

## A Supplementary calculations

The main idea of the ESS model is to calculate the local elastic energy for a number of allowed director structures, comparing them, and picking the structure with the lowest elastic energy. To calculate the elastic energy density based on the expression for the director, the Oseen-Frank energy is used. This calculation is detailed below. First the divergence and curl of the director are calculated, followed by the splay, twist and bend terms of the Oseen-Frank energy.

### A.1 Divergence

The divergence in cylindrical coordinates is defined as

$$\nabla \cdot \hat{\mathbf{n}} = \frac{1}{r} \frac{\partial(rn_r)}{\partial r} + \frac{1}{r} \frac{\partial n_\theta}{\partial \theta} + \frac{\partial n_z}{\partial z}. \quad (1)$$

We calculate each term separately.

$$\frac{1}{r} \frac{\partial(rn_r)}{\partial r} = \frac{1}{r} \frac{\partial}{\partial r} \left( -r \cos \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \sin \left( z \frac{\theta + \Delta}{d} \right) \right) = -\frac{1}{r} \cos \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \sin \left( z \frac{\theta + \Delta}{d} \right) \quad (2)$$

$$\frac{1}{r} \frac{\partial n_\theta}{\partial \theta} = \frac{1}{r} \frac{\partial}{\partial \theta} \left( \cos \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \cos \left( z \frac{\theta + \Delta}{d} \right) \right) = -\frac{z}{rd} \cos \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \sin \left( z \frac{\theta + \Delta}{d} \right) \quad (3)$$

$$\frac{\partial n_z}{\partial z} = \frac{\partial}{\partial z} \sin \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) = -\frac{\alpha_c + \alpha_l}{d} \cos \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \quad (4)$$

We then take the sum of the three terms.

$$\nabla \cdot \hat{\mathbf{n}} = -\frac{d+z}{rd} \cos \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \sin \left( z \frac{\theta + \Delta}{d} \right) - \frac{\alpha_c + \alpha_l}{d} \cos \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \quad (5)$$

### A.2 Curl

The curl in cylindrical coordinates is given by

$$\nabla \times \hat{\mathbf{n}} = \left( \frac{1}{r} \frac{\partial n_z}{\partial \theta} - \frac{\partial n_\theta}{\partial z}, \frac{\partial n_r}{\partial z} - \frac{\partial n_z}{\partial r}, \frac{1}{r} \left( \frac{\partial(rn_\theta)}{\partial r} - \frac{\partial n_r}{\partial \theta} \right) \right). \quad (6)$$

We calculate each component separately. For the  $r$  component, we note that  $\frac{1}{r} \frac{\partial n_z}{\partial \theta} = 0$ .

$$\begin{aligned} (\nabla \times \hat{\mathbf{n}})_r &= \frac{1}{r} \frac{\partial n_z}{\partial \theta} - \frac{\partial n_\theta}{\partial z} = \frac{1}{r} \frac{\partial}{\partial \theta} \sin \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) - \frac{\partial}{\partial z} \cos \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \cos \left( z \frac{\theta + \Delta}{d} \right) = \\ &= 0 - \frac{\alpha_c + \alpha_l}{d} \sin \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \cos \left( z \frac{\theta + \Delta}{d} \right) + \frac{\theta + \Delta}{d} \cos \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \sin \left( z \frac{\theta + \Delta}{d} \right) \end{aligned} \quad (7)$$

For the  $\theta$  component, we can see that  $\frac{\partial n_z}{\partial r} = 0$ .

$$\begin{aligned} (\nabla \times \hat{\mathbf{n}})_\theta &= \frac{\partial n_r}{\partial z} - \frac{\partial n_z}{\partial r} = -\frac{\partial}{\partial z} \cos \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \sin \left( z \frac{\theta + \Delta}{d} \right) - \frac{\partial}{\partial r} \sin \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) = \\ &= -\frac{\alpha_c + \alpha_l}{d} \sin \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \sin \left( z \frac{\theta + \Delta}{d} \right) - \frac{\theta + \Delta}{d} \cos \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \cos \left( z \frac{\theta + \Delta}{d} \right) - 0 \end{aligned} \quad (8)$$

For the  $z$  component, both terms end up on the form  $x \cdot \cos\left(\alpha_c - z\frac{\alpha_c + \alpha_l}{d}\right) \cos\left(z\frac{\theta + \Delta}{d}\right)$ , where  $x$  is  $1/r$  in the first term and  $z/dr$  in the second term, allowing us to add the two terms together.

$$\begin{aligned}
(\nabla \times \hat{\mathbf{n}})_z &= \frac{1}{r} \left( \frac{\partial(rn_\theta)}{\partial r} - \frac{\partial n_r}{\partial \theta} \right) = \\
&= \frac{1}{r} \frac{\partial}{\partial r} r \cos\left(\alpha_c - z\frac{\alpha_c + \alpha_l}{d}\right) \cos\left(z\frac{\theta + \Delta}{d}\right) + \frac{1}{r} \frac{\partial}{\partial \theta} \cos\left(\alpha_c - z\frac{\alpha_c + \alpha_l}{d}\right) \sin\left(z\frac{\theta + \Delta}{d}\right) = \\
&= \frac{d+z}{dr} \cos\left(\alpha_c - z\frac{\alpha_c + \alpha_l}{d}\right) \cos\left(z\frac{\theta + \Delta}{d}\right)
\end{aligned} \tag{9}$$

We now have all three components of  $(\nabla \times \hat{\mathbf{n}})$ .

### A.3 Splay term

For the splay term of the energy density, we need to calculate  $(\nabla \cdot \hat{\mathbf{n}})^2$ . We take advantage of having calculated  $\nabla \cdot \hat{\mathbf{n}}$  earlier.

$$\begin{aligned}
(\nabla \cdot \hat{\mathbf{n}})^2 &= \left( -\frac{d+z}{rd} \cos\left(\alpha_c - z\frac{\alpha_c + \alpha_l}{d}\right) \sin\left(z\frac{\theta + \Delta}{d}\right) - \frac{\alpha_c + \alpha_l}{d} \cos\left(\alpha_c - z\frac{\alpha_c + \alpha_l}{d}\right) \right)^2 = \\
&= \cos^2\left(\alpha_c - z\frac{\alpha_c + \alpha_l}{d}\right) \left( \frac{(d+z)^2}{r^2 d^2} \sin^2\left(z\frac{\theta + \Delta}{d}\right) + \frac{2(d+z)(\alpha_c + \alpha_l)}{d^2 r} \sin\left(z\frac{\theta + \Delta}{d}\right) + \frac{(\alpha_c + \alpha_l)^2}{d^2} \right)
\end{aligned} \tag{10}$$

This can then be multiplied by  $K_{11}/2$  to get the splay term.

### A.4 Twist term

For the twist term we need to calculate  $(\hat{\mathbf{n}} \cdot (\nabla \times \hat{\mathbf{n}}))^2$ . We have already calculated  $\nabla \times \hat{\mathbf{n}}$ , so the first step will be  $\hat{\mathbf{n}} \cdot (\nabla \times \hat{\mathbf{n}})$ . From

$$\hat{\mathbf{n}} \cdot (\nabla \times \hat{\mathbf{n}}) = n_r (\nabla \times \hat{\mathbf{n}})_r + n_\theta (\nabla \times \hat{\mathbf{n}})_\theta + n_z (\nabla \times \hat{\mathbf{n}})_z, \tag{11}$$

we first calculate one term at a time.

$$\begin{aligned}
n_r (\nabla \times \hat{\mathbf{n}})_r &= -\cos\left(\alpha_c - z\frac{\alpha_c + \alpha_l}{d}\right) \sin\left(z\frac{\theta + \Delta}{d}\right) \cdot \\
&\cdot \left( -\frac{\alpha_c + \alpha_l}{d} \sin\left(\alpha_c - z\frac{\alpha_c + \alpha_l}{d}\right) \cos\left(z\frac{\theta + \Delta}{d}\right) + \frac{\theta + \Delta}{d} \cos\left(\alpha_c - z\frac{\alpha_c + \alpha_l}{d}\right) \sin\left(z\frac{\theta + \Delta}{d}\right) \right) = \\
&= \frac{\alpha_c + \alpha_l}{d} \sin\left(\alpha_c - z\frac{\alpha_c + \alpha_l}{d}\right) \cos\left(\alpha_c - z\frac{\alpha_c + \alpha_l}{d}\right) \cos\left(z\frac{\theta + \Delta}{d}\right) \sin\left(z\frac{\theta + \Delta}{d}\right) - \\
&\quad - \frac{\theta + \Delta}{d} \cos^2\left(\alpha_c - z\frac{\alpha_c + \alpha_l}{d}\right) \sin^2\left(z\frac{\theta + \Delta}{d}\right)
\end{aligned} \tag{12}$$

$$\begin{aligned}
n_\theta (\nabla \times \hat{\mathbf{n}})_\theta &= \cos\left(\alpha_c - z\frac{\alpha_c + \alpha_l}{d}\right) \cos\left(z\frac{\theta + \Delta}{d}\right) \cdot \\
&\cdot \left( -\frac{\alpha_c + \alpha_l}{d} \sin\left(\alpha_c - z\frac{\alpha_c + \alpha_l}{d}\right) \sin\left(z\frac{\theta + \Delta}{d}\right) - \frac{\theta + \Delta}{d} \cos\left(\alpha_c - z\frac{\alpha_c + \alpha_l}{d}\right) \cos\left(z\frac{\theta + \Delta}{d}\right) \right) =
\end{aligned}$$

$$\begin{aligned}
&= -\frac{\alpha_c + \alpha_l}{d} \sin\left(\alpha_c - z\frac{\alpha_c + \alpha_l}{d}\right) \cos\left(\alpha_c - z\frac{\alpha_c + \alpha_l}{d}\right) \sin\left(z\frac{\theta + \Delta}{d}\right) \cos\left(z\frac{\theta + \Delta}{d}\right) - \\
&\quad -\frac{\theta + \Delta}{d} \cos^2\left(\alpha_c - z\frac{\alpha_c + \alpha_l}{d}\right) \cos^2\left(z\frac{\theta + \Delta}{d}\right)
\end{aligned} \tag{13}$$

$$n_z (\nabla \times \hat{\mathbf{n}})_z = \frac{d+z}{dr} \sin\left(\alpha_c - z\frac{\alpha_c + \alpha_l}{d}\right) \cos\left(\alpha_c - z\frac{\alpha_c + \alpha_l}{d}\right) \cos\left(z\frac{\theta + \Delta}{d}\right) \tag{14}$$

We then take the sum of these three terms.

$$\begin{aligned}
&\hat{\mathbf{n}} \cdot (\nabla \times \hat{\mathbf{n}}) = \\
&= \frac{\alpha_c + \alpha_l}{d} \sin\left(\alpha_c - z\frac{\alpha_c + \alpha_l}{d}\right) \cos\left(\alpha_c - z\frac{\alpha_c + \alpha_l}{d}\right) \cos\left(z\frac{\theta + \Delta}{d}\right) \sin\left(z\frac{\theta + \Delta}{d}\right) - \\
&\quad -\frac{\theta + \Delta}{d} \cos^2\left(\alpha_c - z\frac{\alpha_c + \alpha_l}{d}\right) \sin^2\left(z\frac{\theta + \Delta}{d}\right) - \\
&\quad -\frac{\alpha_c + \alpha_l}{d} \sin\left(\alpha_c - z\frac{\alpha_c + \alpha_l}{d}\right) \cos\left(\alpha_c - z\frac{\alpha_c + \alpha_l}{d}\right) \sin\left(z\frac{\theta + \Delta}{d}\right) \cos\left(z\frac{\theta + \Delta}{d}\right) - \\
&\quad -\frac{\theta + \Delta}{d} \cos^2\left(\alpha_c - z\frac{\alpha_c + \alpha_l}{d}\right) \cos^2\left(z\frac{\theta + \Delta}{d}\right) + \\
&\quad + \frac{d+z}{dr} \sin\left(\alpha_c - z\frac{\alpha_c + \alpha_l}{d}\right) \cos\left(\alpha_c - z\frac{\alpha_c + \alpha_l}{d}\right) \cos\left(z\frac{\theta + \Delta}{d}\right) = A1 - A2 - A3 - A4 + A5
\end{aligned} \tag{15}$$

The terms A1 and A3 cancel each other, and the terms A2 and A4 can be added together using the Pythagorean identity. We then get

$$\begin{aligned}
&\hat{\mathbf{n}} \cdot (\nabla \times \hat{\mathbf{n}}) = \\
&= \frac{d+h}{dr} \sin\left(\alpha_c - z\frac{\alpha_c + \alpha_l}{d}\right) \cos\left(\alpha_c - z\frac{\alpha_c + \alpha_l}{d}\right) \cos\left(z\frac{\theta + \Delta}{d}\right) - \frac{\theta + \Delta}{d} \cos^2\left(\alpha_c - z\frac{\alpha_c + \alpha_l}{d}\right)
\end{aligned} \tag{16}$$

Next, we take the absolute value of this.

$$\begin{aligned}
&|\hat{\mathbf{n}} \cdot (\nabla \times \hat{\mathbf{n}})| = \sqrt{(\hat{\mathbf{n}} \cdot (\nabla \times \hat{\mathbf{n}}))^2} \\
&= \sqrt{\left(\frac{d+z}{dr} \sin\left(\alpha_c - z\frac{\alpha_c + \alpha_l}{d}\right) \cos\left(\alpha_c - z\frac{\alpha_c + \alpha_l}{d}\right) \cos\left(z\frac{\theta + \Delta}{d}\right) - \frac{\theta + \Delta}{d} \cos^2\left(\alpha_c - z\frac{\alpha_c + \alpha_l}{d}\right)\right)^2}
\end{aligned} \tag{17}$$

Finally, we subtract  $q$  and take the square.

$$\begin{aligned}
&(|\hat{\mathbf{n}} \cdot (\nabla \times \hat{\mathbf{n}})| - q)^2 = (\hat{\mathbf{n}} \cdot (\nabla \times \hat{\mathbf{n}}))^2 - 2q\sqrt{(\hat{\mathbf{n}} \cdot (\nabla \times \hat{\mathbf{n}}))^2} + q^2 = \\
&= \left(\frac{d+z}{dr} \sin\left(\alpha_c - z\frac{\alpha_c + \alpha_l}{d}\right) \cos\left(\alpha_c - z\frac{\alpha_c + \alpha_l}{d}\right) \cos\left(z\frac{\theta + \Delta}{d}\right) - \frac{\theta + \Delta}{d} \cos^2\left(\alpha_c - z\frac{\alpha_c + \alpha_l}{d}\right)\right)^2 - \\
&- 2q\sqrt{\left(\frac{d+z}{dr} \sin\left(\alpha_c - z\frac{\alpha_c + \alpha_l}{d}\right) \cos\left(\alpha_c - z\frac{\alpha_c + \alpha_l}{d}\right) \cos\left(z\frac{\theta + \Delta}{d}\right) - \frac{\theta + \Delta}{d} \cos^2\left(\alpha_c - z\frac{\alpha_c + \alpha_l}{d}\right)\right)^2} + \\
&\quad + q^2
\end{aligned} \tag{18}$$

This can then be multiplied by  $K_{22}/2$  to get the twist term.

## A.5 Bend term

The bend term is  $|\hat{\mathbf{n}} \times (\nabla \times \hat{\mathbf{n}})|^2$ . Once again, we use our already calculated  $\nabla \times \hat{\mathbf{n}}$ . We have

$$(\hat{\mathbf{n}} \times (\nabla \times \hat{\mathbf{n}})) = (n_\theta (\nabla \times \hat{\mathbf{n}})_z - n_z (\nabla \times \hat{\mathbf{n}})_\theta, n_z (\nabla \times \hat{\mathbf{n}})_r - n_r (\nabla \times \hat{\mathbf{n}})_z, n_r (\nabla \times \hat{\mathbf{n}})_\theta - n_\theta (\nabla \times \hat{\mathbf{n}})_r), \quad (19)$$

and begin by calculating each of the three components separately.

$$\begin{aligned} (\hat{\mathbf{n}} \times (\nabla \times \hat{\mathbf{n}}))_r &= n_\theta (\nabla \times \hat{\mathbf{n}})_z - n_z (\nabla \times \hat{\mathbf{n}})_\theta = \frac{d+z}{dr} \cos^2 \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \cos^2 \left( z \frac{\theta + \Delta}{d} \right) + \\ &\quad + \frac{\alpha_c + \alpha_l}{d} \sin^2 \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \sin \left( z \frac{\theta + \Delta}{d} \right) + \\ &\quad + \frac{\theta + \Delta}{d} \cos \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \sin \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \cos \left( z \frac{\theta + \Delta}{d} \right) \end{aligned} \quad (20)$$

$$\begin{aligned} (\hat{\mathbf{n}} \times (\nabla \times \hat{\mathbf{n}}))_\theta &= n_z (\nabla \times \hat{\mathbf{n}})_r - n_r (\nabla \times \hat{\mathbf{n}})_z = \\ &= \frac{d+z}{dr} \cos^2 \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \sin \left( z \frac{\theta + \Delta}{d} \right) \cos \left( z \frac{\theta + \Delta}{d} \right) - \\ &\quad - \frac{\alpha_c + \alpha_l}{d} \sin^2 \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \cos \left( z \frac{\theta + \Delta}{d} \right) + \\ &\quad + \frac{\theta + \Delta}{d} \cos \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \sin \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \sin \left( z \frac{\theta + \Delta}{d} \right) \end{aligned} \quad (21)$$

For the  $z$  component, the terms B2 and B4 cancel each other, and the Pythagorean identity can be applied to the terms B1 and B3, as follows:

$$\begin{aligned} (\hat{\mathbf{n}} \times (\nabla \times \hat{\mathbf{n}}))_z &= n_r (\nabla \times \hat{\mathbf{n}})_\theta - n_\theta (\nabla \times \hat{\mathbf{n}})_r = \\ &= \frac{\alpha_c + \alpha_l}{d} \sin \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \cos \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \sin^2 \left( z \frac{\theta + \Delta}{d} \right) + \\ &\quad + \frac{\theta + \Delta}{d} \cos^2 \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \sin \left( z \frac{\theta + \Delta}{d} \right) \cos \left( z \frac{\theta + \Delta}{d} \right) + \\ &\quad + \frac{\alpha_c + \alpha_l}{d} \sin \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \cos \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \cos^2 \left( z \frac{\theta + \Delta}{d} \right) + \\ &\quad - \frac{\theta + \Delta}{d} \cos^2 \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \sin \left( z \frac{\theta + \Delta}{d} \right) \cos \left( z \frac{\theta + \Delta}{d} \right) = B1 + B2 + B3 - B4 = \\ &= \frac{\alpha_c + \alpha_l}{d} \sin \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \cos \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \end{aligned} \quad (22)$$

The next step is to take the square of each component.

$$\begin{aligned} (\hat{\mathbf{n}} \times (\nabla \times \hat{\mathbf{n}}))_r^2 &= \frac{(d+z)^2}{d^2 r^2} \cos^4 \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \cos^4 \left( z \frac{\theta + \Delta}{d} \right) + \\ &\quad + \frac{(\alpha_c + \alpha_l)^2}{d^2} \sin^4 \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \sin^2 \left( z \frac{\theta + \Delta}{d} \right) + \end{aligned}$$

$$\begin{aligned}
& + \frac{(\theta + \Delta)^2}{d^2} \cos^2 \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \sin^2 \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \cos^2 \left( z \frac{\theta + \Delta}{d} \right) + \\
& + 2 \frac{(d+z)(\alpha_c + \alpha_l)}{d^2 r} \sin \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \cos \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \sin \left( z \frac{\theta + \Delta}{d} \right) \cos \left( z \frac{\theta + \Delta}{d} \right) + \\
& + 2 \frac{(d+z)(\theta + \Delta)}{d^2 r} \sin \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \cos^3 \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \cos^3 \left( z \frac{\theta + \Delta}{d} \right) + \\
& + 2 \frac{(\alpha_c + \alpha_l)(\theta + \Delta)}{d^2} \sin^3 \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \cos \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \sin \left( z \frac{\theta + \Delta}{d} \right) \cos \left( z \frac{\theta + \Delta}{d} \right) = \\
& = C1 + C2 + C3 + C4 + C5 + C6
\end{aligned} \tag{23}$$

$$\begin{aligned}
(\hat{\mathbf{n}} \times (\nabla \times \hat{\mathbf{n}}))_\theta^2 &= \frac{(d+z)^2}{d^2 r^2} \cos^4 \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \sin^2 \left( z \frac{\theta + \Delta}{d} \right) \cos^2 \left( z \frac{\theta + \Delta}{d} \right) + \\
& + \frac{(\alpha_c + \alpha_l)^2}{d^2} \sin^4 \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \cos^2 \left( z \frac{\theta + \Delta}{d} \right) + \\
& + \frac{(\theta + \Delta)^2}{d^2} \cos^2 \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \sin^2 \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \sin^2 \left( z \frac{\theta + \Delta}{d} \right) + \\
& - 2 \frac{(d+z)(\alpha_c + \alpha_l)}{d^2 r} \sin \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \cos \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \sin \left( z \frac{\theta + \Delta}{d} \right) \cos \left( z \frac{\theta + \Delta}{d} \right) + \\
& + 2 \frac{(d+z)(\theta + \Delta)}{d^2 r} \sin \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \cos^3 \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \sin^2 \left( z \frac{\theta + \Delta}{d} \right) \cos \left( z \frac{\theta + \Delta}{d} \right) + \\
& - 2 \frac{(\alpha_c + \alpha_l)(\theta + \Delta)}{d^2} \sin^3 \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \cos \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \sin \left( z \frac{\theta + \Delta}{d} \right) \cos \left( z \frac{\theta + \Delta}{d} \right) = \\
& = C7 + C8 + C9 + C10 + C11 + C12
\end{aligned} \tag{24}$$

$$(\hat{\mathbf{n}} \times (\nabla \times \hat{\mathbf{n}}))_z^2 = \frac{(\alpha_c + \alpha_l)^2}{d^2} \sin^2 \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \cos^2 \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) = C13 \tag{25}$$

When adding these squared components together, we see that the terms C4, C6, C10 and C12 will cancel each other. We also see that each pair of terms (C1, C7), (C2, C8), (C3, C9) and (C5, C11) can be combined using the Pythagorean identity. Thus, the sum will have only five terms.

$$\begin{aligned}
|\hat{\mathbf{n}} \times (\nabla \times \hat{\mathbf{n}})|^2 &= \frac{(d+z)^2}{d^2 r^2} \cos^4 \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \cos^2 \left( z \frac{\theta + \Delta}{d} \right) + \\
& + \frac{(\alpha_c + \alpha_l)^2}{d^2} \sin^4 \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) + \frac{(\theta + \Delta)^2}{d^2} \cos^2 \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \sin^2 \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) + \\
& + 2 \frac{(d+z)(\theta + \Delta)}{d^2 r} \sin \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \cos^3 \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \cos \left( z \frac{\theta + \Delta}{d} \right) + \\
& + \frac{(\alpha_c + \alpha_l)^2}{d^2} \sin^2 \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \cos^2 \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) = D1 + D2 + D3 + D4 + D5
\end{aligned} \tag{26}$$

We further notice that the Pythagorean identity can be applied to the terms D2 and D5, giving us

$$\begin{aligned}
|\hat{\mathbf{n}} \times (\nabla \times \hat{\mathbf{n}})|^2 &= \frac{(d+z)^2}{d^2 r^2} \cos^4 \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \cos^2 \left( z \frac{\theta + \Delta}{d} \right) + \\
&+ \frac{(\alpha_c + \alpha_l)^2}{d^2} \sin^2 \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) + \frac{(\theta + \Delta)^2}{d^2} \cos^2 \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \sin^2 \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) + \\
&+ 2 \frac{(d+z)(\theta + \Delta)}{d^2 r} \sin \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \cos^3 \left( \alpha_c - z \frac{\alpha_c + \alpha_l}{d} \right) \cos \left( z \frac{\theta + \Delta}{d} \right), \tag{27}
\end{aligned}$$

which can then be multiplied by  $K_{33}/2$  to get the bend term.

## A.6 Numerical integration

The derivation so far provides a volume energy density. However, in order to make a useful comparison we must integrate over  $z$  to get an area energy density. This is done numerically in Matlab, according to

$$E_{OF}(r, \theta) = \sum_{i=1}^{1000} \frac{d}{1000} E_{OF}(r, \theta, z_i) \tag{28}$$

We do this for each allowed amount of twist, compare, and chose the one that yields the lowest area energy density locally. The process is repeated for each point  $(r, \theta)$ . The Matlab code used to run the model is available on Github [1].

## References

- [1] Ebba Grönfors. *ebba-gronfors/ESS-model: Electrostatic Stiffening Model v1.0.0*. 2025. DOI: 10.5281/zenodo.17779401.