Appendix for On the Social Cost of Orbital Debris

Abstract

This appendix collects the equations of the ISEM, solves the central planner maximization problem, and derives the expression for the computation of the Social Cost of Orbital Debris (SCOD).

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Technical appendix

This appendix collects the equations of the ISEM, solves the central planner maximization problem, and derives the closed-form expression for the computation of the Social Cost of Orbital Debris (SCOD).

Appendix A. The model

The economic sub-model is formed by the following equations:

$$max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho}\right)^t U(\hat{c}_t, N_t) \tag{A.1}$$

$$c_t + i_t + h_t = y_t \tag{A.2}$$

$$y_t = a_t f(k_t, s_t, N_t) \tag{A.3}$$

$$k_{t+1} = (1-\delta_k)k_t + i_t \tag{A.4}$$

$$s_{t+1} = (1-\delta_s)s_t + q_t h_t - x_t \tag{A.5}$$

The sources of economic growth are,

$$a_{t+1} = exp(g_{a,t})a_t \tag{A.6}$$

$$g_{a,t} = g_{a,0}exp(-\delta_a t) \tag{A.7}$$

$$q_{t+1} = exp(g_{q,t})q_t \tag{A.8}$$

$$g_{q,t} = g_{q,0}exp(-\delta_q t) \tag{A.9}$$

$$N_{t+1} = N_t \left(\frac{N^*}{N_t}\right)^{\zeta} \tag{A.10}$$

The damage function linking the economic sub-model with the debris sub-model is the following,

$$x_t = \theta D_t s_t \tag{A.11}$$

The debris sub-model is given by,

$$D_{t+1} = (1-\delta_d)D_t + Z_t \tag{A.12}$$

$$Z_t = \omega L_t + \gamma X_t \tag{A.13}$$

The mapping between economic and physical variables is given by,

$$S_t = \mu s_t \tag{A.14}$$

$$X_t = \mu x_t \tag{A.15}$$

$$H_t = \mu q_t h_t \tag{A.16}$$

$$S_{t+1} = (1-\delta_s)S_t + H_t - X_t \tag{A.17}$$

$$X_t = \theta D_t S_t \tag{A.18}$$

Finally, the launch sub-module is composed by the following equations:

$$H_t = \eta L_t \tag{A.19}$$

$$h_t = \frac{\eta}{\mu q_t} L_t \tag{A.20}$$

We assume that the aggregate production function can be represented by the following Cobb-Douglas technology:

$$y_t = a_t k_t^{\alpha_1} s_t^{\alpha_2} N_t^{1-\alpha_1-\alpha_2} \tag{A.21}$$

The households' instantaneous utility function is assumed to be a CRRA-type:

$$U(\hat{c}_t, N_t) = \frac{\hat{c}_t^{1-\sigma} - 1}{1-\sigma} N_t \tag{A.22}$$

Appendix B. Central planner maximization problem

This section states the planning problem that maximizes social welfare. The central planner chooses consumption, investment in capital, and investment in satellites to maximize social welfare,

$$\max_{c_t, i_t, h_t, D_t} E_0 \sum_{t=0}^{\infty} \beta^t N_t U(\hat{c}_t) \tag{B.1}$$

subject to

$$c_t + i_t + h_t = y_t \tag{B.2}$$

$$y_t = a_t f(k_t, s_t, N_t) \tag{B.3}$$

$$k_{t+1} = (1 - \delta_k)k_t + i_t \tag{B.4}$$

$$s_{t+1} = (1 - \delta_s)s_t + q_t h_t - x_t \tag{B.5}$$

$$x_t = \theta D_t s_t \tag{B.6}$$

$$D_{t+1} = (1 - \delta_d)D_t + Z_t \tag{B.7}$$

$$Z_t = \omega L_t + \gamma X_t \tag{B.8}$$

$$L_t = \frac{\mu}{\eta} q_t h_t \tag{B.9}$$

$$X_t = \mu x_t \tag{B.10}$$

Additionally, the mapping from the "economic" model to the "physical" model is given by the following expressions:

$$S_{t+1} = (1 - \delta_s)S_t + H_t - X_t \tag{B.11}$$

$$X_t = \theta D_t S_t \tag{B.12}$$

$$S_t = \mu s_t \tag{B.13}$$

$$H_t = \eta L_t \tag{B.14}$$

The maximization problem can be defined as:

$$\begin{aligned} \mathcal{L} = E_t \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho} \right)^t U(\hat{c}_t) N_t \\ - \sum_{t=0}^{\infty} \lambda_{1,t} \left[c_t + k_{t+1} - (1 - \delta_k)k_t + \frac{\eta}{q_t \mu} L_t - a_t k_t^{\alpha_1} s_t^{\alpha_2} N_t^{1-\alpha_1-\alpha_2} \right] \\ - \sum_{t=0}^{\infty} \lambda_{2,t} \left[s_{t+1} - (1 - \delta_s)s_t - \frac{\eta}{\mu} L_t + \theta D_t s_t \right] \\ - \sum_{t=0}^{\infty} \lambda_{3,t} [D_{t+1} - (1 - \delta_d)D_t - \omega L_t - \gamma X_t] \end{aligned} \tag{B.15}$$

The Lagrangian multipliers for each constraint in period t are $\lambda_{1,t}$, $\lambda_{2,t}$, and $\lambda_{3,t}$, $\lambda_{1,t}$ is the standard shadow price of consumption. $\lambda_{2,t}$ is the price of satellite assets. $\lambda_{3,t}$ is the cost of the stock of orbital debris. We are interested in computing $\lambda_{1,t}$ and $\lambda_{3,t}$, which are the component of SCOD. Note that in the text, $\lambda_{1,t}$ is $\lambda_{c,t}$, equal to $\left(\frac{1}{1+\rho}\right)^t c_t^{-\sigma}$. Similarly, $\lambda_{3,t}$ is $\lambda_{z,t}$.

First order conditions from the maximization problem, for $t = 0, 1, ..., \infty$ are,

$$\frac{\partial \mathcal{L}}{\partial c_t} = \left(\frac{1}{1+\rho}\right)^t \hat{c}_t^{-\sigma} - \lambda_{1,t} = 0 \tag{B.16}$$

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} = -\lambda_{1,t} + \lambda_{1,t+1}\left[1 - \delta_k + \alpha_1 \frac{y_{t+1}}{k_{t+1}}\right] = 0 \tag{B.17}$$

$$\frac{\partial \mathcal{L}}{\partial s_{t+1}} = \lambda_{1,t+1}\alpha_2 \frac{y_{t+1}}{s_{t+1}} - \lambda_{2,t} + \lambda_{2,t+1}(1 - \delta_s - \theta D_{t+1}) + \lambda_{3,t+1}\gamma\theta\mu D_{t+1} = 0 \tag{B.18}$$

$$\frac{\partial \mathcal{L}}{\partial L_t} = -\lambda_{1,t}\frac{\eta}{\mu q_t} + \lambda_{2,t}\frac{\eta}{\mu} + \lambda_{3,t}\omega = 0 \tag{B.19}$$

$$\frac{\partial \mathcal{L}}{\partial D_{t+1}} = -\lambda_{2,t+1}\theta s_{t+1} - \lambda_{3,t} + \lambda_{3,t+1}(1 - \delta_d + \gamma\mu\theta s_{t+1}) = 0 \tag{B.20}$$

Appendix C. The closed-form expression for the social cost of orbital debris (SCOD)

The closed-form expresion for SCOD is derived as follows. From the first first-order condition (expression B.16) we obtain the first component (the denominator) of the SCOD,

$$\lambda_{1,t} = \left(\frac{1}{1+\rho}\right)^t \hat{c}_t^{-\sigma} \tag{C.1}$$

The Euler equation for investment in capital other than satellites.

$$\hat{c}_t^{-\sigma} = \left(\frac{1}{1+\rho}\right) E_t \hat{c}_{t+1}^{-\sigma}\left[1 - \delta_k + \alpha_1 \frac{y_{t+1}}{k_{t+1}})\right] \tag{C.2}$$

Next, from the first-order condition (B.19), we obtain that,

$$\lambda_{2,t} = \frac{\left(\frac{1}{1+\rho}\right)^t \hat{c}_t^{-\sigma}}{q_t} - \frac{\mu\omega}{\eta}\lambda_{3,t} = 0 \tag{C.3}$$

and moving one period ahead previous expression,

$$\lambda_{2,t+1} = \frac{\left(\frac{1}{1+\rho}\right)^{t+1} \hat{c}_{t+1}^{-\sigma}}{q_{t+1}} - \frac{\mu\omega}{\eta}\lambda_{3,t+1} = 0 \tag{C.4}$$

By substituting $\lambda_{2,t+1}$ into the first-order condition (B.20), we obtain that,

$$-\left[\frac{\left(\frac{1}{1+\rho}\right)^{t+1} \hat{c}_{t+1}^{-\sigma}}{q_{t+1}} - \frac{\mu\omega}{\eta}\lambda_{3,t+1}\right]\theta s_{t+1} = \lambda_{3,t} - \lambda_{3,t+1}(1 - \delta_d + \gamma\mu\theta s_{t+1}) \tag{C.5}$$

Operating and solving for $\lambda_{3,t+1}$ yields,

$$\lambda_{3,t+1} = \frac{1}{1-\delta_d + \gamma\mu\theta s_{t+1} + \frac{\mu\omega}{\eta}\theta s_{t+1}} \left[\lambda_{3,t} + \frac{\left(\frac{1}{1+\rho}\right)^{t+1} \hat{c}_{t+1}^{-\sigma}}{q_{t+1}} \theta s_{t+1} \right] \tag{C.6}$$

Substituting the expressions for $\lambda_{1,t}$, $\lambda_{2,t}$, and $\lambda_{2,+1}$ in the first-order condition (B.18), we have,

$$\begin{aligned} \frac{\left(\frac{1}{1+\rho}\right)^{t} \hat{c}_{t}^{-\sigma}}{q_{t}} - \frac{\mu\omega}{\eta}\lambda_{3,t} &= \left(\frac{1}{1+\rho}\right)^{t+1} \hat{c}_{t+1}^{-\sigma}\alpha_2 \frac{y_{t+1}}{s_{t+1}} \\ + \left[\frac{\left(\frac{1}{1+\rho}\right)^{t+1} \hat{c}_{t+1}^{-\sigma}}{q_{t+1}} - \frac{\mu\omega}{\eta}\lambda_{3,t+1} \right] &(1-\delta_s-\theta D_{t+1}) + \lambda_{3,t+1}\gamma\mu\theta D_{t+1} \end{aligned} \tag{C.7}$$

and collecting terms,

$$\begin{aligned} \frac{\left(\frac{1}{1+\rho}\right)^{t} \hat{c}_{t}^{-\sigma}}{q_{t}} - \frac{\mu\omega}{\eta}\lambda_{3,t} &= \left(\frac{1}{1+\rho}\right)^{t+1} \hat{c}_{t+1}^{-\sigma}\alpha_2 \frac{y_{t+1}}{s_{t+1}} \\ + \left[\frac{\left(\frac{1}{1+\rho}\right)^{t+1} \hat{c}_{t+1}^{-\sigma}}{q_{t+1}} \right] (1-\delta_s-\theta D_{t+1}) &- \lambda_{3,t+1} \left[\frac{\mu\omega}{\eta}(1-\delta_s-\theta D_{t+1}) - \gamma\mu\theta D_{t+1} \right] \end{aligned} \tag{C.8}$$

Next, by substituting the expression for $\lambda_{3,t+1}$,

$$\begin{aligned} \frac{\left(\frac{1}{1+\rho}\right)^{t} \hat{c}_{t}^{-\sigma}}{q_{t}} - \frac{\mu\omega}{\eta}\lambda_{3,t} = \left(\frac{1}{1+\rho}\right)^{t+1} \hat{c}_{t+1}^{-\sigma}\alpha_2 \frac{y_{t+1}}{s_{t+1}} + \left[\frac{\left(\frac{1}{1+\rho}\right)^{t+1} \hat{c}_{t+1}^{-\sigma}}{q_{t+1}} \right] (1-\delta_s-\theta D_{t+1}) \\ - \frac{1}{1-\delta_d + \gamma\mu\theta s_{t+1} + \frac{\mu\omega}{\eta}\theta s_{t+1}} \left[\lambda_{3,t} + \frac{\left(\frac{1}{1+\rho}\right)^{t+1} \hat{c}_{t+1}^{-\sigma}}{q_{t+1}} \theta s_{t+1} \right] \\ \left[\frac{\mu\omega}{\eta}(1-\delta_s-\theta D_{t+1}) - \gamma\mu\theta D_{t+1} \right] \end{aligned} \tag{C.9}$$

Collecting terms, we find,

$$\left(\frac{\mu\omega}{\eta}-\frac{1}{1-\delta_d+\gamma\mu\theta s_{t+1}+\frac{\mu\omega}{\eta}\theta s_{t+1}}\left[\frac{\mu\omega}{\eta}(1-\delta_s-\theta D_{t+1})-\gamma\mu\theta D_{t+1}\right]\right)\lambda_{3,t}=$$

$$\frac{\left(\frac{1}{1+\rho}\right)^t\hat{c}_t^{-\sigma}}{q_t}-\left(\frac{1}{1+\rho}\right)^{t+1}\hat{c}_{t+1}^{-\sigma}\alpha_2\frac{y_{t+1}}{s_{t+1}}-\left[\frac{\left(\frac{1}{1+\rho}\right)^{t+1}\hat{c}_{t+1}^{-\sigma}}{q_{t+1}}\right](1-\delta_s-\theta D_{t+1})$$

$$+\frac{1}{1-\delta_d+\gamma\mu\theta s_{t+1}+\frac{\mu\omega}{\eta}\theta s_{t+1}}\frac{\left(\frac{1}{1+\rho}\right)^{t+1}\hat{c}_{t+1}^{-\sigma}}{q_{t+1}}\theta s_{t+1}\left[\frac{\mu\omega}{\eta}(1-\delta_s-\theta D_{t+1})-\gamma\mu\theta D_{t+1}\right] \quad \text{(C.10)}$$

Finally, the SCOD is defined as,

$$SCOD_t=-\frac{\lambda_{z,t}}{\lambda_{c,t}} \quad \text{(C.11)}$$

Hence, the exact expression for the computation of the SCOD is given by,

$$SCOD_t=-\frac{1}{\left(\frac{1}{1+\rho}\right)^t\hat{c}_t^{-\sigma}}\frac{1}{\frac{\mu\omega}{\eta}-\frac{1}{1-\delta_d+\gamma\mu\theta s_{t+1}+\frac{\mu\omega}{\eta}\theta s_{t+1}}\left[\frac{\mu\omega}{\eta}(1-\delta_s-\theta D_{t+1})-\gamma\mu\theta D_{t+1}\right]}$$

$$\left[\frac{\left(\frac{1}{1+\rho}\right)^t\hat{c}_t^{-\sigma}}{q_t}-\left(\frac{1}{1+\rho}\right)^{t+1}\hat{c}_{t+1}^{-\sigma}\alpha_2\frac{y_{t+1}}{s_{t+1}}-\left[\frac{\left(\frac{1}{1+\rho}\right)^{t+1}\hat{c}_{t+1}^{-\sigma}}{q_{t+1}}\right](1-\delta_s-\theta D_{t+1})\right.$$

$$\left.+\frac{1}{1-\delta_d+\gamma\mu\theta s_{t+1}+\frac{\mu\omega}{\eta}\theta s_{t+1}}\frac{\left(\frac{1}{1+\rho}\right)^{t+1}\hat{c}_{t+1}^{-\sigma}}{q_{t+1}}\theta s_{t+1}\left[\frac{\mu\omega}{\eta}(1-\delta_s-\theta D_{t+1})-\gamma\mu\theta D_{t+1}\right]\right] \quad \text{(C.12)}$$