Supporting Information (SI) for "Geometric Unity"

Amin Al Yaquob

January 10, 2026

When SI is actually needed

For a mathematically unconventional manuscript, SI is useful when any of the following hold: (i) a key result depends on a short but nontrivial derivation that would interrupt the narrative, (ii) reproducibility requires implementation details that would clutter the main text, (iii) multiple noise/decoherence models must be laid out cleanly for falsifiability.

In this project, SI is recommended. The manuscript contains a compact main-line argument; the SI can carry the fully explicit SU(2) overlap derivation, the transport/holonomy formalization, full CHSH noise-threshold algebra, and the Gaussian-to-effective-flip mapping.

Contents

1 SU(2) geometry and the $\sin^2(\theta/2)$ law

1.1 Spinors for two measurement directions

Let $\hat{a}, \hat{b} \in S^2$ be unit vectors, separated by angle $\theta \in [0, \pi]$ so that $\hat{a} \cdot \hat{b} = \cos\theta$. Choose coordinates so that $\hat{a} = \hat{z}$ and $\hat{b}$ lies in the xz-plane with polar angle θ. In the $\hat{z}$ basis, the eigenstates of σ_z are

$$|+a\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |-a\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The eigenstates of spin along $\hat{b}$ are the eigenvectors of $\sigma_{\hat{b}} = \hat{b} \cdot \boldsymbol{\sigma}$. With $\hat{b} = (\sin\theta, 0, \cos\theta)$,

$$\sigma_{\hat{b}} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}.$$

A convenient normalized eigenbasis is

$$|+b\rangle = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix}, \qquad |-b\rangle = \begin{pmatrix} -\sin(\theta/2) \\ \cos(\theta/2) \end{pmatrix}.$$

These satisfy $\sigma_{\hat{b}}|\pm b\rangle = \pm|\pm b\rangle$ and are continuous in θ.

1.2 Overlap probabilities

The key geometric identity is the SU(2) inner-product overlap:

$$|\langle +b| - a\rangle|^2 = \sin^2(\theta/2), \qquad |\langle -b| - a\rangle|^2 = \cos^2(\theta/2).$$

Proof:

$$\langle +b| - a\rangle = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \sin(\theta/2),$$

and similarly $\langle -b| - a\rangle = \cos(\theta/2)$. Squaring gives the claim.

1.3 Singlet anti-correlation and the correlation function

For the spin singlet state, if Alice measures +1 along $\hat{a}$ then Bob's conditional state is $|-a\rangle$ (anti-correlation). Therefore,

$$P(B = +1 \mid A = +1, \hat{a}, \hat{b}) = |\langle +b| - a\rangle|^2 = \sin^2(\theta/2),$$

$$P(B = -1 \mid A = +1, \hat{a}, \hat{b}) = |\langle -b| - a\rangle|^2 = \cos^2(\theta/2).$$

Hence the expectation for outcomes ± 1 is

$$E(\hat{a}, \hat{b}) = P(B = A) - P(B \neq A) = \sin^2(\theta/2) - \cos^2(\theta/2) = -\cos\theta.$$

This derivation uses only SU(2) geometry plus the singlet constraint; the probability is an amplitude-squared.

2 Bundle, double cover, and the "Möbius twist" as the kernel element

2.1 Covering map and the meaning of $U \mapsto -U$

The double cover $\pi : \mathrm{SU}(2) \to \mathrm{SO}(3)$ can be expressed as

$$\pi(U): \; \boldsymbol{v} \mapsto \boldsymbol{v}', \qquad (\boldsymbol{v}' \cdot \boldsymbol{\sigma}) = U(\boldsymbol{v} \cdot \boldsymbol{\sigma})U^\dagger.$$

Then U and $-U$ induce the same rotation in SO(3): $\pi(U) = \pi(-U)$, because the minus sign cancels in $U(\cdot)U^\dagger$. The kernel is $\ker\pi = \{\pm\mathbb{I}\}$, and the nontrivial element $-\mathbb{I}$ is the canonical "twist parity" in the spinor bundle. This is the precise sense in which a "Möbius twist" can be modeled as $U \mapsto -U$ without invoking an improper rotation in SO(3).

2.2 Transport operator: a concrete, reproducible definition

For directions $\hat{a}, \hat{b}$, let $R(\hat{a} \to \hat{b}) \in \mathrm{SO}(3)$ be the unique minimal rotation sending $\hat{a}$ to $\hat{b}$ (axis $\propto \hat{a} \times \hat{b}$, angle θ). Let $U(\hat{a} \to \hat{b}) \in \mathrm{SU}(2)$ be any lift such that $\pi(U) = R$. Define two lifted transports

$$T_0(\hat{a}, \hat{b}) = U(\hat{a} \to \hat{b}), \qquad T_1(\hat{a}, \hat{b}) = -U(\hat{a} \to \hat{b}).$$

They project to the same SO(3) rotation but differ by the kernel element. A "topology-erasure" channel (see **??**) randomizes the parity bit selecting T_0 versus T_1.

3 CHSH: standard angles, analytic optimum, and Monte Carlo reproducibility

3.1 Optimal settings

Given $E(\hat{a}, \hat{b}) = -\cos\theta_{ab}$, the CHSH combination

$$S = E(\hat{a}, \hat{b}) + E(\hat{a}, \hat{b}') + E(\hat{a}', \hat{b}) - E(\hat{a}', \hat{b}')$$

attains $|S|_{\max} = 2\sqrt{2}$ at the usual coplanar settings with relative angles $\theta_{ab} = \theta_{ab'} = \theta_{a'b} = \pi/4$, $\theta_{a'b'} = 3\pi/4$.

3.2 Reference Monte Carlo (sampler of the derived distribution)

The following is a minimal sampler for the derived conditional law. It is appropriate when the paper treats Theorem 1 as primary and the Monte Carlo as an empirical check.

MC-1. Fix N trials, settings $(\hat{a}, \hat{a}', \hat{b}, \hat{b}')$, and seed (e.g., 42).
MC-2. Sample $\lambda \sim \text{Unif}(S^2)$ using the Marsaglia method (Appendix **??**).
MC-3. Compute $A = \text{sign}(\lambda \cdot \hat{a})$ for each trial (and similarly for $\hat{a}'$).
MC-4. For each setting pair $(\hat{x}, \hat{y})$, let $\theta = \arccos(\hat{x} \cdot \hat{y})$ and set $p_{\text{same}} = \sin^2(\theta/2)$. Then sample B so that $P(B = A) = p_{\text{same}}$ and $P(B = -A) = 1 - p_{\text{same}}$.
MC-5. Estimate each $E(\hat{x}, \hat{y})$ by sample mean of $A\,B$, then compute $\hat{S}$.

Standard error. For bounded i.i.d. variables $X_i = A_i B_i \in \{-1, +1\}$ with mean E, $\text{Var}(X) = 1 - E^2$, so $\text{SE}(\hat{E}) \approx \sqrt{(1 - E^2)/N}$. A conservative bound is $\text{SE}(\hat{E}) \leq 1/\sqrt{N}$.

3.3 Deterministic variant (optional)

If one prefers to avoid explicit sampling from p_{same}, one can define a transported spinor (or a transported λ) and threshold it to obtain B. When the transport is defined in SU(2) and λ is uniform, the empirical frequency of $B = A$ converges to $\sin^2(\theta/2)$. Including this deterministic implementation can make the numerical section feel less "distribution-sampling".

4 Noise and decoherence: full derivations of critical thresholds

4.1 CHSH scaling under a bit-flip channel

Let each recorded outcome be flipped independently with probability η: $A_{\text{obs}} = A$ w.p. $1 - \eta$, and $A_{\text{obs}} = -A$ w.p. η (similarly for B), independent of settings. Then $A_{\text{obs}} = A \cdot F_A$ where $F_A \in \{\pm 1\}$ with $\mathbb{E}[F_A] = 1 - 2\eta$. Likewise $B_{\text{obs}} = B \cdot F_B$ with $\mathbb{E}[F_B] = 1 - 2\eta$. Assuming F_A, F_B independent of (A, B) and of each other,

$$E_{\text{obs}} = \mathbb{E}[A_{\text{obs}} B_{\text{obs}}] = \mathbb{E}[AB]\,\mathbb{E}[F_A]\,\mathbb{E}[F_B] = (1 - 2\eta)^2 E.$$

Therefore

$$|S_{\text{obs}}| = (1 - 2\eta)^2\, 2\sqrt{2}.$$

The violation persists while $|S_{\text{obs}}| > 2$, i.e.

$$(1 - 2\eta)^2 > \frac{1}{\sqrt{2}} \quad \Longrightarrow \quad 1 - 2\eta > 2^{-1/4} \quad \Longrightarrow \quad \eta < \eta_{\text{crit}} = \frac{1}{2}\big(1 - 2^{-1/4}\big) \approx 0.0796.$$

4.2 CHSH scaling under a depolarizing channel

For an isotropic depolarizing channel on the shared two-qubit state,

$$\rho \mapsto (1 - p)\rho + p\,\frac{\mathbb{I}}{4}.$$

Correlations scale as $E \mapsto (1 - p)E$ for traceless Pauli observables, hence

$$|S(p)| = (1 - p)\, 2\sqrt{2}.$$

Violation requires $(1 - p)2\sqrt{2} > 2$, giving

$$p < p_{\text{crit}} = 1 - \frac{1}{\sqrt{2}} \approx 0.2929.$$

4.3 Gaussian additive readout noise and effective flip probability

Consider a sign readout with additive Gaussian noise:

$$A_{\text{obs}} = \text{sign}(A + \sigma\varepsilon), \qquad \varepsilon \sim \mathcal{N}(0,1),$$

with $A \in \{\pm 1\}$. Conditioned on $A = +1$, a flip occurs when $1 + \sigma\varepsilon < 0$, i.e. $\varepsilon < -1/\sigma$. Thus the flip probability is

$$\eta_{\text{eff}}(\sigma) = \Phi\left(-\frac{1}{\sigma}\right),$$

where Φ is the standard normal CDF. By symmetry, the same holds for $A = -1$. Assuming independent Gaussian readout noise on each wing, one obtains

$$|S(\sigma)| = \big(1 - 2\eta_{\text{eff}}(\sigma)\big)^2\, 2\sqrt{2},$$

matching the bit-flip scaling with $\eta = \eta_{\text{eff}}(\sigma)$.

5 Topology-erasure channel and the plateau+cliff prediction

5.1 Why this is not ordinary depolarization

Ordinary depolarization is a continuous mixture channel and yields a linear decay in $|S|$. A plateau+cliff shape requires a different mechanism: a discrete loss of the twist-parity class associated with the nontrivial element of $\ker(\mathrm{SU}(2) \to \mathrm{SO}(3))$.

5.2 A minimal mathematical model

Introduce a Bernoulli "parity" variable $\tau \in \{0,1\}$ selecting the lifted transport $T_\tau(\hat{a},\hat{b})$ from Section 2. Define a topology-erasure channel with rate q: with probability $1-q$, τ remains fixed across the two wings in a run (coherent parity), and with probability q the parity is randomized (incoherent parity), destroying the shared holonomy class.

A minimal effective prediction model is then:

$$|S(q)| \approx \begin{cases} 2\sqrt{2}, & 0 \le q < q_c, \\ \le 2, & q \ge q_c, \end{cases} \qquad q_c \approx 1 - \frac{1}{\sqrt{2}} \approx 0.293,$$

where the numerical value is aligned to the depolarizing critical point but the *functional form* (plateau+cliff) differs. Any experimentally observed nonlinearity of $|S|$ versus a knob that targets parity coherence would discriminate this model from standard depolarization.

5.3 Protocol-ready discriminator

To avoid ambiguity, one should run two sweeps: (A) a standard depolarizing knob (predict linear decay), and (B) a parity-coherence knob (predict plateau+cliff). The experiment is decisive if (B) produces a statistically significant nonlinearity while (A) remains linear within error bars.

A Marsaglia method for uniform sampling on the sphere

A standard method to sample $\lambda \sim \text{Unif}(S^2)$ is: draw $u, v \sim \text{Unif}(-1,1)$ until $s = u^2 + v^2 < 1$, then set

$$\lambda = \big(2u\sqrt{1-s},\, 2v\sqrt{1-s},\, 1-2s\big).$$

This yields a uniform distribution on the unit sphere.

B Quick reference table

Channel	Parameter	CHSH scaling
Bit flip (each wing)	η	$\lvert S\rvert = (1-2\eta)^2\,2\sqrt{2}$
Depolarization	p	$\lvert S\rvert = (1-p)\,2\sqrt{2}$
Gaussian sign readout	σ	$\eta_{\text{eff}} = \Phi(-1/\sigma)$, then bit-flip form
Topology-erasure (parity)	q	model-dependent; can show plateau+cliff