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Research Article

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Posted Date: March 3rd, 2026

DOI: https://doi.org/10.21203/rs.3.rs-8717765/v1

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Additional Declarations: No competing interests reported.

Version of Record: A version of this preprint was published at Evolutionary Intelligence on July 18th, 2026. See the published version at https://doi.org/10.1007/s12065-026-01235-x.

A unified uncertainty-aware decision framework for air quality assessment

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Abstract

Complex real-world decision-making problems are usually distinguished by uncertain, imprecise and multi-source information, especially in environmental assessment cases, as in air quality assessment. To deal with such issues it is important to have strong mathematical models that can incorporate interval uncertainty, expert hesitation, and multiple criteria in a single decision-making framework. This paper generalizes the development of an uncertainty-based decision-making framework based on a cubic intuitionistic multi-fuzzy soft set environment. The proposed model becomes effective in representing interval-valued membership/non-membership data and simplifying a number of expert judgments and uncertainty, which is parameterized. Weighted arithmetic and geometric aggregation operators are developed to facilitate multi-criteria decision-making and the key bases of these aggregation operators are evaluated. A systematic decision-making algorithm is based on these operators. The proposed framework's applicability and effectiveness are illustrated using an air quality assessment. Besides, comparative structural and numerical methods with the current fuzzy soft set determinists show that the presented approach offers a better representation of information and consistent ranking ability in the presence of uncertainty. The findings indicate that the suggested framework can be used as a credible, versatile mathematical instrument in complicated environmental decision-making issues.

Keywords: Multi-criteria decision making; Uncertain information; Advanced uncertainty modeling; Entropy-based weighting; Distance measures; Air quality assessment

1 Introduction

In real-world settings, making decisions often is complicated by the existence of various conflicting criteria, vague information, and diverse viewpoints on the part of the experts. Problem [1, 2] of the environmental assessment, especially air-quality analysis, involves the joint action of quantitative and qualitative values, as well as the presupposition of the expert knowledge, in the circumstances of uncertainty and incompleteness of data. Crisp-value or exact probability distributions are considered to be too inflexible to deal with such uncertainty, and alternative forms of decision-making, such as Monte Carlo analysis, may be used to find unreliable, unstable, or biased evaluation results. As a result, there is an increasing need for versatile and robust mathematical modeling that could be successfully used to capture uncertainty and assist in sound multi-criteria decision-making (MCDM) [3, 4].
In order to reduce uncertainty in decision-making, fuzzy set theory [5] and its variations have been widely used in MCDM problems [6, 7]. In 1999, Molodtsov [8] coined the term "soft sets" as a general parameterized mathematical mechanism of operating with uncertainty without any defined membership functions. Uncertainty in soft set theory is represented by a mapping of a set of parameters to subsets of the universal set and can thus be used to represent imprecise information in a flexible and intuitive way by decision-makers. In order to extend the expressiveness of soft sets, several extensions are suggested. Neutrosophic soft sets were developed by Deli et al. [9] to enable using indeterminacy in addition to truth and falsity information. Afterward, scientists worked on fuzzy soft sets, which combine the notion of fuzzy membership with the parameterized soft set apparatus. Specifically, the study by Feng et al. [10] showed that fuzzy soft sets are useful in enhancing decision-making performance, as they are able to provide gradual membership information in the face of uncertainty. Demir [11] made use of Fermatean fuzzy soft set-based MCDM methods in the application of infectious diseases and has shown their efficacy in intricate health-related choice settings. Dalkilicc et al. [12] explored the topological structure of virtual fuzzy parameterized fuzzy soft sets and this has been applied to the theoretical background of fuzzy soft set models. In addition, intuitionistic fuzzy sets proposed by Atanassov [13] and intuitionistic fuzzy soft sets by Maji et al. [14] enable the joint representation of membership and non-membership levels, which offers a more sophisticated representation of hesitation in the process of decision-making. Memics et al. [15] explored adaptive machine learning techniques on intuitionistic fuzzy parameterized intuitionistic fuzzy soft matrices, whereas Zheng et al. [16] provided a product ranking scheme on the basis of intuitionistic fuzzy soft sets to address variability in online review sites. Jun et al. [17] presented the idea of cubic sets by combining zones of fuzzy sets with interval information to provide interval-valued membership data and exact fuzzy membership values in a single framework. They also developed internal and external cubic sets, which presented a systematic results mechanism of describing uncertainty ranges and precise calculations. As cubic sets are flexible models in that they retain an interval uncertainty and accurate membership information, they are especially applicable in complex decision-making problems. In order to facilitate a real-world use, the decision analysis operators suggested by the authors of this paper, Khan et al. [18],

are specialized cubic aggregation operators. Sajid et al. [19] proposed a new multi-attribute selection algorithm of suppliers in healthcare based on cosine similarity of single-valued neutrosophic cubic hypersoft sets. Saqlain et al. [20] created distance measures of cubic intuitionistic fuzzy soft sets and proved their adequacy in decision-making. Mahmood et al. [21] considered cubic hesitant fuzzy sets and the operations of aggregation, whereas Tehreem et al. [22] presented the concept of spherical cubic fuzzy sets as an even more generalized form. Saeed et al. [23] made extensions of interval-valued and Pythagorean fuzzy sets into cubic Pythagorean fuzzy soft sets. Later Saeed et al. [24] designed the Hausdorff and Hamming distance between cubic intuitionistic fuzzy hypersoft sets and used it to regulate cement quality. Similarity measures of interval-valued intuitionistic fuzzy hypersoft sets were considered in Naveed et al. [25] to solve complex decision-making problems, whereas Masadeh et al. [26] applied an extended TOPSIS methodology to the interval of cubic vague soft set relation to assess renewable energy. In uncertain conditions, Tanoli et al. [27] used Einstein aggregation operators in the multicriteria group decision-making of a fuzzy sets cubic picture under the multicriteria.

Sebastian and Ramakrishnan [28] present a concept of multi-fuzzy sets, which is a sophisticated extension of fuzzy sets where the fuzzy membership of an element is described as a series of ordered fuzzy membership values. Complex phenomena that would not be feasible to model with only one degree of membership can be modeled using this structure. Multi-fuzzy sets have been observed to be useful in real life, like in terms of pixel color representation and pattern recognition. Yang et al. [29] also applied this idea to multi-fuzzy soft sets and discussed their use in decision-making. The multi-fuzzy hypersoft sets were developed by Saeed et al. [30] and the entropy and similarity measures were applied to MCDM problems. Arshad et al. [31] offered a strong decision model for the optimal choice of masks to address the COVID-19 pandemic through interval-valued multi-fuzzy hypersoft set aggregation operators.

Although these developments have taken place, current fuzzy soft set-based decision-making representation models tend to have a small level of expressiveness since their ability to represent interval uncertainty, membership-non-membership hesitation, and multiple expert judgments in a single model tends to be limited. Therefore, it may end up losing invaluable information of uncertainty during aggregation, which has an adverse impact on decision robustness. The cubic intuitionistic fuzzy sets are a stronger representation that combines interval-valued fuzzy data with accurate intuitionistic membership and non-membership values. In a multi-fuzzy setting, these models can be used to systematically introduce more than one expert appraisal, which increases the stability and reliability of the results of the decision-making process.

Table 1 shows a comparative overview of representative MCDM-based air-quality assessment studies. The table reveals that the majority of the existing models are based on traditional fuzzy or intuitionistic models, they often use subjective weighting schemes, and they often make use of a single aggregation or distance metric. In addition to this, the topic of air quality is not directly met with or borne out in terms of robustness validation. These constraints encourage the creation of a more expressive, objective and stable decision-making framework, which is specifically conditioned to solve problems of complex assessment of the environment.

Table 1: Comparative summary of representative MCDM-based studies for air-quality assessment and monitoring

Study (Year)	**Geographic**	**Aim**	**Air-Quality Parameters**	**Data Source**	**MCDM Method(s)**	**Weighting Strategy**	**Integration & Validation**
Hacıoğlu et al. [32]	Kütahya, Türkiye (urban & rural sites)	Air-quality monitoring station siting	SO_2, PM, NO_x, CO, VOCs	AERMOD dispersion outputs + expert input	AHP, ELECTRE III	Expert pairwise comparisons	AERMOD–MCDM integration validated via cross-method consistency, with logistics and security considered.
Wang et al. [33]	Chinese city case study	Assess air pollution impact on sustainability	Multiple air-pollution indicators	City-level datasets (small sample)	Entropy–BP neural network + TOPSIS	Objective entropy + Bayesian-regularized BP	Improved weight stability under data scarcity; no explicit GIS or dispersion modeling.
Hsueh & Cheng [34]	Community-level policy evaluation	Evaluate air-quality improvement policies	Policy and qualitative criteria	Expert panels (Delphi)	Delphi + AHP + fuzzy inference	Expert-driven (Delphi, AHP)	Demonstrated expert-based policy evaluation; pollutant-level benchmarking not reported.
Kandakoglu et al. [35]	Global review (sustainable development)	Survey MCDM in environmental domains	Environmental indicators (incl. air quality)	Literature synthesis	AHP, ANP, TOPSIS, VIKOR, ELECTRE, PROMETHEE, fuzzy/grey	Expert-based and objective methods reviewed	Highlights need for uncertainty modeling, sensitivity analysis, and validation in environmental MCDM.
Yannis et al. [36]	Transport-sector review (transferable methods)	Review MCDM variants and validation tools	Methodological focus	Literature synthesis	TOPSIS, ELECTRE, PROMETHEE, fuzzy/grey hybrids	Expert and hybrid weighting	Recommends sensitivity analysis, GAIA visualization, and stakeholder-specific MCDM workflows.
Hajduk & Jelonek [37]	Smart cities / urban energy (review)	Rank and assess smart cities and urban energy performance	Emissions, energy, socioeconomic and climatic indicators	Secondary datasets (WCCD)	TOPSIS, AHP, DEA; hybrid approaches	Various (expert, objective, hybrid)	Shows prevalence of hybrid MCDM; air-quality evaluation is indirect and validation protocols remain inconsistent.
Suresh et al. [38]	Indian cities (Bangalore, Mysore, Hubli-Dharwad)	Fuzzy AQI computation and city comparison	PM_{10}, NO_2, SO_2	Monitoring station data (2017–2019)	Fuzzy MCDM; defuzzification (Hwang & Yoon)	Expert linguistic weighting	Produced interpretable AQI rankings; Bangalore ranked worst; limited external validation and benchmarking.
Guo et al. [39]	Methodological (AQI-related applications)	Evaluate AQI improvement via novel aggregation	$PM_{2.5}$ (VOCs, NO_x, SO_x referenced)	Numerical and comparative datasets	Circular intuitionistic fuzzy Heronian means	Not explicitly specified	Demonstrates robustness of new aggregation operator; validation limited to numerical comparison examples.
Proposed Study	Multi-city air-quality assessment	Robust ranking of cities based on air-quality performance under uncertainty	$PM_{2.5}$, PM_{10}, NO_2, SO_2, O_3, CO	Multi-source air-quality data and expert inputs	**CIMFSS with CIMFWAA and CIMFWGA operators**	**Entropy-based objective weighting**	**Introduces a unified cubic intuitionistic multi-fuzzy framework that models interval uncertainty, expert diversity, and decision attitude, and ensures robust rankings using dual aggregation and distance-based validation.**

This paper presents a proposal to overcome these obstacles by proposing the framework of a cubic intuitionistic multi-fuzzy soft set (CIMFSS) to use in the assessment of the air quality in multi-city. The suggested method incorporates interval-valued uncertainty, uncertainty in terms of intuitionism membership and non-membership data, and multi-expert ratings into a single mathematical framework. Objective entropy-based weighting is used to minimize subjectivity in criteria importance measurement, whereas dual aggregation operators CIMFWAA and CIMFWGA are presented that can be used to model various attitudes of the decision maker, such as the optimistic and the conservative attitudes. Moreover, both weighted Euclidean and weighted Hamming distance measurements are combined to increase the strength and permanence of ranking.

1.1 Research Gap

In spite of the noticeable advancement in fuzzy soft set-based decision-making models, there are several research gaps that are not adequately filled:

- The current fuzzy and intuitionistic fuzzy soft set models do not have the potential of depicting interval uncertainty, membership and non-membership hesitation, and multi-expert data under a single framework.
- Many CIMFSS-based decision models use subjective criteria weights, potentially biasing the results of the rankings.
- Aggregation operators use most operators with fixed compensation schemes, which do not allow exploring various decision attitudes, such as optimistic (compensatory) and pessimistic (conservative) attitudes.
- The results of ranking depend on the metric used to compute the distance or similarity, making the ranking susceptible to the choice of metric and decreasing the level of robustness.
- The thin empirical research supports the frameworks of CIMFSS in the intricate environmental assessment issues, especially in the air-quality assessment settings.

Motivation

This study is motivated by the necessity of a powerful, objective, and adaptive decision-making model, which can be concerned with intrinsic uncertainty, professional diversification, and conflicting standards in the real-life environmental issues. Unstable or biased decision findings can result in poor policy development and escalated health hazards for the population in the context of air-quality evaluation. Consequently, it is imperative to come up with a methodology that maintains a substantial amount of uncertainty, lowers the subjective effects of weighting, and provides reliable ranking results across varying aggregation vistas. The study could be used as a reliable decision-support tool in complex and high-stakes applications by involving entropy-based objective weighting, dual aggregation operators with different attitudes towards decisions and complementary distance estimates in the framework of CIMFSS.

The key findings of this paper can be summed up as follows:

- The determination of major shortcomings of the current fuzzy soft set-based decision-making methods in terms of uncertainty modeling, subjectivity of weights, and ranking of strength.
- Creation of a CIMFSS-based MCDM model that incorporates cubic intuitionistic information, as well as multi-fuzzy expert views.
- Implementation of the entropy-based objective weighting in order to minimize subjectivity in criteria importance measurement.
- Introduction of CIMFWAA and CIMFWGA operators in order to support various attitudes toward decision-making in the same aggregation framework.
- The use of weighted Euclidean distance and weighted Hamming distance in order to get the robust and stable ranking measurements.
- A multi-city air-quality assessment case study is used to evaluate the proposed framework.

This flowchart 1 shows the entire flow of the proposed study in terms of the orderly process of work starting with identifying the problem and the CIMFSS modeling to aggregation, ranking by distance, comparative analysis, and justifying the final decision.

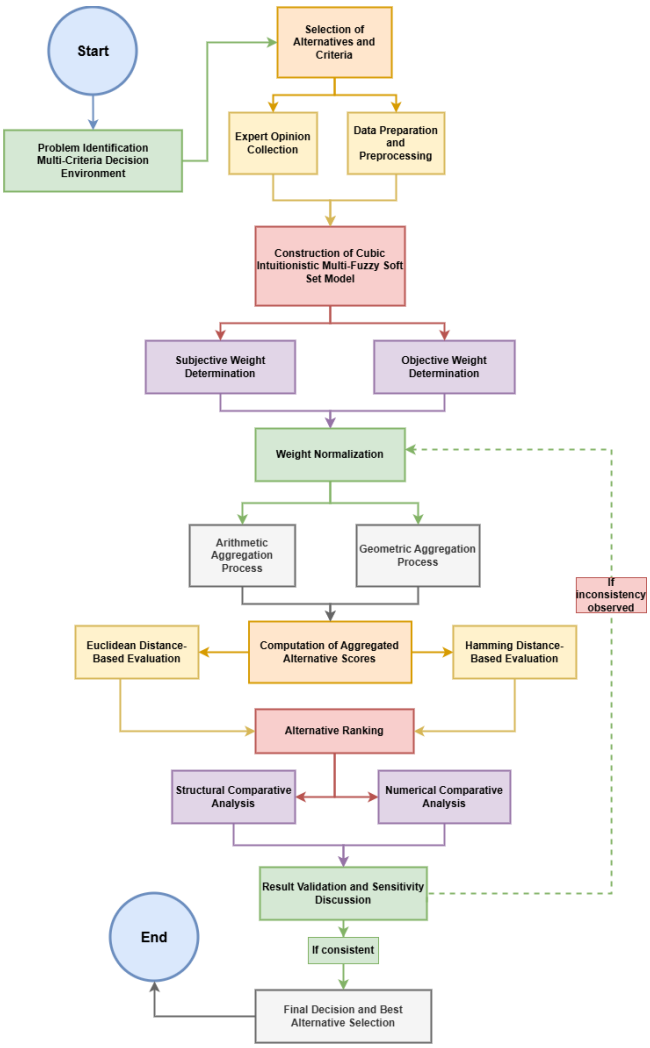


Fig. 1: Workflow of the proposed CIMFSS-based multi-criteria decision-making framework

The rest of this paper will be structured in the following way. Section 2 outlines the preliminaries and basic ideas associated with CIMFSS. Section 3 provides the proposed CIMFSS structure. Section 4 builds up the aggregation operators, whereas Section 5 displays the construction of distance measures. The methodology of the proposed decision-making is discussed in Section 6. Section 7 contains the case study on air quality monitoring, and Section 8 talks about the implementation details. The results and their discussion are presented in Section 9 and a numerical analysis is presented in Section 10 to confirm the soundness of the results obtained. Lastly, Section 11 provides a conclusion to the paper and suggestions for future research directions.

Abbreviations and Symbols

Table 2 is the list of the main abbreviations and mathematical symbols that will be applied in the paper to make it clear and consistent.

Table 2: List of abbreviations and symbols used in this paper

Abbreviations		**Symbols**	
Term	**Description**	**Symbol**	**Description**
MCDM	Multi-criteria decision making	$\mathcal{U}$	Set of alternatives
DM	Decision maker	$\mathcal{A}$	Set of criteria
CIMFSS	Cubic intuitionistic multi-fuzzy soft set	β_i	i-th alternative
CIMFWAAO	Weighted arithmetic aggregation operator	e_j	j-th criterion
CIMFWGAO	Weighted geometric aggregation operator	k	Number of decision makers
WED	Weighted Euclidean distance	w_j	Criterion weight
WHD	Weighted Hamming distance	ω_j	Normalized weight
AQI	Air quality index	$\nu_L(\beta)$	Lower membership degree
		$\nu_U(\beta)$	Upper membership degree
		$\omega_L(\beta)$	Lower non-membership degree
		$\omega_U(\beta)$	Upper non-membership degree
		$\sigma_c(\beta)$	Representative membership value
		$\tau_c(\beta)$	Representative non-membership value
		$\mathcal{S}_i$	Aggregated score of β_i
		$\mathcal{R}_i$	Rank of β_i
		$\oplus$	CIMFSS aggregation operator
		$\otimes$	CIMFSS multiplication operator

2 Preliminaries

This section presents the basic definitions of the CIMFSS structure. These definitions are what provide the foundation of the theoretical framework and further analysis that was carried out within this study, based on the CIMFSS.

Definition 1. *[5] A fuzzy set $\varpi_{\mathcal{G}}$ over the universe $\mathcal{U}$ is defined as*

$$\varpi_{\mathcal{G}} = \{\mathcal{R}_{\varpi_{\mathcal{G}}}(\eta)/\eta : \eta \in \mathcal{U}\},$$

where $\mathcal{R}_{\varpi_{\mathcal{G}}} : \mathcal{U} \to [0,1]$ is the membership function that assigns to each element $\eta \in \mathcal{U}$ its belonging degree in $\varpi_{\mathcal{G}}$.

Definition 2. *[8] Let $\mathcal{U}$ be an initial universe and $\mathcal{E}$ a set of parameters. Suppose $\mathcal{A} \subseteq \mathcal{E}$. A pair $(\mathcal{F}, \mathcal{A})$ is called a **soft set** over the universe $\mathcal{U}$ and can be expressed*

as a collection of ordered pairs

$$(\mathcal{F}, \mathcal{A}) = \{\mathcal{F}(e) \in P(\mathcal{U}) : e \in \mathcal{E},\ \mathcal{F}(e) = \varnothing \text{ if } e \notin \mathcal{A}\},$$

where $\mathcal{F}$ is a mapping $\mathcal{F} : \mathcal{A} \to P(\mathcal{U})$. In simpler terms, a soft set is a parameterized collection of subsets of $\mathcal{U}$, where each parameter in $\mathcal{A}$ maps to a subset of $\mathcal{U}$ through $\mathcal{F}$.

Definition 3. *[40] Let $\mathcal{U}$ be a universal set and $\mathcal{E}$ a set of parameters. Denote by $PF(\mathcal{U})$ the collection of all fuzzy subsets of $\mathcal{U}$. A pair $(\mathcal{F}, \mathcal{A})$, where $\mathcal{A} \subseteq \mathcal{E}$, is called a fuzzy soft set over $\mathcal{U}$ if $\mathcal{F} : \mathcal{A} \to PF(\mathcal{U})$ is a mapping that assigns to each parameter $e \in \mathcal{A}$ a fuzzy subset $\mathcal{F}(e)$ of $\mathcal{U}$, i.e. $\mathcal{F}(e) : \mathcal{U} \to [0, 1]$.*

Definition 4. *[14] Let $\mathcal{U}$ be a universal set and $\mathcal{E}$ a set of parameters. Let $\mathcal{A} \subseteq \mathcal{E}$. A pair $(\mathcal{F}, \mathcal{A})$ is called an intuitionistic fuzzy soft set (IFSS) over $\mathcal{U}$ if*

$$\mathcal{F} : \mathcal{A} \to IF(\mathcal{U})$$

*is a mapping that assigns to each parameter $e \in \mathcal{A}$ an **intuitionistic fuzzy set** over $\mathcal{U}$, i.e.*

$$\mathcal{F}(e) = \big\{(\beta, \mu_{\mathcal{F}(e)}(\beta), \nu_{\mathcal{F}(e)}(\beta)) : \beta \in \mathcal{U}\big\},$$

where

- $\mu_{\mathcal{F}(e)}(\beta) \in [0, 1]$ *is the membership degree of* β,
- $\nu_{\mathcal{F}(e)}(\beta) \in [0, 1]$ *is the non-membership degree of* β,
- $0 \leq \mu_{\mathcal{F}(e)}(\beta) + \nu_{\mathcal{F}(e)}(\beta) \leq 1$ *for all* $\beta \in \mathcal{U}$.

Definition 5. *[17] A cubic fuzzy set $\mathcal{A}$ over the universal set $\mathcal{U}$ is defined as*

$$\mathcal{A} = \Big\{\big(\beta, [\upsilon^L_{\mathcal{A}}(\beta), \upsilon^U_{\mathcal{A}}(\beta)], \sigma_{\mathcal{A}}(\beta)\big) : \beta \in \mathcal{U}\Big\},$$

where

- $[\upsilon^L_{\mathcal{A}}(\beta), \upsilon^U_{\mathcal{A}}(\beta)] \subseteq [0, 1]$ *is the interval-valued membership function of* β,
- $\sigma_{\mathcal{A}}(\beta) \in [0, 1]$ *is the single-valued fuzzy membership of* β.

Definition 6. *[41] Let $\mathcal{U}$ be an initial universe and $\mathcal{E}$ a set of parameters. A cubic soft set over $\mathcal{U}$ is a pair $(\mathcal{F}, \mathcal{A})$, where $\mathcal{A} \subseteq \mathcal{E}$ and $\mathcal{F} : \mathcal{A} \to C^{\mathcal{U}}$ is a mapping assigning a cubic set to each parameter in $\mathcal{A}$. Equivalently, the cubic soft set can be represented as*

$$(\mathcal{F}, \mathcal{A}) = \big\{\mathcal{F}(e) \mid e \in \mathcal{A}\big\}, \quad \text{where} \quad \mathcal{F}(e) = \big(\upsilon_{\mathcal{F}(e)}, \sigma_{\mathcal{F}(e)}\big).$$

Here, $\upsilon_{\mathcal{F}(e)}$ represents the interval-valued membership function and $\sigma_{\mathcal{F}(e)}$ the (single-valued) fuzzy membership function for the cubic set corresponding to parameter e.

Definition 7. *[20] Let $\mathcal{U}$ be a universe and $\mathcal{E}$ a set of parameters with $\mathcal{A} \subseteq \mathcal{E}$. A pair $(\mathcal{F}, \mathcal{A})$ is called a* cubic intuitionistic fuzzy soft set *over $\mathcal{U}$ if $\mathcal{F} : \mathcal{A} \to CIF(\mathcal{U})$, where for each $e \in \mathcal{A}$,*

$$\mathcal{F}(e) = \Big\{\big(\beta, [\mu^L(\beta), \mu^U(\beta)], [\nu^L(\beta), \nu^U(\beta)]\big) : \beta \in \mathcal{U}\Big\},$$

with $[\mu^L, \mu^U]$ *the interval-valued membership,* $[\nu^L, \nu^U]$ *the interval-valued non-membership, and* $0 \leq \mu^U(\beta) + \nu^U(\beta) \leq 1 \quad \forall \beta \in \mathcal{U}$.

Definition 8. *[28] Let* k *be a positive integer. A multi-fuzzy set* $\mathcal{A}$ *over the universe* $\mathcal{U}$ *is a collection of ordered sequences*

$$\mathcal{A} = \left\{ \beta \;/\; \big(\upsilon_1(\beta), \upsilon_2(\beta), \ldots, \upsilon_j(\beta), \ldots, \upsilon_k(\beta)\big) : \beta \in \mathcal{U} \right\},$$

where each $\upsilon_j : \mathcal{U} \to [0,1]$ *for* $j = 1, 2, \ldots, k$. *The vector function* $\upsilon_{\mathcal{A}} = (\upsilon_1, \upsilon_2, \ldots, \upsilon_k)$ *is called the multi-membership function of the multi-fuzzy set* $\mathcal{A}$. *The integer* k *is called the dimension of* $\mathcal{A}$. *The set of all multi-fuzzy sets of dimension* k *over* $\mathcal{U}$ *is denoted by* $M^k FS(\mathcal{U})$.

Definition 9. *[29] A pair* $(\mathcal{F}, \mathcal{Z})$ *is called a* ***multi-fuzzy soft set*** *of dimension* k *over* $\mathcal{Z}$, *where* $\mathcal{F}$ *is a mapping given by* $\mathcal{F} : \mathcal{Z} \to M^k FS(\mathcal{P})$. *A multi-fuzzy soft set is a mapping from parameters to* $M^k FS(\mathcal{P})$. *It is a parameterized family of multi-fuzzy subsets of* $\mathcal{P}$. *For* $t \in z$, $\mathcal{F}(t)$ *may be considered as the set of* t*-approximate elements of the multi-fuzzy soft set* $(\mathcal{F}, \mathcal{Z})$.

3 Cubic Intuitionistic Multi-Fuzzy Soft Set (CIMFSS)

This section introduces the key principles of the CIMFSS framework that combines cubic representations, intuitionistic appraisals, and multi-fuzzy information in a single framework. CIMFSS enhances the mathematical capabilities of interval-valued membership, non-membership, and multi-expert evaluation to provide more accurate and flexible models of complex, uncertain, and multi-dimensional decision environments. These definitions form the theoretical basis for the following methodologies and case studies.

Definition 10. *A pair* $(\mathcal{F}, \mathcal{A})$ *is called a CIMFSS of dimension* k *over the universal set* $\mathcal{U}$ *if* $\mathcal{F} : \mathcal{A} \to M^k PFS(C^{\mathcal{U}})$ *where* $M^k PFHS(C^{\mathcal{U}})$ *denotes the collection of all* ***CIMFSS*** *of dimension* k *over* $\mathcal{U}$.

For each $e \in \mathcal{A}$, *the mapping* $\mathcal{F}(e)$ *assigns a CIMFSS*

$$\mathcal{F}(e) = \left\{ \left\langle \beta, \big([\upsilon_j^L(\beta), \upsilon_j^U(\beta)], [\omega_j^L(\beta), \omega_j^U(\beta)], \sigma_j(\beta), \tau_j(\beta)\big) \right\rangle : \beta \in \mathcal{U} \right\},$$

where $[\upsilon_j^L(\beta), \upsilon_j^U(\beta)] = \big([\upsilon_1^L(\beta), \upsilon_1^U(\beta)], \ldots, [\upsilon_k^L(\beta), \upsilon_k^U(\beta)]\big)$, $[\omega_j^L(\beta), \omega_j^U(\beta)] = \big([\omega_1^L(\beta), \omega_1^U(\beta)], \ldots, [\omega_k^L(\beta), \omega_k^U(\beta)]\big)$, $\sigma_j(\beta) = \big(\sigma_1(\beta), \ldots, \sigma_k(\beta)\big)$, $\tau_j(\beta) = \big(\tau_1(\beta), \ldots, \tau_k(\beta)\big)$, *with each interval* $[\upsilon_j^L(\beta), \upsilon_j^U(\beta)] \subseteq [0,1]$ *representing the interval-valued membership function, each interval* $[\omega_j^L(\beta), \omega_j^U(\beta)] \subseteq [0,1]$ *the interval-valued non-membership function, and each* $\sigma_j(\beta), \tau_j(\beta) \in [0,1]$ *denoting the corresponding membership and non-membership degrees respectively, for* $j = 1, 2, \ldots, k$. *These values satisfy the intuitionistic constraint* $0 \leq \upsilon_j^U(\beta) + \omega_j^U(\beta) \leq 1, \quad 0 \leq \sigma_j(\beta) + \tau_j(\beta) \leq 1 \quad for\ all\ \beta \in \mathcal{U},\ j = 1, \ldots, k$.

Definition 11. *A CIMFSS* $(\mathcal{F}, \mathcal{A})$ *over* $\mathcal{U}$ *is called an* ***external CIMFSS*** *if, for each* $e \in \mathcal{A}$ *and* $\beta \in \mathcal{U}$,

$$\sigma_j(\beta) \notin [\upsilon_j^L(\beta), \upsilon_j^U(\beta)] \quad or \quad \tau_j(\beta) \notin [\omega_j^L(\beta), \omega_j^U(\beta)] \quad for\ all\ j = 1, 2, \ldots, k.$$

That is, at least one of the membership or non-membership degrees lies outside its corresponding interval-valued bound.

Definition 12. *A CIMFSS $(\mathcal{F}, \mathcal{A})$ over $\mathcal{U}$ is called an **internal CIMFSS** if, for each $e \in \mathcal{A}$ and $\beta \in \mathcal{U}$,*

$$\sigma_j(\beta) \in [\upsilon_j^L(\beta), \upsilon_j^U(\beta)] \quad \text{and} \quad \tau_j(\beta) \in [\omega_j^L(\beta), \omega_j^U(\beta)] \quad \text{for all } j = 1, 2, \ldots, k.$$

That is, all membership and non-membership degrees lie within their respective interval-valued bounds.

Definition 13. *Let $(\mathcal{F}_1, \mathcal{A}_1)$ and $(\mathcal{F}_2, \mathcal{A}_2)$ be two CIMFSSs of dimension k over the same universal set $\mathcal{U}$. We say that $(\mathcal{F}_1, \mathcal{A}_1)$ is a **subset** of $(\mathcal{F}_2, \mathcal{A}_2)$, written as*

$$(\mathcal{F}_1, \mathcal{A}_1) \tilde{\subseteq} (\mathcal{F}_2, \mathcal{A}_2),$$

if

1. *$\mathcal{A}_1 \subseteq \mathcal{A}_2$, and*
2. *for every $e \in \mathcal{A}_1$ and every $\beta \in \mathcal{U}$,*

$$\begin{aligned} \upsilon_{1j}^L(\beta) &\geq \upsilon_{2j}^L(\beta), & \upsilon_{1j}^U(\beta) &\geq \upsilon_{2j}^U(\beta), \\ \omega_{1j}^L(\beta) &\leq \omega_{2j}^L(\beta), & \omega_{1j}^U(\beta) &\leq \omega_{2j}^U(\beta), \\ \sigma_{1j}(\beta) &\leq \sigma_{2j}(\beta), & \tau_{1j}(\beta) &\geq \tau_{2j}(\beta), \end{aligned}$$

for all $j = 1, 2, \ldots, k$.

That is, the membership degrees of $(\mathcal{F}_1, \mathcal{A}_1)$ are less than or equal to those of $(\mathcal{F}_2, \mathcal{A}_2)$, and its non-membership degrees are greater than or equal to those of $(\mathcal{F}_2, \mathcal{A}_2)$.

Definition 14. *A CIMFSS $(\mathcal{F}_\varnothing, \mathcal{A})$ over $\mathcal{U}$ is called a **null (or empty) set** if, for each $e \in \mathcal{A}$ and $\beta \in \mathcal{U}$,*

$$[\upsilon_j^L(\beta), \upsilon_j^U(\beta)] = [0, 0], \quad [\omega_j^L(\beta), \omega_j^U(\beta)] = [1, 1],$$

$$\sigma_j(\beta) = 0, \quad \tau_j(\beta) = 1 \quad \text{for all } j = 1, 2, \ldots, k.$$

That is, every element has complete non-membership and zero membership.

Definition 15. *A CIMFSS $(\mathcal{F}_\mathcal{U}, \mathcal{A})$ over $\mathcal{U}$ is called an **absolute set** if, for each $e \in \mathcal{A}$ and $\beta \in \mathcal{U}$,*

$$[\upsilon_j^L(\beta), \upsilon_j^U(\beta)] = [1, 1], \quad [\omega_j^L(\beta), \omega_j^U(\beta)] = [0, 0],$$

$$\sigma_j(\beta) = 1, \quad \tau_j(\beta) = 0 \quad \text{for all } j = 1, 2, \ldots, k.$$

This represents complete membership and zero non-membership for every element.

Definition 16. *Let $(\mathcal{F}, \mathcal{A})$ be a CIMFSS of dimension k over $\mathcal{U}$. The **complement**, denoted $(\mathcal{F}, \mathcal{A})^c$, is defined as*

$$(\mathcal{F}, \mathcal{A})^c = \left\{ \left\langle \beta, \left([\omega_j^L(\beta), \omega_j^U(\beta)], [\upsilon_j^L(\beta), \upsilon_j^U(\beta)], \tau_j(\beta), \sigma_j(\beta)\right) \right\rangle : \beta \in \mathcal{U} \right\},$$

for each $e \in \mathcal{A}$. Thus, the membership and non-membership components are interchanged for every element and parameter.

4 Weighted Aggregation Operators of CIMFSNs

Weighted aggregation operators have been used as indispensable tools in decision-making applications that involve Cubic Intuitionistic Multi-Fuzzy Soft Numbers (CIMFSNs) in aggregating various intuitionistic and interval-based information. Such operators allow a versatile combination of various evaluations by experts without losing hesitation, multi-dimensional uncertainty, and the cubic structure of the data. The weighted arithmetic and weighted geometric aggregation operators are among them, as they offer two supplementary mechanisms. the former is the additive contribution of each CIMFSN given its significance weight, and the latter is the multiplicative or conjunctive interactions between them that indicate more persistent consensus behavior. Both operators uphold core mathematical characteristics, including idempotency, monotonicity and boundedness to guarantee reliable, consistent, and interpretable aggregate results in CIMFSS-based decision models.

Definition 17. *Let $\mathcal{A}(\beta)$ and $\mathcal{B}(\beta)$ be two CIMFSNs of dimension k defined as:*

$$\mathcal{A}(\beta) = \big([\upsilon_j^L(\beta), \upsilon_j^U(\beta)], [\omega_j^L(\beta), \omega_j^U(\beta)], \sigma_j(\beta), \tau_j(\beta)\big),$$

$$\mathcal{B}(\beta) = \big([\mu_j^L(\beta), \mu_j^U(\beta)], [\eta_j^L(\beta), \eta_j^U(\beta)], \lambda_j(\beta), \rho_j(\beta)\big).$$

Then, the basic arithmetic operations between $\mathcal{A}(\beta)$ and $\mathcal{B}(\beta)$ are defined as follows:

1. ***Addition***

$$\mathcal{A}(\beta) \oplus \mathcal{B}(\beta) = \begin{pmatrix} [\upsilon_j^L(\beta) + \mu_j^L(\beta) - \upsilon_j^L(\beta)\mu_j^L(\beta),\ \upsilon_j^U(\beta) + \mu_j^U(\beta) - \upsilon_j^U(\beta)\mu_j^U(\beta)], \\ [\omega_j^L(\beta)\eta_j^L(\beta),\ \omega_j^U(\beta)\eta_j^U(\beta)], \sigma_j(\beta) + \lambda_j(\beta) - \sigma_j(\beta)\lambda_j(\beta),\ \tau_j(\beta)\rho_j(\beta) \end{pmatrix}.$$

2. ***Multiplication***

$$\mathcal{A}(\beta) \otimes \mathcal{B}(\beta) = \begin{pmatrix} [\upsilon_j^L(\beta)\mu_j^L(\beta),\ \upsilon_j^U(\beta)\mu_j^U(\beta)], \\ [\omega_j^L(\beta) + \eta_j^L(\beta) - \omega_j^L(\beta)\eta_j^L(\beta),\ \omega_j^U(\beta) + \eta_j^U(\beta) - \omega_j^U(\beta)\eta_j^U(\beta)], \\ \sigma_j(\beta)\lambda_j(\beta),\ \tau_j(\beta) + \rho_j(\beta) - \tau_j(\beta)\rho_j(\beta) \end{pmatrix}.$$

3. ***Probabilistic Sum***

$$\tau\,\mathcal{A}(\beta) = \big([1 - (1 - \upsilon_j^L(\beta))^\tau,\ 1 - (1 - \upsilon_j^U(\beta))^\tau], [(\omega_j^L(\beta))^\tau, (\omega_j^U(\beta))^\tau], 1 - (1 - \sigma_j(\beta))^\tau,\ (\tau_j(\beta))^\tau\big).$$

4. ***Power Operation***

$$\mathcal{A}(\beta)^\tau = \big([(\upsilon_j^L(\beta))^\tau, (\upsilon_j^U(\beta))^\tau], [1 - (1 - \omega_j^L(\beta))^\tau,\ 1 - (1 - \omega_j^U(\beta))^\tau], (\sigma_j(\beta))^\tau,\ 1 - (1 - \tau_j(\beta))^\tau\big).$$

All values satisfy the intuitionistic constraint

$$0 \leq \upsilon_j^U(\beta) + \omega_j^U(\beta) \leq 1, \quad 0 \leq \sigma_j(\beta) + \tau_j(\beta) \leq 1, \quad \forall\, \beta \in \mathcal{U},\ j = 1, 2, \ldots, k.$$

Definition 18. *The* ***score*** *and* ***accuracy*** *functions for a CIMFSN are defined as follows:*

1. ***Score Function***

$$S(\mathcal{A}(\beta)) = \frac{1}{k}\sum_{j=1}^{k}\left\langle \frac{1}{3}\left(1 + [\upsilon_j^L(\beta) + \upsilon_j^U(\beta) - \omega_j^L(\beta) - \omega_j^U(\beta)] + \sigma_j(\beta) - \tau_j(\beta)\right)\right\rangle. \tag{1}$$

2. ***Accuracy Function***

$$H(\mathcal{A}(\beta)) = \frac{1}{k}\sum_{j=1}^{k}\left\langle \frac{1}{2}\left(\frac{[\upsilon_j^L(\beta) + \upsilon_j^U(\beta) - \omega_j^L(\beta) - \omega_j^U(\beta)]}{2} + \sigma_j(\beta) - \tau_j(\beta)\right)\right\rangle.$$

Definition 19. *Let $\mathcal{A}(\beta)$ and $\mathcal{B}(\beta)$ be two CIMFSNs. Their comparison is defined as follows:*

- *If $S(\mathcal{A}(\beta)) > S(\mathcal{B}(\beta))$, then $\mathcal{A}(\beta) \succ \mathcal{B}(\beta)$.*
- *If $S(\mathcal{A}(\beta)) = S(\mathcal{B}(\beta))$ and $H(\mathcal{A}(\beta)) > H(\mathcal{B}(\beta))$, then $\mathcal{A}(\beta) \succ \mathcal{B}(\beta)$.*
- *Otherwise, $\mathcal{A}(\beta) \sim \mathcal{B}(\beta)$ or they are incomparable.*

4.1 Weighted Arithmetic Averaging Operator of CIMFSNs

Let $\mathcal{A}(\beta) = \left([\upsilon_j^L(\beta), \upsilon_j^U(\beta)], [\omega_j^L(\beta), \omega_j^U(\beta)], \sigma_j(\beta), \tau_j(\beta)\right)$ be the CIMFSNs for $j = 1, 2, \ldots, k$. The following aggregation operators are defined on the set of CIMFSNs. Let w_j be the weight associated with $\mathcal{A}(\beta)$ such that $\sum_{j=1}^{k} w_j = 1$ and let $CIMFSWAA : \mathcal{G}^k \to \mathcal{G}$. Then the weighted arithmetic averaging operator of CIMFSNs is defined as:

$$CIMFSWAA(\mathcal{A}_1(\beta), \ldots, \mathcal{A}_k(\beta)) = \bigoplus_{j=1}^{k} w_j \mathcal{A}_j(\beta)$$

Theorem 1. *Set $\mathcal{A}_j(\beta) = \left([\upsilon_j^L(\beta), \upsilon_j^U(\beta)], [\omega_j^L(\beta), \omega_j^U(\beta)], \sigma_j(\beta), \tau_j(\beta)\right)$ $(j = 1, 2, \ldots, k)$ be CIMFSNs with their weight vector $W = (w_1, w_2, \ldots, w_k)$ for $0 \leq w_j \leq 1$ and $\sum_{j=1}^{k} w_j = 1$. Then, the aggregated value of the CIMFSWAA operator is also CIMFSN, which is given by the following formula:*

$$CIMFSWAA(\mathcal{A}_1(\beta), \ldots, \mathcal{A}_k(\beta)) = \begin{pmatrix} \left[1 - \prod_{j=1}^{k}(1 - \upsilon_j^L(\beta))^{w_j}, \right. \\ \left. 1 - \prod_{j=1}^{k}(1 - \upsilon_j^U(\beta))^{w_j}\right], \\ \left[\prod_{j=1}^{k}(\omega_j^L(\beta))^{w_j}, \prod_{j=1}^{k}(\omega_j^U(\beta))^{w_j}\right], \\ 1 - \prod_{j=1}^{k}(1 - \sigma_j(\beta))^{w_j}, \prod_{j=1}^{k}(\tau_j(\beta))^{w_j} \end{pmatrix} \tag{2}$$

***Proof**:*

Regarding mathematical induction, Equation 2 is verified below:

(1) Set $k = 2$. Corresponding to the operational relations in Definition 17, we can obtain the following aggregation result:

$$CIMFSWAA(\mathcal{A}_1(\beta), \mathcal{A}_2(\beta)) = w_1\mathcal{A}_1(\beta) \oplus w_2\mathcal{A}_2(\beta)$$

$$= \begin{pmatrix} \begin{bmatrix} 1-(1-\upsilon_1^L(\beta))^{w_1} + 1-(1-\mu_2^L(\beta))^{w_2} \\ -(1-(1-\upsilon_1^L(\beta))^{w_1})(1-(1-\mu_2^L(\beta))^{w_2}), \\ 1-(1-\upsilon_1^U(\beta))^{w_1} + 1-(1-\mu_2^U(\beta))^{w_2} \\ -(1-(1-\upsilon_1^U(\beta))^{w_1})(1-(1-\mu_2^U(\beta))^{w_2}), \end{bmatrix} \\ \begin{bmatrix} \omega_1^L(\beta)^{w_1}\eta_2^L(\beta)^{w_2}, \\ \omega_1^U(\beta)^{w_1}\eta_2^U(\beta)^{w_2} \end{bmatrix} \\ 1-(1-\sigma_1(\beta))^{w_1} + 1-(1-\lambda_2(\beta))^{w_2} \\ -(1-(1-\sigma_1(\beta))^{w_1})(1-(1-\lambda_2(\beta))^{w_2}), \\ \tau_1(\beta)^{w_1}\rho_2(\beta)^{w_2} \end{pmatrix}$$

$$= \begin{pmatrix} \left[1-\prod_{j=1}^2(1-\upsilon_j^L(\beta))^{w_j}, 1-\prod_{j=1}^2(1-\upsilon_j^U(\beta))^{w_j}\right], \\ \left[\prod_{j=1}^2(\omega_j^L(\beta))^{w_j}, \prod_{j=1}^2(\omega_j^U(\beta))^{w_j}\right], \\ \left(1-\prod_{j=1}^2(1-\sigma_j(\beta))^{w_j}\right), \\ \prod_{j=1}^2(\tau_j(\beta))^{w_j} \end{pmatrix} \tag{3}$$

(2) Let $k = l$. Equation 2 can then be expressed as follows:

$$CIMFSWAA(\mathcal{A}_1(\beta), \mathcal{A}_2(\beta), \ldots, \mathcal{A}_l(\beta)) = \bigoplus_{j=1}^{l} w_j\mathcal{A}_j(\beta)$$

$$= \begin{pmatrix} \left[1-\prod_{j=1}^l(1-\upsilon_j^L(\beta))^{w_j}, 1-\prod_{j=1}^l(1-\upsilon_j^U(\beta))^{w_j}\right], \\ \left[\prod_{j=1}^l(\omega_j^L(\beta))^{w_j}, \prod_{j=1}^l(\omega_j^U(\beta))^{w_j}\right], \\ 1-\prod_{j=1}^l(1-\sigma_j(\beta))^{w_j}, \prod_{j=1}^l(\tau_j(\beta))^{w_j} \end{pmatrix} \tag{4}$$

(3) Let $k = l + 1$. Considering the operational relations in Definition 17 and combining Equation 4 with Equation 2, we obtain the following aggregation:

$$CIMFSWAA(\mathcal{A}_1(\beta), \mathcal{A}_2(\beta), \ldots, \mathcal{A}_{l+1}(\beta)) = \bigoplus_{j=1}^{l} w_j\mathcal{A}_j(\beta) \oplus w_{l+1}\mathcal{A}_{l+1}(\beta)$$

$$= \begin{pmatrix} \left[1-\prod_{j=1}^l(1-\upsilon_j^L(\beta))^{w_j}, 1-\prod_{j=1}^l(1-\upsilon_j^U(\beta))^{w_j}\right], \\ \left[\prod_{j=1}^l(\omega_j^L(\beta))^{w_j}, \prod_{j=1}^l(\omega_j^U(\beta))^{w_j}\right], \\ 1-\prod_{j=1}^l(1-\sigma_j(\beta))^{w_j}, \prod_{j=1}^l(\tau_j(\beta))^{w_j} \end{pmatrix} \oplus w_{l+1}\mathcal{A}_{l+1}(\beta)$$

$$= \begin{pmatrix} \begin{bmatrix} 1-\prod_{j=1}^{l}(1-\upsilon_j^L(\beta))^{w_j}\cdot(1-\upsilon_{l+1}^L(\beta))^{w_{l+1}} \\ 1-\prod_{j=1}^{l}(1-\upsilon_j^U(\beta))^{w_j}\cdot(1-\upsilon_{l+1}^U(\beta))^{w_{l+1}} \end{bmatrix}, \\ \left[\prod_{j=1}^{l}(\omega_j^L(\beta))^{w_j}\cdot(\omega_{l+1}^L(\beta))^{w_{l+1}}, \prod_{j=1}^{l}(\omega_j^U(\beta))^{w_j}\cdot(\omega_{l+1}^U(\beta))^{w_{l+1}}\right], \\ 1-\prod_{j=1}^{l}(1-\sigma_j(\beta))^{w_j}\cdot(1-\sigma_{l+1}(\beta))^{w_{l+1}}, \\ \prod_{j=1}^{l}(\tau_j(\beta))^{w_j}\cdot(\tau_{l+1}(\beta))^{w_{l+1}} \end{pmatrix}$$

$$= \begin{pmatrix} \left[1-\prod_{j=1}^{l+1}(1-\upsilon_j^L(\beta))^{w_j}, 1-\prod_{j=1}^{l+1}(1-\upsilon_j^U(\beta))^{w_j}\right], \\ \left[\prod_{j=1}^{l+1}(\omega_j^L(\beta))^{w_j}, \prod_{j=1}^{l+1}(\omega_j^U(\beta))^{w_j}\right], \\ 1-\prod_{j=1}^{l+1}(1-\sigma_j(\beta))^{w_j}, \prod_{j=1}^{l+1}(\tau_j(\beta))^{w_j} \end{pmatrix}$$

Thus, Equation 2 holds true for all l.

Theorem 2. *The CIMFSWAA operator of Equation 2 contains the following properties:*

- ***Idempotency:*** *Set* $\mathcal{A}_j(\beta) = \left([\upsilon_j^L(\beta), \upsilon_j^U(\beta)], [\omega_j^L(\beta), \omega_j^U(\beta)], \sigma_j(\beta), \tau_j(\beta)\right)$ $(j = 1, 2, \ldots, k)$ *be CIMFSNs.*
  If $\mathcal{A}_j(\beta) = \mathcal{A}(\beta) = \left([\upsilon^L(\beta), \upsilon^U(\beta)], [\omega^L(\beta), \omega^U(\beta)], \sigma(\beta), \tau(\beta)\right)$ $(j = 1, 2, \ldots, k)$, *then*

$$CIMFSWAA(\mathcal{A}_1(\beta), \mathcal{A}_2(\beta), \ldots, \mathcal{A}_k(\beta)) = \mathcal{A}(\beta)$$

- ***Boundedness:*** *Set* $\mathcal{A}_j(\beta) = \left([\upsilon_j^L(\beta), \upsilon_j^U(\beta)], [\omega_j^L(\beta), \omega_j^U(\beta)], \sigma_j(\beta), \tau_j(\beta)\right)$ $(j = 1, 2, \ldots, k)$ *be CIMFSNs and let*

$$\mathcal{A}_{\min}(\beta) = \begin{pmatrix} \left[\min_j(\upsilon_j^L(\beta)), \min_j(\upsilon_j^U(\beta))\right], \left[\min_j(\omega_j^L(\beta)), \min_j(\omega_j^U(\beta))\right], \\ \min_j(\sigma_j(\beta)), \max_j(\tau_j(\beta)) \end{pmatrix}$$

and

$$\mathcal{A}_{\max}(\beta) = \begin{pmatrix} \left[\max_j(\upsilon_j^L(\beta)), \max_j(\upsilon_j^U(\beta))\right], \left[\max_j(\omega_j^L(\beta)), \max_j(\omega_j^U(\beta))\right], \\ \max_j(\sigma_j(\beta)), \min_j(\tau_j(\beta)) \end{pmatrix}$$

be the minimum CIMFSN and the maximum CIMFSN, respectively. Then, there exists

$$\mathcal{A}_{\min}(\beta) \leq CIMFSWAA(\mathcal{A}_1(\beta), \mathcal{A}_2(\beta), \ldots, \mathcal{A}_k(\beta)) \leq \mathcal{A}_{\max}(\beta).$$

- ***Monotonicity:*** *Set* $\mathcal{A}_j(\beta) = \left([\upsilon_j^L(\beta), \upsilon_j^U(\beta)], [\omega_j^L(\beta), \omega_j^U(\beta)], \sigma_j(\beta), \tau_j(\beta)\right)$, *and* $\mathcal{A}_j^*(\beta) = \left([\upsilon_j^{L*}(\beta), \upsilon_j^{U*}(\beta)], [\omega_j^{L*}(\beta), \omega_j^{U*}(\beta)], \sigma_j^*(\beta), \tau_j^*(\beta)\right)$ $(j = 1, 2, \ldots, k)$ *as two groups of CIMFSNs. Then, there exists*

$$CIMFSWAA(\mathcal{A}_1(\beta), \mathcal{A}_2(\beta), \ldots, \mathcal{A}_k(\beta)) \leq$$
$$CIMFSWAA(\mathcal{A}_1^*(\beta), \mathcal{A}_2^*(\beta), \ldots, \mathcal{A}_k^*(\beta)) \text{ for } \mathcal{A}_j(\beta) \leq \mathcal{A}_j^*(\beta).$$

Proof:

(1) Owing to $\mathcal{A}_j(\beta) = \mathcal{A}(\beta)\,(j = 1, 2, \ldots, k)$*, one obtains the following result:*

$$CIMFSWAA(\mathcal{A}_1(\beta), \mathcal{A}_2(\beta), \ldots, \mathcal{A}_k(\beta)) = \bigoplus_{j=1}^{k} w_j \mathcal{A}_j(\beta)$$

$$= \begin{pmatrix} \left[1 - \prod_{j=1}^{k}(1 - \upsilon_j^L(\beta))^{w_j}, 1 - \prod_{j=1}^{k}(1 - \upsilon_j^U(\beta))^{w_j}\right], \\ \left[\prod_{j=1}^{k}(\omega_j^L(\beta))^{w_j}, \prod_{j=1}^{k}(\omega_j^U(\beta))^{w_j}\right], \\ 1 - \prod_{j=1}^{k}(1 - \sigma_j(\beta))^{w_j}, \prod_{j=1}^{k}(\tau_j(\beta))^{w_j} \end{pmatrix}$$

$$= \begin{pmatrix} \left[1 - (1 - \upsilon^L(\beta))^{\sum_{j=1}^{k} w_j}, 1 - (1 - \upsilon^U(\beta))^{\sum_{j=1}^{k} w_j}\right], \\ \left[(\omega^L(\beta))^{\sum_{j=1}^{k} w_j}, (\omega^U(\beta))^{\sum_{j=1}^{k} w_j}\right], \\ 1 - (1 - \sigma(\beta))^{\sum_{j=1}^{k} w_j}, (\tau(\beta))^{\sum_{j=1}^{k} w_j} \end{pmatrix}$$

$$= \left([\upsilon^L(\beta), \upsilon^U(\beta)], [\omega^L(\beta), \omega^U(\beta)], \sigma(\beta), \tau(\beta)\right) = \mathcal{A}(\beta)$$

(2) The inequality $\mathcal{A}_{\min}(\beta) \leq \mathcal{A}_j(\beta) \leq \mathcal{A}_{\max}(\beta)$ *can exist since* $\mathcal{A}_{\min}(\beta)$ *and* $\mathcal{A}_{\max}(\beta)$ *are the minimum and maximum CIMFSNs. Then,*

$$\bigoplus_{j=1}^{k} w_j \mathcal{A}_{\min}(\beta) \leq \bigoplus_{j=1}^{k} w_j \mathcal{A}_j(\beta) \leq \bigoplus_{j=1}^{k} w_j \mathcal{A}_{\max}(\beta)$$

can also exist. Based on the above property (1), $\mathcal{A}_{\min}(\beta) \leq \bigoplus_{j=1}^{k} w_j \mathcal{A}_{\min}(\beta) \leq \mathcal{A}_{\max}(\beta)$ *can hold, namely*

$$\mathcal{A}_{\min}(\beta) \leq CIMFSWAA(\mathcal{A}_1(\beta), \mathcal{A}_2(\beta), \ldots, \mathcal{A}_k(\beta)) \leq \mathcal{A}_{\max}(\beta)$$

(3) Owing to $\mathcal{A}_j(\beta) \leq \mathcal{A}_j^*(\beta)$*, there is*

$$\bigoplus_{j=1}^{k} w_j \mathcal{A}_j(\beta) \leq \bigoplus_{j=1}^{k} w_j \mathcal{A}_j^*(\beta)$$

namely

$$CIMFSWAA(\mathcal{A}_1(\beta), \mathcal{A}_2(\beta), \ldots, \mathcal{A}_k(\beta)) \leq CIMFSWAA(\mathcal{A}_1^*(\beta), \mathcal{A}_2^*(\beta), \ldots, \mathcal{A}_k^*(\beta))$$

Therefore, all the above properties are true.

4.2 Weighted Geometric Averaging Operator of CIMFSNs

Let $\mathcal{A}_j(\beta) = \left([\upsilon_j^L(\beta), \upsilon_j^U(\beta)], [\omega_j^L(\beta), \omega_j^U(\beta)], \sigma_j(\beta), \tau_j(\beta)\right)$ be the CIMFSNs for $j = 1, 2, \ldots, k$. The following aggregation operators are defined on the set of CIMFSNs. Let w_j be the weight associated with $\mathcal{A}_j(\beta)$ such that $\sum_{j=1}^{k} w_j = 1$ and

let $CIMFSWGA : \mathcal{G}^k \to \mathcal{G}$. Then the weighted geometric averaging operator of CIMFSNs is defined as:

$$CIMFSWGA(\mathcal{A}_1(\beta), \mathcal{A}_2(\beta), \ldots, \mathcal{A}_k(\beta)) = \bigotimes_{j=1}^{k} (\mathcal{A}_j(\beta))^{w_j}. \tag{5}$$

Theorem 3. *Set* $\mathcal{A}_j(\beta) = \left([\upsilon_j^L(\beta), \upsilon_j^U(\beta)], [\omega_j^L(\beta), \omega_j^U(\beta)], \sigma_j(\beta), \tau_j(\beta)\right)$ $(j = 1, 2, \ldots, k)$ *be CIMFSNs with their weight vector* $W = (w_1, w_2, \ldots, w_k)$ *for* $0 \leq w_j \leq 1$ *and* $\sum_{j=1}^{k} w_j = 1$. *Then, the aggregated value of the CIMFSWGA operator is also CIMFSN, which is given by the following formula:*

$$CIMFSWGA(\mathcal{A}_1(\beta), \mathcal{A}_2(\beta), \ldots, \mathcal{A}_k(\beta)) = \begin{pmatrix} \left[\prod_{j=1}^{k} (\upsilon_j^L(\beta))^{w_j}, \prod_{j=1}^{k} (\upsilon_j^U(\beta))^{w_j}\right], \\ \left[\prod_{j=1}^{k} (\omega_j^L(\beta))^{w_j}, \prod_{j=1}^{k} (\omega_j^U(\beta))^{w_j}\right], \\ \prod_{j=1}^{k} (\sigma_j(\beta))^{w_j}, \prod_{j=1}^{k} (\tau_j(\beta))^{w_j} \end{pmatrix}$$

***Proof**:*
Using a proof approach analogous to that employed in Theorem 4.1 for CIMFSNs, and taking into account the multiplicative operations of CIMFSNs (componentwise geometric aggregation for the membership interval, non-membership interval, and the scalar functions $\sigma_j(\beta)$ *and* $\tau_j(\beta)$*), Theorem 3 can be readily confirmed. Therefore, the detailed proof is not presented here.*

Theorem 4. *The CIMFSWGA operator of Equation 5 contains the following properties:*

- ***Idempotency:*** *Set* $\mathcal{A}_j(\beta) = \left([\upsilon_j^L(\beta), \upsilon_j^U(\beta)], [\omega_j^L(\beta), \omega_j^U(\beta)], \sigma_j(\beta), \tau_j(\beta)\right)$ $(j = 1, 2, \ldots, k)$ *be CIMFSNs.*
  If $\mathcal{A}_j(\beta) = \mathcal{A}(\beta) = \left([\upsilon^L(\beta), \upsilon^U(\beta)], [\omega^L(\beta), \omega^U(\beta)], \sigma(\beta), \tau(\beta)\right)$ $(j = 1, 2, \ldots, k)$, *then*
  $$CIMFSWGA(\mathcal{A}_1(\beta), \mathcal{A}_2(\beta), \ldots, \mathcal{A}_k(\beta)) = \mathcal{A}(\beta).$$
- ***Boundedness:*** *Set* $\mathcal{A}_j(\beta) = \left([\upsilon_j^L(\beta), \upsilon_j^U(\beta)], [\omega_j^L(\beta), \omega_j^U(\beta)], \sigma_j(\beta), \tau_j(\beta)\right)$ $(j = 1, 2, \ldots, k)$ *be CIMFSNs and let*
  $$\mathcal{A}_{\min}(\beta) = \begin{pmatrix} \left[\min_j(\upsilon_j^L(\beta)), \min_j(\upsilon_j^U(\beta))\right], \\ \left[\min_j(\omega_j^L(\beta)), \min_j(\omega_j^U(\beta))\right], \min_j(\sigma_j(\beta)), \min_j(\tau_j(\beta)) \end{pmatrix}$$
  and
  $$\mathcal{A}_{\max}(\beta) = \begin{pmatrix} \left[\max_j(\upsilon_j^L(\beta)), \max_j(\upsilon_j^U(\beta))\right], \\ \left[\max_j(\omega_j^L(\beta)), \max_j(\omega_j^U(\beta))\right], \max_j(\sigma_j(\beta)), \max_j(\tau_j(\beta)) \end{pmatrix}$$
  be the minimum CIMFSN and the maximum CIMFSN, respectively. Then, there exists
  $$\mathcal{A}_{\min}(\beta) \leq CIMFSWGA(\mathcal{A}_1(\beta), \mathcal{A}_2(\beta), \ldots, \mathcal{A}_k(\beta)) \leq \mathcal{A}_{\max}(\beta).$$

- ***Monotonicity:*** *Set* $\mathcal{A}_j(\beta) = \left([\upsilon_j^L(\beta), \upsilon_j^U(\beta)], [\omega_j^L(\beta), \omega_j^U(\beta)], \sigma_j(\beta), \tau_j(\beta)\right)$, *and* $\mathcal{A}_j^*(\beta) = \left([\upsilon_j^{L*}(\beta), \upsilon_j^{U*}(\beta)], [\omega_j^{L*}(\beta), \omega_j^{U*}(\beta)], \sigma_j^*(\beta), \tau_j^*(\beta)\right)$ $(j = 1, 2, \ldots, k)$ *as two groups of CIMFSNs. Then, there exists*

$$CIMFSWGA(\mathcal{A}_1(\beta), \mathcal{A}_2(\beta), \ldots, \mathcal{A}_k(\beta)) \leq CIMFSWGA(\mathcal{A}_1^*(\beta), \mathcal{A}_2^*(\beta), \ldots, \mathcal{A}_k^*(\beta)) \text{ for } \mathcal{A}_j(\beta) \leq \mathcal{A}_j^*(\beta).$$

Proof*:*
Theorem 4 can likewise be validated using a reasoning approach analogous to that of Theorem 2, by applying the explicit form of CIMFSWGA in Theorem 3 componentwise to each of the four CIMFSN components and using the monotonicity of the geometric mean on $[0, 1]$*. Accordingly, the detailed proof is omitted for conciseness.*

Remark 1. ***(Interpretation)****: The nature of the decision problem under cubic intuitionistic multi-fuzzy soft information has determined the decision made between the CIMFSWAA and CIMFSWGA operators. The CIMFSWAA operator is particularly suitable where one is interested in achieving a compensatory effect among criteria or expert judgments; in other words, those that are stronger are able to offset those that are weaker. On the other hand, the CIMFSWGA operator is more suited where the aggregation requires a conjunctive or consensus-based procedure whereby a low rating in any of the dimensions would highly influence the overall outcome. Thus, CIMFSWAA is an accrual model of uncertainty and trustworthiness, and CIMFSWGA functions as a compensatory or punitive model that focuses on reinforcement and punishments. All these operators are mutually complementary, and when applied together, they can assist decision-makers in choosing the appropriate mechanism for applications in uncertain situations.*

5 Development of distance measures utilizing CIMFSS

In this section, we generalize the development of distance measures to the context of CIMFSS. In the CIMFSS framework, interval-valued membership and interval-valued non-membership functions are combined with the relative scalar membership and non-membership degrees and then provide a more detailed description of uncertainty. To compare the dissimilarity of CIMFSS-based measurements, we design specific weighted Euclidean distance (WED) and weighted Hamming distance (WHD) metrics. We also prove that the proposed measures meet the four fundamental axioms of metrics non-negativity, identity of indiscernibles, symmetry and triangle inequality that guarantee their strength and consistency in the CIMFSS based analyses.

5.1 Proposed distance measures for CIMFSS

Let $\mathcal{A}_j(\beta)$ and $\mathcal{B}_j(\beta)$ be two CIMFSS over the universe $\mathcal{U}$. For each parameter index $j = 1, 2, \ldots, k$ and element $\beta \in \mathcal{U}$ denote

$$\mathcal{A}_j(\beta) = \left([\upsilon_j^L(\beta), \upsilon_j^U(\beta)], [\omega_j^L(\beta), \omega_j^U(\beta)], \sigma_j(\beta), \tau_j(\beta)\right),$$

$$\mathcal{B}_j(\beta) = \big([\mu_j^L(\beta), \mu_j^U(\beta)], [\eta_j^L(\beta), \eta_j^U(\beta)], \lambda_j(\beta), \rho_j(\beta)\big)\,.$$

where all interval endpoints and scalar degrees lie in $[0,1]$ and satisfy the intuitionistic constraints $\upsilon_j^U(\beta) + \omega_j^U(\beta) \le 1,\ \sigma_j(\beta) + \tau_j(\beta) \le 1$, and similarly for $\mathcal{B}_j(\beta)$. Let $W = (w_1, \ldots, w_k)$ be a weight vector with $0 \le w_j \le 1$ and $\sum_{j=1}^k w_j = 1$. Let $n = |\mathcal{U}|$ denote the number of elements of the universe.

Definition 20. *The WED measure for CIMFSS is defined as:*

$$\mathcal{E}_E(\mathcal{A}_j(\beta), \mathcal{B}_j(\beta)) = \left(\frac{1}{6n} \sum_{j=1}^{k} w_j \left(\begin{array}{c} |\,\upsilon_j^L(\beta) - \mu_j^L(\beta)\,|^2 + |\,\upsilon_j^U(\beta) - \mu_j^U(\beta)\,|^2 \\ + |\,\omega_j^L(\beta) - \eta_j^L(\beta)\,|^2 + |\,\omega_j^U(\beta) - \eta_j^U(\beta)\,|^2 \\ + |\,\sigma_j(\beta) - \lambda_j(\beta)\,|^2 + |\,\tau_j(\beta) - \rho_j(\beta)\,|^2 \end{array} \right) \right)^{\frac{1}{2}}.$$

Here, n denotes the total number of elements in the universal set.

Theorem 5. *Let $\mathcal{E}_E\big(\mathcal{A}_j(\beta), \mathcal{B}_j(\beta)\big)$ denote the WED between two CIMFSS as defined in Definition (20). Then, $\mathcal{E}_E$ is a valid distance measure on CIMFSS if it satisfies the following four properties:*

1. *Non-negativity:* $0 \le \mathcal{E}_E\big(\mathcal{A}_j(\beta), \mathcal{B}_j(\beta)\big) \le 1$,
2. *Identity of indiscernibles:* $\mathcal{E}_E\big(\mathcal{A}_j(\beta), \mathcal{B}_j(\beta)\big) = 0 \iff \mathcal{A}_j(\beta) = \mathcal{B}_j(\beta)$,
3. *Symmetry:* $\mathcal{E}_E\big(\mathcal{A}_j(\beta), \mathcal{B}_j(\beta)\big) = \mathcal{E}_E\big(\mathcal{B}_j(\beta), \mathcal{A}_j(\beta)\big)$,
4. *Triangular / monotonic property: If* $\mathcal{A}_j(\beta) \subseteq \mathcal{B}_j(\beta) \subseteq \mathcal{C}_j(\beta)$, *then*

$$\mathcal{E}_E\big(\mathcal{A}_j(\beta), \mathcal{B}_j(\beta)\big) \le \mathcal{E}_E\big(\mathcal{A}_j(\beta), \mathcal{C}_j(\beta)\big), \quad \mathcal{E}_E\big(\mathcal{B}_j(\beta), \mathcal{A}_j(\beta)\big) \le \mathcal{E}_E\big(\mathcal{A}_j(\beta), \mathcal{C}_j(\beta)\big).$$

Proof. *Let*

$$\mathcal{A}_j(\beta) = ([\upsilon_j^L(\beta), \upsilon_j^U(\beta)], [\omega_j^L(\beta), \omega_j^U(\beta)], \sigma_j(\beta), \tau_j(\beta)),$$
$$\mathcal{B}_j(\beta) = ([\mu_j^L(\beta), \mu_j^U(\beta)], [\eta_j^L(\beta), \eta_j^U(\beta)], \lambda_j(\beta), \rho_j(\beta)).$$

1) Non-negativity: *Since all components of CIMFSS lie in $[0,1]$, each squared difference in the sum is nonnegative and bounded by 1. The sum of all $6n$ terms weighted by w_j is thus bounded by $6n$, and dividing by $6n$ ensures*

$$0 \le \mathcal{E}_E\big(\mathcal{A}_j(\beta), \mathcal{B}_j(\beta)\big) \le 1.$$

2) Identity of indiscernibles: *If $\mathcal{E}_E\big(\mathcal{A}_j(\beta), \mathcal{B}_j(\beta)\big) = 0$, then each squared term must vanish, implying*

$$\upsilon_j^L(\beta) = \mu_j^L(\beta),\ \upsilon_j^U(\beta) = \mu_j^U(\beta),\ \omega_j^L(\beta) = \eta_j^L(\beta),\ \omega_j^U(\beta) = \eta_j^U(\beta),\ \sigma_j(\beta) = \lambda_j(\beta),\ \tau_j(\beta) = \rho_j(\beta)\ \forall j,$$

which means $\mathcal{A}_j(\beta) = \mathcal{B}_j(\beta)$. The converse is immediate.

3) Symmetry: *Since $(\mathcal{A}_j(\beta) - \mathcal{B}_j(\beta))^2 = (\mathcal{B}_j(\beta) - \mathcal{A}_j(\beta))^2$ for all real $\mathcal{A}_j(\beta), \mathcal{B}_j(\beta)$,*

$$\mathcal{E}_E\big(\mathcal{A}_j(\beta), \mathcal{B}_j(\beta)\big) = \mathcal{E}_E\big(\mathcal{B}_j(\beta), \mathcal{A}_j(\beta)\big).$$

4) Triangular / monotonic property: *Assume* $\mathcal{A}_j(\beta) \subseteq \mathcal{B}_j(\beta) \subseteq \mathcal{C}_j(\beta)$. *Then for each component and each* j,

$$\big|(\mathcal{A}_j(\beta)) - (\mathcal{B}_j(\beta))\big| \leq \big|(\mathcal{A}_j(\beta)) - (\mathcal{C}_j(\beta))\big|.$$

Squaring, weighting, summing, and taking the square root preserves the inequality:

$$\mathcal{E}_E(\mathcal{A}_j(\beta), \mathcal{B}_j(\beta)) \leq \mathcal{E}_E(\mathcal{A}_j(\beta), \mathcal{C}_j(\beta)), \quad \mathcal{E}_E(\mathcal{B}_j(\beta), \mathcal{A}_j(\beta)) \leq \mathcal{E}_E(\mathcal{A}_j(\beta), \mathcal{C}_j(\beta)).$$

Hence, $\mathcal{E}_E$ *satisfies all four metric properties and is a valid distance measure.*

Definition 21. *The WHD measure for CIMFSS is defined as:*

$$\mathcal{E}_H(\mathcal{A}_j(\beta), \mathcal{B}_j(\beta)) = \frac{1}{6n} \sum_{j=1}^{k} w_j \begin{pmatrix} \big|\, \upsilon_j^L(\beta) - \mu_j^L(\beta) \,\big| + \big|\, \upsilon_j^U(\beta) - \mu_j^U(\beta) \,\big| \\ + \big|\, \omega_j^L(\beta) - \eta_j^L(\beta) \,\big| + \big|\, \omega_j^U(\beta) - \eta_j^U(\beta) \,\big| \\ + |\, \sigma_j(\beta) - \lambda_j(\beta) \,| + |\, \tau_j(\beta) - \rho_j(\beta) \,| \end{pmatrix}.$$

Here, n *denotes the total number of elements in the universal set.*

Theorem 6. *Let* $\mathcal{E}_H\big(\mathcal{A}_j(\beta), \mathcal{B}_j(\beta)\big)$ *denote the WHD between two CIMFSS as defined in Definition 21. Then,* $\mathcal{E}_H$ *is a valid distance measure on CIMFSS if it satisfies the following four properties:*

1. *Non-negativity:* $0 \leq \mathcal{E}_H\big(\mathcal{A}_j(\beta), \mathcal{B}_j(\beta)\big) \leq 1$,
2. *Identity of indiscernibles:* $\mathcal{E}_H\big(\mathcal{A}_j(\beta), \mathcal{B}_j(\beta)\big) = 0 \iff \mathcal{A}_j(\beta) = \mathcal{B}_j(\beta)$,
3. *Symmetry:* $\mathcal{E}_H\big(\mathcal{A}_j(\beta), \mathcal{B}_j(\beta)\big) = \mathcal{E}_H\big(\mathcal{B}_j(\beta), \mathcal{A}_j(\beta)\big)$,
4. *Triangular / monotonic property: If* $\mathcal{A}_j(\beta) \subseteq \mathcal{B}_j(\beta) \subseteq \mathcal{C}_j(\beta)$, *then*

$$\mathcal{E}_H\big(\mathcal{A}_j(\beta), \mathcal{B}_j(\beta)\big) \leq \mathcal{E}_H\big(\mathcal{A}_j(\beta), \mathcal{C}_j(\beta)\big), \quad \mathcal{E}_H\big(\mathcal{B}_j(\beta), \mathcal{A}_j(\beta)\big) \leq \mathcal{E}_H\big(\mathcal{A}_j(\beta), \mathcal{C}_j(\beta)\big).$$

Proof. *Let*

$$\mathcal{A}(\beta) = \big\langle([\upsilon_j^L(\beta), \upsilon_j^U(\beta)], [\omega_j^L(\beta), \omega_j^U(\beta)], \sigma_j(\beta), \tau_j(\beta))\big\rangle,$$
$$\mathcal{B}(\beta) = \big\langle([\mu_j^L(\beta), \mu_j^U(\beta)], [\eta_j^L(\beta), \eta_j^U(\beta)], \lambda_j(\beta), \rho_j(\beta))\big\rangle.$$

1) Non-negativity: *Since all component values of CIMFSS lie in* $[0, 1]$, *each absolute difference*

$$|\mathcal{A}_j(\beta) - \mathcal{B}_j(\beta)| \in [0, 1], \quad \mathcal{A}_j(\beta), \mathcal{B}_j(\beta) \in \{\upsilon_j^L(\beta), \upsilon_j^U(\beta), \omega_j^L(\beta), \omega_j^U(\beta), \sigma_j(\beta), \tau_j(\beta)\},$$

and hence the weighted sum

$$\sum_{j=1}^{k} w_j \Big(|\upsilon_j^L(\beta) - \mu_j^L(\beta)| + |\upsilon_j^U(\beta) - \mu_j^U(\beta)| + |\omega_j^L(\beta) - \eta_j^L(\beta)| + |\omega_j^U(\beta) - \eta_j^U(\beta)| + |\sigma_j(\beta) - \lambda_j(\beta)| + |\tau_j(\beta) - \rho_j(\beta)|\Big)$$

is nonnegative. Dividing by $6n$ *ensures*

$$0 \leq \mathcal{E}_H\big(\mathcal{A}_j(\beta), \mathcal{B}_j(\beta)\big) \leq 1.$$

2) Identity of indiscernibles: *If* $\mathcal{E}_H\big(\mathcal{A}_j(\beta), \mathcal{B}_j(\beta)\big) = 0$*, then the weighted sum of absolute differences is zero. Since each term is nonnegative, every term must vanish, i.e.,*

$$\begin{aligned}&\upsilon_j^L(\beta) = \mu_j^L(\beta), \upsilon_j^U(\beta) = \mu_j^U(\beta), \omega_j^L(\beta) = \eta_j^L(\beta), \omega_j^U(\beta) = \eta_j^U(\beta),\\ &\sigma_j(\beta) = \lambda_j(\beta), \tau_j(\beta) = \rho_j(\beta), for\ all\ j = 1, ..., k.\end{aligned}$$

Hence, $\mathcal{A}_j(\beta) = \mathcal{B}_j(\beta)$*. Conversely, if the two CIMFSS are equal, all differences vanish and* $\mathcal{E}_H = 0$*.*

3) Symmetry: *Since* $|\mathcal{A}_j(\beta) - \mathcal{B}_j(\beta)| = |\mathcal{B}_j(\beta) - \mathcal{A}_j(\beta)|$ *for all real* $\mathcal{A}_j(\beta), \mathcal{B}_j(\beta)$*, it immediately follows that*

$$\mathcal{E}_H\big(\mathcal{A}_j(\beta), \mathcal{B}_j(\beta)\big) = \mathcal{E}_H\big(\mathcal{B}_j(\beta), \mathcal{A}_j(\beta)\big).$$

4) Triangular / monotonic property: *Assume* $\mathcal{A}_j(\beta) \subseteq \mathcal{B}_j(\beta) \subseteq \mathcal{C}_j(\beta)$*. Then for each component and each* j*,*

$$|\upsilon_j^L(\beta) - \mu_j^L(\beta)| \leq |\upsilon_j^L(\beta) - \gamma_j^L(\beta)|, \quad |\upsilon_j^U(\beta) - \mu_j^U(\beta)| \leq |\upsilon_j^U(\beta) - \gamma_j^U(\beta)|,$$

$$|\omega_j^L(\beta) - \eta_j^L(\beta)| \leq |\omega_j^L(\beta) - \theta_j^L(\beta)|, \quad |\omega_j^U(\beta) - \eta_j^U(\beta)| \leq |\omega_j^U(\beta) - \theta_j^U(\beta)|,$$
$$|\sigma_j(\beta) - \lambda_j(\beta)| \leq |\sigma_j(\beta) - \phi_j(\beta)|, \quad |\tau_j(\beta) - \rho_j(\beta)| \leq |\tau_j(\beta) - \psi_j(\beta)|.$$

Multiplying by the nonnegative weights w_j*, summing over all* j*, and dividing by* $6n$ *gives*

$$\mathcal{E}_H(\mathcal{A}_j(\beta), \mathcal{B}_j(\beta)) \leq \mathcal{E}_H(\mathcal{A}_j(\beta), \mathcal{C}_j(\beta)), \quad \mathcal{E}_H(\mathcal{B}_j(\beta), \mathcal{A}_j(\beta)) \leq \mathcal{E}_H(\mathcal{A}_j(\beta), \mathcal{C}_j(\beta)).$$

Hence, $\mathcal{E}_H$ *satisfies all four metric properties and is a valid distance measure.*

6 Entropy-Based Objective Weighting and Distance Measures in CIMFSS Decision-Making

In this section, two distance-based decision-making algorithms within the scope of the CIMFSS are proposed. The overall aim of these algorithms is to compare and rank a finite set of alternatives with a tune of measuring their closeness to an ideal solution in terms of suitable CIMFSS-based distance measures.

In the CIMFSS setting, Algorithm 1 presents an entropy-based model objective weighting scheme for deciding the relative importance of decision parameters. Entropy measure measures the amount of uncertainty or dispersion in each parameter in the set of alternatives. Parameters with greater informational diversity are given heavier weights, whilst those with smaller variability are given lighter weights. The objective

weighting process removes subjectivity that may be created through the use of expert judgment and increases the quality and strength of the decision-making process.
The algorithm 2 is a single CIMFSS based distance decision-making model that employs a combination of entropy-based parameter weights and aggregate of expert opinions as well as the distance analysis. There are two aggregation operators that are taken into account i.e. CIMFSWAA operator and the CIMFSWG operator. Furthermore, the calculation of the other rankings relies on either the WED or WHD measure. The weighting based on entropy and the distance-based evaluation makes the result objective, consistent, and effective in the process of making decisions under uncertainty.

Algorithm 1 Entropy-Based Objective Weight Determination under CIMFSS

Require: Universal set of alternatives $\mathcal{U} = \{\beta_1, \beta_2, \ldots, \beta_n\}$;
Set of parameters $\mathcal{A} = \{e_1, e_2, \ldots, e_m\}$;
CIMFSS mapping $\mathcal{F} : \mathcal{A} \to CIM^kFS(C^{\mathcal{U}})$;
CIMFSS decision matrix $\mathcal{D} = \left[\mathcal{F}(e_j)(\beta_i)\right]_{n\times m}$.

Ensure: Entropy-based parameter weight vector $W = \{w_1, w_2, \ldots, w_m\}$.

1: Compute score matrix $S = [S_{ij}]_{n\times m}$ using the CIMFSS score function (Definition 1).
(**Note:** Since the CIMFSS score function yields non-negative real values, the normalization is well-defined.)
2: Normalize the score matrix: $p_{ij} = \frac{S_{ij}}{\sum_{i=1}^{n} S_{ij}}$
3: Compute entropy values: $E_j = -\frac{1}{\ln n}\sum_{i=1}^{n} p_{ij}\ln(p_{ij}), \quad j = 1, 2, \ldots, m$
4: Determine divergence degrees: $d_j = 1 - E_j$
5: Compute entropy-based weights: $w_j = \frac{d_j}{\sum_{j=1}^{m} d_j}$
6: **return** W

Algorithm 2 Unified CIMFSS Distance-Based Decision-Making Algorithm

Require: $\mathcal{U} = \{\mathcal{U}_1, \ldots, \mathcal{U}_n\}$ (Alternatives);
$\mathcal{A} = \{e_1, \ldots, e_m\}$ (Parameters);
`CIMFSS_Info_Table` (CIMFSS information system);
`OpType` $\in \{$`WAA`, `WGA`$\}$;
`DistType` $\in \{$`WED`, `WHD`$\}$.
Ensure: Ranked list of alternatives.

Step 1: Construct Ideal CIMFSS Matrix
`Ideal_Matrix` $\leftarrow$ `Construct_Ideal_CIMFSS_Matrix`(`CIMFSS_Info_Table`) (**Note:** The ideal CIMFSS matrix assigns the best possible membership and non-membership degrees for all parameters.)
Step 2: Construct CIMFSS Decision Matrix
`Decision_Matrix` $\leftarrow$ `Construct_CIMFSS_Matrix`(`CIMFSS_Info_Table`)
Step 3: Determine Objective Parameter Weights
Compute entropy-based weight vector $\omega = \{\omega_1, \omega_2, \ldots, \omega_m\}$ using Algorithm 1.
Step 4: Aggregate Expert Opinions (see Operators 4.1,5)
for each parameter e_i **do**
  if `OpType` = `WAA` **then**
    Apply CIMFSWAA operator to aggregate expert opinions.
  else if `OpType` = `WGA` **then**
    Apply CIMFSWG operator to aggregate expert opinions.
  end if
end for
Step 5: Compute Weighted Distance from Ideal Solution (see Definitions 20,21)
for each alternative $\mathcal{U}_p$ **do**
  if `DistType` = `WED` **then**
    Compute weighted Euclidean distance using ω.
  else if `DistType` = `WHD` **then**
    Compute weighted Hamming distance using ω.
  end if
end for
Step 6: Ranking
Rank alternatives in ascending order of distance values.
return Ranked list of alternatives.

The graphical workflow of the proposed methodology, beginning with the raw input data and entropy-based objective weighting and including aggregation, distance assessment, and final ranking is shown in Figure 2. The framework combines CIMFWAA/CIMFWGA operators with distances of WED and WHD so as to have robust and uncertainty-conscious decision-making.

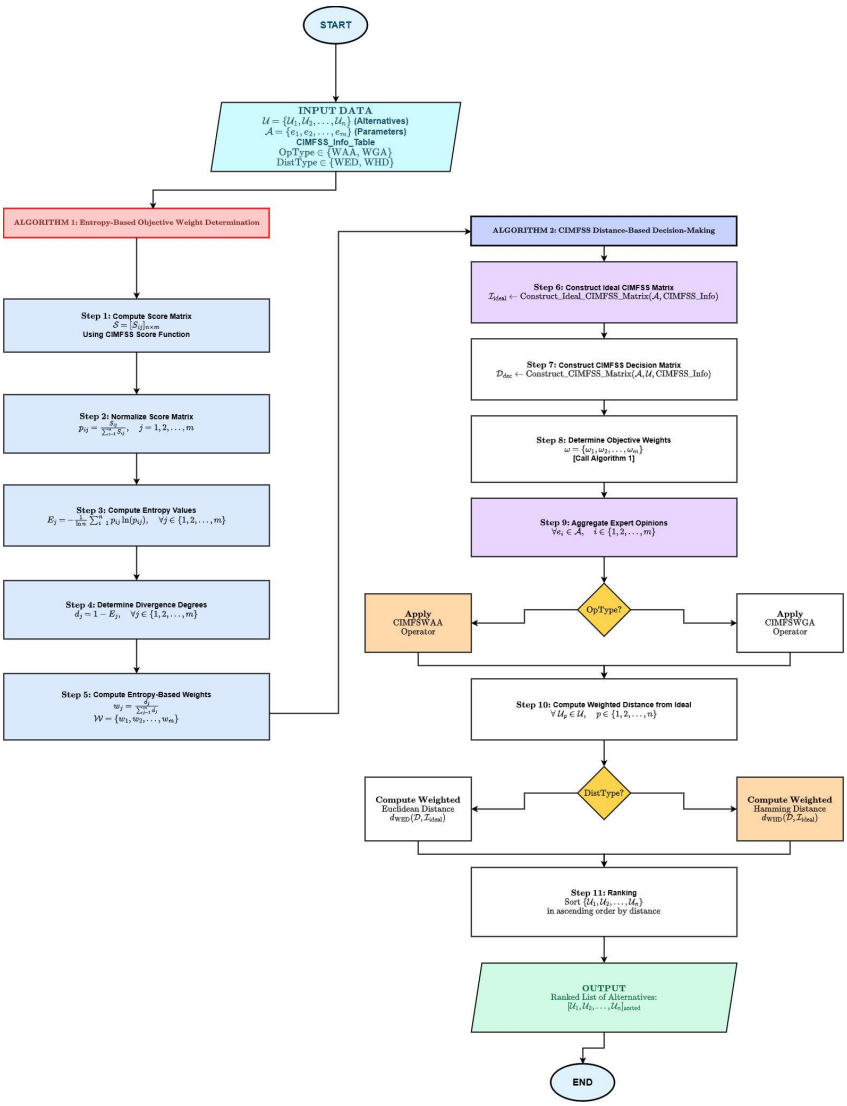


Fig. 2: Computational flow of the proposed CIMFSS-based decision-making framework.

7 Case Study: Uncertainty-Aware Air Quality Evaluation Using CIMFSS

The growth of cities, the increase in industrial production, and the growth of car emissions have all contributed to the growth of the air pollution level in most areas of the world, which is why the evaluation of air quality becomes one of the most important elements of environmental management. Monitoring networks are regularly used

by cities to monitor the level of pollutants, issue citizen health warnings and assist in the regulation of the environment. Nevertheless, the particularities of air-quality assessment that include changing meteorological conditions, non-homogeneous sources of emissions, spatial heterogeneity, and incomplete or unequal sensor measurements make this process inherently difficult. As a result, the correct determination of the pollution rates in different cities is now a multi-criteria and high-uncertainty problem, particularly when several types of pollutants and multi-sensory readings have to be considered at the same time.

Conventional modelling methods are typically not able to handle interval-valued uncertainties, sensor conflicts, expert differences, and so on.parameters of multi-dimensional pollution. Moreover, traditional decision support systems commonly assume accurate and independent pollutant readings, an assumption rarely true in practical monitoring systems due to factors like equipment calibration, spatial position, weather, and sampling rate. The above constraints prompt the need to have a mathematically sound and uncertainty-sensitive modelling framework capable of incorporating multi-source information in a coherent and understandable way.

In order to overcome these issues, this paper presents a CIMFSS-based decision-support model in the review of air quality in various cities. The offered framework (i) introduces the uncertainty, hesitation, and variability of pollutant measurements explicitly; (ii) integrates measurements of pollutants among k independent monitoring stations in a single city; (iii) allows a hierarchical structure of parameters to cover several pollutants; and (iv) results in readings that can be interpreted and reproduced to serve comparative evaluation of the environment.

A group of cities is present in this case study

$$\mathcal{C} = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4\}$$

is measured with the help of six major air-quality parameters indicating the multi-dimensionality of urban air pollution. These parameters and their relevance to the environment are highlighted in Table 3. All these parameters represent the unique health or ecological risk and their combined consideration is a prerequisite of a comprehensive evaluation of air quality.

Table 3: Primary air-quality parameters and their description.

Parameter	**Symbol**	**Description**
Fine Particulate Matter	PM2.5	Concentration of particles smaller than 2.5 µm; major contributor to respiratory diseases.
Coarse Particulate Matter	PM10	Larger airborne particles influencing visibility and public health.
Nitrogen Dioxide	NO_2	Indicator of traffic-related pollution and combustion processes.
Ozone	O_3	Secondary pollutant formed by photochemical reactions; harmful to lungs.
Sulfur Dioxide	SO_2	Pollutant produced from industrial combustion and fossil fuels.
Carbon Monoxide	CO	Gas arising from incomplete combustion; affects cardiovascular health.

To have viable and spatially representative assessments, a multi-source monitoring setup is fitted in every city, comprising four stations

$$\mathcal{S} = \{\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4\},$$

as detailed in Table 4. These stations are different urban environments, such as traffic congested areas, residential areas, industrial areas, and rather clean peri-urban areas.

Table 4: Sensor configuration used for multi-source air-quality monitoring.

Station	Location Type	Measurement Context
$\mathcal{S}_1$	Urban Center	High-density traffic and commercial emissions.
$\mathcal{S}_2$	Residential Zone	Population exposure and domestic burning activities.
$\mathcal{S}_3$	Industrial Fringe	Manufacturing, power plants, and heavy industry.
$\mathcal{S}_4$	Green/Peri-Urban Belt	Baseline background pollution levels.

In the CIMFSS model, pollution data for every city is structured as an information system. The air-quality observations of some monitoring station $\mathcal{S}_k \in \mathcal{S}$, and parameter $\mathcal{P}_j \in \mathcal{P}$, in some city $\mathcal{C}_i \in \mathcal{C}$ are cubic intuitionistic fuzzy values of the form

$$\left([\mu_{ijk}^L, \mu_{ijk}^U], [\nu_{ijk}^L, \nu_{ijk}^U], \mu_{ijk}, \nu_{ijk}\right),$$

when interval-valued membership and non-membership elements describe measurement uncertainty and variability, the intuitionistic crisp values represent the general pollution level based upon sensor values and can be deduced.
Figure 3 represents the entire application process of the suggested solution, beginning with the development of the multi-city air-quality problem and the determination of the locations of uncertainties up to the systematized data collection in cities, stations, and pollutants. The information obtained is represented by CIMFSSs and integrated to come up with city-level quality scores. Lastly, comparative rankings are created to enable sound policy-based decision-making in complex and uncertain settings.

8 Implementation of the CIMFSS-based Algorithms

This section includes systematic and algorithm-consistent implementation of the proposed CIMFSS framework. Its main goal is to transform the real-world air quality into a mathematically organized decision environment that will be transportable to further distance-based analysis. In order to guarantee the methodological transparency and reproducibility, the implementation is carried out in strict accordance with the given stepwise algorithmic sequence. The process will be initiated by the development of the CIMFSS information system, and subsequently, an ideal CIMFSS reference matrix will be developed. The two steps determine the basic decision framework through identification of the evaluation space and benchmark solution, respectively. The later aggregation, distance calculation and ranking processes are based on this basic representation.
The algorithmic steps are followed meticulously based on the predefined CIMFSS

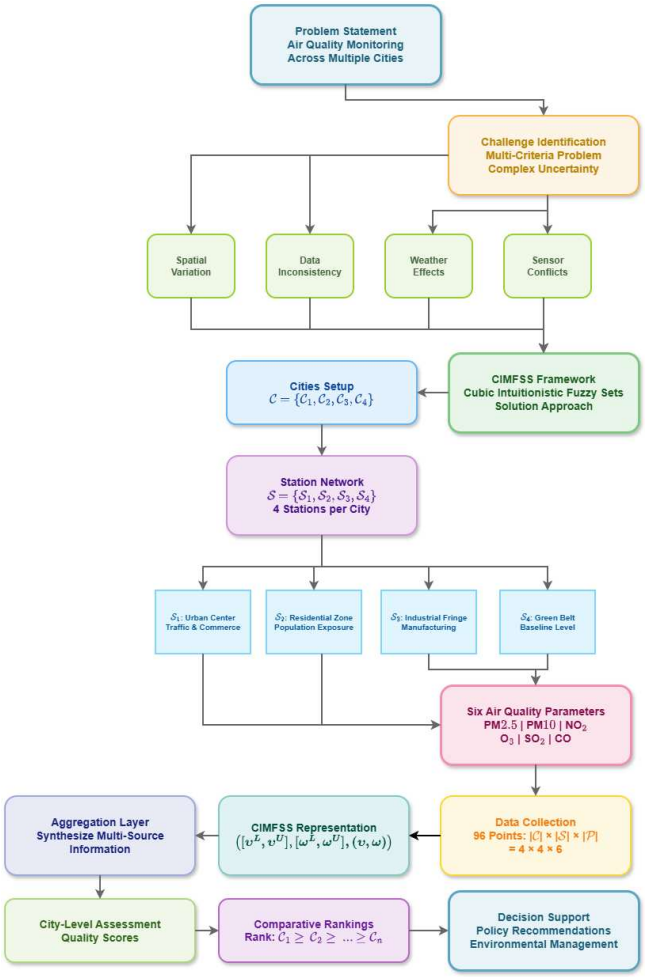


Fig. 3: Application-oriented flowchart of the proposed CIMFSS-based framework

datasets and weighted in-built to provide methodological consistency. Greater membership levels in the air-quality case study suggest more conformity to acceptable air-quality standards and not higher pollution levels. Therefore, CIMFSS modeling is preceded by the interpretation of raw pollutant concentrations in air-quality acceptability.

The step 1 involves modeling air-quality observations with CIMFSS representations. The interval-valued membership and non-membership degrees are used to reflect uncertainty due to imprecision of measurements, temporal variances and spatial heterogeneity at the monitoring stations. The accompanying intuitive crisp intuition is a reflection of the concerning trend of dominance of pollution based on the observed data. With this formulation, a CIMFSS information system is developed in every city. Tables 5–8 shows the full CIMFSS matrices of City $\mathcal{C}_1$–$\mathcal{C}_4$. Each table has the same

structure format in the sense that it presents the same form of information and therefore provides a standardized platform against which comparative analysis should be done in the next decision-making process.

Table 5: CIMFSS dataset for City $\mathcal{C}_1$ across all air-quality parameters and monitoring stations.

Parameter	Station	$\left([\upsilon^L, \upsilon^U], [\omega^L, \omega^U], (\upsilon, \omega)\right)$
$PM_{2.5}$	$\mathcal{S}_1$	$([0.65, 0.78], [0.12, 0.22], (0.72, 0.18))$
	$\mathcal{S}_2$	$([0.52, 0.66], [0.20, 0.30], (0.60, 0.25))$
	$\mathcal{S}_3$	$([0.70, 0.85], [0.08, 0.18], (0.80, 0.12))$
	$\mathcal{S}_4$	$([0.35, 0.48], [0.35, 0.45], (0.42, 0.40))$
PM_{10}	$\mathcal{S}_1$	$([0.60, 0.72], [0.15, 0.25], (0.68, 0.20))$
	$\mathcal{S}_2$	$([0.50, 0.62], [0.22, 0.32], (0.58, 0.27))$
	$\mathcal{S}_3$	$([0.68, 0.80], [0.10, 0.20], (0.75, 0.15))$
	$\mathcal{S}_4$	$([0.38, 0.50], [0.32, 0.42], (0.45, 0.38))$
NO_2	$\mathcal{S}_1$	$([0.58, 0.70], [0.18, 0.28], (0.65, 0.22))$
	$\mathcal{S}_2$	$([0.45, 0.60], [0.25, 0.35], (0.52, 0.30))$
	$\mathcal{S}_3$	$([0.62, 0.75], [0.15, 0.25], (0.70, 0.18))$
	$\mathcal{S}_4$	$([0.30, 0.45], [0.40, 0.50], (0.38, 0.45))$
O_3	$\mathcal{S}_1$	$([0.48, 0.60], [0.25, 0.35], (0.55, 0.30))$
	$\mathcal{S}_2$	$([0.42, 0.55], [0.30, 0.40], (0.48, 0.35))$
	$\mathcal{S}_3$	$([0.55, 0.68], [0.20, 0.30], (0.62, 0.25))$
	$\mathcal{S}_4$	$([0.35, 0.48], [0.35, 0.45], (0.42, 0.40))$
SO_2	$\mathcal{S}_1$	$([0.40, 0.55], [0.30, 0.40], (0.48, 0.35))$
	$\mathcal{S}_2$	$([0.35, 0.50], [0.35, 0.45], (0.42, 0.40))$
	$\mathcal{S}_3$	$([0.50, 0.65], [0.20, 0.30], (0.58, 0.25))$
	$\mathcal{S}_4$	$([0.28, 0.40], [0.45, 0.55], (0.35, 0.50))$
CO	$\mathcal{S}_1$	$([0.35, 0.50], [0.35, 0.45], (0.42, 0.40))$
	$\mathcal{S}_2$	$([0.30, 0.45], [0.40, 0.50], (0.38, 0.45))$
	$\mathcal{S}_3$	$([0.45, 0.60], [0.25, 0.35], (0.52, 0.30))$
	$\mathcal{S}_4$	$([0.25, 0.38], [0.45, 0.55], (0.32, 0.50))$

After the CIMFSS information system has been implemented, Step 2, an optimal CIMFSS matrix, is determined to act as a reference solution. This is a matrix that is the theoretically optimum air-quality scenario used to evaluate all alternatives. The optimal value of the CIMFSS is set to $([1, 1], [0, 0], (1, 0))$ in all parameters and monitoring stations.

Membership degree 1 and non-membership degree 0 are not assigned randomly. In the intuitionistic and cubic fuzzy theory, this arrangement is the full satisfaction of the desired state and the full rejection of the undesired state. Any deviation from these extreme values would mean some partial pollution or uncertainty, which is contrary to the conceptual role of an ideal reference point. The pair $(1, 0)$ therefore makes the optimal CIMFSS matrix an absolute benchmark that is most desirable and ambiguous. The obtained optimal CIMFSS matrix is presented in Table 9. This matrix will not change during the analysis but will act as the reference point afterwards where all the cities' CIMFSS evaluations will be compared in the distance-based assessments.

Table 6: CIMFSS dataset for City $\mathcal{C}_2$ across all air-quality parameters and monitoring stations.

Parameter	Station	$\big([\upsilon^L, \upsilon^U], [\omega^L, \omega^U], (\upsilon, \omega)\big)$
$PM_{2.5}$	$\mathcal{S}_1$	$([0.55, 0.68], [0.18, 0.28], (0.62, 0.22))$
	$\mathcal{S}_2$	$([0.48, 0.60], [0.22, 0.32], (0.55, 0.27))$
	$\mathcal{S}_3$	$([0.60, 0.75], [0.15, 0.25], (0.68, 0.20))$
	$\mathcal{S}_4$	$([0.32, 0.45], [0.40, 0.50], (0.38, 0.45))$
PM_{10}	$\mathcal{S}_1$	$([0.52, 0.65], [0.20, 0.30], (0.58, 0.25))$
	$\mathcal{S}_2$	$([0.45, 0.58], [0.25, 0.35], (0.52, 0.30))$
	$\mathcal{S}_3$	$([0.62, 0.75], [0.15, 0.25], (0.68, 0.20))$
	$\mathcal{S}_4$	$([0.35, 0.48], [0.35, 0.45], (0.42, 0.40))$
NO_2	$\mathcal{S}_1$	$([0.50, 0.62], [0.22, 0.32], (0.56, 0.27))$
	$\mathcal{S}_2$	$([0.42, 0.55], [0.30, 0.40], (0.48, 0.35))$
	$\mathcal{S}_3$	$([0.58, 0.70], [0.18, 0.28], (0.65, 0.22))$
	$\mathcal{S}_4$	$([0.30, 0.42], [0.42, 0.52], (0.36, 0.47))$
O_3	$\mathcal{S}_1$	$([0.45, 0.58], [0.25, 0.35], (0.52, 0.30))$
	$\mathcal{S}_2$	$([0.38, 0.50], [0.32, 0.42], (0.45, 0.38))$
	$\mathcal{S}_3$	$([0.55, 0.68], [0.20, 0.30], (0.62, 0.25))$
	$\mathcal{S}_4$	$([0.32, 0.45], [0.38, 0.48], (0.38, 0.43))$
SO_2	$\mathcal{S}_1$	$([0.38, 0.50], [0.32, 0.42], (0.45, 0.38))$
	$\mathcal{S}_2$	$([0.32, 0.45], [0.38, 0.48], (0.38, 0.43))$
	$\mathcal{S}_3$	$([0.48, 0.62], [0.22, 0.32], (0.55, 0.27))$
	$\mathcal{S}_4$	$([0.25, 0.38], [0.45, 0.55], (0.32, 0.50))$
CO	$\mathcal{S}_1$	$([0.32, 0.45], [0.38, 0.48], (0.38, 0.43))$
	$\mathcal{S}_2$	$([0.28, 0.40], [0.42, 0.52], (0.34, 0.47))$
	$\mathcal{S}_3$	$([0.42, 0.55], [0.30, 0.40], (0.48, 0.35))$
	$\mathcal{S}_4$	$([0.22, 0.35], [0.48, 0.58], (0.28, 0.52))$

8.1 Implementation of the CIMFSS distance-based decision-making using the CIMFWAA operator

The suggested distance-based decision-making process, based on the ideal CIMFSS matrix developed, is carried out with the help of the CIMFWAA aggregation operator in the presence of the CIMFSS information system. In this subsection, Steps 3-6 of the algorithm are presented in a linear and organized way.

In the implementation phase, the score function is applied with the sole purpose of obtaining objective weights by using entropy. The aggregated CIMFSS values are then evaluated and ranked directly by using distance measures, and in doing this, the multidimensional fuzzy information is not lost in the final evaluation and ranking of the alternatives.

Moreover, the entropy-based weighting process reflects the relative discriminating ability of parameters among the alternatives, and the CIMFWAA aggregation operator does information fusion in the structure of CIMFSS of each parameter; these two procedures have different positions in the implementation line and are not interchangeable.

In Step 3, the score function specified on CIMFSSs is used to convert CIMFSS evaluations into a real-valued score matrix. This change allows a quantitative comparison of alternatives and also does not lose the underlying intuitionistic and interval-valued properties. The final score table for all the cities in relation to the discussed air-quality

Table 7: CIMFSS dataset for City $\mathcal{C}_3$ across all air-quality parameters and monitoring stations.

Parameter	Station	$\left([\upsilon^L, \upsilon^U], [\omega^L, \omega^U], (\upsilon, \omega)\right)$
$PM_{2.5}$	$\mathcal{S}_1$	$([0.45, 0.58], [0.25, 0.35], (0.52, 0.30))$
	$\mathcal{S}_2$	$([0.40, 0.52], [0.30, 0.40], (0.46, 0.35))$
	$\mathcal{S}_3$	$([0.55, 0.68], [0.20, 0.30], (0.62, 0.25))$
	$\mathcal{S}_4$	$([0.28, 0.40], [0.45, 0.55], (0.34, 0.50))$
PM_{10}	$\mathcal{S}_1$	$([0.42, 0.55], [0.28, 0.38], (0.48, 0.33))$
	$\mathcal{S}_2$	$([0.38, 0.50], [0.32, 0.42], (0.45, 0.38))$
	$\mathcal{S}_3$	$([0.55, 0.68], [0.20, 0.30], (0.62, 0.25))$
	$\mathcal{S}_4$	$([0.30, 0.42], [0.42, 0.52], (0.36, 0.47))$
NO_2	$\mathcal{S}_1$	$([0.40, 0.52], [0.30, 0.40], (0.46, 0.35))$
	$\mathcal{S}_2$	$([0.35, 0.48], [0.35, 0.45], (0.42, 0.40))$
	$\mathcal{S}_3$	$([0.50, 0.62], [0.22, 0.32], (0.56, 0.27))$
	$\mathcal{S}_4$	$([0.28, 0.40], [0.45, 0.55], (0.34, 0.50))$
O_3	$\mathcal{S}_1$	$([0.38, 0.50], [0.32, 0.42], (0.45, 0.38))$
	$\mathcal{S}_2$	$([0.32, 0.45], [0.38, 0.48], (0.38, 0.43))$
	$\mathcal{S}_3$	$([0.48, 0.60], [0.25, 0.35], (0.55, 0.30))$
	$\mathcal{S}_4$	$([0.25, 0.38], [0.45, 0.55], (0.32, 0.50))$
SO_2	$\mathcal{S}_1$	$([0.32, 0.45], [0.38, 0.48], (0.38, 0.43))$
	$\mathcal{S}_2$	$([0.28, 0.40], [0.42, 0.52], (0.34, 0.47))$
	$\mathcal{S}_3$	$([0.45, 0.58], [0.25, 0.35], (0.52, 0.30))$
	$\mathcal{S}_4$	$([0.22, 0.35], [0.48, 0.58], (0.28, 0.52))$
CO	$\mathcal{S}_1$	$([0.28, 0.40], [0.42, 0.52], (0.34, 0.47))$
	$\mathcal{S}_2$	$([0.25, 0.38], [0.45, 0.55], (0.32, 0.50))$
	$\mathcal{S}_3$	$([0.40, 0.52], [0.30, 0.40], (0.46, 0.35))$
	$\mathcal{S}_4$	$([0.20, 0.32], [0.50, 0.60], (0.26, 0.55))$

parameters is presented in Table 10. This table displays each city's pollution level compared to other cities for each parameter.

A weighted score of the objective criteria is calculated on the basis of the entropy measure based on the received score matrix. Entropy is a measure of the amount of variation of information on each criterion among alternatives. Criteria that are more variable add more information to the decision process and thus bigger weights are attributed to them. Table 11 shows the computed values of entropy, associated divergence values, and normalized weights based on the entropy.

During Step 4, the CIMFWAA operator will be used to combine the CIMFSS assessment of the various monitoring stations within each city. This operator amalgamates interval-valued membership degrees, non-membership degrees, and crisp intuitionistic components into one representative CIMFSS value of each parameter integrated with the entropy-based weights.
Table 9 reported the ideal decision matrix. This equation is the reference solution that is used in the distance computation step. On the same note, Tables 12–15 below provides the aggregated CIMFSS decision matrices of Cities $\mathcal{C}_1$–$\mathcal{C}_4$, obtained with the CIMFWAA operator, respectively. They are matrices that represent the multi-station air-quality data of every city in a single CIMFSS representation.

At Step 5, the distance between the aggregated CIMFSS decision matrix of each city and the aggregated ideal CIMFSS matrix is calculated. Both WED and WHD measures are used to make the evaluation robust enough. These measures of distance

Table 8: CIMFSS dataset for City $\mathcal{C}_4$ across all air-quality parameters and monitoring stations.

Parameter	Station	$\left([\upsilon^L, \upsilon^U], [\omega^L, \omega^U], (\upsilon, \omega)\right)$
$PM_{2.5}$	$\mathcal{S}_1$	$([0.35, 0.48], [0.35, 0.45], (0.42, 0.40))$
	$\mathcal{S}_2$	$([0.30, 0.42], [0.42, 0.52], (0.36, 0.47))$
	$\mathcal{S}_3$	$([0.45, 0.58], [0.25, 0.35], (0.52, 0.30))$
	$\mathcal{S}_4$	$([0.22, 0.35], [0.48, 0.58], (0.28, 0.52))$
PM_{10}	$\mathcal{S}_1$	$([0.32, 0.45], [0.38, 0.48], (0.38, 0.43))$
	$\mathcal{S}_2$	$([0.28, 0.40], [0.42, 0.52], (0.34, 0.47))$
	$\mathcal{S}_3$	$([0.45, 0.58], [0.25, 0.35], (0.52, 0.30))$
	$\mathcal{S}_4$	$([0.25, 0.38], [0.45, 0.55], (0.32, 0.50))$
NO_2	$\mathcal{S}_1$	$([0.30, 0.42], [0.42, 0.52], (0.36, 0.47))$
	$\mathcal{S}_2$	$([0.25, 0.38], [0.45, 0.55], (0.32, 0.50))$
	$\mathcal{S}_3$	$([0.42, 0.55], [0.28, 0.38], (0.48, 0.33))$
	$\mathcal{S}_4$	$([0.20, 0.32], [0.50, 0.60], (0.26, 0.55))$
O_3	$\mathcal{S}_1$	$([0.28, 0.40], [0.42, 0.52], (0.34, 0.47))$
	$\mathcal{S}_2$	$([0.25, 0.38], [0.45, 0.55], (0.32, 0.50))$
	$\mathcal{S}_3$	$([0.40, 0.52], [0.30, 0.40], (0.46, 0.35))$
	$\mathcal{S}_4$	$([0.20, 0.32], [0.50, 0.60], (0.26, 0.55))$
SO_2	$\mathcal{S}_1$	$([0.25, 0.38], [0.45, 0.55], (0.32, 0.50))$
	$\mathcal{S}_2$	$([0.22, 0.35], [0.48, 0.58], (0.28, 0.52))$
	$\mathcal{S}_3$	$([0.38, 0.50], [0.32, 0.42], (0.45, 0.38))$
	$\mathcal{S}_4$	$([0.18, 0.30], [0.52, 0.62], (0.24, 0.57))$
CO	$\mathcal{S}_1$	$([0.22, 0.35], [0.48, 0.58], (0.28, 0.52))$
	$\mathcal{S}_2$	$([0.20, 0.32], [0.50, 0.60], (0.26, 0.55))$
	$\mathcal{S}_3$	$([0.35, 0.48], [0.35, 0.45], (0.42, 0.40))$
	$\mathcal{S}_4$	$([0.15, 0.28], [0.55, 0.65], (0.20, 0.60))$

Table 9: Ideal CIMFSS dataset representing optimal air quality conditions.

Parameter	Station(s)	$\left([\upsilon^L, \upsilon^U], [\omega^L, \omega^U], (\upsilon, \omega)\right)$
$PM_{2.5}$, PM_{10}, NO_2, O_3, SO_2, CO	$\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4$	$([1, 1], [0, 0], (1, 0))$

Table 10: Score matrix of alternatives with respect to criteria

Alternative	CO	NO_2	O_3	PM_{10}	$PM_{2.5}$	SO_2
$\mathcal{C}_1$	0.3308	0.5992	0.5233	0.6900	0.7233	0.4133
$\mathcal{C}_2$	0.2617	0.5158	0.4850	0.5958	0.6017	0.3633
$\mathcal{C}_3$	0.2100	0.3975	0.3542	0.4533	0.4667	0.2825
$\mathcal{C}_4$	0.1067	0.2258	0.2100	0.2975	0.3033	0.1617

are used to evaluate the degree to which every city is out of the desired air-quality state in the weighted CIMFSS environment. Lastly, during Step 6, alternatives are ranked based on a distance value in an ascending manner. A smaller distance means proximity to the optimal CIMFSS and, thus, a better overall performance of air quality. Table 16 presents the value of the distance calculated and the rankings derived by WED and WHD. The reliability and consistency of the proposed CIMFSS distance-based

Table 11: Entropy values, divergence measures, and entropy-based weights of criteria

Criterion	Entropy	Divergence	Weight
CO	0.9494	0.0506	0.2173
NO_2	0.9595	0.0405	0.1740
O_3	0.9620	0.0380	0.1635
PM_{10}	0.9678	0.0322	0.1383
$PM_{2.5}$	0.9664	0.0336	0.1442
SO_2	0.9621	0.0379	0.1628

Table 12: Aggregated decision matrix for city $\mathcal{C}_1$ obtained using the CIMFWAA operator

Parameter	$\left([\upsilon^L, \upsilon^U], [\omega^L, \omega^U], (\upsilon, \omega)\right)$
CO	$([0.3418, 0.4890], [0.3543, 0.4562], (0.4147, 0.4054))$
NO_2	$([0.5021, 0.6416], [0.2280, 0.3327], (0.5796, 0.2704))$
O_3	$([0.4550, 0.5840], [0.2692, 0.3708], (0.5235, 0.3201))$
PM_{10}	$([0.5537, 0.6788], [0.1803, 0.2863], (0.6313, 0.2355))$
$PM_{2.5}$	$([0.5746, 0.7236], [0.1610, 0.2704], (0.6624, 0.2156))$
SO_2	$([0.3879, 0.5338], [0.3118, 0.4151], (0.4643, 0.3637))$

Table 13: Aggregated decision matrix for city $\mathcal{C}_2$ obtained using the CIMFWAA operator

Parameter	$\left([\upsilon^L, \upsilon^U], [\omega^L, \omega^U], (\upsilon, \omega)\right)$
CO	$([0.3140, 0.4426], [0.3894, 0.4905], (0.3744, 0.4379))$
NO_2	$([0.4596, 0.5847], [0.2658, 0.3695], (0.5242, 0.3144))$
O_3	$([0.4316, 0.5615], [0.2792, 0.3814], (0.5006, 0.3327))$
PM_{10}	$([0.4947, 0.6282], [0.2264, 0.3297], (0.5602, 0.2783))$
$PM_{2.5}$	$([0.4977, 0.6358], [0.2208, 0.3253], (0.5708, 0.2704))$
SO_2	$([0.3632, 0.4955], [0.3312, 0.4340], (0.4316, 0.3854))$

Table 14: Aggregated decision matrix for city $\mathcal{C}_3$ obtained using the CIMFWAA operator

Parameter	$\left([\upsilon^L, \upsilon^U], [\omega^L, \omega^U], (\upsilon, \omega)\right)$
CO	$([0.2865, 0.4097], [0.4103, 0.5119], (0.3492, 0.4612))$
NO_2	$([0.3879, 0.5116], [0.3193, 0.4219], (0.4508, 0.3708))$
O_3	$([0.3632, 0.4890], [0.3420, 0.4438], (0.4316, 0.3957))$
PM_{10}	$([0.4199, 0.5479], [0.2945, 0.3972], (0.4865, 0.3484))$
$PM_{2.5}$	$([0.4282, 0.5564], [0.2866, 0.3899], (0.4951, 0.3385))$
SO_2	$([0.3230, 0.4521], [0.3720, 0.4744], (0.3868, 0.4214))$

decision-making framework under the CIMFWAA aggregation scheme is confirmed by the consistency of the ranking results obtained by the two distance measures.

Table 15: Aggregated decision matrix for city $\mathcal{C}_4$ obtained using the CIMFWAA operator

Parameter	$\left([\upsilon^L, \upsilon^U], [\omega^L, \omega^U], (\upsilon, \omega)\right)$
CO	$([0.2337, 0.3622], [0.4636, 0.5648], (0.2949, 0.5119))$
NO_2	$([0.2975, 0.4240], [0.4033, 0.5053], (0.3603, 0.4545))$
O_3	$([0.2865, 0.4097], [0.4103, 0.5119], (0.3492, 0.4612))$
PM_{10}	$([0.3296, 0.4586], [0.3661, 0.4682], (0.3955, 0.4173))$
$PM_{2.5}$	$([0.3353, 0.4643], [0.3644, 0.4669], (0.4015, 0.4138))$
SO_2	$([0.2615, 0.3872], [0.4354, 0.5369], (0.3274, 0.4871))$

Table 16: Distance measures and ranking of alternatives

Alternative	WED	Rank	WHD	Rank
$\mathcal{C}_1$	0.3993	1	0.3807	1
$\mathcal{C}_2$	0.4379	2	0.4220	2
$\mathcal{C}_3$	0.4913	3	0.4785	3
$\mathcal{C}_4$	0.5628	4	0.5518	4

8.2 Implementation of the CIMFSS distance-based decision-making using the CIMFAGA operator

The weights of the parameters at this stage of the decision-making process have been done and included by the use of CIMFWAAO as has been noted in the previous subsection, Table 1. Hence, there is no further weight elicitation and normalization done here. This subsection is solely focused on aggregation and distance-based evaluation with the help of the already established weighted structure.

The decision process follows the aggregation and distance-based assessment once the CIMFSS information systems have been constructed and the optimal CIMFSS matrix has been developed in Table 9. It is realized in terms of strict adherence to the algorithm steps, pre-defined CIMFSS datasets and estimated weights to guarantee methodological consistency.

At this point, the CIMFWGA operator will be used to combine the CIMFSS assessments of several monitoring stations of the individual parameters within a city. The operator sums the intervals valued membership degrees, the intervals valued non-membership degrees and the corresponding crisp intuitionistic pairs into one representative cubic intuitionistic multi-fuzzy value of each parameter. Such an aggregation maintains the natural uncertain nature of the data but generates a small and understandable decision table at the city level. Under this process, aggregated CIMFSS decision matrices are derived individually for Cities $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, and $\mathcal{C}_4$. The resulting aggregated matrices are reported in Tables 12–15, all having a similar structural format in order to easily compare alternatives with each other.

Based on algorithm steps 5 and 6, once the aggregated decision matrices are determined, weighted distance measures are used to assess the differences between these matrices and the ideal CIMFSS matrix. Both WED and WHD are used in this research. These scales assess the similarity of the aggregate CIMFSS matrix of each

Table 17: Aggregated decision matrix for city $\mathcal{C}_1$ obtained using the CIMFWGA operator

Parameter	$\left([\upsilon^L, \upsilon^U], [\omega^L, \omega^U], (\upsilon, \omega)\right)$
CO	$([0.3297, 0.4759], [0.3543, 0.4562], (0.4037, 0.4054))$
NO_2	$([0.4694, 0.6136], [0.2280, 0.3327], (0.5476, 0.2704))$
O_3	$([0.4438, 0.5729], [0.2692, 0.3708], (0.5120, 0.3201))$
PM_{10}	$([0.5277, 0.6500], [0.1803, 0.2863], (0.6040, 0.2355))$
$PM_{2.5}$	$([0.5364, 0.6770], [0.1610, 0.2704], (0.6172, 0.2156))$
SO_2	$([0.3742, 0.5171], [0.3118, 0.4151], (0.4498, 0.3637))$

Table 18: Aggregated decision matrix for city $\mathcal{C}_2$ obtained using the CIMFWGA operator

Parameter	$\left([\upsilon^L, \upsilon^U], [\omega^L, \omega^U], (\upsilon, \omega)\right)$
CO	$([0.3016, 0.4314], [0.3894, 0.4905], (0.3630, 0.4379))$
NO_2	$([0.4372, 0.5627], [0.2658, 0.3695], (0.5008, 0.3144))$
O_3	$([0.4165, 0.5458], [0.2792, 0.3814], (0.4846, 0.3327))$
PM_{10}	$([0.4747, 0.6070], [0.2264, 0.3297], (0.5417, 0.2783))$
$PM_{2.5}$	$([0.4745, 0.6092], [0.2208, 0.3253], (0.5448, 0.2704))$
SO_2	$([0.3476, 0.4798], [0.3312, 0.4340], (0.4165, 0.3854))$

Table 19: Aggregated decision matrix for city $\mathcal{C}_3$ obtained using the CIMFWGA operator

Parameter	$\left([\upsilon^L, \upsilon^U], [\omega^L, \omega^U], (\upsilon, \omega)\right)$
CO	$([0.2736, 0.3988], [0.4103, 0.5119], (0.3377, 0.4612))$
NO_2	$([0.3742, 0.4988], [0.3193, 0.4219], (0.4379, 0.3708))$
O_3	$([0.3476, 0.4759], [0.3420, 0.4438], (0.4165, 0.3957))$
PM_{10}	$([0.4028, 0.5294], [0.2945, 0.3972], (0.4686, 0.3484))$
$PM_{2.5}$	$([0.4080, 0.5352], [0.2866, 0.3899], (0.4739, 0.3385))$
SO_2	$([0.3069, 0.4372], [0.3720, 0.4744], (0.3703, 0.4214))$

Table 20: Aggregated decision matrix for city $\mathcal{C}_4$ obtained using the CIMFWGA operator

Parameter	$\left([\upsilon^L, \upsilon^U], [\omega^L, \omega^U], (\upsilon, \omega)\right)$
CO	$([0.2192, 0.3503], [0.4636, 0.5648], (0.2796, 0.5119))$
NO_2	$([0.2817, 0.4094], [0.4033, 0.5053], (0.3463, 0.4545))$
O_3	$([0.2736, 0.3988], [0.4103, 0.5119], (0.3377, 0.4612))$
PM_{10}	$([0.3169, 0.4463], [0.3661, 0.4682], (0.3829, 0.4173))$
$PM_{2.5}$	$([0.3193, 0.4498], [0.3644, 0.4669], (0.3852, 0.4138))$
SO_2	$([0.2477, 0.3758], [0.4354, 0.5369], (0.3136, 0.4871))$

city and the desirable reference by jointly referring to interval limits and crisp intuitionistic elements. A lower value of the distance implies that an entity is the one that is more similar to the ideal air-quality state and, thus, the one that can be characterized as having a better environmental performance. Table 21 presents the calculated WED and WHD values for each city.The alternatives are given ascending gradiently

with regard to their distances to the optimal CIMFSS, based on the values of the calculated distances. The pattern of consistent ranking is exhibited within both WED and WHD measurements, which proves the stability and the strength of the suggested model of CIMFSS-based distance decision-making. The most preferable alternative city regarding the overall air quality is the one with the minimum distance.

Table 21: Distance measures and ranking of alternatives

Alternative	WED	Rank	WHD	Rank
$\mathcal{C}_1$	0.4118	1	0.3924	1
$\mathcal{C}_2$	0.4483	2	0.4312	2
$\mathcal{C}_3$	0.5003	3	0.4863	3
$\mathcal{C}_4$	0.5707	4	0.5586	4

9 Results and Discussion

In this section, the theoretical results of the proposed CIMFSS-based multi-criteria decision-making framework are presented and the performance of alternatives with respect to entrepreneurship is discussed in terms of entropy-weighted aggregation and distance-based evaluation. The offered methodology combines the objective weight calculation with the help of the entropy method, the process of aggregation with CIMFWAA and CIMFWGA operators, and the ultimate ranking with the help of weighted Euclidean and weighted Hamming distance measures.
Figure 4 shows how an objective criteria weight is computed, based on the entropy approach. The difference in the values of entropy is due to the differences in the degree of information dispersion of the criterion, and this implies that not all the parameters make the same contribution to the decision-making process. Those criteria that have more variability receive comparatively more weight, thus giving them more emphasis on their ability to discriminate. Subjective bias during the weighting spike is alleviated by the entropy-based weighting scheme, which also offers a data-driven foundation for the rank of criteria in terms of their relative importance.

Following the weight calculation, the CIMFWAA and CIMFWGA operators calculate the weighted CIMFSS. In Figures 5a and 5b, the WED and WHD of the alternatives to the ideal solution when using WAA- and WGA- Aggregation schemes are given respectively. The ideal solution in this study is the alternative, whose membership degree and non-membership degree, among all the criteria, have the highest and lowest values, respectively. In both instances of aggregation, the lower magnitude of distance would imply closer closeness to the ideal solution and therefore a higher performance in general.
It is noted that the CIMFWGA operator produces slightly bigger distance values than the CIMFWAA operator. This tendency is in line with the fact that the geometric aggregation of the criteria is multiplicative and, as a result, compensatory effects between criteria are minimized, producing a more conservative judgment. Conversely,

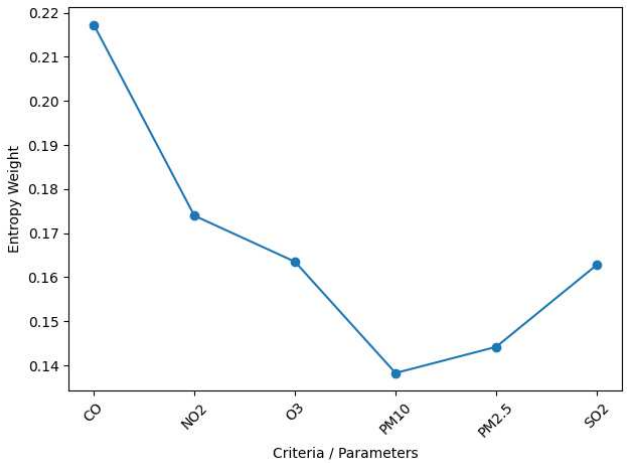


Fig. 4: Computation of objective criteria weights using the entropy approach

the arithmetic aggregation permits part compensation, and thus distance measures are relatively smaller.

The weighted Euclidean and weighted Hamming distances between each of the two aggregation operators are compared and shown in Figures 5c and 5d. Although the absolute values of the distances vary in the two aggregations of weighted averages and weighted grades, the relative positions of the alternatives are the same on both measures. This ranking consistency shows that the proposed framework is stable in terms of aggregation attitude and the choice of distance measures.

Figure 2 below 5e once again consolidates the comparison of WED and WHD measures and it can be verified that both similarity measures provide coherent and consistent patterns of evaluation. The WED records magnitude-based deviations of the ideal solution and the WHD records component-based mismatches. The correlation of these complementary measures makes the findings of the ranking more reliable.

Tables 16 and 21 present the numerical data of WED and WHD and the respective ranking of alternatives. According to both distance measures as well as both aggregation operators, the ranking order concluded is $\mathcal{C}_1 \succ \mathcal{C}_2 \succ \mathcal{C}_3 \succ \mathcal{C}_4$. Alternative $\mathcal{C}_1$ always has the closest distance to the optimal solution, which means that $\mathcal{C}_4$ is the most compliant with the reasonable air-quality standards, but it has the lowest total performance.

The comparison in rankings exemplified in Figure 5f suggests that WAA- and WGA-based appraisals are in full accord. This stability supports the fact that the suggested CIMFSS-based framework maintains the consistency of the decisions and is flexible in the modeling of compensatory and conservative aggregation behavior. Small changes in the distance values do not affect the end ranking, which means that the framework is insensitive to aggregation attitude and choice of distance measures. Therefore, the same ranking results are to be assumed as a strength confirmation but not redundancy.

The findings confirm that the suggested CIMFWAA and CIMFWGA operators, in conjunction with entropy-based objective weighting and distance-based evaluation, are effective in offering a sound, interpretable and stable decision-making framework. The process is useful in capturing multidimensional uncertainty, diversity of experts and the interaction of parameters, which is inherent in complex air-quality assessment problems and other real-world decision situations.

The **advantages** of the suggested CIMFSS-based framework are that it is able to maintain the rich cubic intuitionistic and multi-fuzzy information, as well as unite the multiple expert views. Entropy based weighting scheme is more objective, as it is based on the data dispersion instead of human judgment and the flexibility of the scheme is achieved by using both CIMFWAA and CIMFWGA operators to represent the compensating and conservative decision strategies. The high similarity of the ranking results given by the various distance measures also confirms the soundness of the framework.

The **Pros and Cons** Compared to current fuzzy soft set-based techniques, the given method has a greater level of informational grain in the course of aggregation and ranking. The explicit conceptualization of weighting, aggregation, and distance-based evaluation enhances the methodological clarity and facilitates the consistent decision results when the aggregation attitudes are different. The greater expressive power of the CIMFSS structure does come at the cost of increased computational complexity, however, which can be detrimental in large-scale decision problems. Furthermore, entropy-based weighting is sensitive to the variability of data and can underestimate criteria that have a low dispersion and a high conceptual value.

Limitations and Implications

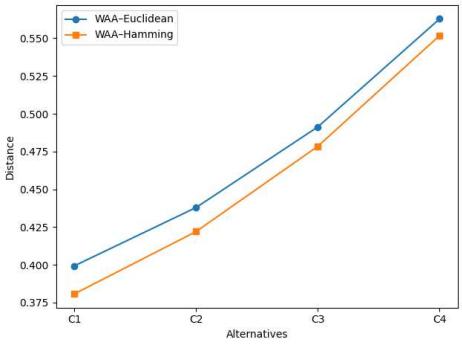
This subsection identifies the limitations of the proposed framework that are related to its methodology and explains the practical and theoretical implications of the results for applying the findings to real-life decision-making.

Limitations.

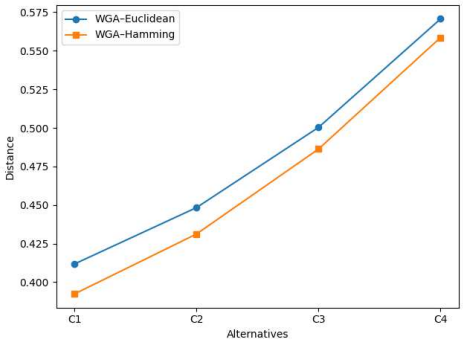
- The framework presupposes the same reliability of expert sources and does not directly involve expert credibility and confidence rates.
- The computational complexity is dependent on the number of criteria, alternatives and the opinions of the experts.
- Respect to entropy-derived weights, sensitivity analysis has not been done.
- The validation only corresponds to a case study of air-quality assessment.

Implications.

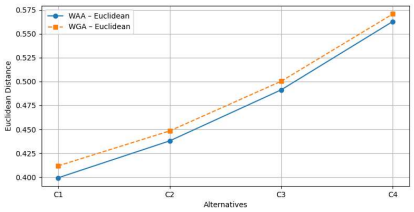
- The suggested framework is a powerful and objective decision support mechanism in environment assessment and planning of policy.
- Depending on feasible needs, decision-makers can choose compensatory CIMFWAA and conservative CIMFWGA aggregation strategies.
- The approach can be applied to other fields of application, such as sustainability measurement, disaster management, and prioritization in healthcare provision.



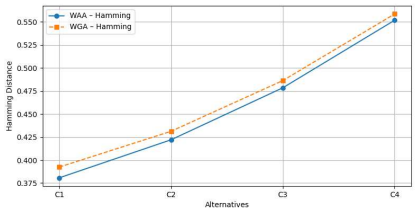
(a) WAA-based Weighted Euclidean and Hamming distances



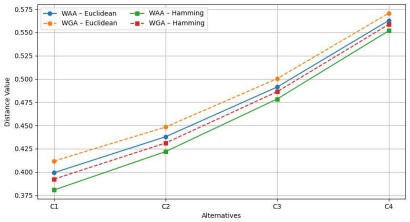
(b) WGA-based Weighted Euclidean and Hamming distances



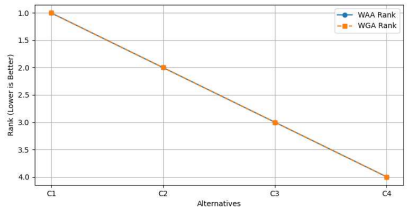
(c) Weighted Euclidean distance comparison (WAA vs WGA)



(d) Weighted Hamming distance comparison (WAA vs WGA)



(e) Combined weighted Euclidean and Hamming distances comparison



(f) Ranking comparison of alternatives

Fig. 5: Distance-based evaluation and ranking comparison of alternatives using WAA and WGA operators

- Entropy-based weighting when integrated with the CIMFSS structure is useful in making transparent, reproducible and data-driven decisions, which are needed in policy-oriented and regulatory applications.

Table 22: Structural comparative analysis of existing fuzzy soft set models and the proposed CIMFWAAO and CIMFWGAO frameworks

Feature	IFSS	IVIFSS	CIFSS	CIMFWAAO	CIMFWGAO
Information representation	Intuitionistic	Interval-valued	Cubic	Cubic IMF	Cubic IMF
Multi-membership handling	×	×	×	✓	✓
Uncertainty modeling level	Low	Moderate	High	Very high	Very high
Aggregation operator	Classical	Classical	Classical	WAA	WGA
Objective weight integration	×	×	×	✓	✓
Distance-based ranking	✓	✓	✓	✓	✓
Expert sensitivity	Limited	Limited	Moderate	High	High
Decision reliability	Moderate	Moderate	High	Very high	Very high

- The suggested framework offers the basis for future extensions, such as sensitivity analysis, expert credibility modeling, and a hybrid combination of machine learning methods to estimate adaptive weights.

10 Comparative Analysis

In this section, the proposed CIMFWAAO and CIMFWGAO frameworks are compared with the known fuzzy soft set-based models to examine their effectiveness and applicability to decision-making problems. The comparison brings out the unique differences and the benefits of the proposed approaches in practice.

10.1 Structural Comparative Analysis

Table 22 shows that there is an evident development between the traditional models of fuzzy soft sets and the new CIMFWAAO and CIMFWGAO frameworks. IFSS and IVIFSS use single or range-based intuitionistic representations, which restrict their ability to handle complicated uncertainties. CIFSS is effective in single-expert decision-making, though it lacks multi-membership capability and therefore is not as effective in multi-expert decision-making as its cubic representation attempts to make it.

The proposed frameworks overcome this shortcoming by using cubic intuitionistic multi-fuzzy information, which allows the representation of various membership degrees at the same time. This structural addition is of enormous benefit to uncertainty modeling and sensitivity to experts. Besides, the combination of objective weights, the WAA, and the WGA aggregation operators enhances the strength of a decision-making process and minimizes its subjectivity. Subsequently, the CIMFWAAO and CIMFWGAO methods become more reliable for decision-making while still maintaining distance-dependent ranking mechanisms, making them more suitable for complex multi-criteria group decision-making issues.

10.2 Numerical Comparative Analysis

A numerical comparison of the proposed decision-making operators of CIMFSS with some of the existing fuzzy soft set models based on the weighted Hamming and weighted Euclidean distance measures is provided in Table 23. The findings suggest that the ranking of alternatives in each of the distance metrics is the same in all the discussed models. This consistency indicates a high level of decision robustness and proves that the proposed CIMFSS framework does not violate preference ordering and is able to give a more complete representation of uncertainty. Though minor differences are seen in the calculated values of the scores obtained, the variation is a result of the nature of the aggregation operators and distance formulations and not the result of the alteration of the overall decision result. Specifically, the operators of CIMFWAAO and CIMFWGAO can be noted to have a consistent pattern of discrimination using both distance measures, which indicates the reliability, metric independence, and generality of the suggested CIMFSS-based method. Therefore, the comparative numerical analysis confirms the new framework to be a consistent and significant generalization of the current fuzzy soft set models instead of an alternative ranking mechanism.
To evaluate even further the stability of the offered decision-making framework, a qualitative sensitivity analysis is conducted without any further numerical experiments. The implicit sensitivity test is done using different distance measures (WED and WHD) and aggregation operators (CIMFWAAO and CIMFWGAO) in the same test environment. The fact that the same alternative rankings are maintained across these methodological differences means that the suggested CIMFSS-based method is not too sensitive to the manipulations of the parameters of the analysis. Additionally, the observed slight variations in the score values do not affect the overall preference structure, indicating that the decision results remain robust despite significant perturbations in the model formulations. Such evidence supports the view that the proposed framework yields stable and reliable results, which supports its applicability to complex multi-criteria decision-making problems that feature uncertainty and vagueness.

In Figure 6, a visual comparison of the results of both weighted Euclidean and weighted Hamming distance measures is made on the outcome of the decisions using IFSS, IVIFSS, CIFSS and the proposed CIMFSS-based operators. As can be seen in Figures 6a and 6b, though there are minor differences in the value of distance scores across the considered model, the relative ranking of alternatives is the same under both distance formulations. This graphical consistency helps to confirm the numerical results, which are provided in Table 23 and also proves the soundness of the decision results. Besides, the integrated presentation presented in Fig. 6c reveals that the results of the Euclidean and Hamming distance-based measures are highly similar, which means that the proposed CIMFSS-based operators remain stable and consistent when measured based on various distance measures. The graphical analysis supports the consistency that the proposed framework supports the consistency in decisions and supports a richer model of uncertainty, hence, supporting the reliability and applicability of the multi-criteria decision-making environment.

Table 23: Comparative numerical analysis of distance measures and ranking of alternatives under existing and proposed CIMFSS models

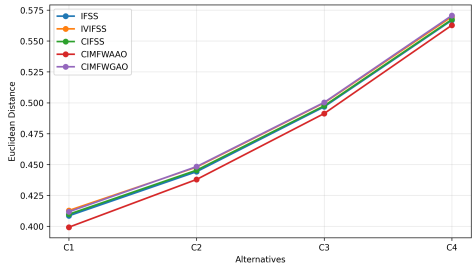
	WHD									
	IFSS		IVIFSS		CIFSS		CIMFWAAO		CIMFWGAO	
Alternative	Score	Rank	Score	Rank	Score	Rank	Score	Rank	Score	Rank
$\mathcal{C}_1$	0.3946	1	0.3946	1	0.3938	1	0.3807	1	0.3924	1
$\mathcal{C}_2$	0.4317	2	0.4317	2	0.4315	2	0.4220	2	0.4312	2
$\mathcal{C}_3$	0.4866	3	0.4866	3	0.4864	3	0.4785	3	0.4863	3
$\mathcal{C}_4$	0.5581	4	0.5581	4	0.5580	4	0.5518	4	0.5586	4
	WED									
	IFSS		IVIFSS		CIFSS		CIMFWAAO		CIMFWGAO	
Alternative	Score	Rank	Score	Rank	Score	Rank	Score	Rank	Score	Rank
$\mathcal{C}_1$	0.4085	1	0.4128	1	0.4098	1	0.3993	1	0.4118	1
$\mathcal{C}_2$	0.4442	2	0.4479	2	0.4454	2	0.4379	2	0.4483	2
$\mathcal{C}_3$	0.4968	3	0.5000	3	0.4976	3	0.4913	3	0.5003	3
$\mathcal{C}_4$	0.5668	4	0.5697	4	0.5675	4	0.5628	4	0.5707	4

11 Conclusion

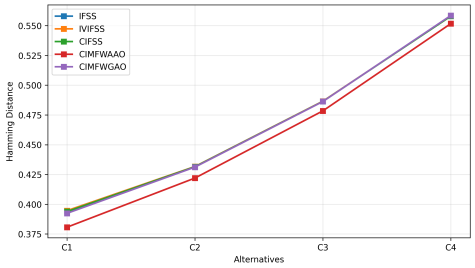
The paper tackled the issue of making decisions in a complex and uncertain environment using multi-criteria by suggesting a cubic intuitionistic multi-fuzzy soft set model. The model presented was to eliminate the structural constraints of the developed fuzzy soft set and hypersoft set methods in the description of interval-valued membership, non-membership, and hesitation data and also combine several opinions of the experts. The proposed approach is more expressive and adaptable in representing uncertainty by incorporating cubic and intuitionistic multi-fuzzy properties into a soft set model.

Two aggregation operators, the cubic intuitionistic multi-fuzzy weighted arithmetic aggregation operator and the cubic intuitionistic multi-fuzzy weighted geometric aggregation operator, were created to facilitate the use of decision-making applications and the basic mathematical properties of these two operators were rigorously proven. A systematic decision-making algorithm was developed, so the implementation of the proposed framework is systematic. The structural and numerical comparative analyses revealed that the operators proposed provide consistent ranking results in addition to providing better information preservation than those of existing fuzzy soft set-based models.

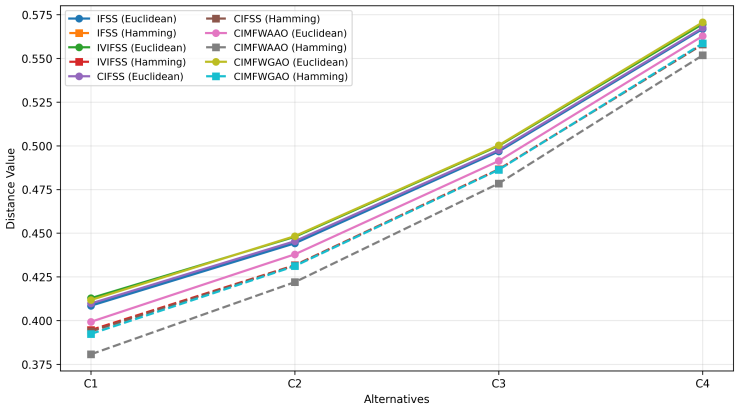
The utility and usefulness of the cubic intuitionistic multi-fuzzy soft set-based decision model were explained by applying the air-quality evaluation issue on the basis of real environmental parameters. The received findings proved the fact that the suggested strategy is able to process multi-parameter and complicated environmental data and delivers reliable and interpretable decision-related results in the conditions of uncertainty. These results have indicated that the proposed framework can be effectively applied in environmental decision-support problems where the conventional fuzzy models can be affected by the loss of information or lack of modeling potential.



(a) Weighted Euclidean distance comparison



(b) Weighted Hamming distance comparison



(c) Combined Weighted Euclidean and Hamming distance comparison of the considered models

Fig. 6: Comparative evaluation of Weighted Euclidean and Hamming distance measures across IFSS, IVIFSS, CIFSS, CIMFWAAO, and CIMFWGAO-based models

Although it is effective, there are several research directions that could be addressed by more studies. Future work may focus on:

- applying the intuitionistic multi-fuzzy soft set framework that is proposed to dynamic decision-making contexts whereby criteria weights and expert opinions change with time;
- establishing hybrid and adaptive weighting plans through the combination of objective and subjective weighting plans;
- This includes incorporating additional distance measures and indices of uncertainty to enhance the discrimination ability in complex decision-making situations.
- showing how the suggested framework can be applied to other real-life fields, including those related to disaster management, resource allocation, sustainability assessment, and risk analysis;
- This involves considering how to combine cubic intuitionistic multi-fuzzy soft sets with other multi-criteria decision-making frameworks to expand their applicability.

Statements and Declarations

The authors have no relevant financial or non-financial interests to disclose.

Funding Statement

The authors declare that no funds, grants, or other support were received during the preparation of this manuscript.

Ethical Approval

The author hereby approves that principles of ethical and professional conduct have been followed in the study.

Consent to Participate

The author agreed to participate in the research.

Consent to Publish

The author agreed to publish the manuscript.

Data Availability

The authors confirm that the data supporting the findings of this study are available within the article.

Authors Contributions

Fatima Razaq: Conceptualization, Formal Analysis, Methodology, Writing – original draft.
Muhammad Saeed: Data curation, Supervision, Validation, Writing – review & editing.

Competing Interests

The author declares no competing interests.

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