ remains in X (see (4.2)), for any $\bar{\xi} \in \mathcal{X}$ and $t \geqslant 0$. Write

$$\Psi(t, 0, \bar{\xi}) = (x_1(t), x_2(t), \cdots, x_n(t)) \ \text{ and } \ \bar{\xi} = \left(\bar{\xi}_1, \bar{\xi}_2, \cdots, \bar{\xi}_n\right).$$

It is easy to see that $\Psi(t, 0, \bar{\xi}) \in \mathbb{R}^n_+$ (see [19, Chapter 5, Theorem 2.1]), whenever $\bar{\xi} \in \mathcal{C}$. Since $g(x_n(t-r)) - \alpha_1 x_1 \leqslant g(0) - \alpha_1 x_1$ and $\bar{\xi}_1 \leqslant g(0)\alpha_1^{-1}$, the comparison theorem (see [13, Chapter 6, Theorem 6.9.4]) implies that

$$x_1(t) \leqslant e^{-\alpha_1 t}g(0)\alpha_1^{-1} + g(0)\alpha_1^{-1}\left(1 - e^{-\alpha_1 t}\right) = g(0)\alpha_1^{-1}.$$

Similarly, one has

$$x_i(t) \leqslant g(0)\alpha_1^{-1}\alpha_2^{-1}\cdots\alpha_i^{-1}, \quad \text{for any } \ 2 \leqslant i \leqslant n-1.$$

Note that $x_{n-1}(t-r) - \alpha_n(t)x_n \leqslant g(0)\alpha_1^{-1}\alpha_2^{-1}\cdots\alpha_{n-1}^{-1} - \alpha_n x_n$, we obtain

$$x_n(t) \leqslant g(0)\alpha_1^{-1}\alpha_2^{-1}\cdots\alpha_n^{-1}.$$

Thus, $\Psi(t, 0, \bar{\xi})$ remains in X for any $t \geqslant 0$.

Next, we verify that system (4.1) satisfies (2.3) in $(\mathbf{A4})'$. Let

$$U = [0, g(0)] \ \text{ in } \ \mathbb{R} \ \text{ and } \ \mathcal{U} = [\tilde{0}, \tilde{g}(0)] \ \text{ in } \ \mathcal{C}_U.$$

Then, system (4.1) can be imbedded in the extended symmetric system

$$\begin{cases} \dot{x}_1 = g(y_n(t-r)) - \alpha_1 x_1, \\ \dot{x}_i = x_{i-1}(t-r) - \alpha_i x_i, \\ \dot{x}_n = x_{n-1}(t-r) - \alpha_n(t) x_n, \\ \dot{y}_1 = g(x_n(t-r)) - \alpha_1 y_1, \\ \dot{y}_i = y_{i-1}(t-r) - \alpha_i y_i, \\ \dot{y}_n = y_{n-1}(t-r) - \alpha_n(t) y_n. \end{cases} \quad 2 \leqslant i \leqslant n-1, \tag{4.3}$$

Let $\left(x^0(t), y^0(t)\right) = \Phi(t, 0, (\tilde{x}_0, \tilde{y}_0))$ denote the solution of system (4.3). Then, for any $\theta \in [0, \tau]$, one has

$$\int_0^\theta f(t, y_t^0, \mathbf{h}(x_t^0))\,\mathrm{d}t = y^0(\theta) - y_0 \leqslant 0,$$

and

$$\int_0^\theta f(t, x_t^0, \mathbf{h}(y_t^0))\,\mathrm{d}t = x^0(\theta) - x_0 \geqslant 0.$$

This verifies (2.3) in $(\mathbf{A4})'$.

Finally, we prove that system (4.1) satisfies (2.4) in $(\mathbf{A4})'$. Let $a(\cdot)$, $b(\cdot) \in C\left([-r, \tau], X\right)$ be two τ-periodic continuous functions satisfying $a(\cdot) \leqslant b(\cdot)$ and

$$\int_0^\tau f(t, a_t, \mathbf{h}(b_t))\,\mathrm{d}t = 0 = \int_0^\tau f(t, b_t, \mathbf{h}(a_t))\,\mathrm{d}t. \tag{4.4}$$

We will show that $a(t) = b(t)$ for any $t \in [0, \tau]$. To this end, we write

$$a(t) = (a_1(t), a_2(t), \cdots, a_n(t))^\top \quad \text{and} \quad b(t) = (b_1(t), b_2(t), \cdots, b_n(t))^\top.$$

Then (4.4) implies that

$$\begin{aligned} &\int_0^\tau g(b_n(t-r))\,\mathrm{d}t = \alpha_1 \cdot \int_0^\tau a_1(t)\,\mathrm{d}t, \\ &\int_0^\tau a_i(t-r)\,\mathrm{d}t = \alpha_{i+1} \cdot \int_0^\tau a_{i+1}(t)\,\mathrm{d}t, \quad 1 \leqslant i \leqslant n-2, \\ &\int_0^\tau a_{n-1}(t-r)\,\mathrm{d}t = \int_0^\tau \alpha_n(t) a_n(t)\,\mathrm{d}t. \end{aligned}$$

Hence, by iterating these equalities, one has

$$\int_0^\tau g(b_n(t))\,\mathrm{d}t = \prod_{i=1}^{n-1} \alpha_i \cdot \int_0^\tau \alpha_n(t) a_n(t)\,\mathrm{d}t.$$

Similarly, one can also obtain symmetrically

$$\int_0^\tau g(a_n(t))\,\mathrm{d}t = \prod_{i=1}^{n-1} \alpha_i \cdot \int_0^\tau \alpha_n(t) b_n(t)\,\mathrm{d}t.$$

Consequently,

$$\int_0^\tau \Big[g(a_n(t)) + \prod_{i=1}^{n-1} \alpha_i \cdot \alpha_n(t) a_n(t)\Big]\,\mathrm{d}t = \int_0^\tau \Big[g(b_n(t)) + \prod_{i=1}^{n-1} \alpha_i \cdot \alpha_n(t) b_n(t)\Big]\,\mathrm{d}t. \tag{4.5}$$

Now, define the following functional on $C([-r,\tau],\mathbb{R})$ as

$$J : C([-r,\tau],\mathbb{R}) \to \mathbb{R},\ \xi \mapsto \int_0^\tau \Big[g(\xi(t)) + \prod_{i=1}^{n-1} \alpha_i \cdot \alpha_n(t)\xi(t)\Big]\,\mathrm{d}t.$$

Then, the Frechét derivative of J at ξ is

$$\langle DJ(\xi), \varphi\rangle = \int_0^\tau \Big[g'(\xi(t)) + \prod_{i=1}^{n-1} \alpha_i \cdot \alpha_n(t)\Big]\varphi(t)\,\mathrm{d}t. \tag{4.6}$$

Together (4.5), there holds

$$\begin{aligned} 0 &\overset{(4.5)}{=\!=} J(b_n) - J(a_n) \\ &=\!= \int_0^1 \langle DJ\left((1-s)a_n + sb_n\right), b_n - a_n\rangle\,\mathrm{d}s \\ &\overset{(4.6)}{=\!=} \int_0^1\int_0^\tau \Big[g'\left((1-s)a_n(t) + sb_n(t)\right) + \prod_{i=1}^{n-1} \alpha_i \cdot \alpha_n(t)\Big]\left(b_n(t) - a_n(t)\right)\mathrm{d}t\,\mathrm{d}s. \end{aligned} \tag{4.7}$$

Suppose $a_n(\cdot) \neq b_n(\cdot)$. Then one can find an interval $[c,d] \subset [0,\tau]$ such that

$$a_n(t) < b_n(t), \ \text{ for } \ t \in [c,d].$$

Then the assumption (**H1**) entails that

$$\int_c^d \Big[g'\left((1-s)a_n(t) + sb_n(t)\right) + \prod_{i=1}^{n-1} \alpha_i \cdot \alpha_n(t)\Big]\left(b_n(t) - a_n(t)\right)\mathrm{d}t > 0,$$

contradicting (4.7), which completes the proof. □

Next, we present some numerical simulations to illustrate our main results in system (4.1). We take $n = 3$, $\alpha_1 = 2$, $\alpha_2 = 1$, $\alpha_3(t) = 2 - \frac{4}{5}\sin(\frac{2}{5}\pi t)$ and $g(u) = \frac{2}{1+u}$. Let $x_0 = (0,0,0)^\top$, $y_0 = (1,1,\frac{5}{6})^\top$, $X = [x_0, y_0]$ and $\mathcal{X} = [\tilde{x}_0, \tilde{y}_0]$. Then, Theorem 4.1 yields that there exists a 5-periodic solution $r(t) \in X$ such that for any $\bar{\xi} \in X$,

$$\|\Psi(t,0,\bar{\xi}) - r(t)\| \to 0, \quad \text{as } \ t \to \infty.$$

Take $r = 1$, $\bar{\xi}(s) = \left(\frac{k}{4}, \frac{k}{4}, \frac{5k}{24}\right)^\top$, for any $s \in [-1,0]$ and $k = 0, 1, \cdots, 4$. Our simulations are as follows:

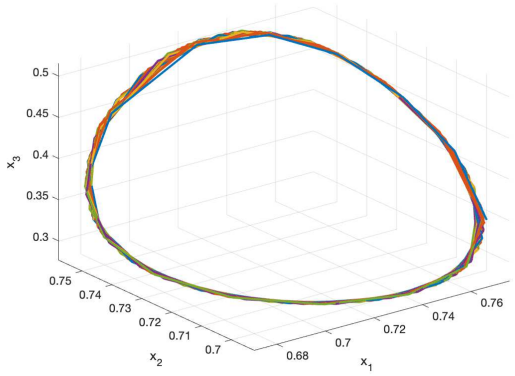


Figure 1: Trajectories of $\Psi(t, 0, \bar{\xi})$.

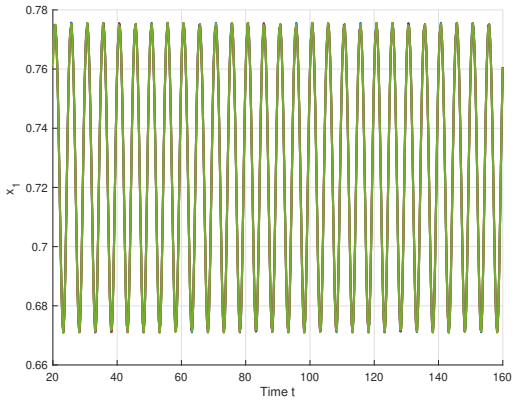


Figure 2: Trajectories of $x_1(t)$.

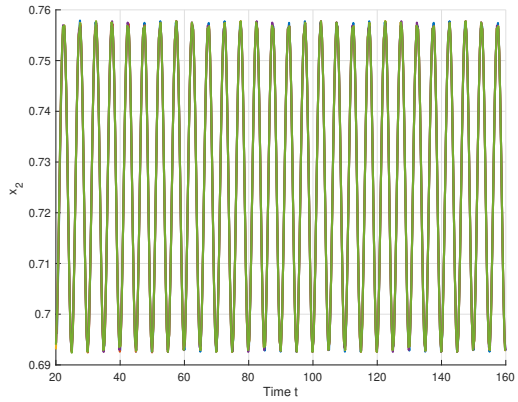


Figure 3: Trajectories of $x_2(t)$.

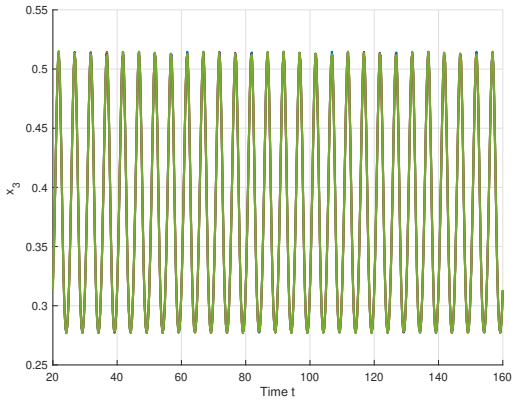


Figure 4: Trajectories of $x_3(t)$.

Take $r = 8$, $\bar{\xi}(s) = \left(\frac{k}{4}, \frac{k}{4}, \frac{5k}{24}\right)^\top$, for any $s \in [-8, 0]$ and $k = 0, 1, \cdots, 4$. Our simulations are as follows:

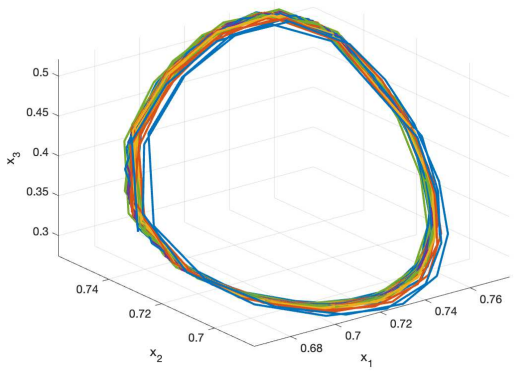


Figure 5: Trajectories of $\Psi(t, 0, \bar{\xi})$.

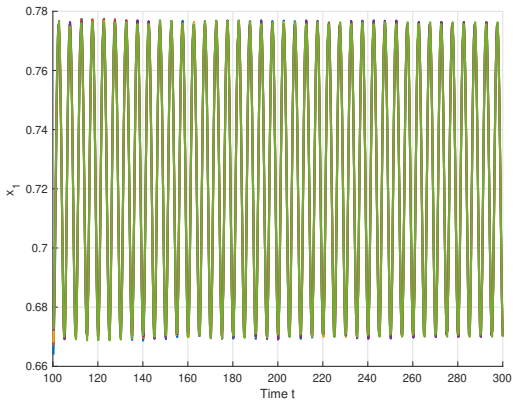


Figure 6: Trajectories of $x_1(t)$.

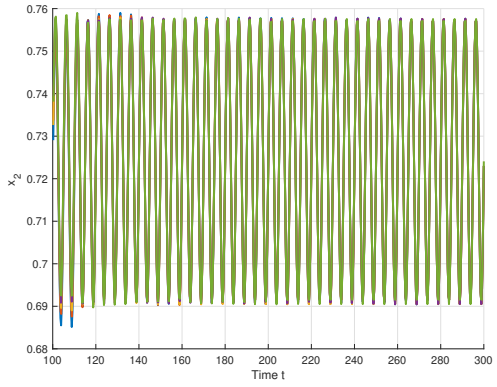


Figure 7: Trajectories of $x_2(t)$.

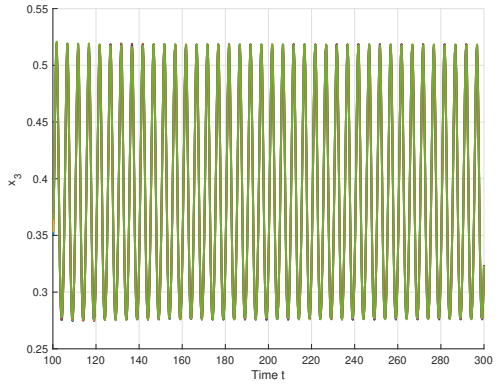


Figure 8: Trajectories of $x_3(t)$.

Figure 1 and Figure 5 respectively show that the solutions $\Psi(t,0,\bar{\xi})$ to system (4.1), initialized at $\bar{\xi}$, ultimately converge to one common periodic orbit. Figures 2-4 and 6-8 respectively present the trajectory graphs for the three corresponding components of $\Psi(t,0,\bar{\xi}) = (x_1(t), x_2(t), x_3(t))$. These figures collectively demonstrate that each component asymptotically approaches a 5-periodic solution whose period is consistent with the period of the time-dependent vector field.

Remark 4.2. *For the autonomous case of* (4.1) *with* $r = 0$, *Griffith considered* $g(u) = \dfrac{1}{1+u^m}$ *and proved that the unique positive equilibrium is asymptotically stable when* $m \leqslant 8$ *(see [6]). Here, we take* $\alpha_1 = \alpha_2 = 0.1$ *and* $m = 9$. *Firstly, taking* $\alpha_3 = 0.1$, *we obtain that the positive equilibrium is unstable and the periodic oscillatory phenomenon has emerged (see* ***Case 1.****). Moreover, when the autonomous case of* (4.1) *with* $r = 0$ *is introduced by external periodic driving (e.g., take* $\alpha_3(t) = 2.1 + 2\sin(0.1\pi t)$*), the system exhibits sub-harmonic solutions (see* ***Case 2.****). Furthermore, solutions of* (4.1) *with* $r > 0$ *may not be sub-harmonic and even display more complex dynamical behaviors as the delay* r *varies(see* ***Case 3. and 4.****).*

Case 1. Take $\alpha_1 = \alpha_2 = \alpha_3 = 0.1$ and $g(u) = \dfrac{1}{1+u^9}$. Let $x_0 = (0,0,0)^\top$, $y_0 = (1,1,\frac{5}{6})^\top$. Take initial value $\bar{x} = x_0$ and y_0, our simulations are as follows:

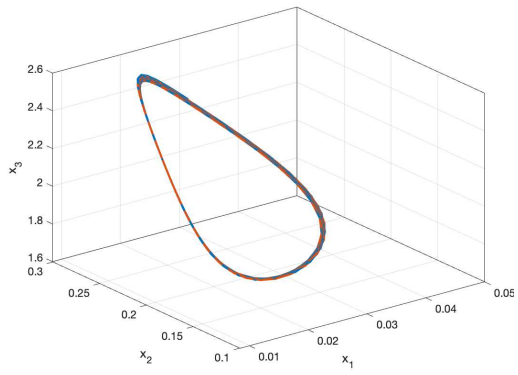


Figure 9: Trajectories of $\Psi(t,\bar{x})$.

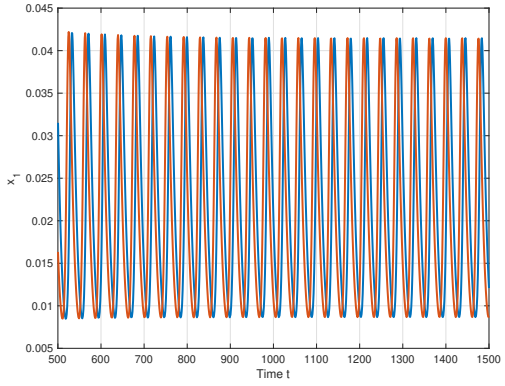


Figure 10: Trajectories of $x_1(t)$.

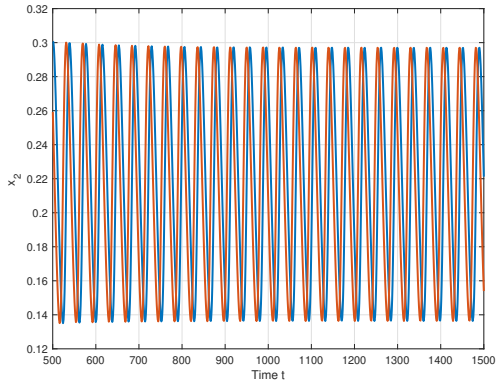


Figure 11: Trajectories of $x_2(t)$.

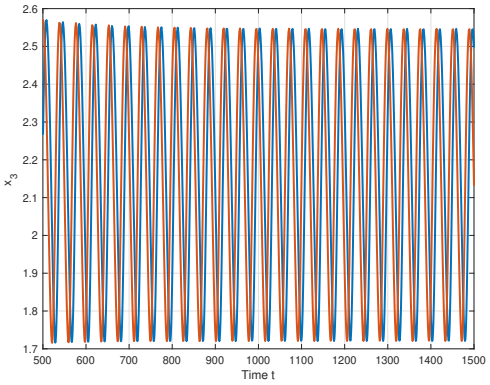


Figure 12: Trajectories of $x_3(t)$.

Case 2. Take $\alpha_1 = \alpha_2 = 0.1$, $\alpha_3(t) = 2.1 + 2\sin(0.1\pi t)$ and $g(u) = \dfrac{1}{1+u^9}$. Let $x_0 = (0,0,0)^\top$, $y_0 = (1,1,\frac{5}{6})^\top$. Take initial value $\bar{x} = x_0$ and y_0, our simulations are as follows:

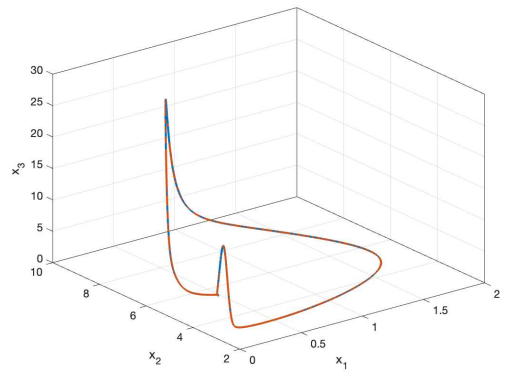


Figure 13: Trajectories of $\Psi(t,0,\bar{x})$.

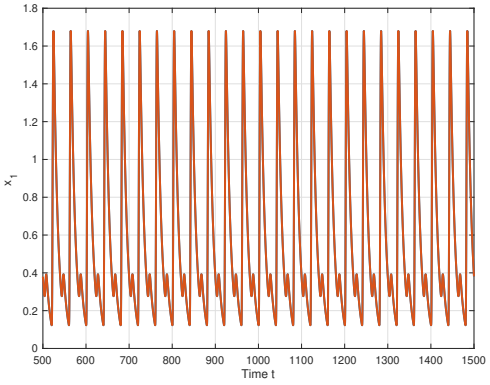


Figure 14: Trajectories of $x_1(t)$.

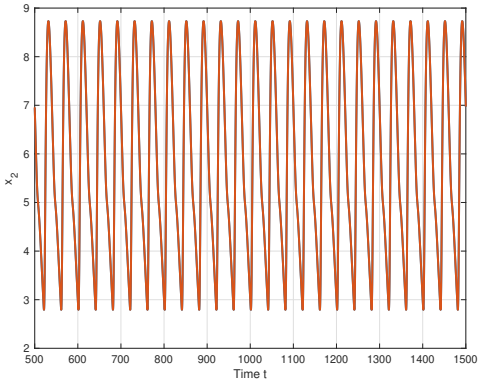


Figure 15: Trajectories of $x_2(t)$.

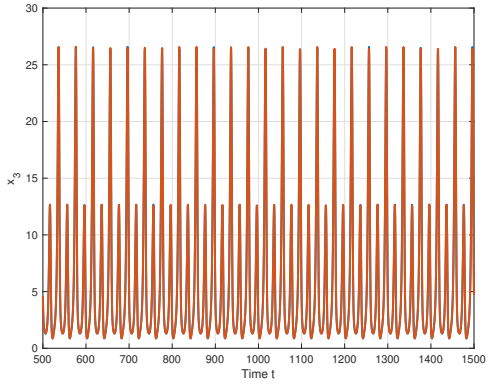


Figure 16: Trajectories of $x_3(t)$.

Case 3. Take $\alpha_1 = \alpha_2 = 0.1$, $\alpha_3(t) = 2.1 + 2\sin(0.1\pi t)$ and $g(u) = \dfrac{1}{1+u^9}$. Let $x_0 = (0,0,0)^\top$, $y_0 = (1,1,\frac{5}{6})^\top$. Take $r = 0.1$, initial value $\bar{\xi}(s) = \left(\frac{k}{4}, \frac{k}{4}, \frac{5k}{24}\right)^\top$, for any $s \in [-0.1, 0]$ and $k = 0, 1, \cdots, 4$. Our simulations are as follows:

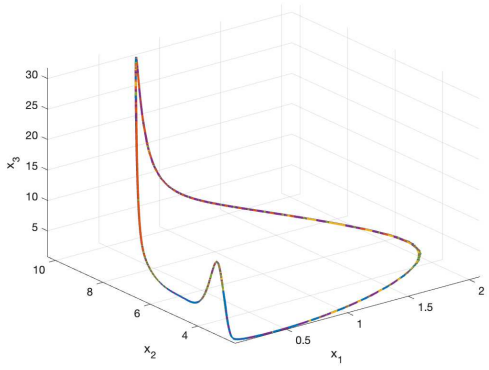


Figure 17: Trajectories of $\Psi(t, 0, \bar{\xi})$.

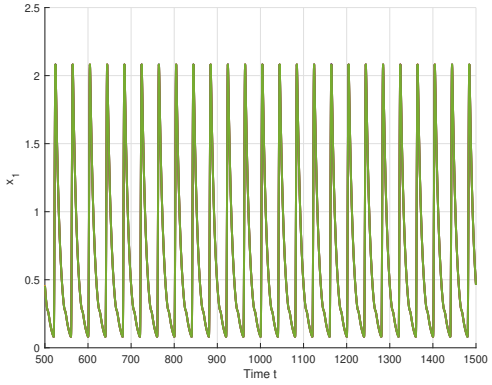


Figure 18: Trajectories of $x_1(t)$.

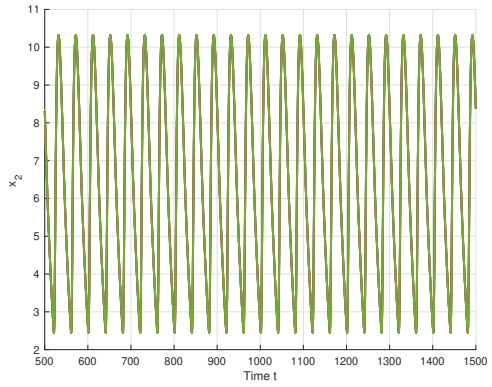


Figure 19: Trajectories of $x_2(t)$.

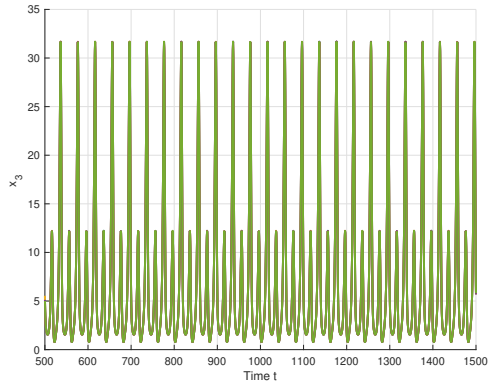


Figure 20: Trajectories of $x_3(t)$.

Case 4. Take $\alpha_1 = \alpha_2 = 0.1$, $\alpha_3(t) = 2.1 + 2\sin(0.1\pi t)$ and $g(u) = \dfrac{1}{1+u^9}$. Let $x_0 = (0,0,0)^\top$, $y_0 = (1,1,\frac{5}{6})^\top$. Take $r = 1$, initial value $\bar{\xi}(s) = \left(\frac{k}{4}, \frac{k}{4}, \frac{5k}{24}\right)^\top$, for any $s \in [-1, 0]$ and $k = 0, 1, \cdots, 4$. Our simulations are as follows:

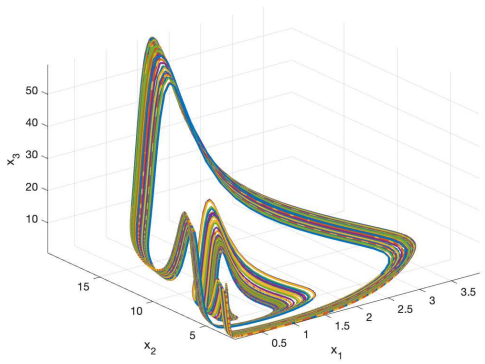


Figure 21: Trajectories of $\Psi(t, 0, \bar{\xi})$.

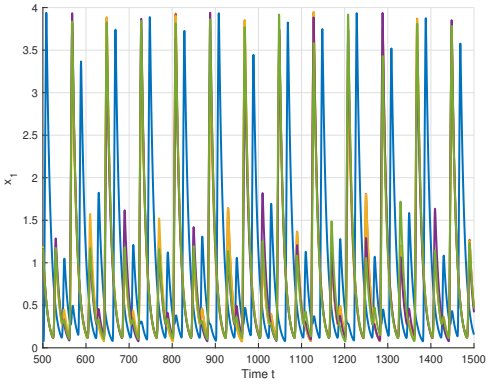


Figure 22: Trajectories of $x_1(t)$.

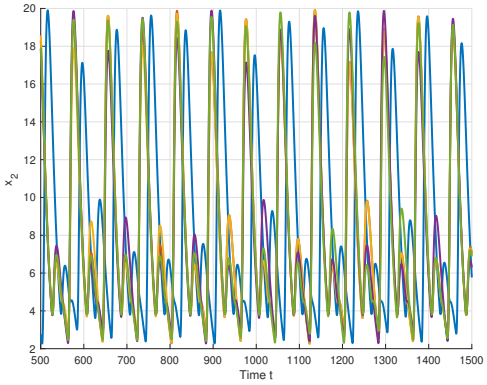


Figure 23: Trajectories of $x_2(t)$.

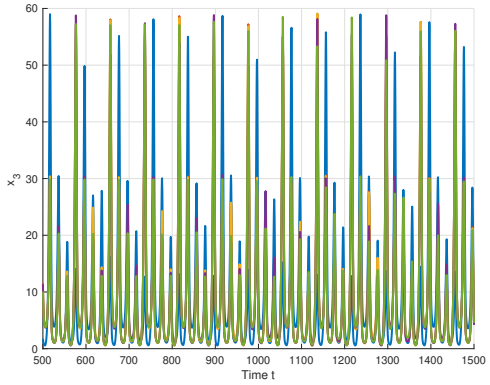


Figure 24: Trajectories of $x_3(t)$.

4.2. Time-periodic gene regulatory models with distribution delay

Now, we consider the gene regulatory system with distribution delay

$$\begin{cases} \dot{x}_1 = g(L_n x_n^t) - \alpha_1 x_1, \\ \dot{x}_i = L_{i-1} x_{i-1}^t - \alpha_i x_i, \qquad 2 \leqslant i \leqslant n-1, \\ \dot{x}_n = L_{n-1} x_{n-1}^t - \alpha_n(t) x_n. \end{cases} \tag{4.8}$$

Here $\alpha_i > 0$ for any $i = 1, \cdots, n-1$, and the τ-periodic function $\alpha_n(t) = \alpha_n(t+\tau) > 0$ is continuous, and $g : \mathbb{R}_+ \to \mathbb{R}_+$ is continuously differentiable and satisfies $g(0) > 0$ and $g' < 0$. See [18] for more details. Let

$$L_i x_i^t = \frac{1}{r} \int_{-r}^{0} x_i(t+s)\, \mathrm{d}s, \ 1 \leqslant i \leqslant n,$$

where $x = (x_1, x_2, \cdots, x_n)^\top, x_t = (x_1^t, x_2^t, \cdots, x_n^t)^\top$ and $x_t(s) = x(t+s)$, $-r \leqslant s \leqslant 0$.

Let $\mathcal{C} = C\left([-r,0],\mathbb{R}^n_+\right)$ and $\mathcal{C}_U = C\left([-r,0],\mathbb{R}_+\right)$. The map $\mathbf{h}:\mathcal{C}\to\mathcal{C}_U$ is defined by $\mathbf{h}(\xi)=\tilde{g}(L_n\xi_n)$, that is, $\mathbf{h}(\xi)$ is a constant function. With these hypotheses, the time-dependent vector field of system (4.8) naturally satisfies assumptions $(\mathbf{A1})'$-$(\mathbf{A3})'$. Take the cone $K=\mathbb{R}^n_+$. Let

$$X=[x_0,y_0]\triangleq\left[0,g(0)\left(\alpha_1^{-1},\alpha_1^{-1}\alpha_2^{-1},\cdots,\alpha_1^{-1}\alpha_2^{-1}\cdots\alpha_n^{-1}\right)^\top\right],\ \mathcal{X}=[\tilde{x}_0,\tilde{y}_0]. \tag{4.9}$$

Given delay r, denote

$$N=\begin{cases}1, & \text{if } \tau\geqslant r,\\ \left[\frac{r}{\tau}\right]+1, & \text{if } \tau<r.\end{cases}$$

be the minimal positive integer such that $N\tau\geqslant r$ and introduce the assumption

$$(\mathbf{H2})\quad \max\left\{-g'(u):0\leqslant u\leqslant\alpha^{-1}g(0)\right\}<\frac{\alpha r}{N\tau},\quad \text{where}\ \ \alpha=\prod_{i=1}^n\alpha_i\ \ \text{and}\ \ \alpha_n=\min_{0\leqslant t\leqslant\tau}\alpha_n(t).$$

By applying Corollary 2.2 to the delayed gene regulatory model (4.8), we obtain

Theorem 4.3. *Assume* $(\mathbf{H2})$ *holds. Then system* (4.8) *has a unique* τ*-periodic solution* $r(t)$ *in* X *such that for any* $\bar{\xi}\in\mathcal{X}$,

$$\|\Psi(t,0,\bar{\xi})-r(t)\|\to 0,\quad \text{as } t\to\infty,$$

where $\Psi(t,0,\bar{\xi})$ *is the solution of system* (4.8) *satisfying* $\Psi_0(0,\bar{\xi})=\bar{\xi}$.

Proof. We first show that the solution $\Psi(t,0,\bar{\xi})$ remains in X (see (4.9)), for any $\bar{\xi}\in\mathcal{X}$ and $t\geqslant 0$. Write

$$\Psi(t,0,\bar{\xi})=(x_1(t),x_2(t),\cdots,x_n(t))\ \ \text{and}\ \ \bar{\xi}=\left(\bar{\xi}_1,\bar{\xi}_2,\cdots,\bar{\xi}_n\right).$$

It is easy to see that $\Psi(t,0,\bar{\xi})\in\mathbb{R}^n_+$ (see [19, Chapter 5, Theorem 2.1]), whenever $\bar{\xi}\in\mathcal{C}$. Since $g(L_nx_n^t)-\alpha_1x_1\leqslant g(0)-\alpha_1x_1$ and $\bar{\xi}_1\leqslant g(0)\alpha_1^{-1}$, the comparison theorem (see [13, Chapter 6, Theorem 6.9.4]) implies that

$$x_1(t)\leqslant e^{-\alpha_1t}g(0)\alpha_1^{-1}+g(0)\alpha_1^{-1}\left(1-e^{-\alpha_1t}\right)=g(0)\alpha_1^{-1}.$$

Similarly, one has

$$x_i(t)\leqslant g(0)\alpha_1^{-1}\alpha_2^{-1}\cdots\alpha_i^{-1},\quad \text{for any } 2\leqslant i\leqslant n-1.$$

Note that $L_{n-1}x_{n-1}^t-\alpha_n(t)x_n\leqslant g(0)\alpha_1^{-1}\alpha_2^{-1}\cdots\alpha_{n-1}^{-1}-\alpha_nx_n$, we obtain

$$x_n(t)\leqslant g(0)\alpha_1^{-1}\alpha_2^{-1}\cdots\alpha_n^{-1}.$$

Thus, $\Psi(t,0,\bar{\xi})$ remains in X for any $t\geqslant 0$.

Next, we verify that system (4.8) satisfies (2.3) in $(\mathbf{A4})'$. Let

$$U=[0,g(0)] \ \text{ in } \ \mathbb{R} \ \text{ and } \ \mathcal{U}=[\tilde{0},\tilde{g}(0)] \ \text{ in } \ \mathcal{C}_U.$$

Then, system (4.8) can be imbedded in the extended symmetric system

$$\begin{cases} \dot{x}_1 = g(L_n y_n^t) - \alpha_1 x_1, \\ \dot{x}_i = L_{i-1} x_{i-1}^t - \alpha_i x_i, \\ \dot{x}_n = L_{n-1} x_{n-1}^t - \alpha_n(t) x_n, \\ \dot{y}_1 = g(L_n x_n^t) - \alpha_1 y_1, \\ \dot{y}_i = L_{i-1} y_{i-1}^t - \alpha_i y_i, \\ \dot{y}_n = L_{n-1} y_{n-1}^t - \alpha_n(t) y_n. \end{cases} \quad 2 \leqslant i \leqslant n-1, \tag{4.10}$$

Let $\left(x^0(t), y^0(t)\right) = \Phi(t, 0, (\tilde{x}_0, \tilde{y}_0))$ denote the solution of system (4.10). Then, for any $\theta \in [0,\tau]$, one has

$$\int_0^\theta f(t, y_t^0, \mathbf{h}(x_t^0))\,\mathrm{d}t = y^0(\theta) - y_0 \leqslant 0,$$

and

$$\int_0^\theta f(t, x_t^0, \mathbf{h}(y_t^0))\,\mathrm{d}t = x^0(\theta) - x_0 \geqslant 0.$$

This verifies (2.3) in $(\mathbf{A4})'$.

Finally, we prove that system (4.8) satisfies (2.4) in $(\mathbf{A4})'$. Let $a(\cdot)$, $b(\cdot) \in C\left([0,\tau], X\right)$ be two τ-periodic continuous functions satisfying $a(\cdot) \leqslant b(\cdot)$ and

$$\int_0^\tau f(t, a_t, \mathbf{h}(b_t))\,\mathrm{d}t = 0 = \int_0^\tau f(t, b_t, \mathbf{h}(a_t))\,\mathrm{d}t. \tag{4.11}$$

We will show that $a(t) = b(t)$ for any $t \in [0,\tau]$. To this end, we write

$$a(t) = (a_1(t), a_2(t), \cdots, a_n(t))^\top \ \text{ and } \ b(t) = (b_1(t), b_2(t), \cdots, b_n(t))^\top.$$

Then (4.11) implies that

$$\begin{aligned} \int_0^\tau g\left(\frac{1}{r}\int_{-r}^0 b_n(t+s)\,\mathrm{d}s\right)\mathrm{d}t &= \alpha_1 \cdot \int_0^\tau a_1(t)\,\mathrm{d}t, \\ \int_0^\tau \frac{1}{r}\int_{-r}^0 a_i(t+s)\mathrm{d}s\,\mathrm{d}t &= \alpha_{i+1} \cdot \int_0^\tau a_{i+1}(t)\,\mathrm{d}t, \quad 1 \leqslant i \leqslant n-2, \\ \int_0^\tau \frac{1}{r}\int_{-r}^0 a_{n-1}(t+s)\mathrm{d}s\,\mathrm{d}t &= \int_0^\tau \alpha_n(t) a_n(t)\,\mathrm{d}t. \end{aligned}$$

Hence, by iterating these equalities, one has

$$\int_0^\tau g\left(\frac{1}{r}\int_{t-r}^t b_n(v)\,\mathrm{d}v\right)\mathrm{d}t = \prod_{i=1}^{n-1}\alpha_i \cdot \int_0^\tau \alpha_n(t) a_n(t)\,\mathrm{d}t.$$

Similarly, one can also obtain symmetrically

$$\int_0^\tau g\left(\frac{1}{r}\int_{t-r}^t a_n(v)\,\mathrm{d}v\right)\mathrm{d}t=\prod_{i=1}^{n-1}\alpha_i\cdot\int_0^\tau \alpha_n(t)b_n(t)\,\mathrm{d}t.$$

Consequently,

$$\begin{aligned}&\int_0^\tau g\left(\frac{1}{r}\int_{t-r}^t a_n(v)\,\mathrm{d}v\right)\mathrm{d}t+\prod_{i=1}^{n-1}\alpha_i\cdot\int_0^\tau \alpha_n(t)a_n(t)\,\mathrm{d}t\\ =&\int_0^\tau g\left(\frac{1}{r}\int_{t-r}^t b_n(v)\,\mathrm{d}v\right)\mathrm{d}t+\prod_{i=1}^{n-1}\alpha_i\cdot\int_0^\tau \alpha_n(t)b_n(t)\,\mathrm{d}t.\end{aligned}\tag{4.12}$$

Now, define the following functional on $C\left([-r,\tau],\mathbb{R}\right)$ as

$$J:C\left([-r,\tau],\mathbb{R}\right)\to\mathbb{R},\,\xi\mapsto\int_0^\tau g\left(\frac{1}{r}\int_{t-r}^t \xi(v)\,\mathrm{d}v\right)\mathrm{d}t+\prod_{i=1}^{n-1}\alpha_i\cdot\int_0^\tau \alpha_n(t)\xi(t)\,\mathrm{d}t.$$

Then, the Frechét derivative of J at ξ is

$$\langle DJ(\xi),\varphi\rangle=\int_0^\tau\left[g'\left(\frac{1}{r}\int_{t-r}^t \xi(v)\,\mathrm{d}v\right)\cdot\frac{1}{r}\int_{t-r}^t\varphi(v)\,\mathrm{d}v\right]\mathrm{d}t+\prod_{i=1}^{n-1}\alpha_i\cdot\int_0^\tau \alpha_n(t)\varphi(t)\,\mathrm{d}t.\tag{4.13}$$

Together (4.12), there holds

$$\begin{aligned}0&\overset{(4.12)}{=\!=}J(b_n)-J(a_n)\\ &=\!=\int_0^1\langle DJ\left((1-s)a_n+sb_n\right),b_n-a_n\rangle\,\mathrm{d}s\\ &\overset{(4.13)}{=\!=}\int_0^1\left[\int_0^\tau\left[g'\left(u(t,s)\right)\cdot\frac{1}{r}\int_{t-r}^t\left[b_n(v)-a_n(v)\right]\mathrm{d}v\right]\mathrm{d}t+\prod_{i=1}^{n-1}\alpha_i\cdot\int_0^\tau\alpha_n(t)\left(b_n(t)-a_n(t)\right)\mathrm{d}t\right]\mathrm{d}s,\end{aligned}\tag{4.14}$$

where $u(t,s)=\dfrac{1}{r}\displaystyle\int_{t-r}^t\Big[(1-s)a_n(v)+sb_n(v)\Big]\,\mathrm{d}v$.

Case 1. If $\tau\geqslant r$, then one has

$$\begin{aligned}&\int_0^\tau\left[g'\left(u(t,s)\right)\cdot\frac{1}{r}\int_{t-r}^t\left[b_n(v)-a_n(v)\right]\mathrm{d}v\right]\mathrm{d}t+\prod_{i=1}^{n-1}\alpha_i\cdot\int_0^\tau\alpha_n(t)\left(b_n(t)-a_n(t)\right)\mathrm{d}t\\ \geqslant&\int_0^\tau\left[g'\left(u(t,s)\right)\cdot\frac{1}{r}\int_0^\tau\left[b_n(v)-a_n(v)\right]\mathrm{d}v\right]\mathrm{d}t+\prod_{i=1}^{n-1}\alpha_i\cdot\int_0^\tau\alpha_n(t)\left(b_n(t)-a_n(t)\right)\mathrm{d}t\\ \geqslant&\left(\frac{1}{r}\int_0^\tau g'\left(u(t,s)\right)\mathrm{d}t+\prod_{i=1}^{n}\alpha_i\right)\cdot\int_0^\tau\Big[b_n(t)-a_n(t)\Big]\mathrm{d}t.\end{aligned}$$

Case 2. If $\tau \leqslant r$, then one has

$$\begin{aligned}
&\int_0^\tau \left[g'\left(u(t,s)\right)\cdot\frac{1}{r}\int_{t-r}^{t}\left[b_n(v)-a_n(v)\right]\mathrm{d}v\right]\mathrm{d}t+\prod_{i=1}^{n-1}\alpha_i\cdot\int_0^\tau \alpha_n(t)\left(b_n(t)-a_n(t)\right)\mathrm{d}t\\
\geqslant&\int_0^\tau \left[g'\left(u(t,s)\right)\cdot\frac{1}{r}\int_{0}^{N\tau}\left[b_n(v)-a_n(v)\right]\mathrm{d}v\right]\mathrm{d}t+\prod_{i=1}^{n-1}\alpha_i\cdot\int_0^\tau \alpha_n(t)\left(b_n(t)-a_n(t)\right)\mathrm{d}t\\
\geqslant&\left(\frac{N}{r}\int_0^\tau g'\left(u(t,s)\right)\mathrm{d}t+\prod_{i=1}^{n}\alpha_i\right)\cdot\int_0^\tau\Big[b_n(t)-a_n(t)\Big]\mathrm{d}t.
\end{aligned}$$

Suppose $a_n(\cdot)\neq b_n(\cdot)$. Then one can find an interval $[c,d]\subset[0,\tau]$ such that

$$a_n(t)<b_n(t),\ \text{ for any }\ t\in[c,d].$$

Then the assumption (**H2**) entails that

$$\int_0^\tau \left[g'\left(u(t,s)\right)\cdot\frac{1}{r}\int_{t-r}^{t}\left[b_n(v)-a_n(v)\right]\mathrm{d}v\right]\mathrm{d}t+\prod_{i=1}^{n-1}\alpha_i\cdot\int_0^\tau \alpha_n(t)\left(b_n(t)-a_n(t)\right)\mathrm{d}t>0,$$

contradicting (4.14), which completes the proof. □

For system (4.8), we present some numerical simulations to illustrate our main results. Take $n=3$, $\alpha_1=4$, $\alpha_2=3$, $\alpha_3(t)=2-\frac{4}{5}\sin(\frac{2}{5}\pi t)$ and $g(u)=\frac{2}{1+u}$. Let $x_0=(0,0,0)^\top$, $y_0=(\frac{1}{2},\frac{1}{6},\frac{5}{36})^\top$, $X=[x_0,y_0]$ and $\mathcal{X}=[\tilde{x}_0,\tilde{y}_0]$. Then, Theorem 4.3 yields that there exists a 5-periodic solution $r(t)\in X$ such that for any $\bar{\xi}\in\mathcal{X}$,

$$\|\Psi(t,0,\bar{\xi})-r(t)\|\to 0,\quad \text{as }\ t\to\infty.$$

Take $r=1$, $\bar{\xi}(s)=\left(\frac{k}{8},\frac{k}{24},\frac{5k}{144}\right)^\top$, for any $s\in[-1,0]$ and $k=0,1,\cdots,4$. Our simulations are as follows:

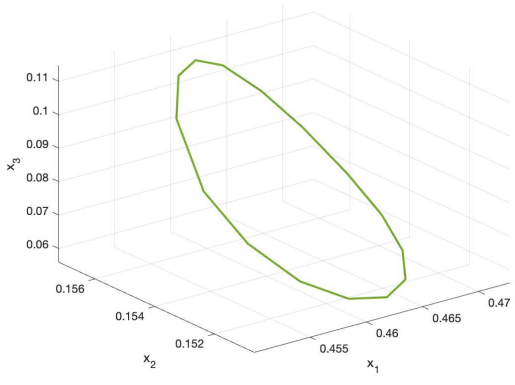


Figure 25: Trajectories of $\Psi(t,0,\bar{\xi})$.

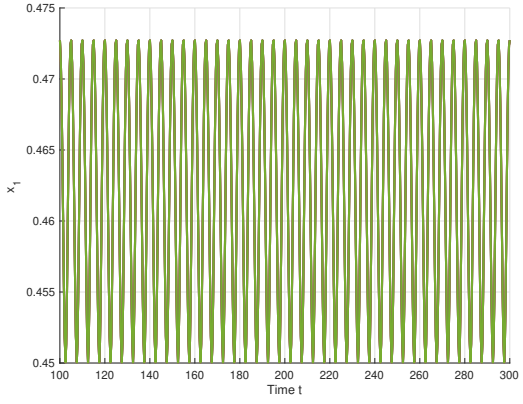


Figure 26: Trajectories of $x_1(t)$.

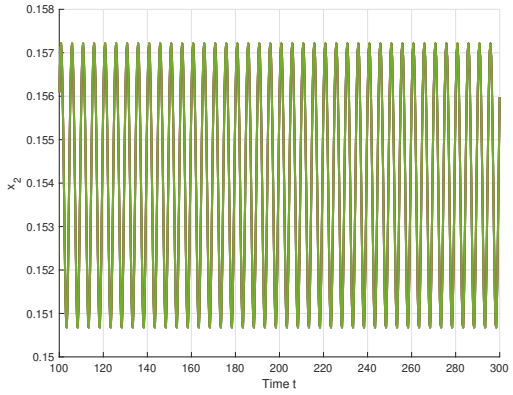


Figure 27: Trajectories of $x_2(t)$.

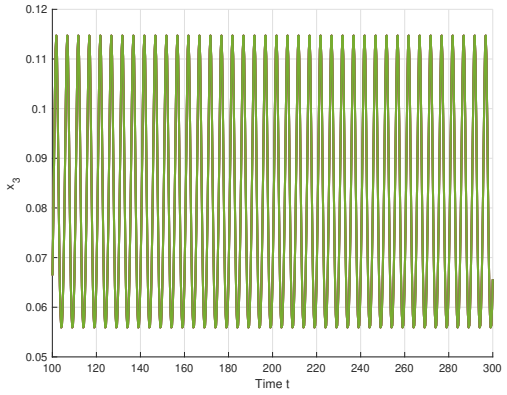


Figure 28: Trajectories of $x_3(t)$.

Take $r = 8$, $\bar{\xi}(s) = \left(\frac{k}{8}, \frac{k}{24}, \frac{5k}{144}\right)^\top$, for any $s \in [-8, 0]$ and $k = 0, 1, \cdots, 4$. Our simulations are as follows:

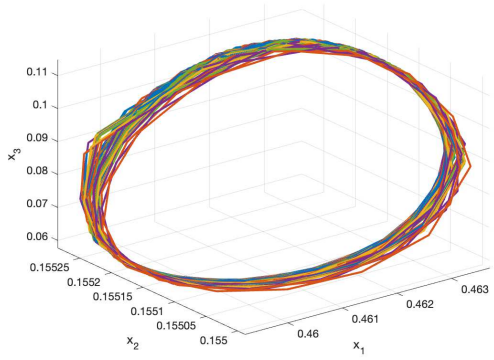


Figure 29: Trajectories of $\Psi(t, 0, \bar{\xi})$.

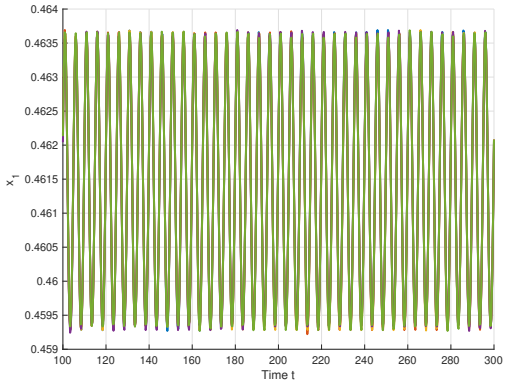


Figure 30: Trajectories of $x_1(t)$.

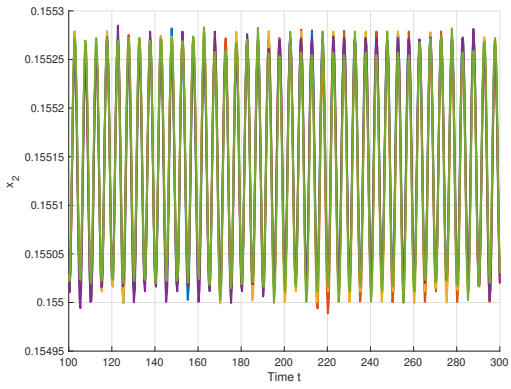


Figure 31: Trajectories of $x_2(t)$.

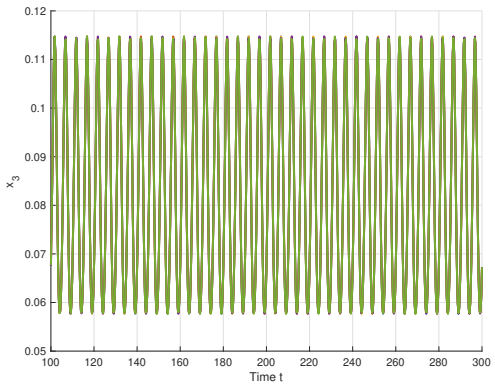


Figure 32: Trajectories of $x_3(t)$.

Figure 25 and Figure 29 show that the solution $\Psi(t,0,\bar{\xi})$ to system (4.8), initialized at $\bar{\xi}$, ultimately converge to one common periodic orbit. Figures 26-28 and 30-32 respectively present the trajectory graphs for the three corresponding components of $\Psi(t,0,\bar{\xi}) = (x_1(t), x_2(t), x_3(t))$. These figures collectively demonstrate that each component asymptotically approaches a 5-periodic solution whose period is consistent with the period of the time-dependent vector field.

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