Supplemental Material:
Consistency of Dalgarno–Lewis perturbation theory for confining two-body systems

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I. DALGARNO–LEWIS FORMALISM FOR CONFINING TWO-BODY SYSTEMS

This Supplemental Material provides technical details underlying the implementation of second-order Dalgarno–Lewis (DL) perturbative corrections employed in the main text. The DL method allows wavefunction corrections to be obtained without explicit summation over intermediate eigenstates and is particularly useful for few-body bound-state problems in which the parent Hamiltonian admits analytic solutions.

The formalism is presented here for a confining two-body system described by a Cornell-type interaction. The total Hamiltonian is decomposed as

$$H = H_0 + H', \tag{1}$$

with

$$H_0 = -\frac{\nabla^2}{2\mu} + br, \qquad H' = -\frac{4\alpha_s}{3r} + c, \tag{2}$$

where μ denotes the reduced mass of the two-body system. The constant energy shift c does not contribute to normalized spatial moments and therefore does not enter the DL correction functions or any observable derived from $\langle r^n \rangle$. It is retained only for completeness.

II. PARENT AIRY-FUNCTION SOLUTION

For the linear confining parent Hamiltonian H_0, the reduced radial Schrödinger equation admits Airy-function solutions. For the S-wave ground state, the reduced radial wavefunction is written as

$$u_0(r) = \mathcal{N}\,\mathrm{Ai}(kr + \rho_{01}), \tag{3}$$

where

$$k = (2\mu b)^{1/3}, \qquad \rho_{01} = -2.3381\ldots \tag{4}$$

is the first zero of the Airy function. The normalization constant $\mathcal{N}$ is fixed by

$$\int_0^\infty |u_0(r)|^2\, dr = 1. \tag{5}$$

This Airy-function basis captures the dominant long-range confinement physics and provides a transparent analytic reference state for examining perturbative convergence in confining two-body systems.

III. FIRST-ORDER DALGARNO–LEWIS CORRECTION

The first-order correction to the radial wavefunction is written as

$$u_1(r) = f(r)\, u_0(r), \tag{6}$$

where the DL function $f(r)$ satisfies an inhomogeneous equation driven by the perturbing interaction H'. Its formal solution can be expressed as

$$f(r) = -\frac{4\alpha_s}{3}\int_0^r \frac{dr'}{u_0^2(r')}\int_0^{r'} \frac{u_0^2(r'')}{r''}\, dr''. \tag{7}$$

The integration constants are fixed by imposing the orthogonality condition

$$\langle 0|1\rangle = 0, \tag{8}$$

which eliminates any admixture of the unperturbed ground state and ensures consistency of the perturbative expansion at first order.

IV. SECOND-ORDER DALGARNO–LEWIS CORRECTION

The second-order correction is written in an analogous form,

$$u_2(r) = h(r)\, u_0(r), \tag{9}$$

with the DL function $h(r)$ given by

$$h(r) = -\frac{4\alpha_s}{3} \int_0^r \frac{dr'}{u_0^2(r')} \int_0^{r'} \frac{u_0^2(r'')}{r''}\, f(r'')\, dr''. \tag{10}$$

At second order, the orthogonality condition

$$\langle 0|2\rangle = -\frac{1}{2}\, \langle 1|1\rangle \tag{11}$$

is imposed to ensure proper normalization of the perturbed state and to guarantee consistency between first- and second-order corrections.

The fully corrected radial wavefunction up to second order is therefore written as

$$u(r) = \mathcal{N}\, \mathrm{Ai}(kr + \rho_{01}) \left[1 + f(r) + h(r) - \frac{1}{2}\, \langle 1|1\rangle\right], \tag{12}$$

with $\mathcal{N}$ chosen such that

$$\int_0^\infty |u(r)|^2\, dr = 1. \tag{13}$$

V. EVALUATION OF SPATIAL MOMENTS

The spatial moments entering the analysis are evaluated as

$$\langle r^n\rangle = \int_0^\infty |u(r)|^2 r^n\, dr, \tag{14}$$

using the normalized second-order corrected wavefunction. These moments determine observables probing the spatial structure of the two-body bound state, including

- the slope $\rho^2 = \mu^2\langle r^2\rangle$,
- the curvature $C = \mu^4\langle r^4\rangle/6$,
- the geometric root-mean-square radius r_{rms},
- the electromagnetic charge radius within the impulse approximation.

All numerical integrations are performed using sufficiently dense radial grids to ensure percent-level stability of the quoted quantities. The relation between $\langle r^2\rangle$ and the electromagnetic charge radius follows the standard impulse-approximation treatment commonly employed in few-body potential and light-front approaches.

VI. REMARKS ON MODEL DEPENDENCE

All results presented in the main text and in this Supplemental Material are obtained within a nonrelativistic potential-model framework. The overlap functions, spatial moments, and radii should therefore be interpreted as model-dependent quantities rather than as first-principles predictions. Their utility lies in providing transparent analytic benchmarks and in clarifying the role of higher-order perturbative corrections in confining two-body bound-state problems.