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Research Article

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From Theory to Practice: The Extended Alpha Power Weibull Model for Complex Hazard Shapes in Reliability Engineering

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Abstract

Flexible lifetime models that accommodate both monotonic and non-monotonic failure-rate behaviors, such as bathtub-shaped hazard functions, are essential in contemporary reliability and survival analysis. We introduce the Extended Alpha Power Weibull (EAPW) distribution, a four-parameter family derived by applying an Extended Alpha Power generator to the Weibull distribution. The EAPW applies compounding at both the cumulative distribution function (CDF) and hazard-function levels to produce a parsimonious yet highly flexible family of models capable of representing increasing, decreasing, and bathtub-shaped hazard rates. Closed-form expressions for the cumulative distribution and probability density functions are presented, and the quantile function is obtained in closed form using the Lambert W function. Analytical results are established for survival and hazard functions, order statistics, mean residual life, and entropy. Conditions that produce heavy-tailed behavior, including cases in which conventional moments fail to exist, are examined, together with their implications for extreme-value analysis and reliability inference. Parameter estimation is conducted via maximum likelihood, and finite-sample performance is assessed through Monte Carlo simulations, reporting bias, root mean square error (RMSE), and confidence interval coverage across several scenarios. Empirical performance is demonstrated using two engineering datasets characterized by bathtub hazard rates: active repair times and air-conditioning system failures. The EAPW achieves superior fit compared to benchmark models, including the Weibull, Alpha Power Weibull, and related generator families, as evaluated by information criteria and graphical diagnostics. Practical limitations, such as potential overparameterization and reduced performance under heavy censoring, are discussed, and recommendations for EAPW application are provided. The EAPW constitutes a theoretically robust and practically valuable model for complex lifetime data.

Keywords: Extended Alpha Power Weibull distribution; Bathtub hazard rate; Reliability modeling; Lifetime data; Maximum likelihood estimation; Survival analysis; Heavy-tailed distribution.

1. Introduction

The development of efficient and reliable tools for data analysis is a constant endure in statistical community to help making well inform decisions in real life domain. Modeling lifetime or reliability data often requires distributions that can capture a wide range of hazard (failure-rate) behaviors. Lifetime data appear across numerous disciplines, including economics, survival analysis, banking, and reliability engineering, where selecting an appropriate probability distribution is crucial. Although widely used in engineering and medical research, the exponential and Weibull distributions come with constraints, the exponential presumes a constant risk over time, and the two-parameter Weibull is limited to modeling only increasing or decreasing hazard trends. Non-monotonic data is data that varies, usually increasing and decreasing at different intervals rather than always increasing or decreasing. In engineering, such data emerges in many circumstances due to intricate system behaviors, external factors, or dynamic processes. For example, in mechanical engineering, vibration amplitudes vary over time in systems containing oscillating mechanical parts, such as bridges, engines, or buildings. The data include peaks and troughs caused by resonant frequencies, damping, and external pressures. In electrical engineering, AC voltage and current vary sinusoidally, with the waveform alternating between positive and negative peaks, making the data intrinsically non-monotonic. Structures bearing variable loads exhibit complex load-displacement relationships in civil engineering, especially as the structure reaches its elastic limit. The displacement may rise or decrease depending on the form of material failure. Some catalytic reactions exhibit rate oscillations or fluctuations due to reaction circumstances or catalyst deactivation/activation cycles, resulting in non-monotonic behavior in reaction rate against time. Fig 1 illustrates a canonical bathtub hazard function, composed of early decreasing, middle constant, and late increasing phases. Such non-monotonic hazards cannot be captured by the standard Weibull model alone.

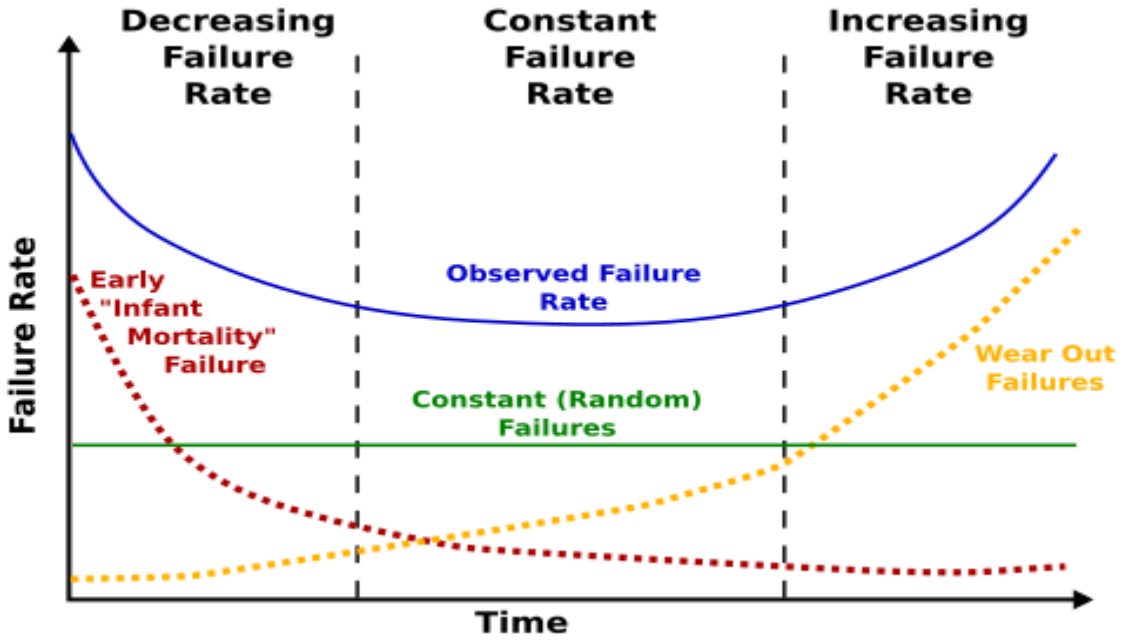


Figure 1: The classic bathtub hazard function: high initial (infant) failure rate, a constant intermediate period, and rising wear-out failures in later life[1]

Consequently, there has been substantial recent interest in generating new distribution families that extend classical models to allow greater flexibility in hazard shapes. Researchers often introduce a parameterized transformation or generator applied to a baseline distribution's CDF or hazard. Several recent works have introduced new distribution families and their special cases. Researchers have introduced a variety of new distribution families. Alzaatreh et al. inaugurated the generalized Cauchy class [1], Cordeiro et al. extended this to an odd-half Cauchy variant [2], and Al-Aqtash et al. devised a logit-based framework that subsumes the Gumbel–Weibull as a special case [3]. Ijaz et al. presented the Gull Alpha Power family along with its Weibull extension [4], while Alzaatreh et al. also explored a Gamma–X series grounded on the Normal distribution [5]. Nasir et al. formulated a Burr generalization via its quantile function [6], Abid & Abdulrazak proposed a truncated Fréchet–G class [7], and Alzaareh et al. introduced the T–Normal family [8]. Finally, Aldeni et al. crafted quantile-driven generalized lambda distributions [9], Alzaghal et al. developed a T–X framework [10], and Muurya et al. offered an exponential-derived distribution class [11].

This paper aims to introduce a new probability model capable of capturing lifetime data characterized by either monotonic or non-monotonic hazard-rate behavior. For this purpose, we have introduced a new generator and applied it to the Weibull distribution [12] to develop a new model to accomplish supreme flexibility. Some remarkable work in capturing the reality of a process behavior in various real life domain are given by [13–23]. Existing distributions often

lack the flexibility to model non-monotonic behaviors effectively. To bridge this gap, we introduce the Extended Alpha Power (EAP) distribution family. The Extended Power Alpha Weibull (EPAW) distribution, a special case based on the Weibull model, effectively captures both monotonic and non-monotonic hazard rates. We derive its key theoretical properties, including the r-th moments, survival and hazard functions, and mean residual life. The model's flexibility and applicability are further validated through simulated and real data analyses.

The Novelty and Significance of the study lies in the following bullets:

- Introduces the Extended Alpha Power (EAP) generator, a novel transformation framework that unifies CDF-based and hazard-based compounding within a single, coherent family of probability distributions.
- Proposed the four-parameter Extended Alpha Power Weibull (EAPW) distribution, an advancement of the classical Weibull model that includes an additional shape parameter, allowing it to model both monotonic and bathtub-shaped hazard rates.
- Offers a rigorous theoretical framework, establishing CDF validity, deriving the explicit PDF, analyzing survival and hazard functions, performing quantile inversion via the Lambert W function, and proving model identifiability.
- Establishes heavy-tail properties (infinite moment-generating function) with practical implications for reliability forecasting and highlights when and how infinite-tail behavior may occur.
- Conducts a simulation study assessing MLE bias, RMSE, and AIC/BIC consistency, enhancing finite-sample insights for complex four-parameter lifetime models.
- Applies the EAPW distribution to two distinct engineering data sets with bathtub hazards, outperforming several established competitors (Weibull, GAPW, APW, EIFW) through both numerical criteria and graphical diagnostics (Q–Q, P–P, TTT), thus showcasing its practical utility.

The structure of the paper is organized in the following sections:

In the next section (2) we will comprehensively discuss the Extended Alpha Power Family of Distributions. The third section is dedicated to Parameter Estimation using Maximum Likelihood Estimation. The Shannon Entropy is also discussed to theoretically validate the proposed methodology. Section 4 showcases the practical use of our proposed distribution by analyzing two real-world data sets: Active Repair Times and Air Conditioning System Failures. To empirically validate the performance of our proposed methodology, we have conducted a comprehensive simulation study in section 5. The sixth section is dedicated to the Discussion and the finding and future recommendation of the proposed methodology is discussed in the

sixth section Conclusion. At the end of the paper References of citations are given at the end of this article.

2. Family of Extended Alpha Power Distributions

This section presents the Extended Alpha Power (EAP) family, a new class of continuous probability distributions, via the following cumulative distribution function (CDF) for the random variable Z:

$$F_{EAP}(z) = \frac{\alpha F(z)^{H(z)}}{\alpha^{F(z)}}, if \alpha > 1, z > 0 \quad (1)$$

Where α is the shape parameter; $F(z)$ is the CDF of the baseline distribution; and $H(z)$ is its hazard-rate function.

2.1 Extend Alpha Power Weibull Distribution

We now specialize to the case in which the baseline cumulative distribution function (CDF) is represented by a two-parameter Weibull distribution, while the baseline hazard h(z) is taken to be exponential. Let the two-parameter Weibull CDF be:

$$F_w(z) = 1 - e^{-\varphi z^{\gamma}}, z > 0 \quad (2)$$

Let ϕ and γ represent the scale and shape parameters, respectively. The hazard rate function corresponding to the exponential distribution can be expressed as follows:

$$H_E(z) = \frac{f(z)}{1 - F(z)} = \frac{\varphi e^{-\varphi z}}{1 - (1 - e^{-\varphi z})} \quad (3)$$

The Weibull distribution has found widespread application across various research domains. For instance, it has been employed to model brittle material failure Fok et al. [15], wind speed and earthquake dat a Al-Hasan & Nigmatullin [24], the fracture strength of glass Keshavan et al. [25], electrical failure in multilayer ceramic capacitors (Wang et al. [26]), and the drying kinetics of Pepino fruit Uribe et al. [27]. Despite its versatility, the Weibull distribution is constrained to modeling monotonically increasing, decreasing, or constant failure rates. Let X be a continuous random variable that follows the EAPW distribution, defined by the following cumulative distribution function (CDF) and probability density function (PDF):

$$F(x) = \frac{\alpha\left(1 - e^{-\beta x^{\gamma}}\right)^{\theta}}{\alpha^{1 - e^{-\beta x^{\gamma}}}}, x > 0, \alpha, \beta, \gamma, \theta > 0 \quad (4)$$

$$f(x) = \alpha^{e^{-\beta x^{\gamma}}} \beta\gamma\theta x^{\gamma-1}\left(1 - e^{-\beta x^{\gamma}}\right)^{\theta-1} e^{-\beta x^{\gamma}} - \alpha^{e^{-\beta x^{\gamma}}} log(a)\beta\gamma x^{\gamma-1}\left(1 - e^{-\beta x^{\gamma}}\right)^{\theta} e^{-\beta x^{\gamma}}$$

The proposed distribution is defined by four strictly positive parameters: α, β, γ, and θ. Figs 2 and 3 illustrate how variations in these parameters affect the PDF and CDF.

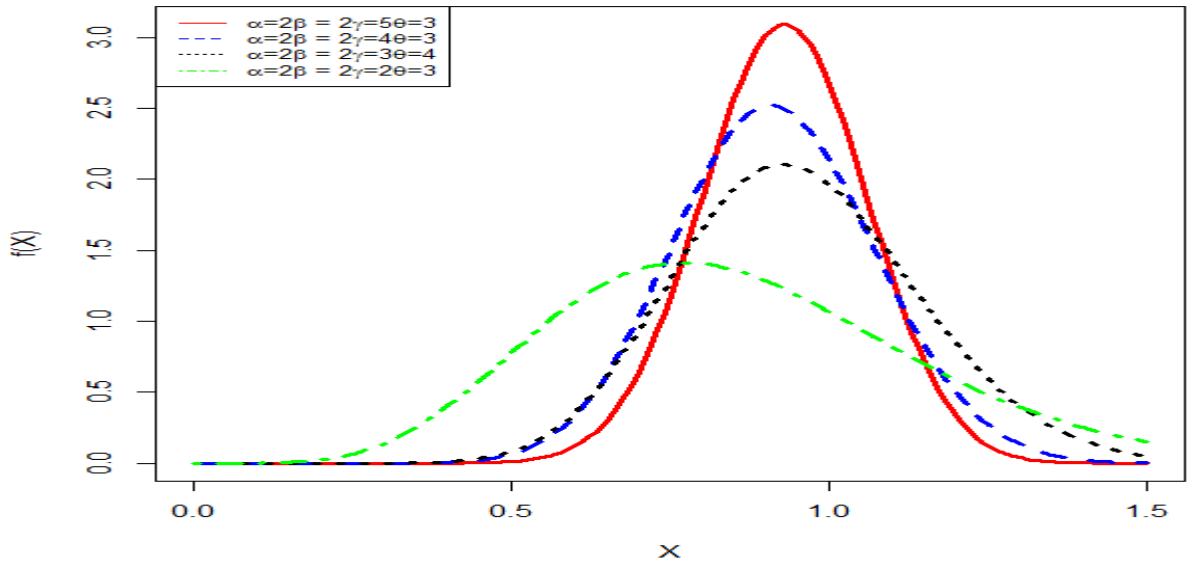


Fig-2: Probability Density Function of EAPW

2.2. Survival and Hazard rate function

For a continuous random variable X that follows the EAPW distribution, the corresponding survival function is given by

$$S(x;\alpha,\beta,\gamma,\lambda) \;=\; 1 - F(x).$$

Substituting the expression from equation (4) then gives:

$$S(x;\alpha,\beta,\gamma,\lambda) = 1 - \left(\frac{\alpha\left(1-e^{-\beta x^{\gamma}}\right)^{\theta}}{\alpha^{1-e^{-\beta x^{\gamma}}}}\right)$$

$$S(x;\alpha,\beta,\gamma,\lambda) = 1 - \alpha^{e^{-\beta x^{\gamma}}}\left(1-e^{-\beta x^{\gamma}}\right)^{\theta}$$

Analogously, the EAPW hazard rate is defined by

$$H(x) = \frac{f(x)}{1-F(x)}.$$

Substituting expressions (4) and (5) into this formula then produces the desired result.

$$H(x) = \frac{\alpha^{e^{-\beta x^{\gamma}}}\beta\gamma\theta x^{\gamma-1}\left(1-e^{-\beta x^{\gamma}}\right)^{\theta-1}e^{-\beta x^{\gamma}} - \alpha^{e^{-\beta x^{\gamma}}}\,log(\,a)\beta\gamma x^{\gamma-1}\left(1-e^{-\beta x^{\gamma}}\right)^{\theta}e^{-\beta x^{\gamma}}}{1-\alpha^{e^{-\beta x^{\gamma}}}(1-e^{-\beta x^{\gamma}})^{\theta}}$$

2.3. The Quantile Function

The quantile function is crucial for calculating positional summaries like the median, quartiles, deciles, and percentiles, and is also essential in simulation studies. To obtain the quantile function of the WPE, substitute F(x)=u into equation (4).

$$\frac{\alpha\left(1-e^{-\beta x^{\gamma}}\right)^{\theta}}{\alpha^{1-e^{-\beta x^{\gamma}}}} = u \qquad (6)$$

Where $u(0,1)$ is the uniform random number. Simplifying (6), we get the following result

$$x = \left(\frac{log(u) + d\, log\left(-\frac{u^{-1/d}\, log(a)}{d} \right)}{bd} \right)^{1/c} \tag{7}$$

The Lambert W (or product-log) function, denoted W(z), is the function implicitly defined by the equation.

$$W(z)\, e^{W(z)} = z.$$

$$W(z) = \sum_{n=1}^{\infty} \frac{(-1)^n n^{n-2}}{(n-1)!} z^n$$

Putting $u = 0.5$ in (7), we get the median.

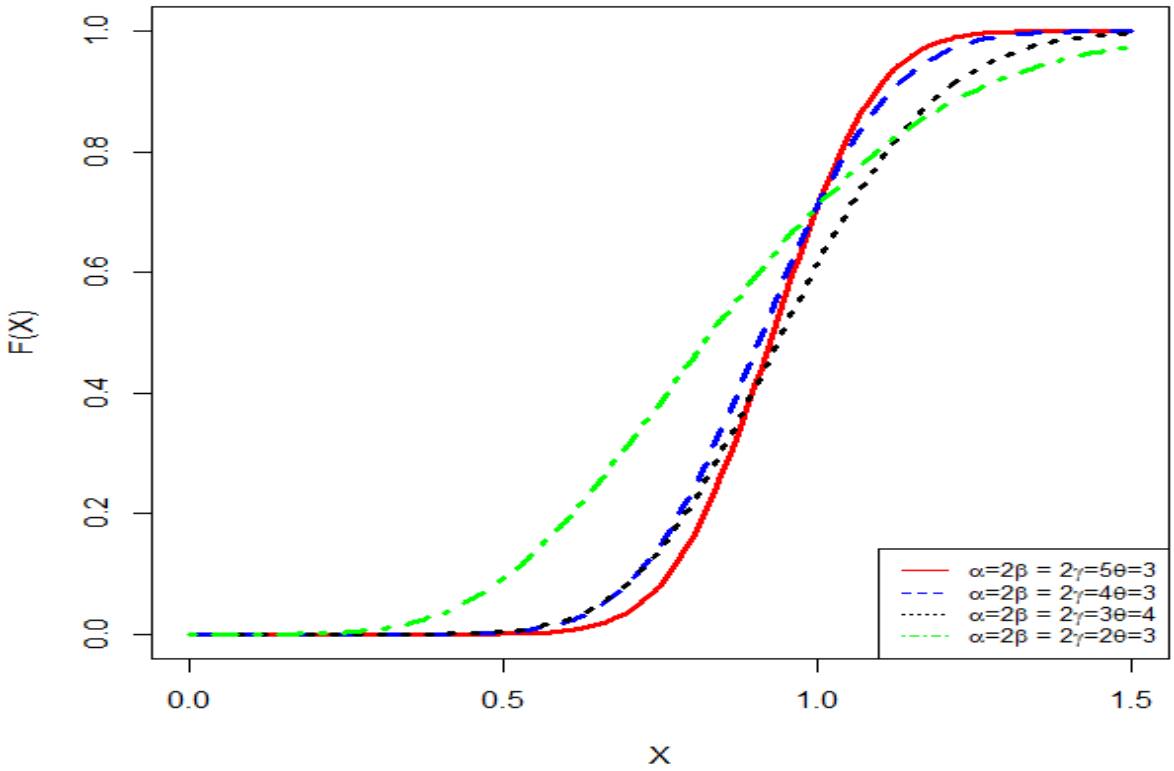


Fig 3: Cumulative Distribution Function of EAPW

2.4. The rth Moments

For a random variable X that follows a WPE distribution with parameters α,β,γ, and θ, the r^{th} raw moment is expressed as:

$$\mu_r' = E(X^r) = \int_0^{\infty} x^r f(x)\, dx \tag{8}$$

Using expression (5), we have

$$\mu_r' = \int_0^{\infty} \underbrace{x^r(\alpha^{e^{-\beta x^\gamma}} \beta\gamma\theta x^{\gamma-1}\left(1-e^{-\beta x^\gamma}\right)^{\theta-1} e^{-\beta x^\gamma}}_{I_1}$$

$$\underbrace{- \alpha^{e^{-\beta x^\gamma}} log(a)\beta\gamma x^{\gamma-1}\left(1-e^{-\beta x^\gamma}\right)^{\theta} e^{-\beta x^\gamma})}_{I_2} dx \qquad (9)$$

We have divided the above expression into two parts. Now, we will solve each part separately. The first part of (9) is

$$I_1 = \int_0^{\infty} x^r\, \alpha^{e^{-\beta x^\gamma}} \beta\gamma\theta x^{\gamma-1}\left(1-e^{-\beta x^\gamma}\right)^{\theta-1} e^{-\beta x^\gamma} dx \qquad (10)$$

Let $z = e^{-\beta x^\gamma}, -dz = \beta\gamma x^{\gamma-1} e^{-\beta x^\gamma}$ and $x = \left(\frac{-log(z)}{b}\right)^{1/\gamma}$, where the range of variable x and z are $0 < x < \infty, 1 < z < 0$

Equation (10), becomes

$$I_1 = \int_0^1 \left(\frac{-log(z)}{b}\right)^{1/\gamma} \alpha^z \theta(1-z)^{\theta-1} dz$$

$$I_1 = (1/b)^{r/\gamma} \int_0^1 (-log(z))^{r/\gamma}\, \alpha^z \theta(1-z)^{\theta-1} dz \qquad (11)$$

Where $(-log(z))^{r/\gamma} = \left(\sum_{k=1}^{\infty} \frac{(-1)^k (z-1)^k}{k}\right)^{r/\gamma}$

Equation (10), take the form

$$I_1 = \left(\sum_{k=1}^{\infty} \frac{(-1)^k}{\beta k}\right)^{r/\gamma} \int_0^1 (z-1)^{kr/\gamma}\, \alpha^z \theta(1-z)^{\theta-1} dz$$

Finally, we obtained the following result

$$I_1 = \left(\sum_{k=1}^{\infty} \frac{(-1)^k}{\beta k}\right)^{r/\gamma} \frac{(-1)^{\theta} \alpha\theta \overline{\left(\frac{kr+\gamma\theta}{\gamma}, -\log(\alpha)(z-1)\right)}}{\left(-\log(\alpha)\right)^{\frac{kr+\gamma\theta}{\gamma}}} \Big|_0^1 \qquad (12)$$

We have assumed that $-\theta - 1 \neq 0, \theta \neq 0, \theta(2\theta+1) \neq 0, \gamma kr \neq 0, \gamma(kr+\gamma) \neq 0, \gamma(kr+2\gamma) \neq 0,$

$$\gamma(kr+\gamma\theta) \neq 0, (kr+\gamma)(2k+3\gamma) \neq 0, -log(\alpha) - 1 \neq 0, log(\alpha) \neq 0 \qquad (13)$$

The second part of (9) is expressed as

$$I_2 = \int_0^{\infty} x^r \alpha^{e^{-\beta x^\gamma}} log(a)\beta\gamma x^{\gamma-1}\left(1-e^{-\beta x^\gamma}\right)^{\theta} e^{-\beta x^\gamma} dx$$

Supposing the same transformations as earlier, then the above expression takes the form

$$I_2 = log(\alpha)\left(\sum_{k=1}^{\infty} \frac{(-1)^k}{\beta k}\right)^{r/\gamma} \int_0^1 (z-1)^{rk/\gamma} \alpha^z (1-z)^{\theta}\, dz$$

We obtained the following result

$$I_2 = \log(\alpha)\left(\sum_{k=1}^{\infty}\frac{(-1)^k}{\beta k}\right)^{r/\gamma}\left.\frac{(-1)^{\theta}\alpha\overline{\left(\left(\frac{kr+\gamma\theta+\gamma}{\gamma},-\log(\alpha)(z-1)\right)\right)}}{\left(-\log(\alpha)\right)^{\frac{kr+\gamma\theta+\gamma}{\gamma}}}\right|_0^1 \qquad (14)$$

Assuming that, $-\theta-2\neq 0, -\theta-1\neq 0, (\theta+1)(2\theta+3)\neq 0, \gamma(kr+\gamma)\neq 0, \gamma(kr+2\gamma)\neq 0,$

$$\gamma(kr+\gamma(\theta+1))\neq 0, (kr+\gamma)(2k+3\gamma)\neq 0, -log(\alpha)-1\neq 0, log(\alpha)\neq 0$$

By combining (12) and (14), we get

$$\mu_r' = \left(\sum_{k=1}^{\infty}\frac{(-1)^k}{\beta k}\right)^{r/\gamma}\left(\left.\frac{(-1)^{\theta}\alpha\theta\overline{\left(\left(\frac{kr+\gamma\theta}{\gamma},-\log(\alpha)(z-1)\right)\right)}}{\left(-\log(\alpha)\right)^{\frac{kr+\gamma\theta}{\gamma}}}\right|_0^1 - \log(\alpha)\left.\frac{(-1)^{\theta}\alpha\overline{\left(\left(\frac{kr+\gamma\theta+\gamma}{\gamma},-\log(\alpha)(z-1)\right)\right)}}{\left(-\log(\alpha)\right)^{\frac{kr+\gamma\theta+\gamma}{\gamma}}}\right|_0^1\right)$$

It has been observed that the r^{th} moments fail to exist.

2.5. The Mode

The mode is a key measure of central tendency. For the EAPW distribution, it can be determined by solving the equation

$$f(x)' = 0 \text{ or } \frac{df(x)}{dx} = 0$$

Substituting equation (5) into the preceding expression yields:

$$\frac{d}{dx}\left(\alpha^{e^{-\beta x^{\gamma}}}\beta\gamma\theta x^{\gamma-1}\left(1-e^{-\beta x^{\gamma}}\right)^{\theta-1}e^{-\beta x^{\gamma}} - \alpha^{e^{-\beta x^{\gamma}}}log(\alpha)\beta\gamma x^{\gamma-1}\left(1-e^{-\beta x^{\gamma}}\right)^{\theta}e^{-\beta x^{\gamma}}\right) = 0$$

$$\frac{\beta\gamma x^{\gamma-1}\left(1-e^{-\beta x^{\gamma}}\right)e^{-\beta x^{\gamma}}\alpha^{e^{-\beta x^{\gamma}}}\left((\theta-log(\alpha))e^{\beta x^{\gamma}}+log(\alpha)\right)}{e^{\beta x^{\gamma}}-1} = 0$$

$$(\theta-log(\alpha))e^{\beta x^{\gamma}}+log(\alpha) = 0$$

$$e^{\beta x^{\gamma}} = \frac{-log(\alpha)}{\theta-log(\alpha)}$$

Taking log on both sides, we have

$$log(e^{\beta x^{\gamma}}) = log\left(\frac{-log(\alpha)}{\theta-log(\alpha)}\right)$$

Hence, the mode of the EAPW distribution becomes

$$x = \left(\frac{log\left(\frac{-log(\alpha)}{\theta-log(\alpha)}\right)}{\beta}\right)^{1/\gamma}$$

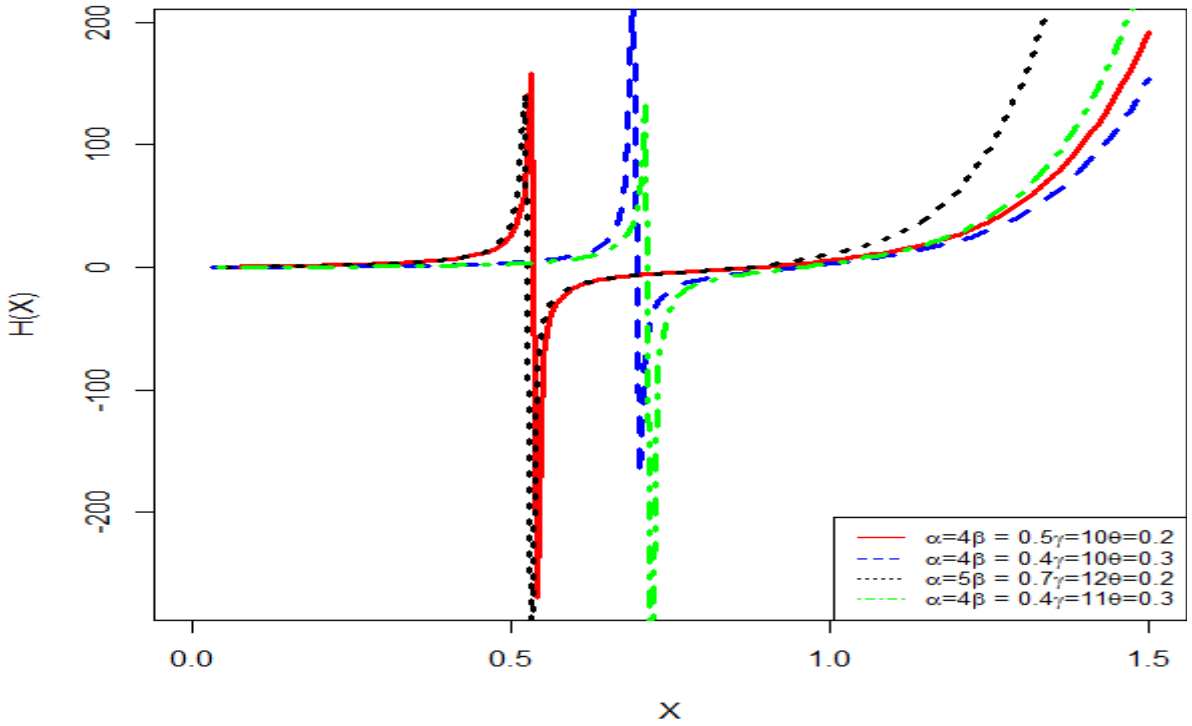


Fig-4: Hazard Rate function of EAPW

2.6. Skewness and Kurtosis

Skewness and kurtosis are fundamental measures that quantify a distribution's asymmetry and tail heaviness, respectively. The mathematical formulas defined by Galton Skewness and Moors kurtosis are given as

$$Sk = \frac{Q(3/4) + Q(1/4) - 2Q(1/2)}{Q(3/4) - Q(1/4)}$$

$$Kr = \frac{Q(7/8) + Q(3/8) - Q(5/8) - Q(1/3)}{Q(6/8) - Q(2/8)}$$

The quantile function, denoted as Q(.), is used, and Table 1 presents the numerical values of skewness and kurtosis for various parameter settings.

Table-1: Numerical values of Skewness and Kurtosis

α	β	γ	θ	Skewness	Kurtosis
0.9	0.6	1	1.2	-0.23797	1.245305
0.2	0.1	0.5	0.6	0.208064	0.76255
0.9	0.21	0.25	0.36	-0.79981	2.380455
0.8	0.3	0.2	0.4	-0.94307	4.262381
0.1	0.2	0.3	0.4	0.376291	1.029656
0.4	0.3	0.2	0.1	0.966604	6.072883
0.8	0.7	0.5	0.05	-0.30262	1.006018

1	1	1	1	-0.26186	1.30627
0.9	0.6	1	1	-0.22717	1.220157
0.1	0.2	0.6	0.9	-0.05155	0.96281

2.7. Order Statistics

Let $x_{1:n} \leq x_{2:n} \leq \cdots \leq x_{n:n}$ denote the ordered sample drawn from the EAPW distribution, with corresponding probability density function $f(x)$ and cumulative distribution function $F(x)$. The probability density function of the i^{th} order statistic, $x_{i:n}$, is given by

$$(x) = \frac{n!}{(i-1)!\,(n-i)!}\,[F(x)]^{\,i-1}\,[1-F(x)]^{\,n-i}\,f(x). \tag{15}$$

In particular, setting $i = 1\ or\ i = n$ recovers the formulas for the minimum and maximum statistics (Equations 4 and 5).

$$f_{(1,n)}(x) = n\alpha^{e^{-\beta x^{\gamma}}}\beta\gamma\theta x^{\gamma-1}\left(1-e^{-\beta x^{\gamma}}\right)^{\theta-1}e^{-\beta x^{\gamma}} - \alpha^{e^{-\beta x^{\gamma}}}\,log(\,a)\beta\gamma x^{\gamma-1}\left(1-e^{-\beta x^{\gamma}}\right)^{\theta}e^{-\beta x^{\gamma}}\left[1-\alpha^{e^{-\beta x^{\gamma}}}\left(1-e^{-\beta x^{\gamma}}\right)^{\theta}\right]^{(n-1)}$$

$$f_{(n,n)}(x) = n\left(\alpha^{e^{-\beta x^{\gamma}}}\left(1-e^{-\beta x^{\gamma}}\right)^{\theta}\right)^{(n-1)}\alpha^{e^{-\beta x^{\gamma}}}\beta\gamma\theta x^{\gamma-1}\left(1-e^{-\beta x^{\gamma}}\right)^{\theta-1}e^{-\beta x^{\gamma}} - \alpha^{e^{-\beta x^{\gamma}}}\,log(\,a)\beta\gamma x^{\gamma-1}\left(1-e^{-\beta x^{\gamma}}\right)^{\theta}e^{-\beta x^{\gamma}}$$

2.8. Mean Residual Life (MRL)

For the EAPW distribution characterized by the parameters α, β, γ, and θ, the mean residual life (MRL) function is expressed as:

$$\mu(t) = \frac{\int_t^{\infty} xf(x)dx - t}{S(t)} \tag{16}$$

Let f(x) denote the probability density function (pdf) and S(t) the corresponding survival function of the proposed model. Under this notation, the numerator of equation (16) may be recast as:

$$\int_t^{\infty} xf(x)dx = \int_t^{\infty} x\alpha^{e^{-\beta x^{\gamma}}}\beta\gamma\theta x^{\gamma-1}\left(1-e^{-\beta x^{\gamma}}\right)^{\theta-1}e^{-\beta x^{\gamma}}dx - \int_t^{\infty} x\,\alpha^{e^{-\beta x^{\gamma}}}\,log(\,a)\beta\gamma x^{\gamma-1}\left(1-e^{-\beta x^{\gamma}}\right)^{\theta}e^{-\beta x^{\gamma}}dx \tag{17}$$

Above expression can be solved into two parts. The first part is given as

$$I_1 = \int_t^{\infty} x\,\alpha^{e^{-\beta x^{\gamma}}}\beta\gamma\theta x^{\gamma-1}\left(1-e^{-\beta x^{\gamma}}\right)^{\theta-1}e^{-\beta x^{\gamma}}dx \tag{18}$$

Let $z = e^{-\beta x^{\gamma}}, -dz = \beta\gamma x^{\gamma-1}e^{-\beta x^{\gamma}}$ and $x = \left(\frac{-\,log(z)}{b}\right)^{1/\gamma}$, where, the range of variable x and z are $0 < x < \infty, e^{-\beta t^{\gamma}} < z < 1$, then (18) takes the form

(19)

$$I_1 = (1/\beta)^{1/\gamma} \int_1^{e^{-\beta t^{\gamma}}} (-log(z))^{1/\gamma} \alpha^z \theta (1-z)^{\theta-1}\, dz$$

Where $(-log(z))^{r/\gamma} = \left(\sum_{k=1}^{\infty} \frac{(-1)^k (z-1)^k}{k}\right)^{r/\gamma}$

So, (19) becomes

$$I_1 = \left(\sum_{k=1}^{\infty} \frac{(-1)^k}{\beta k}\right)^{1/\gamma} \int_1^{e^{-\beta t^{\gamma}}} (z-1)^{k/\gamma} \alpha^z \theta (1-z)^{\theta-1}\, dz$$

The above integrals can be written as

$$I_1 = \left(\sum_{k=1}^{\infty} \frac{(-1)^k}{\beta k}\right)^{1/\gamma} \frac{(-1)^{\theta} \alpha \theta \overline{\left(\frac{k+\gamma\theta}{\gamma}, -\log(\alpha)(z-1)\right)}}{\left(-\log(\alpha)\right)^{\frac{k+\gamma\theta}{\gamma}}} \Bigg|_1^{e^{-\beta t^{\gamma}}} \tag{20}$$

Assuming that, $-\theta - 1 \neq 0, \theta \neq 0, \theta(2\theta+1) \neq 0, \gamma k \neq 0, \gamma(k+\gamma) \neq 0, \gamma(k+2\gamma) \neq 0, \gamma(k+\gamma\theta) \neq 0$

$$(k+\gamma)(2k+3\gamma) \neq 0, -log(\alpha) - 1 \neq 0, log(\alpha) \neq 0$$

Now, we will solve the second part of (17) which can be written as

$$I_2 = \int_t^{\infty} x\, \alpha^{e^{-\beta x^{\gamma}}}\, log(a) \beta \gamma x^{\gamma-1} \left(1 - e^{-\beta x^{\gamma}}\right)^{\theta} e^{-\beta x^{\gamma}} dx \tag{21}$$

Using the same transformation as we have used above, then (21) takes the form

$$I_2 = -log(\alpha) \left(\sum_{k=1}^{\infty} \frac{(-1)^k}{\beta k}\right)^{1/\gamma} \int_1^{e^{-\beta t^{\gamma}}} (z-1)^{k/\gamma} \alpha^z \theta (1-z)^{\theta-1}\, dz$$

We obtained the following result

$$I_2 = \log(\alpha) \left(\sum_{k=1}^{\infty} \frac{(-1)^k}{\beta k}\right)^{1/\gamma} \frac{(-1)^{\theta} \alpha \overline{\left(\frac{k+\gamma\theta+\gamma}{\gamma}, -\log(\alpha)(z-1)\right)}}{\left(-\log(\alpha)\right)^{\frac{k+\gamma\theta+\gamma}{\gamma}}} \Bigg|_1^{e^{-\beta t^{\gamma}}} \tag{22}$$

It was assuming that $-\theta - 2 \neq 0, -\theta - 1 \neq 0, (\theta+1)(2\theta+3) \neq 0, \gamma(k+\gamma) \neq 0, \gamma(k+2\gamma) \neq 0, \gamma\left(k+\gamma(\theta+1)\right) \neq 0$

$$(k+\gamma)(2k+3\gamma) \neq 0, -log(\alpha) - 1 \neq 0, log(\alpha) \neq 0$$

Substituting (20) and (22) in (17) then put in (16), we get the final result

$$\mu(t)=\frac{\left(\sum_{k=1}^{\infty}\frac{(-1)^k}{\beta k}\right)^{1/\gamma}\left(\frac{(-1)^{\theta}\alpha\sqrt{\left(\frac{k+\gamma\theta+\gamma}{\gamma},-\log(\alpha)(z-1)\right)}+\theta\sqrt{\left(\frac{k+\gamma\theta}{\gamma},-\log(\alpha)(z-1)\right)}}{\left(-\log(\alpha)\right)^{\frac{k+\gamma\theta}{\gamma}}}\right)\Bigg|_{1}^{e^{-\beta t^{\gamma}}}-t}{1-\alpha^{e^{-\beta t^{\gamma}}}\left(1-e^{-\beta t^{\gamma}}\right)^{\theta}}$$

3. Parameters Estimation

Let $x_1, x_2, \ldots, x_n$ be an independent sample of size n from the EAPW distribution with parameters $\alpha, \beta, \gamma, \theta$. The likelihood function for the sample is:

$$\mathrm{L}(\alpha,\beta,\gamma,\theta)=\prod_{i=1}^{n} f(x,\alpha,\beta,\gamma,\theta)$$

Where

$$\begin{aligned} f(x,\alpha,\beta,\gamma,\theta) &= \alpha^{e^{-\beta x^{\gamma}}}\beta\gamma\theta x^{\gamma-1}\left(1-e^{-\beta x^{\gamma}}\right)^{\theta-1}e^{-\beta x^{\gamma}} \\ &\quad -\alpha^{e^{-\beta x^{\gamma}}}\,log(\,a)\beta\gamma x^{\gamma-1}\left(1-e^{-\beta x^{\gamma}}\right)^{\theta}e^{-\beta x^{\gamma}} \end{aligned} \tag{23}$$

The likelihood function of the proposed model can be expressed as

$$\begin{aligned} L &= \alpha^{e^{-\beta\sum {x_i}^{\gamma}}}(\beta\gamma\theta)^n\sum {x_i}^{\gamma-1}\left(1-e^{-\beta\sum {x_i}^{\gamma}}\right)^{\theta-1}e^{-\beta\sum {x_i}^{\gamma}} \\ &\quad -\alpha^{e^{-\beta\sum {x_i}^{\gamma}}}\,log(\,a)(\beta\gamma)^n\sum {x_i}^{\gamma-1}\left(1-e^{-\beta\sum {x_i}^{\gamma}}\right)^{\theta}e^{-\beta\sum {x_i}^{\gamma}} \end{aligned} \tag{24}$$

The logarithmic function is

$$\begin{aligned} log(\,L) &= e^{-\beta\sum {x_i}^{\gamma}}\,log(\,\alpha)+n\,log(\,\beta\gamma\theta)+(\gamma-1)\,log\sum x_i+(\theta \\ &\quad -1)\,log\left(1-e^{-\beta\sum {x_i}^{\gamma}}\right)-\beta\sum x_i^{\gamma}-e^{-\beta\sum {x_i}^{\gamma}}\,log(\,\alpha)+log(log(\,\alpha)) \\ &\quad +n\,log(\,\beta\gamma)+(\gamma-1)\,log\sum x_i+\theta\,log\left(1-e^{-\beta\sum {x_i}^{\gamma}}\right)-\beta\sum x_i^{\gamma} \end{aligned}$$

$$\begin{aligned} log\,L &= n\,log(\,\beta\gamma\theta)+n\,log(\,\beta\gamma)+log(log(\,\alpha))+2(\gamma-1)\,log\sum x_i-2\beta\sum x_i^{\gamma}+(2\theta \\ &\quad -1)\,log\left(1-e^{-\beta\sum {x_i}^{\gamma}}\right) \end{aligned} \tag{25}$$

Setting the first partial derivatives of the log-likelihood with respect to $\alpha, \beta, \gamma, \theta$ to zero yields the score equations. Let

$$\frac{d\,log\,L}{d\alpha}=\frac{1}{\alpha\,log(\,\alpha)} \tag{26}$$

$$\frac{d\,log\,L}{d\beta}=\frac{(2\theta-1)\sum {x_i}^{\gamma}e^{-\beta\sum {x_i}^{\gamma}}}{1-e^{-\beta\sum {x_i}^{\gamma}}}+\frac{2n}{\beta}-2\sum {x_i}^{\gamma}$$

(28)

$$\frac{d\,log\,L}{d\gamma} = \frac{(2\theta-1)\beta\sum x_i{}^{\gamma}e^{-\beta\sum x_i{}^{\gamma}}\,log(\sum x_i)}{1-e^{-\beta\sum x_i{}^{\gamma}}} + \frac{2n}{\gamma} - 2\beta\sum x_i{}^{\gamma}\,log\left(\sum x_i\right) + 2\,log\left(\sum x_i\right)$$
$$\frac{d\,log\,L}{d\theta} = \frac{n}{2} + 2\,log\left(1-e^{-\beta\sum x_i{}^{\gamma}}\right) \quad (27)$$

Since equations (26)–(29) do not admit closed-form solutions, estimating the unknown parameters is challenging. However, maximum likelihood estimates must be obtained numerically, using techniques such as the bisection method or the Newton–Raphson algorithm.

3.1. Shannon Entropy

The entropy of a random variable X quantifies the average uncertainty associated with its possible outcomes. Introduced by Claude Shannon in 1948, entropy is defined as:

$$S_E(x) = -\int_0^{\infty} f(x)\,log\,f\,(x)\,dx \quad (30)$$

Where

$$f(x) = \alpha^{e^{-\beta x^{\gamma}}}\beta\gamma\theta x^{\gamma-1}\left(1-e^{-\beta x^{\gamma}}\right)^{\theta-1}e^{-\beta x^{\gamma}} - \alpha^{e^{-\beta x^{\gamma}}}\,log(\,a)\beta\gamma x^{\gamma-1}\left(1-e^{-\beta x^{\gamma}}\right)^{\theta}e^{-\beta x^{\gamma}}$$

and

$$log\,f\,(x) = log(\,\theta) - log\left(1-e^{-\beta\sum x_i{}^{\gamma}}\right) - log(log(\,\alpha))$$

Substituting both expressions above in (30), we get the form

$$S_E(x) = \int_0^{\infty} -log\left(1-e^{-\beta\sum x_i{}^{\gamma}}\right)\alpha^{e^{-\beta x^{\gamma}}}\beta\gamma\theta x^{\gamma-1}\left(1-e^{-\beta x^{\gamma}}\right)^{\theta-1}e^{-\beta x^{\gamma}}dx + \int_0^{\infty}\alpha^{e^{-\beta x^{\gamma}}}\,log(\,a)\beta\gamma x^{\gamma-1}\left(1-e^{-\beta x^{\gamma}}\right)^{\theta}e^{-\beta x^{\gamma}}\,log\left(1-e^{-\beta\sum x_i{}^{\gamma}}\right)dx \quad (31)$$

Equation (31) can be solved in parts. The first part of the equation is given as

$$I_1 = \int_0^{\infty} -log\left(1-e^{-\beta\sum x_i{}^{\gamma}}\right)\alpha^{e^{-\beta x^{\gamma}}}\beta\gamma\theta x^{\gamma-1}\left(1-e^{-\beta x^{\gamma}}\right)^{\theta-1}e^{-\beta x^{\gamma}}dx \quad (32)$$

Let $z = e^{-\beta x^{\gamma}}, -dz = \beta\gamma x^{\gamma-1}e^{-\beta x^{\gamma}}$ where, the range of variable z is $1 < z < 0$, then (32) takes the form

$$I_1 = \int_1^0 \alpha^z\theta(1-z)^{\theta-1}\,log(\,1-z)dz \quad (33)$$

Where

$$log(\,1-z) = -\sum_{k=1}^{\infty}\frac{(-1)^k(-z)^k}{k}, for|z| < 1$$

, and

$$\theta(1-z)^{\theta-1} = \sum_{n=0}^{\infty} z^n (-1)^n \left(d \binom{d-1}{n} \right), for |z| > 1$$

Using the above series representations (33) becomes

$$\mathrm{I}_1 = \sum_{\mathrm{k=1}}^{\infty} \sum_{\mathrm{n=0}}^{\infty} \frac{(-1)^{\mathrm{k+n}}}{\mathrm{k}} \left(\mathrm{d} \binom{\mathrm{d}-1}{\mathrm{n}} \right) \int_0^1 \alpha^{\mathrm{z}} \mathrm{z}^{\mathrm{n}} (-\mathrm{z})^{\mathrm{n}} \mathrm{dz}$$

We obtained the following result

$$I_1 = \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{k+n}}{k} \left(d \binom{d-1}{n} \right) \left(\frac{-\log(\alpha)^{-n-k} (-1)^k \Gamma(n+k+1, -\log(\alpha)) - \Gamma(n+k+1,0)}{\log(\alpha)} \right) \quad (34)$$

Now, solving the second part of (31), we have used the same transformations as earlier

$$I_2 = log(\alpha) \int_0^1 \alpha^z (1-z)^\theta \, log(1-z) \, dz \quad (35)$$

Where $(1-z)^\theta = \sum_{k=0}^{\infty} (-z)^k \binom{d}{k}$, and $log(1-z) = -\sum_{k=1}^{\infty} \frac{(-1)^k (-z)^k}{k}, for |z| < 1$

Putting the above expression in (35), we get

$$I_2 = -log(a) \sum_{k=1}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k}{k} \binom{d}{k} \int_0^1 \alpha^z (-z)^{2k} \, dz$$

$$I_2 = -\log(a) \sum_{k=1}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k}{k} \binom{d}{k} \left(\frac{(-1)^{2k} \Gamma(2k+1, -\log(\alpha)) - \Gamma(2k+1,0)}{(-\log(\alpha))^{2k} \log(\alpha)} \right) \quad (36)$$

By combining (34) and (36), we get the final result

$$S_E(x) = \log(a) \sum_{k=1}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k}{k} \binom{d}{k} \left(\frac{(-1)^{2k} \Gamma(2k+1, -\log(\alpha)) - \Gamma(2k+1,0)}{(-\log(\alpha))^{2k} \log(\alpha)} \right)$$
$$- \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{k+n}}{k} \left(d \binom{d-1}{n} \right) \left(\frac{-\log(\alpha)^{-n-k} (-1)^k \Gamma(n+k+1, -\log(\alpha)) - \Gamma(n+k+1,0)}{\log(\alpha)} \right)$$

4. Simulation Study

A Monte Carlo study was performed to assess the finite-sample behavior of parameter estimates within the EAPW model. Using true parameter values $(\alpha, \beta, \gamma, \lambda) = (0.5, 1.0, 1.5, 1.0)$, we generated 1,000 random samples of sizes n = 50, 100, 200 via the quantile inversion approach (outlined in Section 2.3). Maximum likelihood estimates (MLEs) were obtained for each sample, and their performance was evaluated in terms of bias and root mean squared error (RMSE). Results shown in Table 1 indicated that the MLEs are approximately unbiased even for n=50, and the RMSE decreases roughly with $1/\sqrt{n}$ as expected. This suggests that maximum likelihood is a reliable estimation method for the EAPW model in

moderate samples. We also verified that the empirical coverage of asymptotic 95% confidence intervals (based on the Fisher information) was close to nominal for $n \geq 100$.

In addition to parameter recovery, we used simulated data to check the usage of AIC/BIC and graphical diagnostics in distinguishing EAPW from simpler models. In simulations where data were generated from EAPW with a clear non-monotonic hazard as shown in Fig. 4, we found that fitting a standard Weibull or APW gave significantly larger AIC values, confirming that the EAPW can better capture the true shape. Conversely, when data were generated from a plain Weibull ($\alpha = 1$), the EAPW model fitted typically collapsed to $\hat{\alpha} \approx 1$, and AIC selected the simpler Weibull model (with nearly equal AIC). This indicates that the extra parameter α will be used only when needed. Overall, the simulation study supports the practical applicability of EAPW fitting and confirms that the MLE and model selection behave sensibly.

Table 1

$\boldsymbol{\alpha}$	β	γ	λ	n	$\boldsymbol{Bias(\hat{\alpha})}$	$\boldsymbol{Bias(\beta)}$	$\boldsymbol{RMSE(\hat{\alpha})}$	$\boldsymbol{RMSE(\beta)}$
0.5	1.0	1.5	1.0	50	-1.309	0.434	1.315	0.497
0.5	1.0	1.5	1.0	100	-1.302	0.451	1.304	0.483
0.5	1.0	1.5	1.0	200	-1.292	0.466	1.293	0.482
1.2	0.5	0.8	1.0	50	-1.520	0.879	1.525	0.900
1.2	0.5	0.8	1.0	100	-1.524	0.898	1.527	0.909
1.2	0.5	0.8	1.0	200	-1.521	0.910	1.523	0.916

5. Applications

To illustrate its real-world relevance, the proposed probabilistic model was evaluated on two empirical datasets characterized by non-monotonic hazard-rate behavior. Its performance was compared with established models, including the Weibull, GAPW, EIFW, and APW distributions. Model comparison was carried out using standard goodness-of-fit metrics such as AIC, BIC, HQIC, and CAIC. The mathematical expressions for these criteria are as follows:

$$AIC = -2l(\psi; x_i) + 2p$$

In this context, ℓ represents the log-likelihood maximized using the observed data, $\hat{\psi}$ is the MLE of the parameter vector ψ, and p denotes the total number of estimated model parameters.

$$CAIC = -2l + p(ln(n) + 1)$$

$$BIC = pln(n) - 2l(\psi; x_i)$$

$$HQIC = -2l_{max} + 2Pln(ln(n))$$

Where $l_{\max}$ denotes the maximum log-likelihood, p is the number of parameters, and n is the sample size.

Data Set 1: Active Repair Times (hours): This data set consist of the maintenance data reported on active repair times for an airborne communication transceiver taken from Von Alven [27] and also used by Sultan et.al [28] and Dey et.al [29]. The repair times are

0.2, 0.3, 0.5, 0.5, 0.5, 0.5, 0.6, 0.6, 0.7, 0.7, 0.7, 0.8, 0.8, 1, 1, 1, 1, 1.1, 1.3, 1.5, 1.5, 1.5, 1.5, 2, 2, 2.2, 2.5, 2.7, 3, 3, 3.3, 3.3, 4, 4, 4.5, 4.7, 4.7, 5, 5.4, 5.4, 7, 7.5, 8.8, 9, 10.3, 22, 24.5.

Table 2: Maximum Likelihood Estimates for Active Repair Times Data

Model	**Estimates**
WPE	0.08455144, 3.81817861, 0.23790647, 55.14602340
W	0.3250289, 0.9094521
GAPW	33.7248114, 29.1006319, -0.5502714
APW	13.6387337, 0.2290454, 37.8566427
EIFW	18.790997, 1.581064

Table 3: Goodness of Fit Criteria for Active Repair Times Data

Model	**AIC**	**CAIC**	**BIC**	**HQIC**
WPE	213.2798	214.2322	220.6804	216.0647
W	218.3281	218.6009	222.0284	219.7206
GAPW	1346.369	1346.869	1352.223	1348.613
APW	516.4676	516.9783	522.2631	518.6822
EIFW	584.7542	585.0042	588.6179	586.2306

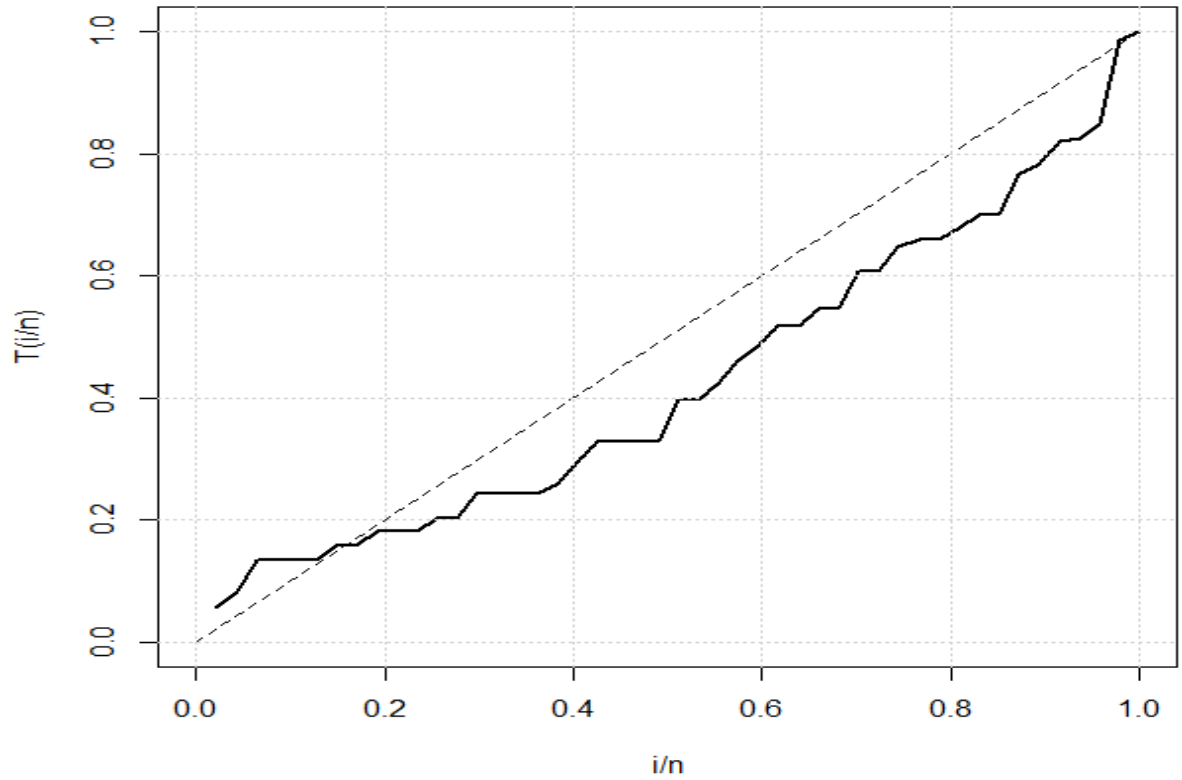


Fig 5: TTT plot of Active Repair Times

Data set 2: Successive Failures for the Air Conditioning System: This data is about the number of successive failures for the air conditioning system of each member in a fleet of 13 Boeing 720 jet airplanes. This data is taken from Micheal et.al [30] also analyzed by Merovei et.al [31]. The data are observed as **:** 194, 413, 90, 74, 55, 23, 97, 50, 359, 50, 130, 487, 57, 102, 15, 14, 10, 57, 320, 261, 51, 44, 9, 254, 493, 33, 18, 209, 41, 58, 60, 48, 56, 87, 11, 102, 12, 5, 14, 14, 29, 37, 186, 29, 104, 7, 4, 72, 270, 283, 7, 61, 100, 61, 502, 220, 120, 141, 22, 603, 35, 98, 54, 100, 11, 181, 65, 49, 12, 239, 14, 18, 39, 3, 12, 5, 32, 9, 438, 43, 134, 184, 20, 386, 182, 71, 80, 188, 230, 152, 5, 36, 79, 59, 33, 246, 1, 79, 3, 27, 201, 84, 27, 156, 21, 16, 88, 130, 14, 118, 44, 15, 42, 106, 46, 230, 26, 59, 153, 104, 20, 206, 5, 66, 34, 29, 26, 35, 5, 82, 31, 118, 326, 12, 54, 36, 34, 18, 25, 120, 31, 22, 18, 216, 139, 67, 310, 3, 46, 210, 57, 76, 14, 111, 97, 62, 39, 30, 7, 44, 11, 63, 23, 22, 23, 14, 18, 13, 34, 16, 18, 130, 90, 163, 208, 1, 24, 70, 16, 101, 52, 208, 95, 62, 11, 191, 14, 71.

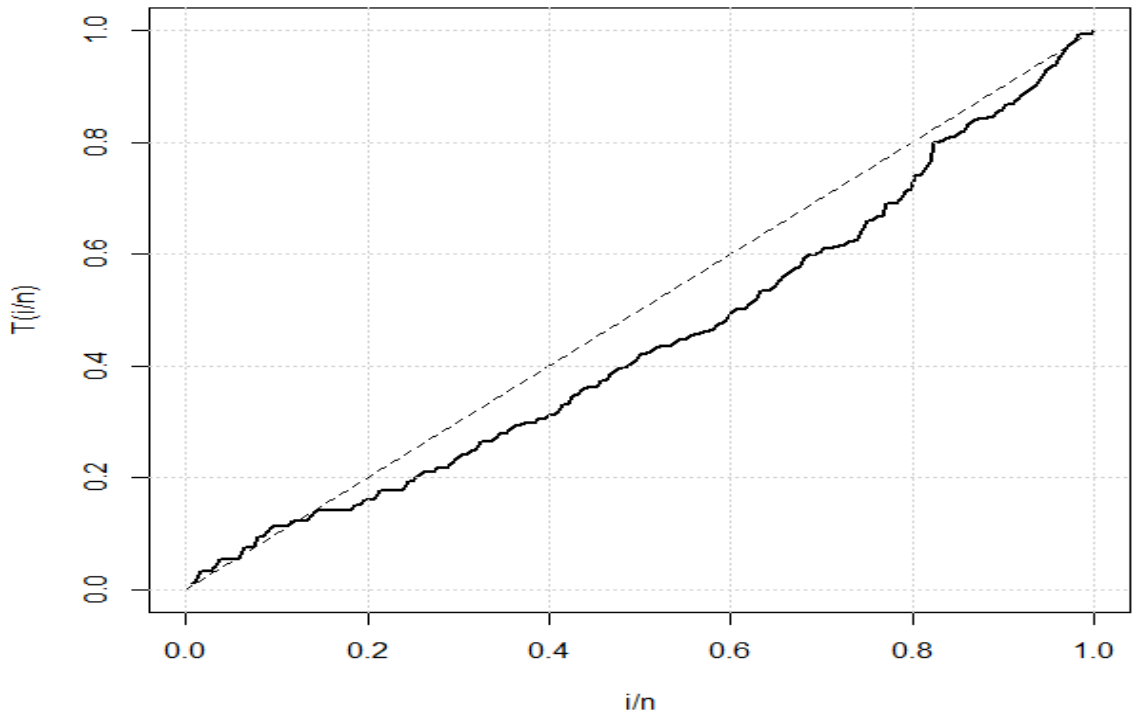


Fig-6: TTT Plot for the Air Conditioning Data Set

Maximum likelihood estimates for both the main and nested models are provided in Tables 2 and 4, whereas model selection criteria (AIC, BIC, CAIC, HQIC) reported in Tables 3 and 5 consistently favor the proposed model over its competitors, including the Weibull, Alpha Power Weibull, Gull Alpha Power Weibull, and Exponentiated Inverse Flexible Weibull distributions. Additionally, the TTT plots in Figures 5 and 6 reveal a bathtub-shaped hazard rate for both datasets, further confirming the suitability of the proposed model.

Table 4: Maximum Likelihood Estimates Failures for the Air Conditioning System

Model	**Estimates**
WPE	2.576081437, 0.002146581, 2.044396730, 7.166527518
W	0.01757475, 1.17697522
GAPW	31.438351, 29.944842, -1.053977
APW	0.357614517, 1.546842656, 0.002276946

Table 5: Goodness of Fit Criteria Failures for the Air Conditioning System

Model	**AIC**	**CAIC**	**BIC**	**HQIC**
WPE	216.2118	217.7502	221.9477	218.0816
W	269.0599	269.4885	271.9279	269.9948
GAPW	363.7817	364.6705	368.0836	365.184
APW	263.9863	264.8751	268.2882	265.3886

6. Discussion and conclusion

This paper aims to develop a new flexible probability model which shape non-monotonic hazard function. We proposed a novel generator for the Weibull distribution, leading to the Extended Alpha Power family of probability models. A key focus was the Extended Alpha Power Weibull (EAPW) distribution, developed using both Weibull and exponential baselines. Key statistical characteristics, such as the survival, hazard, and mean residual life functions, along with the rth moments, were thoroughly explored, and parameters were estimated using maximum likelihood techniques. Model performance, evaluated using AIC, BIC, CAIC, and HQIC criteria, demonstrated that the proposed EAPW model outperforms several well-established distributions.

The Extended Alpha Power Weibull (EAPW) distribution extends the Weibull model via an alpha-power generator, forming a flexible four-parameter family for complex hazard-rate shapes. The study derives clear expressions for key reliability measures, including survival and hazard rates, mean residual life, and higher moments, enabling streamlined and effective likelihood-based inference. Empirical results show that the EAPW outperforms classical Weibull and related models, yielding lower AIC and BIC values for reliability data with bathtub-shaped hazards. These findings indicate that the EAPW is a powerful tool for modeling lifetime data with complex failure patterns.

However, the EAPW's added flexibility introduces challenges, as its four-parameter structure can complicate estimation, leading to potential multi-modal likelihoods and requiring larger samples for stable inference. In contexts where the true hazard is strictly monotonic or follows a simpler pattern, the full EAPW may be unnecessary or risk overfitting relative to a more parsimonious model. Additionally, our validation focused on complete-sample bathtub-shaped data; the EAPW's performance under alternative scenarios (such as heavy censoring, competing risks, or hazard shapes outside those tested) remains to be explored. Practitioners should therefore apply the EAPW model with care, using goodness-of-fit diagnostics and considering simpler alternatives when appropriate. Several avenues for future research arise from this work. A potential direction is to extend the alpha-power construction to other baseline distributions, such as the gamma or log-normal, to develop new flexible families of lifetime models. Embedding the EAPW within regression frameworks (for example, accelerated failure-time or proportional-hazards models) would broaden its applicability to covariate-driven reliability and survival analyses. Further theoretical investigation, of parameter

identifiability, asymptotic behavior, or information-theoretic characteristics (such as entropy), would clarify the model's scope and limitations. Methodologically, Bayesian or penalized estimation approaches could enhance inference robustness in highly parameterized settings. Finally, testing the EAPW on a wider range of engineering reliability problems (including censored or truncated data, system reliability configurations, and multi-component lifetimes) will validate its generality and uncover any additional practical constraints.

Declarations

Ethical Approval

Not applicable. This study did not involve human participants or animals.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Availability of Data

The data used and analyzed in this article are available in the published article as cited against each data sets in the numerical section of this article

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Competing interests

The author(s) declares no competing interests.

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