The Appendix is organized as follows. We begin by introducing the notations and reviewing related work in Section A and Section B. Then, we present our theoretical analysis in three steps:

- **First step: Derivation of the effective Lindbladian dynamics.** The main result is stated in Theorem 7, which is a rigorous version of Theorem 4, with the proof given in Section C 1.

  After deriving the Lindblad dynamics, we demonstrate that two close CPTP maps have close fixed points and mixing times in Section D Theorem 8, which provides a useful tool for analyzing the fixed point and mixing time of Φ.

- **Second step: Fixed point error bounds for thermal and ground state preparation.** The main results are presented in Section E Theorem 9 and Theorem 10, corresponding to the thermal and ground states, respectively, with proofs provided in Section E 1 and Section E 2. Combining the results from the first two steps, we show that the fixed point of Φ is close to the target thermal or ground state when properly adjusting the parameters.

- **Third step: Mixing time and End-to-end efficiency analysis.** We present mixing-time results for several physically relevant models in Theorem 17, Theorem 18, Theorem 19, and Theorem 20 and derive end-to-end runtime estimates for our state preparation algorithm in Theorem 21. The proofs of these results are collected in Section G–Section K.

Appendix A: Notations and detailed balance condition

For a matrix $A \in \mathbb{C}^{N\times N}$, let $A^*, A^T, A^\dagger$ be the complex conjugation, transpose, and Hermitian transpose (or adjoint) of A, respectively. $\|A\|_p = \mathrm{Tr}\left(\left(\sqrt{A^\dagger A}\right)^p\right)^{1/p}$ denotes the Schatten p-norm. The Schatten 1-norm $\|A\|_1$ is also called the trace norm, the Schatten 2-norm $\|A\|_2$ is also called the Hilbert–Schmidt norm (or Frobenius norm for matrices), and the Schatten ∞-norm $\|A\|_\infty$ is the same as the operator norm $\|A\|$. The trace distance between two states ρ, σ is $D(\rho,\sigma) := \frac{1}{2}\|\rho-\sigma\|_1$. Given a superoperator $\Phi : \mathbb{C}^{N\times N} \to \mathbb{C}^{N\times N}$, we define the induced trace norm as

$$\|\Phi\|_{1\leftrightarrow 1} = \sup_{\|A\|_1=1} \|\Phi(A)\|_1 .$$

We denote eigenstates of the Hamiltonian H by $\{|\psi_i\rangle\}$ and the corresponding eigenvalues by $\{\lambda_i\}$. Each difference of eigenvalues $\lambda_i - \lambda_j$ is called a Bohr frequency, and $B(H)$ denotes the set of all Bohr frequencies. Also, given $\nu \in B(H)$ and a matrix A, we define

$$A(\nu) = \sum_{\lambda_j - \lambda_i = \nu} |\psi_j\rangle \langle\psi_j| A |\psi_i\rangle \langle\psi_i| , \tag{A1}$$

where $|\psi_i\rangle$ is an eigenvector of H with eigenvalue λ_i.

Given the thermal state $\sigma_\beta \propto \exp(-\beta H)$, we define the s-inner product on operator space as

$$\langle A, B\rangle_{s,\sigma_\beta} = \mathrm{Tr}\left(A^\dagger \sigma_\beta^{1-s} B \sigma_\beta^s\right)$$

for $0 < s < 1$. Given a Lindbladian operator $\mathcal{L}$, we say $\mathcal{L}$ satisfies the KMS detailed balance condition (KMS DBC) if $\mathcal{L}^\dagger$ is self-adjoint under $\langle A, B\rangle_{1/2,\sigma_\beta}$ and $\mathcal{L}$ satisfies the GNS detailed balance condition (GNS DBC) if $\mathcal{L}^\dagger$ is self-adjoint under $\langle A, B\rangle_{s,\sigma_\beta}$ for any $s \neq 1/2$. We note that, if $\mathcal{L}$ satisfies GNS DBC, it must also satisfy KMS DBC and take a generic form of the Davies generator. Given $\mathcal{L}$ satisfies GNS DBC or KMS DBC, we define the spectral gap as

$$\mathrm{Gap}(\mathcal{L}) = \inf_{\mathrm{Tr}(A\sigma_\beta)=0, A\neq 0} \frac{-\left\langle A, \mathcal{L}^\dagger(A)\right\rangle_{1/2,\sigma_\beta}}{\langle A, A\rangle_{1/2,\sigma_\beta}} .$$

We adopt the following asymptotic notations beside the usual big $\mathcal{O}$ one. We write $f = \Omega(g)$ if $g = \mathcal{O}(f)$; $f = \Theta(g)$ if $f = \mathcal{O}(g)$ and $g = \mathcal{O}(f)$. The notations $\widetilde{\mathcal{O}}$, $\widetilde{\Omega}$, $\widetilde{\Theta}$ are used to suppress subdominant polylogarithmic factors. If not specified, $f = \widetilde{\mathcal{O}}(g)$ if $f = \mathcal{O}(g\,\mathrm{polylog}(g))$; $f = \widetilde{\Omega}(g)$ if $f = \Omega(g\,\mathrm{polylog}(g))$; $f = \widetilde{\Theta}(g)$ if $f = \Theta(g\,\mathrm{polylog}(g))$. Note that these tilde notations do not remove or suppress dominant polylogarithmic factors. For instance, if $f = \mathcal{O}(\log g \log\log g)$, then we write $f = \widetilde{\mathcal{O}}(\log g)$ instead of $f = \widetilde{\mathcal{O}}(1)$.

In addition, we note that when analyzing the approximate fixed point of Φ in Section E 1 and Section E 2, we define the limiting generator of $\mathcal{L}$ as $\widetilde{\mathcal{L}}$ after letting $T \to \infty$, and set $\widetilde{\Phi} = \mathcal{U}_S \circ \exp\left(\widetilde{\mathcal{L}}\alpha^2\right) \circ \mathcal{U}_S$. Furthermore, in the proofs of the mixing times in Section I and Section H, we further approximate $\widetilde{\mathcal{L}}$ by $\widehat{\mathcal{L}}$, which exactly fixes the thermal state or ground state.

Given a quantum channel Φ, the integer mixing time of Φ describes the minimum number of iterations required so that, starting from any initial state, the evolved state is guaranteed to be ϵ-close to the target state. In this sense, it characterizes the worst-case convergence time over all initial states.

Definition 6. *Given a CPTP map Φ with a unique fixed point $\rho_{\mathrm{fix}}(\Phi)$ and $\epsilon > 0$, the integer mixing time $\tau_{\mathrm{mix},\Phi}(\epsilon)$ is defined as*

$$\tau_{\mathrm{mix},\Phi}(\epsilon) = \min\left\{ t \in \mathbb{N} \middle| \sup_{\rho} \|\Phi^t(\rho) - \rho_{\mathrm{fix}}(\Phi)\|_1 \le \epsilon \right\} . \tag{A2}$$

For Φ that takes the form of (2), the parameter α^2 can be interpreted as the effective Lindbladian evolution time per application, and we define the (rescaled) mixing time as

$$t_{\mathrm{mix},\Phi}(\epsilon) = \alpha^2 \tau_{\mathrm{mix},\Phi}(\epsilon). \tag{A3}$$

Besides Theorem 6, other definitions of the mixing time are also used in the literature such as

$$t_{\mathrm{mix;c}} = \min\left\{ t \in \mathbb{N} \middle| \sup_{\rho_1 \neq \rho_2} \frac{\|\Phi^t(\rho_1) - \Phi^t(\rho_2)\|_1}{\|\rho_1 - \rho_2\|_1} \le \frac{1}{2} \right\} .$$

It is well known that $t_{\mathrm{mix}}(\epsilon) \le t_{\mathrm{mix;c}}\left(\log_2(1/\epsilon) + 1\right)$, indicating that $t_{\mathrm{mix}}(\epsilon)$ scales logarithmically in $1/\epsilon$ whenever $t_{\mathrm{mix;c}} < \infty$ [34].

Appendix B: Related works

In this section, we review the related works on thermal and ground state preparation, focusing on the recent developments in Lindblad dynamics and weak-interaction dissipative systems.

Lindblad dynamics, originally developed to model the evolution of weakly coupled open quantum systems, has garnered significant attention in the past two years as a protocol for preparing thermal [16–19] and ground states [20, 22], due to its mathematical simplicity and analytical tractability. Given a Hamiltonian H, one can construct appropriate Lindblad operators (typically of the form $K = \int_{-\infty}^{\infty} f(s) e^{iHs} A e^{-iHs}\, \mathrm{d}s$) along with a suitable coherent term, such that the resulting dynamics drive any initial state toward the thermal or ground state. The convergence rate is governed by the mixing time of the dynamics. Recently, the mixing time analysis of Lindblad dynamics has been successfully carried out for various physically relevant Hamiltonians in both the thermal [33, 34, 36–41] and ground state [22] regimes. Leveraging well-developed Lindbladian simulation algorithms [17, 20, 43–45], such dynamics can be efficiently simulated on a fault-tolerant quantum computer. However, due to the complexity of the jump operator, most simulation algorithms require a large number of ancilla qubits, controlled or time-reversed Hamiltonian evolutions, and intricate quantum control logic for clock registers, making them unsuitable for near-term quantum devices. To mitigate the cost of simulating the detailed balanced Lindblad dynamics, very recently [23] proposes a variational compilation strategy to construct an approximation to the jump operator and to simulate the Lindblad dynamics using local gates.

In contrast to the Lindblad dynamics, the implementation of weak-interaction dissipative systems is more straightforward. Once the bath and system-bath interaction are specified, the dynamics can be simulated using forward Hamiltonian evolution followed by partial trace (or repeated interactions). Similar to our work, several concurrent works [31, 46–50] have also proposed quantum algorithms for thermal state preparation based on system-bath interaction models. While these works offer valuable insights, they do not provide rigorous end-to-end performance guarantees, and/or may face challenges in early fault-tolerant implementation. In the following, we provide a brief overview of these works that are more relevant to ours and highlight the differences with our approach and summarize them in Table I:

- In [46], the authors study the weak-interaction algorithm in the regime of small α and constant $f(t)$, and rigorously establish its correctness and efficiency for a specific free fermion model. To the best of our knowledge, it remains unclear whether their approach extends to a general Hamiltonian H.

Algorithms	Properties			Remarks
	Fixed-point error bound	Mixing time guarantee	Early-fault tolerant Implementation	
Lindblad dynamics based thermal state preparation [16–18]	✓	✓	✗	Controlled Hamiltonian simulation; Complex logic gates
Discrete dynamics simulating Metropolis-type sampling [9, 25]	✓	?	✗	Controlled Hamiltonian simulation; Complex logic gates
Lindblad dynamics based ground state preparation [19, 20]	✓	✓	?	Time-reversed Hamiltonian simulation
Hahn *et al* [23]	?	?	✓	Variational compilation
Weakly-coupled system bath interaction				
Hagan *et al* [31]	✓	?	?	Haar-random system-bath coupling; Exponential simulation time
Hahn *et al* [47]	✓	?	✓	Only allow small energy transitions
Langbehn *et al* [48]	?	?	✓	Rotating wave approximation
Lloyd *et al* [49]	?	?	✓	Similar structure as [47] and perturbative fixed-point analysis
Scandi *et al* [50]	✓	?	?	Gaussian bath coupling
Shtanko *et al* [52], Chen *et al* [53]	✓	✓	?	ETH hypothesis
This work	✓	✓	✓	Large energy transitions; Can prepare ground state

Table I. Comparison of recent quantum thermal and ground state preparation algorithms based on Lindblad dynamics or weakly coupled system-bath interaction. "Fixed-point error bound" refers to whether there is a rigorous fixed-point error bound for a general Hamiltonian H. "Mixing time guarantee" indicates whether the mixing time of the algorithm can be theoretically established at least for certain interacting Hamiltonians (see Section F).

- In [31], the authors assume Haar-random system-bath coupling and establish a rigorous fixed-point error bound for the thermal state. According to their theoretical results, for general systems, the algorithm may require impractical parameter choices to resolve exponentially close eigenvalues.For instance, as discussed in [31, Section I.A], the required coupling strength α might be exponentially small, which in turn requires the simulation time T in each step to scale exponentially with the number of qubits. Consequently, the total simulation time becomes exponentially long to guarantee the correctness of the fixed point.

- In [47], the authors prove a result similar to Theorem 9 for the thermal state preparation. Although their work presents a result similar to ours in the thermal state setting, the authors do not provide theoretical guarantees on the mixing time—an essential component for establishing the end-to-end complexity of the algorithm (see the detailed discussion in Section G and Theorem 22). In contrast, in Section F, we prove that for commuting local Hamiltonians and free fermion systems, the mixing time admits a well-defined limit as $\sigma \to \infty$, thereby yielding a complete fixed-point error bound for these models, as stated in Corollary 21.

- The algorithmic structure in [49] is similar to that in [47]. In both works, the bath state is initialized as $|0\rangle\langle 0|$, and the interaction function f is carefully tuned so that the resulting jump operator in the approximate Lindblad dynamics satisfies the detailed balance condition. Ref. [49] justifies the fixed-point error bound in the perturbative regime. Although the paper does not provide a fully rigorous error bound, its numerical results support both the efficiency of the algorithm and the validity of the perturbative analysis. We note that, unlike the two works [47, 49], our approach employs a nontrivial initial bath state—specifically, the thermal bath state. This choice ensures that the dissipative part of our approximate Lindbladian dynamics automatically satisfies the detailed balance condition. Consequently, the interaction function f in our framework can be designed with a flexibly tunable variance σ (independent of β), without the need to impose additional constraints or formulation to maintain detailed balance. This differs from the interaction functions used in [49] and [47]. Thanks to this flexibility, our algorithm can accommodate large energy transitions and achieve rigorous mixing times, all while maintaining a provable bound on the fixed-point error.

- In [50], the authors prove a result similar to Theorem 4, showing that the corresponding Lindbladian dynamics approximately satisfy the KMS detailed balance condition. This, in turn, implies Theorem 9 as a corollary. In contrast to our result, their analysis only considers the thermal state preparation and relies on the assumption of a Gaussian bath. Their algorithm also requires a detailed characterization of the two-point correlation functions.

- In [48, 52–54], the authors investigate bath and system–bath interaction models similar to ours. However, the theoretical analyses in [48, 54] are primarily limited to small-scale systems, while [52, 53] rely on the Eigenstate Thermalization Hypothesis (ETH). In particular, under the ETH assumption, [53] demonstrates that the repeated interaction dynamics can be effectively approximated by a Davies generator for thermal state preparation.

- Our choice of f is inspired by [17], where the authors construct a Lindbladian dynamics using the same filter function in the jump operators. Under this framework, they also establish a fixed-point error bound for the thermal state similar to Theorem 9.

- Φ to Lindbladian dynamics: There is extensive literature supporting the convergence of Φ to Lindbladian dynamics under the weak-interaction assumption. Notably, [55, 56] derive the Coarse-Grained Master Equation (CGME) in the presence of a general bath. More recently, [17, Appendix D] rigorously shows that the resulting Lindbladian dynamics with $f(t) = \frac{1}{T}\mathbf{1}_{[-T/2,T/2]}(t)$ approximately fixes the thermal state, yielding a result similar to our Theorem 9. In contrast to the general setting of [55, 56], we provide a simple and explicit choice of bath and coupling operators that allows the Lindbladian dynamics to be derived more easily. Moreover, our use of a Gaussian filter $f(t)$ leads to a better fixed-point error bound compared to the flat choice of f in [17, Appendix D].

- In [20], the authors proposed a Lindbladian-dynamics-based algorithm for ground state preparation. As demonstrated in [22], both theoretically and numerically, the dynamics exhibits rapid mixing for several physical Hamiltonians. We note that the algorithm in [20] simulates the Lindbladian dynamics using a single ancilla qubit but requires time-reversed Hamiltonian evolution. In contrast, our algorithm involves only forward Hamiltonian evolution, which leads to a nontrivial Lamb shift term in the dynamics that must be carefully handled in the convergence analysis.

Appendix C: Derivation of Effective Lindblad dynamics

Recall the time evolution operator by $U_S(t) := \exp(-iHt)$, and the associated superoperator by $\mathcal{U}_S(t)[\rho] = U_S(t)\rho U_S^\dagger(t)$. We then show that the quantum map Φ can be approximated by an effective Lindblad dynamics in the following theorem:

Theorem 7 (Rigorous version of Theorem 4). *Under the choice of $H_E, A_S, B_E, f(t), g(\omega)$ in the main text, ρ_{n+1} can be expressed as*

$$\begin{aligned}
\rho_{n+1/3} =& U_S(T)\rho_n U_S^\dagger(T) = \mathcal{U}_S(T)[\rho_n] \\
\rho_{n+2/3} =& \rho_{n+1/3} + \alpha^2 \underbrace{\mathbb{E}_{A_S,\omega}\left\{-i[H_{\mathrm{LS},A_S}(\omega), \rho_{n+1/3}] + \frac{1}{1+\exp(\beta\omega)}\mathcal{D}_{V_{A_S^\dagger,f,T}(\omega)}(\rho_{n+1/3}) + \frac{1}{1+\exp(-\beta\omega)}\mathcal{D}_{V_{A_S,f,T}(-\omega)}(\rho_{n+1/3})\right\}}_{:=\mathcal{L}[\rho]} , \\
& + \mathcal{O}(\alpha^4\|A_S\|^4 T^4 \|f\|_{L^\infty}^4) \\
=& \exp(\mathcal{L}\alpha^2)\rho_{n+1/3} + \mathcal{O}(\alpha^4\|A_S\|^4 T^4 \|f\|_{L^\infty}^4) \\
\rho_{n+1} =& U_S(T)\rho_{n+2/3} U_S^\dagger(T) = \mathcal{U}_S(T)[\rho_{n+2/3}]
\end{aligned} \tag{C1}$$

where $\gamma(\omega) = (g(\omega)+g(-\omega))/(1+\exp(\beta\omega))$ when $\beta < \infty$, and $\gamma(\omega) = (g(\omega)+g(-\omega))\mathbf{1}_{\omega<0} + g(0)\mathbf{1}_{\omega=0}$ when $\beta = \infty$. Here,

$$H_{\mathrm{LS},A_S}(\omega) = -\mathrm{Im}\left(\frac{\exp(-\beta\omega)}{1+\exp(-\beta\omega)}\mathcal{G}_{A_S^\dagger,f}(\omega) + \frac{1}{1+\exp(-\beta\omega)}\mathcal{G}_{A_S,f}(-\omega)\right),$$

with

$$\mathcal{G}_{A_S,f}(\omega) = \int_{-T}^{T}\int_{-T}^{s_1} f(s_2)f(s_1)A_S^\dagger(s_2)A_S(s_1)\exp(-i\omega(s_1-s_2))\mathrm{d}s_2\mathrm{d}s_1 . \tag{C2}$$

We put the proof of the above theorem in Section C 1. In our work, because we assume A_S is uniformly sampled from $\mathcal{A}=\{A^i,-A^i\}_i$ with the property that $\{(A^i)^\dagger\}_i=\{A^i\}_i$ and ω is sampled from g, we obtain

$$\begin{aligned}\mathcal{L}(\rho)=&\mathbb{E}_{A_S}\left\{-i\int_{-\infty}^{\infty}[g(\omega)H_{\mathrm{LS},A_S}(\omega),\rho_{n+1/3}]\mathrm{d}\omega+\int_{-\infty}^{\infty}\frac{g(\omega)}{1+\exp(\beta\omega)}\mathcal{D}_{V_{A_S^\dagger,f,T}(\omega)}(\rho_{n+1/3})\mathrm{d}\omega\right.\\&\left.+\int_{-\infty}^{\infty}\frac{g(\omega)}{1+\exp(-\beta\omega)}\mathcal{D}_{V_{A_S,f,T}(-\omega)}(\rho_{n+1/3})\mathrm{d}\omega\right\}\\=&\mathbb{E}_{A_S}\left\{-i\int_{-\infty}^{\infty}[g(\omega)H_{\mathrm{LS},A_S}(\omega),\rho_{n+1/3}]\mathrm{d}\omega+\int_{-\infty}^{\infty}\frac{g(\omega)}{1+\exp(\beta\omega)}\mathcal{D}_{V_{A_S,f,T}(\omega)}(\rho_{n+1/3})\mathrm{d}\omega\right.\\&\left.+\int_{-\infty}^{\infty}\frac{g(\omega)}{1+\exp(-\beta\omega)}\mathcal{D}_{V_{A_S,f,T}(-\omega)}(\rho_{n+1/3})\mathrm{d}\omega\right\}\\=&\mathbb{E}_{A_S}\left\{-i\int_{-\infty}^{\infty}[g(\omega)H_{\mathrm{LS},A_S}(\omega),\rho_{n+1/3}]\mathrm{d}\omega+\int_{-\infty}^{\infty}\frac{g(\omega)+g(-\omega)}{1+\exp(\beta\omega)}\mathcal{D}_{V_{A_S,f,T}(\omega)}(\rho_{n+1/3})\mathrm{d}\omega\right\}\end{aligned},$$

This gives (8) in Theorem 4. According to the above theorem, another perspective on our algorithm is that it can be viewed as a simulation method that reproduces (8) using at most two forward evolutions with a single ancilla qubit and randomness. It is worth noting that related results on a given Lindbladian simulation (without forward evolution) have also been obtained in [57–59]. However, we emphasize that our main contribution lies in presenting a particularly simple choice of environment and bath, such that the resulting Lindbladian dynamics naturally generate a jump operator in integral form. This construction eliminates the need for block encoding or explicit decomposition of the jump operator.

1. Proof of Theorem 7

In this section, we prove Theorem 7.

Proof of Theorem 7. Define $\rho_{\mathrm{ini}}=\rho_n\otimes\rho_E$, $\rho(T)=U^\alpha(T)\rho_{\mathrm{ini}}U^\alpha(T)^\dagger$ and $G(t)=f(t)\left(A_S\otimes B_E+A_S^\dagger\otimes B_E^\dagger\right)$. We first expand $U^\alpha(t):=\mathcal{T}\exp\left(-i\int_{-T}^t H_\alpha(s)\,\mathrm{d}s\right)$ into Dyson series:

$$U^\alpha(t)=U_0(t;-T)-i\alpha U_1(t;-T)+(-i\alpha)^2U_2(t;-T)+(-i\alpha)^3U_3(t;-T)+\mathcal{O}\left(\alpha^4T^4\|f\|_{L^\infty}^4\left(\|A_S\|\|B_E\|\right)^4\right).$$

Here $U_0(t;-T)=\exp(-i(H+H_E)(t-(-T)))$. Let $\mathcal{G}(t)=U_0^\dagger(t;-T)G(t)U_0(t;-T)$, which is the evolution of $G(t)$ under the Heisenberg picture. Then,

$$U_n(t;-T)=U_0(t;-T)\int_{-T}^{t}\int_{-T}^{s_1}\cdots\int_{-T}^{s_{n-1}}\mathcal{G}(s_1)\mathcal{G}(s_2)\dots\mathcal{G}(s_n)\mathrm{d}s_n\mathrm{d}s_{n-1}\dots\mathrm{d}s_1\,.$$

According to the above expansion, it is straightforward to see that

$$\begin{aligned}\rho(T)=&U_0(T;-T)\rho_{\mathrm{ini}}U_0^\dagger(T;-T)-i\alpha\underbrace{\left(U_1(T;-T)\rho_{\mathrm{ini}}U_0^\dagger(T;-T)-U_0(T;-T)\rho_{\mathrm{ini}}U_1^\dagger(T;-T)\right)}_{\mathbb{E}(\cdot)=0}\\&+\alpha^2\left(-U_0(T;-T)\rho_{\mathrm{ini}}U_2^\dagger(T;-T)-U_2(T;-T)\rho_{\mathrm{ini}}U_0^\dagger(T;-T)+U_1(T;-T)\rho_{\mathrm{ini}}U_1^\dagger(T;-T)\right),\\&+\alpha^3\underbrace{(\cdots)}_{\mathbb{E}(\cdot)=0}+\mathcal{O}\left(\alpha^4T^4\|f\|_{L^\infty}^4\left(\|A_S\|\|B_E\|\right)^4\right)\end{aligned}$$

Here, for the first order and third order term, we have expectation equals to zero because $\mathbb{E}(G(t))=0$.

Now, we only care about the second order term. Let $\widehat{\rho}(T)=U_0(T;-T)\rho_{\mathrm{ini}}U_0^\dagger(T;-T)$. Then,

$$\begin{aligned}&U_0(T;-T)\rho_{\mathrm{ini}}U_2^\dagger(T;-T)=\widehat{\rho}(T)U_0(T;-T)\int_{-T}^{T}\int_{-T}^{s_1}\mathcal{G}(s_2)\mathcal{G}(s_1)\mathrm{d}s_2\mathrm{d}s_1U_0^\dagger(T;-T)\\=&\widehat{\rho}(T)U_0(T;-T)\frac{1}{2}\int_{-T}^{T}\int_{-T}^{T}\mathcal{G}(s_2)\mathcal{G}(s_1)\mathrm{d}s_2\mathrm{d}s_1U_0^\dagger(T;-T)+\widehat{\rho}(T)U_0(T;-T)\frac{1}{2}\int_{-T}^{T}\int_{-T}^{s_1}[\mathcal{G}(s_2),\mathcal{G}(s_1)]\mathrm{d}s_2\mathrm{d}s_1U_0^\dagger(T;-T)\,,\end{aligned}$$

where we use $\int_{-T}^{T}\int_{-T}^{s_1}\mathcal{G}(s_1)\mathcal{G}(s_2)\mathrm{d}s_2\mathrm{d}s_1=\int_{-T}^{T}\int_{s_1}^{T}\mathcal{G}(s_2)\mathcal{G}(s_1)\mathrm{d}s_2\mathrm{d}s_1$ in the last equality. Similarly,

$$U_2(T;-T)\rho_{\mathrm{ini}}U_0^\dagger(T;-T)=U_0(T;-T)\int_{-T}^{T}\int_{-T}^{s_1}\mathcal{G}(s_1)\mathcal{G}(s_2)\mathrm{d}s_2\mathrm{d}s_1U_0^\dagger(T;-T)\widehat{\rho}(T)$$

$$=U_0(T;-T)\frac{1}{2}\int_{-T}^{T}\int_{-T}^{T}\mathcal{G}(s_1)\mathcal{G}(s_2)\mathrm{d}s_2\mathrm{d}s_1U_0^\dagger(T;-T)\widehat{\rho}(T)+U_0(T;-T)\frac{1}{2}\int_{-T}^{T}\int_{-T}^{s_1}[\mathcal{G}(s_1),\mathcal{G}(s_2)]\mathrm{d}s_2\mathrm{d}s_1U_0^\dagger(T;-T)\widehat{\rho}(T)\,,$$

and

$$U_1(T;-T)\rho_{\mathrm{ini}}U_1^\dagger(T;-T)=\left(U_0(T;-T)\underbrace{\int_{-T}^{T}\mathcal{G}(s_1)\mathrm{d}s_1}_{:=V}U_0^\dagger(T;-T)\right)\widehat{\rho}(T)\left(U_0(T;-T)\int_{-T}^{T}\mathcal{G}(s_2)\mathrm{d}s_2U_0^\dagger(T;-T)\right)^\dagger\,.$$

Combining the above three equalities and noticing $U_0^\dagger(T;-T)\widehat{\rho}(T)U_0(T;-T)=\rho_{\mathrm{ini}}$, this implies

$$\begin{aligned}&\rho(T)=U_0(T;-T)\rho_{\mathrm{ini}}U_0^\dagger(T;-T)\\&+\alpha^2U_0(T;-T)\left(\underbrace{V\rho_{\mathrm{ini}}V^\dagger-\frac{1}{2}\left\{V^\dagger V,\rho_{\mathrm{ini}}\right\}}_{:=\text{Term I}}\underbrace{-i\left[\frac{i}{2}\int_{-T}^{T}\int_{-T}^{s_1}[\mathcal{G}(s_2),\mathcal{G}(s_1)]\mathrm{d}s_2\mathrm{d}s_1,\rho_{\mathrm{ini}}\right]}_{:=\text{Term II}}\right)U_0^\dagger(T;-T)+\mathcal{O}\left(\alpha^4T^4\|f\|_{L^\infty}^4\|A_S\|^4\right)\,.\end{aligned}\tag{C3}$$

Here the expectation is taken over A_S and ω. We notice that $\rho_{n+1}=\mathbb{E}\left(\mathrm{Tr}_E\left(\rho(T)\right)\right)$. Let $\rho_{n+2/3}=U_S^\dagger(T)\rho_{n+1}U_S(T)$ and $\rho_{n+1/3}=U_S(T)\rho_nU_S^\dagger(T)$ as defined in Eq. (C1). Applying $U_0^\dagger(0;-T)[\cdot]U_0(0;-T)$ on both sides of the above equality, tracing out the ancilla qubits, and taking the expectation over A_S,ω, we have

$$\begin{aligned}&\rho_{n+2/3}=\rho_{n+1/3}\\&+\alpha^2\mathbb{E}\left(\mathrm{Tr}_E\left(U_0(0;-T)\left(\underbrace{V\rho_{\mathrm{ini}}V^\dagger-\frac{1}{2}\left\{V^\dagger V,\rho_{\mathrm{ini}}\right\}}_{:=\text{Term I}}\underbrace{-i\left[\frac{i}{2}\int_{-T}^{T}\int_{-T}^{s_1}[\mathcal{G}(s_2),\mathcal{G}(s_1)]\mathrm{d}s_2\mathrm{d}s_1,\rho_{\mathrm{ini}}\right]}_{:=\text{Term II}}\right)U_0^\dagger(0;-T)\right)\right)\\&+\mathcal{O}\left(\alpha^4T^4\|f\|_{L^\infty}^4\|A_S\|^4\right)\,.\end{aligned}\tag{C4}$$

Here, we note $U_0(0;-T)=\exp(-i(H+H_E)T)$.

Now, we deal with two terms separately:

- For the first term, we have

$$\begin{aligned}V=&\int_{-T}^{T}f(t)\exp(i\omega(t-(-T)))\left(A_S(t;-T)\otimes|1\rangle\langle 0|\right)\mathrm{d}t\\&+\int_{-T}^{T}f(t)\exp(-i\omega(t-(-T)))\left(A_S^\dagger(t;-T)\otimes|0\rangle\langle 1|\right)\mathrm{d}t\end{aligned}$$

  where

$$A_S(t;-T)=\exp(iH(t+T))A_S\exp(-iH(t+T))\,,$$

  Let $A_{S,f}(\omega)=\int_{-T}^{T}f(t)A_S(t;-T)\exp(i\omega(t+T))\mathrm{d}t$. We have

$$V=A_{S,f}(\omega)\otimes|1\rangle\langle 0|+A_{S,f}^\dagger(\omega)\otimes|0\rangle\langle 1|\,.$$

  This implies that

$$\begin{aligned}&\mathrm{Tr}_E\left(V\rho_{\mathrm{ini}}V^\dagger-\frac{1}{2}\left\{V^\dagger V,\rho_{\mathrm{ini}}\right\}\right)\\=&\frac{\exp(-\beta\omega)}{1+\exp(-\beta\omega)}\left(A_{S,f}^\dagger(\omega)\rho_nA_{S,f}(\omega)-\frac{1}{2}\left\{A_{S,f}(\omega)A_{S,f}^\dagger(\omega),\rho_n\right\}\right)\\&+\frac{1}{1+\exp(-\beta\omega)}\left(A_{S,f}(\omega)\rho_nA_{S,f}^\dagger(\omega)-\frac{1}{2}\left\{A_{S,f}^\dagger(\omega)A_{S,f}(\omega),\rho_n\right\}\right)\end{aligned}\,.$$

Recall $\rho_{n+1/3}=U_S(T)\rho_n U_S^\dagger(T)$ and $\rho_{\mathrm{ini}}=\rho_n\otimes\rho_E$. We can rewrite the above equality as

$$\begin{aligned}
&\mathrm{Tr}_E\left(U_0(0;-T)\left(V\rho_{\mathrm{ini}}V^\dagger-\frac{1}{2}\left\{V^\dagger V,\rho_{\mathrm{ini}}\right\}\right)U_0^\dagger(0;-T)\right)\\
=&\mathrm{Tr}_E\left(\left(U_0(0;-T)VU_0(0;-T)^\dagger\right)U_0(0;-T)\rho_{\mathrm{ini}}U_0^\dagger(0;-T)\left(U_0(0;-T)V^\dagger U_0(0;-T)^\dagger\right)\right.\\
&\left.-\frac{1}{2}U_0(0;-T)\left\{V^\dagger V,\rho_{\mathrm{ini}}\right\}U_0(0;-T)^\dagger U_0^\dagger(0;-T)\right)\\
=&\frac{\exp(-\beta\omega)}{1+\exp(-\beta\omega)}\left(V_{A_S,f}^\dagger(\omega)\rho_{n+1/3}V_{A_S,f}(\omega)-\frac{1}{2}\left\{V_{A_S,f}(\omega)V_{A_S,f}^\dagger(\omega),\rho_{n+1/3}\right\}\right)\\
&+\frac{1}{1+\exp(-\beta\omega)}\left(V_{A_S,f}(\omega)\rho_{n+1/3}V_{A_S,f}^\dagger(\omega)-\frac{1}{2}\left\{V_{A_S,f}^\dagger(\omega)V_{A_S,f}(\omega),\rho_{n+1/3}\right\}\right)
\end{aligned},$$

Here, $V_{A_S,f}(\omega)=\int_{-T}^{T}f(t)A_S(t;0)\exp(i\omega t)\mathrm{d}t$. This gives the Lindbladian operators in (C1).

- For the second term: We first notice

$$\begin{aligned}
&\int_{-T}^{T}\int_{-T}^{s_1}\mathcal{G}(s_2)\mathcal{G}(s_1)\mathrm{d}s_2\mathrm{d}s_1\\
=&\int_{-T}^{T}\int_{-T}^{s_1}(f(s_2)f(s_1)\exp(i\omega(s_2-s_1)))\left(A_S(s_2;-T)A_S^\dagger(s_1;-T)\otimes|1\rangle\langle 1|\right)\mathrm{d}s_2\mathrm{d}s_1\\
&+\int_{-T}^{T}\int_{-T}^{s_1}(f(s_2)f(s_1)\exp(-i\omega(s_2-s_1)))\left(A_S^\dagger(s_2;-T)A_S(s_1;-T)\otimes|0\rangle\langle 0|\right)\mathrm{d}s_2\mathrm{d}s_1
\end{aligned}$$

We notice that

$$A_S(s_2;-T)=\exp(iH(s_2+T))A_S\exp(-iH(s_2+T)),\quad A_S^\dagger(s_1;-T)=\exp(iH(s_1+T))A_S^\dagger\exp(-iH(s_1+T))\,.$$

This implies

$$\begin{aligned}
&A_S(s_2;-T)A_S^\dagger(s_1;-T)=\exp(iH(s_2+T))A_S\exp(-iH(s_2))\exp(iH(s_1))A_S^\dagger\exp(-iH(s_1+T))\\
=&\exp(iHT)A_S(s_2;0)A_S^\dagger(s_1;0)\exp(-iHT)\,.
\end{aligned}$$

Recall $U_0(0;-T)=\exp(-i(H+H_E)T)$. Therefore, we have

$$U_0(0;-T)\left(A_S(s_2;-T)A_S^\dagger(s_1;-T)\otimes|1\rangle\langle 1|\right)U_0^\dagger(0;-T)=A_S(s_2;0)A_S^\dagger(s_1;0)\otimes|1\rangle\langle 1|$$

and

$$U_0(0;-T)\left(A_S^\dagger(s_2;-T)A_S(s_1;-T)\otimes|0\rangle\langle 0|\right)U_0^\dagger(0;-T)=A_S^\dagger(s_2;0)A_S(s_1;0)\otimes|0\rangle\langle 0|$$

Define

$$\mathcal{G}_{A_S,f}(\omega)=\int_{-T}^{T}\int_{-T}^{s_1}f(s_2)f(s_1)A_S^\dagger(s_2;0)A_S(s_1;0)\exp(i\omega(s_2-s_1))\mathrm{d}s_2\mathrm{d}s_1\,.$$

We then have

$$\begin{aligned}
&U_0(0;-T)\int_{-T}^{T}\int_{-T}^{s_1}\mathcal{G}(s_2)\mathcal{G}(s_1)\mathrm{d}s_2\mathrm{d}s_1U_0^\dagger(0;-T)\\
=&\int_{-T}^{T}\int_{-T}^{s_1}(f(s_2)f(s_1)\exp(i\omega(s_2-s_1)))\left(A_S(s_2;0)A_S^\dagger(s_1;0)\otimes|1\rangle\langle 1|\right)\mathrm{d}s_2\mathrm{d}s_1\\
&+\int_{-T}^{T}\int_{-T}^{s_1}(f(s_2)f(s_1)\exp(-i\omega(s_2-s_1)))\left(A_S^\dagger(s_2;0)A_S(s_1;0)\otimes|0\rangle\langle 0|\right)\mathrm{d}s_2\mathrm{d}s_1\\
=&\mathcal{G}_{A_S^\dagger,f}(\omega)\otimes|1\rangle\langle 1|+\mathcal{G}_{A_S,f}(-\omega)\otimes|0\rangle\langle 0|\,.
\end{aligned}$$

Because $\mathcal{G}(s)$ is a Hermitian matrix, we have We then have

$$U_0(0;-T)\int_{-T}^{T}\int_{-T}^{s_1}\mathcal{G}(s_1)\mathcal{G}(s_2)\mathrm{d}s_2\mathrm{d}s_1U_0^\dagger(0;-T)=\left(U_0(0;-T)\int_{-T}^{T}\int_{-T}^{s_1}\mathcal{G}(s_2)\mathcal{G}(s_1)\mathrm{d}s_2\mathrm{d}s_1U_0^\dagger(0;-T)\right)^\dagger$$
$$=\mathcal{G}^\dagger_{A_S^\dagger,f}(\omega)\otimes|1\rangle\langle 1|+\mathcal{G}^\dagger_{A_S,f}(-\omega)\otimes|0\rangle\langle 0|\ .$$

The above calculation gives

$$\begin{aligned}&\mathrm{Tr}_E\left(U_0(0;-T)\,(\text{Term II})\,U_0^\dagger(0;-T)\right)\\ =&-i\left[\frac{i}{2}\frac{\exp(-\beta\omega)}{1+\exp(-\beta\omega)}\left(\mathcal{G}_{A_S^\dagger,f}(\omega)-\mathcal{G}^\dagger_{A_S^\dagger,f}(\omega)\right)+\frac{1}{1+\exp(-\beta\omega)}\left(\mathcal{G}_{A_S,f}(-\omega)-\mathcal{G}^\dagger_{A_S,f}(-\omega)\right),\rho_{n+1/3}\right]\\ =&-i\left[\frac{i}{2}\left(\frac{\exp(-\beta\omega)}{1+\exp(-\beta\omega)}\mathcal{G}_{A_S^\dagger,f}(\omega)+\frac{1}{1+\exp(-\beta\omega)}\mathcal{G}_{A_S,f}(-\omega)-(\dots)^\dagger\right),\rho_{n+1/3}\right]\\ =&-i\left[-\mathrm{Im}\left(\frac{\exp(-\beta\omega)}{1+\exp(-\beta\omega)}\mathcal{G}_{A_S^\dagger,f}(\omega)+\frac{1}{1+\exp(-\beta\omega)}\mathcal{G}_{A_S,f}(-\omega)\right),\rho_{n+1/3}\right]\end{aligned}.$$

This gives the formula of H_{LS,A_S} in the theorem.

□

Appendix D: Approximate CPTP map has close fixed point and mixing time

In this section, we show that the closeness of two CPTP maps Φ_1 and Φ_2 implies the closeness of their fixed points and mixing times. This provides a crucial link between the fixed point and mixing time of the Lindbladian dynamics in Theorem 4 and those of Φ. The result is summarized in the following:

Theorem 8. *Given two CPTP maps Φ_1, Φ_2 with unique fixed points ρ_1, ρ_2. Let $\tau_{1,\mathrm{mix}}(\epsilon), \tau_{2,\mathrm{mix}}(\epsilon)$ be the mixing time of Φ_1, Φ_2 respectively, defined as Theorem 6. Then*

- *ρ_1, ρ_2 are close if the maps themselves are close: For any $\epsilon > 0$,*

$$\|\rho_1-\rho_2\|_1\leq\epsilon+\tau_{1,\mathrm{mix}}(\epsilon)\|\Phi_1-\Phi_2\|_{1\leftrightarrow 1}\,. \tag{D1}$$

- *ρ_1, ρ_2 are close if $\Phi_1(\rho_2)$ is close to ρ_2: For any $\epsilon > 0$,*

$$\|\rho_1-\rho_2\|_1\leq\epsilon+\tau_{1,\mathrm{mix}}(\epsilon)\|\Phi_1(\rho_2)-\rho_2\|_1\,. \tag{D2}$$

- *Φ_2 has comparable mixing time with Φ_1 if Φ_2 is close to Φ_1: Given any $\epsilon > 0$, if $\tau_{1,\mathrm{mix}}(\epsilon/2)\|\Phi_1-\Phi_2\|_{1\leftrightarrow 1}\leq\epsilon/2$, then*

$$\tau_{2,\mathrm{mix}}(2\epsilon)\leq\tau_{1,\mathrm{mix}}(\epsilon/2). \tag{D3}$$

Importantly, Eq. (D2) makes no reference to the map Φ_2, and applies for an arbitrary state ρ_2. Perturbation bounds for quantum channels and their fixed points have been studied previously in the literature, e.g., in [60]. However, in Theorem 8, we rely only on the mixing time of the quantum channel, which is a weaker assumption than the standard contraction conditions typically used in the literature, such as [60, Theorem 4]. For completeness, we provide a full proof of the theorem below.

Proof of Theorem 8. To prove Eq. (D1), we notice that

$$\begin{aligned}&\|\rho_1-\rho_2\|_1=\left\|\rho_1-\Phi_2^{\tau_{1,\mathrm{mix}}(\epsilon)}(\rho_2)\right\|_1\\ \leq&\left\|\rho_1-\Phi_1^{\tau_{1,\mathrm{mix}}(\epsilon)}(\rho_2)\right\|_1+\left\|\Phi_1^{\tau_{1,\mathrm{mix}}(\epsilon)}(\rho_2)-\Phi_2^{\tau_{1,\mathrm{mix}}(\epsilon)}(\rho_2)\right\|_1\leq\epsilon+\tau_{1,\mathrm{mix}}(\epsilon)\|\Phi_1-\Phi_2\|_{1\leftrightarrow 1}\end{aligned}$$

$$\|\rho_1-\rho_2\|_1 \le \left\|\rho_1-\Phi_1^{\tau_{1,\mathrm{mix}}(\epsilon)}(\rho_2)\right\|_1 + \left\|\Phi_1^{\tau_{1,\mathrm{mix}}(\epsilon)}(\rho_2)-\rho_2\right\|_1$$

$$\le \left\|\rho_1-\Phi_1^{\tau_{1,\mathrm{mix}}(\epsilon)}(\rho_2)\right\|_1 + \sum_{n=0}^{\tau_{1,\mathrm{mix}}(\epsilon)-1} \|\Phi_1^{n+1}(\rho_2)-\Phi_1^{n}(\rho_2)\|_1 \le \epsilon + \tau_{1,\mathrm{mix}}(\epsilon)\|\Phi_1(\rho_2)-\rho_2\|_1\,,$$

where we use $\|\Phi_1\|_{1\leftrightarrow 1}\le 1$ in the last inequality.

Finally, to show the comparable mixing time, we note that for any ρ,

$$\|\Phi_2^{\tau_{1,\mathrm{mix}}(\epsilon/2)}(\rho)-\rho_2\|_1 \le \|\Phi_2^{\tau_{1,\mathrm{mix}}(\epsilon/2)}(\rho)-\Phi_1^{\tau_{1,\mathrm{mix}}(\epsilon/2)}(\rho)\|_1 + \|\Phi_1^{\tau_{1,\mathrm{mix}}(\epsilon/2)}(\rho)-\rho_1\|_1 + \|\rho_1-\rho_2\|_1$$
$$\le \|\Phi_2^{\tau_{1,\mathrm{mix}}(\epsilon/2)}(\rho)-\Phi_1^{\tau_{1,\mathrm{mix}}(\epsilon/2)}(\rho)\|_1 + \|\Phi_1^{\tau_{1,\mathrm{mix}}(\epsilon/2)}(\rho)-\rho_1\|_1 + \tau_{1,\mathrm{mix}}(\epsilon/2)\|\Phi_1-\Phi_2\|_{1\leftrightarrow 1} + \epsilon/2$$
$$\le 2\tau_{1,\mathrm{mix}}(\epsilon/2)\|\Phi_1-\Phi_2\|_{1\leftrightarrow 1} + \epsilon \le 2\epsilon$$

where we use (D1) in the second equality. This concludes the proof.

□

Appendix E: Fixed point error bounds for thermal and ground state preparation

Under of $H_E, A_S, B_E, f(t), g(\omega)$ in the main text, the quantum channel Φ defined in Eq. (2) can be engineered to approximately preserve the thermal or ground state of the system Hamiltonian. The integer mixing time of Φ is defined in Theorem 6. According to Theorem 4, the mixing time of Φ should be governed by the underlying Lindbladian operator $\mathcal{L}$. The quantity $t_{\mathrm{mix},\Phi}(\epsilon)$ approximately captures the total Lindbladian evolution time required for mixing. When α is sufficiently small, this mixing time does not diverge as $\alpha\to 0$, but instead remains bounded above by a finite constant that depends only on properties of the Lindbladian.

The following theorem shows that by properly choosing the parameters σ, T, α related to the mixing time, the fixed point of Φ is approximately the thermal state. We also omit some dependence on $\|H\|$ and $\|A_S\|$ for simplicity. The general version of Theorem 9 is stated in Section E 1 as Theorem 12, followed by the proof of both theorems.

Theorem 9 (Thermal state, informal)**.** *Assume $0\le\beta<\infty$ and $g(\omega)=\frac{1}{\omega_{\max}}\mathbf{1}_{[0,\omega_{\max}]}$. Then, for any $\epsilon>0$, if*

$$\sigma=\widetilde{\Omega}\left(\beta\omega_{\max}^{-1}\epsilon^{-1}t_{\mathrm{mix},\Phi}(\epsilon)\right),\quad T=\Omega(\sigma\log(\sigma/\epsilon))\,,$$

and $\alpha=\mathcal{O}\left(\sigma T^{-2}\epsilon^{1/2}t_{\mathrm{mix},\Phi}^{-1/2}(\epsilon)\right)$, then

$$\|\rho_{\mathrm{fix}}(\Phi)-\rho_\beta\|_1<\epsilon\,.$$

Theorem 9 shows that if we set $\sigma=\widetilde{\Theta}\left(\omega_{\max}^{-1}\beta\epsilon^{-1}t_{\mathrm{mix},\Phi}(\epsilon)\right)$, we ensure that the fixed point is ϵ-close to the thermal state by choosing

$$T=\widetilde{\Theta}\left(\omega_{\max}^{-1}\beta\epsilon^{-1}t_{\mathrm{mix},\Phi}(\epsilon)\right),\ \alpha=\widetilde{\Theta}\left(\omega_{\max}\beta^{-1}t_{\mathrm{mix},\Phi}^{-3/2}\epsilon^{3/2}\right).$$

Analogously, we can establish a corresponding result for the ground state as follows.

Theorem 10 (Ground state, informal)**.** *Assume H has a spectral gap Δ and let $|\psi_0\rangle$ be the ground state of H. Then, for any $\epsilon>0$, if*

$$\sigma=\widetilde{\Omega}\left(\Delta^{-1}\log(\|H\|/\epsilon)\right),\quad T=\Omega(\sigma\log(\sigma/\epsilon))\,,$$

and $\alpha=\mathcal{O}\left(\sigma T^{-2}\epsilon^{1/2}t_{\mathrm{mix},\Phi}^{-1/2}(\epsilon)\right)$, then

$$\|\rho_{\mathrm{fix}}(\Phi)-|\psi_0\rangle\langle\psi_0|\,\|_1<\epsilon.$$

Theorem 10 shows that if we set $\sigma=\widetilde{\Theta}\left(\Delta^{-1}\right)$, it suffices to choose

$$T=\widetilde{\Theta}(\Delta^{-1}),\ \alpha=\widetilde{\Theta}\left(\Delta\epsilon^{1/2}t_{\mathrm{mix},\Phi}^{-1/2}(\epsilon)\right).$$

The rigorous version of Theorem 10 is given in Appendix E 2 as Theorem 16. Theorem 10 shows that if we set $\sigma = \widetilde{\Theta}\left(\Delta^{-1}\right)$, it suffices to choose

$$T = \widetilde{\Theta}(\Delta^{-1}),\ \alpha = \widetilde{\Theta}\left(\Delta\epsilon^{1/2} t_{\mathrm{mix},\Phi}^{-1/2}(\epsilon)\right).$$

The result of Theorem 9 applies to all values of β and does not require $\Delta = \mathrm{poly}(N^{-1})$ for efficient state preparation, whereas Theorem 10 does rely on this assumption to ensure efficient preparation. On the other hand, the dependence on β in Theorem 9 may not be sharp, particularly in the large-β regime. For instance, at very low temperatures, where $\beta = \Omega(\mathrm{poly}(N, 1/\Delta, 1/\epsilon))$, preparing the ϵ-approximate thermal state effectively reduces to preparing the ground state. In such cases, one may directly adopt the parameter choices in Theorem 10 rather than those in Theorem 9.

According to the approximation-error bounds in Theorem 9 and Theorem 10, once the effective mixing time $t_{\mathrm{mix},\Phi_\alpha}$ is upper bounded, an appropriate choice of parameters guarantees that the fixed point $\rho_{\mathrm{fix}}(\Phi)$ can be made arbitrarily close to the target state. However, as discussed in the main text, the main difficulty is that $t_{\mathrm{mix},\Phi_\alpha}$ itself depends on the parameters σ and α used in the construction of Φ. Consequently, it may happen that as σ or α^{-1} tends to $+\infty$, the mixing time $t_{\mathrm{mix},\Phi_\alpha}$ also diverges, causing the conditions in Theorem 10 and Theorem 9 to become unsatisfiable (see Section G and Theorem 22). To circumvent this issue, we carefully design the dissipative protocol that allows large energy transition between eigenvectors, which further ensures that, once σ is sufficiently large, the mixing time $t_{\mathrm{mix},\Phi_\alpha}$ becomes *independent* of σ. In Section G–Section K, we rigorously prove that for certain classes of physical models such as free-fermion systems, and local commuting Hamiltonians, the mixing time does not blow up with σ and can be upper bounded by a quantity that scales polynomially with the number of qubits.

To prove Theorem 9, according to Theorem 8 in Section D, it suffices to bound $\|\Phi(\rho_\beta) - \rho_\beta\|_1$. This consists of two main steps:

1. Approximate the map Φ by choosing $\alpha \ll 1$.
2. Show that the limiting map approximately fixes the thermal or ground state when $\sigma, T \gg 1$.

In the first step, using the result of Theorem 4, we have

$$\|\Phi(\rho_\beta) - \rho_\beta\|_1 \approx \left\|\alpha^2 \mathcal{L}(\rho_\beta)\right\|_1, \quad \alpha \ll 1$$

with the approximation error quantified in (7). Thus, it suffices to show the Lindblad dynamics approximately fix the thermal/ground state. This constitutes the most technical part of the proof. For thermal states, it has been shown that the dissipative part of the Lindbladian $\mathcal{L}$ in (8) is approximately detailed-balanced [17] when $\{(A^i)^\dagger\}_i = \{A^i\}_i$, and therefore approximately fixes the thermal state (see Section E 1 Lemma 14). When $\sigma \gg 1$, we show that the Lamb shift Hamiltonian $H_{\mathrm{LS},A_S}(\omega)$ approximately commutes with the thermal state (see Section E 1 Lemma 13). These two properties together imply that $\left\|\mathcal{L}(\rho_\beta)\right\|_1 \approx 0$.

Note that, in order to ensure a small error ϵ, Theorem 9 requires that the parameters defining our algorithm satisfy conditions that depend on the mixing time $t_{\mathrm{mix},\Phi}$. The mixing time enters the proof because the relationship between $\|\rho_{\mathrm{fix}}(\Phi) - \rho_\beta\|_1$ and $\|\Phi(\rho_\beta) - \rho_\beta\|_1$ involves the mixing time, as shown in Section D.

The proof of Theorem 10 is similar; however, under the spectral gap assumption, the ground state case allows a direct upper bound on $\|\mathcal{L}(\rho_\beta)\|_1$, and the fixed-point error bound is independent of the choice of g. Specifically, the γ-dependent term in $\mathcal{L}(\rho_\beta)$ takes the form $\int \gamma(\omega)\mathcal{E}(\omega)\,\mathrm{d}\omega$ for some error operator $\mathcal{E}(\omega)$, which by normalization of γ satisfies $\int \gamma(\omega)\left\|\mathcal{E}(\omega)\right\|_1\ \mathrm{d}\omega \leq \sup_{\omega\in\mathrm{supp}(\gamma)}\left\|\mathcal{E}(\omega)\right\|_1$. This last term can be bounded directly (see Theorem 16), allowing γ (and g) to be optimized to reduce t_{mix}. In contrast, for thermal state preparation, the Lamb shift term cannot be uniformly bounded for all ω; instead, one must estimate the integral itself to show that it approximately commutes with the thermal state (see Lemma 13).

1. Approximate fixed point – Thermal state

In this section, we provide a rigorous version of Theorem 9 in Theorem 12 and provide the proof. We consider (2) with $f(t) = \frac{1}{(2\pi)^{1/4}\sigma^{1/2}}\exp\left(-\frac{t^2}{4\sigma^2}\right)$. First, we can rewrite $\mathcal{L}$ in Eq. (8) as

$$\mathcal{L}(\rho) = \mathbb{E}_{A_S}\left(\int_{-\infty}^{\infty} -i\left[g(\omega)H_{\mathrm{LS},A_S}(\omega), \rho\right] + \gamma(\omega)\mathcal{D}_{V_{A_S,f(\omega)}}(\rho)\mathrm{d}\omega\right), \tag{E1}$$

where $\gamma(\omega) = (g(\omega) + g(-\omega))/(1+\exp(\beta\omega))$. In the case when $\beta = \infty$, $\gamma(\omega) = (g(\omega)+g(-\omega))\mathbf{1}_{\omega<0} + g(0)\mathbf{1}_{\omega=0}$.

Before presenting the rigorous version of Theorem 9, we first consider a simplified CPTP map $\widetilde{\Phi}$ defined as follows:

$$\widetilde{\Phi} = \mathcal{U}_S(T) \circ \exp\left(\widetilde{\mathcal{L}}\alpha^2\right) \circ \mathcal{U}_S(T)\,. \tag{E2}$$

Compared to Φ in Eq. (2), we omit the error terms in Theorem 4 and take the limit $T \to \infty$ in $\mathcal{L}$. Specifically, as mentioned in Section A,

$$\widetilde{\mathcal{L}}(\rho) = -i\left[\widetilde{H}_{\mathrm{LS}}, \rho\right] + \mathbb{E}_{A_S}\left(\int_{-\infty}^{\infty} \gamma(\omega)\mathcal{D}_{\widetilde{V}_{A_S,f}(\omega)}(\rho)\mathrm{d}\omega\right), \tag{E3}$$

where

$$\widetilde{H}_{\mathrm{LS}} = -\mathbb{E}_{A_S}\left(\mathrm{Im}\left(\int_{-\infty}^{\infty}\gamma(\omega)\widetilde{\mathcal{G}}_{A_S,f}(-\omega)\mathrm{d}\omega\right)\right), \quad \widetilde{V}_{A_S,f}(\omega) = \int_{-\infty}^{\infty} f(t)A_S(t)\exp(-i\omega t)\mathrm{d}t\,,$$

with

$$\widetilde{\mathcal{G}}_{A,f}(\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{s_1} f(s_2)f(s_1)A^\dagger(s_2)A(s_1)\exp(-i\omega(s_1-s_2))\mathrm{d}s_2\mathrm{d}s_1\,. \tag{E4}$$

In the formula of $\widetilde{H}_{\mathrm{LS}}$, we use the fact that $\{(A^i)^\dagger\}_i = \{A^i\}_i$ and $\widetilde{\mathcal{G}}_{A^i,f} = \widetilde{\mathcal{G}}_{-A^i,f}$.

The distance between Φ and $\widetilde{\Phi}$ can be controlled in the following lemma:

Lemma 11. *When $T > \sigma$, we have*

$$\left\|\Phi - \widetilde{\Phi}\right\|_{1\leftrightarrow 1} = \mathcal{O}\left(\alpha^2\sigma\exp\left(-T^2/(4\sigma^2)\right)\mathbb{E}(\|A_S\|^2) + \alpha^4T^4\sigma^{-2}\mathbb{E}\left(\|A_S\|^4\right)\right)$$

Proof of Lemma 11. According to Theorem 7 and $\|\gamma(\omega)\|_{L^1} = 1$, we have

$$\begin{aligned}
&\left\|\Phi - \widetilde{\Phi}\right\|_{1\leftrightarrow 1} \le \alpha^2\|\mathcal{L} - \widetilde{\mathcal{L}}\|_{1\leftrightarrow 1} + \mathcal{O}\left(\alpha^4T^4\sigma^{-2}\mathbb{E}\left(\|A_S\|^4\right)\right) \\
=&\mathcal{O}\left(\alpha^2\sup_\omega\left(\|\mathcal{G}_{A_S,f}(\omega) - \widetilde{\mathcal{G}}_{A_S,f}(\omega)\| + \|V_{A_S,f}(\omega) - \widetilde{V}_{A_S,f}(\omega)\|\underbrace{\|V_{A_S,f}(\omega)\|}_{=\mathcal{O}(\sigma^{1/2}\|A_S\|)}\right)\right). \\
&+\mathcal{O}\left(\alpha^4T^4\sigma^{-2}\mathbb{E}\left(\|A_S\|^4\right)\right)
\end{aligned}$$

Thus, it suffices to consider $\|V_{A_S,f}(\omega) - \widetilde{V}_{A_S,f}(\omega)\|$ and $\|\mathcal{G}_{A_S,f}(\omega) - \widetilde{\mathcal{G}}_{A_S,f}(\omega)\|$. For the first term, we have

$$\begin{aligned}
&\|V_{A_S,f}(\omega) - \widetilde{V}_{A_S,f}(\omega)\| \le \|A_S\|\int_{|t|>T} f(t)\mathrm{d}t = \mathcal{O}\left((\sigma^{3/2}/T)\exp(-T^2/(4\sigma^2))\|A_S\|\right) \\
=&\mathcal{O}\left(\sigma^{1/2}\exp(-T^2/(4\sigma^2))\|A_S\|\right),
\end{aligned}$$

where we use $T > \sigma$ in the second equality. For the second term, we have

$$\|\mathcal{G}_{A_S,f}(\omega) - \widetilde{\mathcal{G}}_{A_S,f}(\omega)\| \le \|A_S\|^2\left(\int_{|s_1|\ge T}\int_{-\infty}^{s_1} + \int_{-T}^{T}\int_{-\infty}^{-T} f(s_2)f(s_1)\mathrm{d}s_2\mathrm{d}s_1\right) = \mathcal{O}\left(\sigma\exp(-T^2/(4\sigma^2))\|A_S\|^2\right).$$

Combining these two bounds, we conclude the proof. □

Using $\widetilde{\Phi}$, we are ready to state the rigorous version of Theorem 9 and provide the proof:

Theorem 12. *Define*

$$R := \int_0^\infty \left|\int_{-\infty}^{\infty}\gamma(\omega)\exp(i\omega\sigma q)\mathrm{d}\omega\right|\exp(-q^2/8)\mathrm{d}q\,. \tag{E5}$$

When $T > \sigma > \beta$, we have

$$\begin{aligned}
&\|\rho_{\mathrm{fix}}(\Phi) - \rho_\beta\|_1 \\
\le&\left(\mathbb{E}_{A_S}\left(\left\|\left[\rho_\beta, \int\gamma(\omega)\widetilde{\mathcal{G}}_{A_S,f}(-\omega)\,\mathrm{d}\omega\right]\right\|_1 + \left\|\left[\rho_\beta, \int\gamma(\omega)\left(\widetilde{\mathcal{G}}_{A_S,f}(-\omega)\right)^\dagger\mathrm{d}\omega\right]\right\|_1 + \left\|\int_{-\infty}^{\infty}\gamma(\omega)\mathcal{D}_{\widetilde{V}_{A_S,f}(\omega)}(\rho_\beta)\right\|_1\mathrm{d}\omega\right)\right) \\
=&\widetilde{\mathcal{O}}\left(\left(\left(R + \|\gamma(\omega)\|_\infty\frac{1}{\sigma}\sqrt{\log(\sigma/\beta)}\right)\beta\mathbb{E}\left(\|A_S\|^2\right) + \sigma\exp\left(-T^2/(4\sigma^2)\right)\mathbb{E}(\|A_S\|^2) + \alpha^2T^4\sigma^{-2}\mathbb{E}\left(\|A_S\|^4\right)\right)\alpha^2\tau_{\mathrm{mix},\Phi}(\epsilon) + \epsilon\right)
\end{aligned}$$

According to Theorem 12, to ensure a small fixed-point error, we require R to vanish as $\sigma \to \infty$. This, in turn, imposes a constraint on the choice of $\gamma(\omega)$ (and hence $g(\omega)$). We prove below that it suffices to choose g to be a uniform distribution. We emphasize that this constraint arises from the need to control the fixed-point error associated with the Lamb shift term in Lemma 13. Specifically, we cannot directly prove that each term in the ω-expansion of $\widetilde{H}_{\rm LS}$ commutes with the thermal state. Instead, we prove that the entire term approximately commutes with the thermal state after integrating over ω.

Before proving Theorem 12, we first use it to prove Theorem 9.

Proof of Theorem 9. When $g(\omega) = \frac{1}{\omega_{\max}}\mathbf{1}_{\omega\in[0,\omega_{\max}]}$ with $\omega_{\max} = \Omega(1)$. In this case, we have $\gamma(\omega) = \frac{1}{\omega_{\max}(1+\exp(\beta\omega))}\mathbf{1}_{\omega\in[-\omega_{\max},\omega_{\max}]}$. Thus, $\|\gamma\|_\infty = \frac{1}{\omega_{\max}}$ and

$$R = \underbrace{\int_0^{(\sigma\omega_{\max})^{-1}} \left|\int_{-\infty}^{\infty}\gamma(\omega)\exp(i\omega\sigma q)\mathrm{d}\omega\right|\exp(-q^2/8)\mathrm{d}q}_{=\mathcal{O}((\sigma\omega_{\max})^{-1})} + \int_{(\sigma\omega_{\max})^{-1}}^{\infty}\left|\int_{-\infty}^{\infty}\gamma(\omega)\exp(i\omega\sigma q)\mathrm{d}\omega\right|\exp(-q^2/8)\mathrm{d}q$$

For the second term, we have

$$\begin{aligned}\left|\int_{-\infty}^{\infty}\gamma(\omega)\exp(i\omega\sigma q)\mathrm{d}\omega\right| &= \left|\frac{1}{i\sigma\omega_{\max}q}\int_{-\omega_{\max}}^{\omega_{\max}}\frac{1}{1+\exp(\beta\omega)}\mathrm{d}\left(\exp(i\omega\sigma q)\right)\right| \\ \le \frac{2}{\omega_{\max}\sigma q} + \frac{1}{\omega_{\max}\sigma q}&\left|\int_{-\omega_{\max}}^{\omega_{\max}}\frac{\beta\exp(\beta\omega)}{(1+\exp(\beta\omega))^2}\exp(i\omega\sigma q)\mathrm{d}\omega\right| = \mathcal{O}\left(\frac{1}{\omega_{\max}\sigma q}\right)\end{aligned}$$

Here, we note $\left|\int_{-\omega_{\max}}^{\omega_{\max}}\frac{\beta\exp(\beta\omega)}{(1+\exp(\beta\omega))^2}\exp(i\omega\sigma q)\mathrm{d}\omega\right| \le \left|\int_{-\infty}^{\infty}\frac{\exp(u)}{(1+\exp(u))^2}\mathrm{d}u\right| = \mathcal{O}(1)$. Plugging this back into the expression for R, we obtain

$$R = \mathcal{O}\left(\frac{1}{\sigma\omega_{\max}}\log(\sigma\omega_{\max})\right)\,.$$

Combining this, Theorem 12, and $\|A_S\| \le 1$, we have

$$\begin{aligned}&\|\rho_{\rm fix}(\Phi)-\rho_\beta\|_1 \\ =&\widetilde{\mathcal{O}}\left(\left(\frac{\beta}{\omega_{\max}\sigma}\left(\sqrt{\log(\sigma/\beta)}+\log(\sigma\omega_{\max})\right)\mathbb{E}(\|A_S\|^2) + \sigma\exp\left(-T^2/(4\sigma^2)\right)\mathbb{E}(\|A_S\|^2) + \alpha^2T^4\sigma^{-2}\mathbb{E}\left(\|A_S\|^4\right)\right)\underbrace{\alpha^2\tau_{\mathrm{mix},\Phi}(\epsilon)}_{=t_{\mathrm{mix},\Phi}} + \epsilon\right)\,.\end{aligned}$$

Now, to achieve ϵ-precision, we first need

$$\left(\frac{\beta}{\omega_{\max}\sigma}\left(\sqrt{\log(\sigma/\beta)}+\log(\sigma\omega_{\max})\right) + \sigma\exp\left(-T^2/(4\sigma^2)\right)\right)\mathbb{E}(\|A_S\|^2)t_{\mathrm{mix},\Phi} = \mathcal{O}(\epsilon)\,,$$

which implies

$$\sigma = \widetilde{\mathcal{O}}\left(\beta\mathbb{E}(\|A_S\|^2)\omega_{\max}^{-1}t_{\mathrm{mix},\Phi}\epsilon^{-1}\right)\,,\quad T = \widetilde{\Omega}\left(\sigma\right)\,.$$

In addition, we also require

$$\alpha^2T^4\sigma^{-2}\mathbb{E}\left(\|A_S\|^4\right)t_{\mathrm{mix},\Phi} = \mathcal{O}(\epsilon)\,,$$

which implies

$$\alpha = \mathcal{O}\left(\sigma T^{-2}t_{\mathrm{mix},\Phi}^{-1/2}\mathbb{E}^{-1/2}\left(\|A_S\|^4\right)\epsilon^{1/2}\right)\,.$$

Plugging in $\sigma = \widetilde{\Theta}\left(\beta\mathbb{E}(\|A_S\|^2)\omega_{\max}^{-1}t_{\mathrm{mix},\Phi}\epsilon^{-1}\right)$, we conclude that

$$\sigma = \widetilde{\Theta}\left(\beta\mathbb{E}(\|A_S\|^2)\omega_{\max}^{-1}t_{\mathrm{mix},\Phi}\epsilon^{-1}\right)\,,\quad T = \widetilde{\Theta}\left(\beta\mathbb{E}(\|A_S\|^2)\omega_{\max}^{-1}t_{\mathrm{mix},\Phi}\epsilon^{-1}\right)\,,$$

and

$$\alpha = \widetilde{\Theta}\left(\sigma^{-1}t_{\mathrm{mix},\Phi}^{-1/2}\mathbb{E}^{-1/2}\left(\|A_S\|^4\right)\epsilon^{1/2}\right) = \widetilde{\Theta}\left(\beta^{-1}\omega_{\max}t_{\mathrm{mix},\Phi}^{-3/2}\epsilon^{3/2}\mathbb{E}^{-1}\left(\|A_S\|^2\right)\mathbb{E}^{-1/2}\left(\|A_S\|^4\right)\right)\,.$$

This concludes Theorem 9. ◻

Next, we prove Theorem 12. According to Theorem 8 Eq. (D2), we need to show the upper bound of $\|\Phi(\rho_\beta)-\rho_\beta\|_1$. According to Lemma 11 and

$$\|\Phi(\rho_\beta)-\rho_\beta\|_1 \le \left\|\Phi-\widetilde{\Phi}\right\|_{1\leftrightarrow 1} + \|\widetilde{\Phi}(\rho_\beta)-\rho_\beta\|_1\,, \tag{E6}$$

it suffices to show $\left\|\widetilde{\Phi}(\rho_\beta)-\rho_\beta\right\|_1$ is small. Let d be the dimension of H and H have an eigendecomposition $\{(\lambda_i,|\psi_i\rangle)\}_{i=0}^{d-1}$ with $\lambda_0 \le \lambda_1 \le \ldots, \lambda_{d-1}$. Because the unitary evolution $\mathcal{U}_S(T)$ preserves the thermal state, we have

$$\left\|\widetilde{\Phi}\left(\rho_\beta\right)-\rho_\beta\right\|_1 \le \alpha^2\left\|\widetilde{\mathcal{L}}\left(\rho_\beta\right)\right\|_1\,, \tag{E7}$$

where $\widetilde{\mathcal{L}}$ is defined in (E3). In $\widetilde{\mathcal{L}}$, we consider the Lamb shift term and dissipative term separately. For the Lamb shift term, we have the following lemma:

Lemma 13. *When $T>\sigma$, we have*

$$\left\|\left[\widetilde{H}_{\mathrm{LS}},\rho_\beta\right]\right\|_1 \le \mathbb{E}_{A_S}\left(\left\|\left[\rho_\beta,\int\gamma(\omega)\widetilde{\mathcal{G}}_{A_S,f}(-\omega)\,\mathrm{d}\omega\right]\right\|_1+\left\|\left[\rho_\beta,\int\gamma(\omega)\left(\widetilde{\mathcal{G}}_{A_S,f}(-\omega)\right)^\dagger\,\mathrm{d}\omega\right]\right\|_1\right)=\mathcal{O}\left(R\beta\mathbb{E}(\|A_S\|^2)\right)$$

For the dissipative term, we have the following lemma:

Lemma 14. *When $T>\sigma>\beta$, we have*

$$\left\|\mathbb{E}_{A_S}\left(\int_{-\infty}^{\infty}\gamma(\omega)\mathcal{D}_{\widetilde{V}_{A_S,f(\omega)}}(\rho_\beta)\mathrm{d}\omega\right)\right\|_1 \le \mathbb{E}_{A_S}\left(\left\|\int_{-\infty}^{\infty}\gamma(\omega)\mathcal{D}_{\widetilde{V}_{A_S,f(\omega)}}(\rho_\beta)\mathrm{d}\omega\right\|_1\right)=\mathcal{O}\left(\|\gamma(\omega)\|_\infty\mathbb{E}\left(\|A_S\|^2\right)\frac{\beta}{\sigma}\sqrt{\log(\sigma/\beta)}\right)$$

We put the proof of the above lemmas in the end of this section. Now, we are ready to prove Theorem 12.

Proof of Theorem 12. Combining Lemma 13 and Lemma 14, we have

$$\left\|\widetilde{\mathcal{L}}\left(\rho_\beta\right)\right\|_1=\mathcal{O}\left(\left(R+\|\gamma(\omega)\|_\infty\frac{1}{\sigma}\sqrt{\log(\sigma/\beta)}\right)\beta\mathbb{E}\left(\|A_S\|^2\right)\right)$$

Plugging this into Eq. (E7) and using Lemma 11 and Theorem 8 with (E6), we conclude the proof. □

Finally, we complete the proof of Lemma 13 and Lemma 14.

Proof of Lemma 13. Recall that

$$\begin{aligned}\widetilde{H}_{\mathrm{LS}}&=-\mathbb{E}_{A_S}\left(\mathrm{Im}\left(\int_{-\infty}^{\infty}\gamma(\omega)\widetilde{\mathcal{G}}_{A_S,f}(-\omega)\mathrm{d}\omega\right)\right)\\&=\frac{-1}{2i}\mathbb{E}_{A_S}\left(\int_{-\infty}^{\infty}\gamma(\omega)\widetilde{\mathcal{G}}_{A_S,f}(-\omega)\mathrm{d}\omega-\int_{-\infty}^{\infty}\gamma(\omega)\left(\widetilde{\mathcal{G}}_{A_S,f}(-\omega)\right)^\dagger\mathrm{d}\omega\right)\end{aligned}.$$

This implies

$$\|[\rho_\beta,\widetilde{H}_{\mathrm{LS}}]\|_1 \le \frac{1}{2}\mathbb{E}_{A_S}\left(\left\|\left[\rho_\beta,\int\gamma(\omega)\widetilde{\mathcal{G}}_{A_S,f}(-\omega)\,\mathrm{d}\omega\right]\right\|_1+\left\|\left[\rho_\beta,\int\gamma(\omega)\left(\widetilde{\mathcal{G}}_{A_S,f}(-\omega)\right)^\dagger\,\mathrm{d}\omega\right]\right\|_1\right)$$

Thus, to show that $\|[\rho_\beta,\widetilde{H}_{\mathrm{LS}}]\|_1$ is small, it suffices to bound

$$\left\|\left[\rho_\beta,\int\gamma(\omega)\widetilde{\mathcal{G}}_{A_S,f}(-\omega)\,\mathrm{d}\omega\right]\right\|_1 \quad\text{and}\quad \left\|\left[\rho_\beta,\int\gamma(\omega)\left(\widetilde{\mathcal{G}}_{A_S,f}(-\omega)\right)^\dagger\,\mathrm{d}\omega\right]\right\|_1 \tag{E8}$$

for all $\|A_S\|\le 1$. The argument proceeds in two steps. First, we show that both

$$\left[H,\int\gamma(\omega)\widetilde{\mathcal{G}}_{A_S,f}(-\omega)\,\mathrm{d}\omega\right] \quad\text{and}\quad \left[H,\int\gamma(\omega)\left(\widetilde{\mathcal{G}}_{A_S,f}(-\omega)\right)^\dagger\,\mathrm{d}\omega\right]$$

are small (we omit the proof of the latter as it is analogous), which implies that $[H, \widetilde{H}_{\mathrm{LS}}]$ is small. Then, we expand ρ_β as a polynomial in H and express the commutators $\|[\rho_\beta, \cdot]\|_1$ as sums of nested commutators, from which we establish the smallness of (E8).

We first calculate $\|[H, \widetilde{H}_{\mathrm{LS}}]\|$. For simplicity, we only consider $\widetilde{\mathcal{G}}_{A_S,f}(\omega)$. The calculation with $\left(\widetilde{\mathcal{G}}_{A_S,f}(\omega)\right)^\dagger$ should be quite similar. Using change of variable $p = (s_1 + s_2)/\sigma$ and $q = (s_1 - s_2)/\sigma$,we have

$$
\begin{aligned}
&\widetilde{\mathcal{G}}_{A_S,f}(\omega) \\
=&\frac{\sigma^2}{2}\int_{-\infty}^{\infty}\mathrm{d}p\int_0^{+\infty}\mathrm{d}q f\left(\frac{\sigma(p+q)}{2}\right) f\left(\frac{\sigma(p-q)}{2}\right) A_S^\dagger\left(\frac{\sigma(p-q)}{2}\right) A_S\left(\frac{\sigma(p+q)}{2}\right)\exp(-i\omega\sigma q)
\end{aligned}
$$

Notice that

$$
\left[H, A_S^\dagger\left(\frac{\sigma(p-q)}{2}\right) A_S\left(\frac{\sigma(p+q)}{2}\right)\right] = \frac{-2i}{\sigma}\frac{\mathrm{d}}{\mathrm{d}p}\left(A_S^\dagger\left(\frac{\sigma(p-q)}{2}\right) A_S\left(\frac{\sigma(p+q)}{2}\right)\right).
$$

Thus,

$$
\begin{aligned}
[H, \widetilde{\mathcal{G}}_{A_S,f}(\omega)] =& \frac{-i\sigma}{2\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_0^{\infty}\exp(-p^2/8)\exp(-q^2/8) \\
&\cdot\frac{2}{\sigma}\frac{\mathrm{d}}{\mathrm{d}p}\left(A_S^\dagger\left(\frac{\sigma(p-q)}{2}\right) A_S\left(\frac{\sigma(p+q)}{2}\right)\right)\exp(-i\omega\sigma q)\mathrm{d}q\mathrm{d}p \\
=&\frac{-i}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{p}{4}\exp(-p^2/8)\exp(-q^2/8) \\
&\cdot A_S^\dagger\left(\frac{\sigma(p-q)}{2}\right) A_S\left(\frac{\sigma(p+q)}{2}\right)\exp(-i\omega\sigma q)\mathrm{d}q\mathrm{d}p
\end{aligned}
$$

We notice that

$$
\left\|A_S^\dagger(\sigma(p-q)/2)A_S(\sigma(p+q)/2)\right\| \leq \|A_S\|^2.
$$

thus,

$$
\left\|\left[H, \int\gamma(\omega)\widetilde{\mathcal{G}}_{A_S,f}(-\omega)\,\mathrm{d}\omega\right]\right\| = \mathcal{O}\left(R\mathbb{E}(\|A_S\|^2)\right).
$$

This implies that

$$
\left\|\left[H, \widetilde{H}_{\mathrm{LS}}\right]\right\| = \mathcal{O}\left(R\mathbb{E}(\|A_S\|^2)\right).
$$

Next, we notice that

$$
\begin{aligned}
&\|[\rho_\beta, \widetilde{H}_{\mathrm{LS}}]\|_1 \leq \|\rho_\beta\|_1\|\rho_\beta^{-1}\widetilde{H}_{\mathrm{LS}}\rho_\beta - \widetilde{H}_{\mathrm{LS}}\| = \|\rho_\beta^{-1}\widetilde{H}_{\mathrm{LS}}\rho_\beta - \widetilde{H}_{\mathrm{LS}}\| \\
=&\mathbb{E}_{A_S}\left\|\mathrm{Im}\left(\int_{-\infty}^{\infty}\gamma(\omega)\left(\rho_\beta^{-1}\widetilde{\mathcal{G}}_{A_S,f}(\omega)\rho_\beta - \widetilde{\mathcal{G}}_{A_S,f}(\omega)\right)\mathrm{d}\omega\right)\right\|.
\end{aligned}
$$

We can use the BCH formula to expand the term $\rho_\beta^{-1}\widetilde{\mathcal{G}}_{A_S,f}(\omega)\rho_\beta$ as a series.

$$
\begin{aligned}
\rho_\beta^{-1}\widetilde{\mathcal{G}}_{A_S,f}(\omega)\rho_\beta - \widetilde{\mathcal{G}}_{A_S,f}(\omega) &= \mathrm{e}^{\beta H}\widetilde{\mathcal{G}}_{A_S,f}(\omega)\mathrm{e}^{-\beta H} - \widetilde{\mathcal{G}}_{A_S,f}(\omega) \\
&= \beta[H, \widetilde{\mathcal{G}}_{A_S,f}(\omega)] + \frac{\beta^2}{2}[H,[H,\widetilde{\mathcal{G}}_{A_S,f}(\omega)]] + \ldots\frac{\beta^n}{n!}[\overbrace{H,[H,\ldots[H}^{nH's},\widetilde{\mathcal{G}}_{A_S,f}(\omega)]..] + \ldots.
\end{aligned}
$$

Using change of variable $p = (s_1 + s_2)/\sigma$ and $q = (s_1 - s_2)/\sigma$, similar to the previous calculation

$$\begin{aligned}\left[H, \widetilde{\mathcal{G}}_{A_S,f}(\omega)\right] =& \frac{\sigma}{2\sqrt{2\pi}} \frac{-2i}{\sigma} \int_{-\infty}^{\infty} \mathrm{d}p \int_0^{+\infty} \mathrm{d}q \exp(-p^2/8)\exp(-q^2/8) \\ &\cdot \exp(-i\omega\sigma q) \frac{\mathrm{d}}{\mathrm{d}p}\left(A_S^\dagger\left(\frac{\sigma(p-q)}{2}\right) A_S\left(\frac{\sigma(p+q)}{2}\right)\right) \\ =& \frac{\sigma}{2\sqrt{2\pi}} \frac{2i}{\sigma} \int_{-\infty}^{\infty} \mathrm{d}p \int_0^{+\infty} \mathrm{d}q \frac{\mathrm{d}}{\mathrm{d}p} \exp(-p^2/8)\exp(-q^2/8) \\ &\cdot \exp(-i\omega\sigma q) A_S^\dagger\left(\frac{\sigma(p-q)}{2}\right) A_S\left(\frac{\sigma(p+q)}{2}\right).\end{aligned}$$

Applying this iteratively, we have the commutator form:

$$\begin{aligned}[\overbrace{H,[H,\dots[H}^{nH's},\widetilde{\mathcal{G}}_{A_S,f}(\omega)]..] =& \frac{\sigma}{2\sqrt{2\pi}}\left(\frac{2i}{\sigma}\right)^n \int_{-\infty}^{\infty}\mathrm{d}p\int_0^{+\infty}\mathrm{d}q \frac{\mathrm{d}^n}{\mathrm{d}p^n}\exp(-p^2/8)\exp(-q^2/8) \\ &\cdot \exp(-i\omega\sigma q) A_S^\dagger\left(\frac{\sigma(p-q)}{2}\right) A_S\left(\frac{\sigma(p+q)}{2}\right)\end{aligned}.$$

Notice that:

$$\left|\frac{\mathrm{d}^n}{\mathrm{d}p^n}\exp(-p^2/8)\right| < 2\sqrt{n!}2^{-n}\exp(-p^2/16)\,.$$

As a result, following the proof of the previous lemma, the n-th term of the series can be bounded by

$$\begin{aligned}\left\|\int_{-\infty}^{\infty}\gamma(\omega)\mathrm{d}\omega\frac{\beta^n}{n!}[\overbrace{H,[H,\dots[H}^{nH's},\widetilde{\mathcal{G}}_{A_S,f}(\omega)]..]\right\| &= \mathcal{O}\left(\frac{\sigma}{2\sqrt{2\pi}}\left(\frac{2\beta}{\sigma}\right)^n\frac{1}{\sqrt{n!}}2^{-n}R\|A_S\|^2\right) \\ &= \mathcal{O}\left(\frac{\beta^{n-1}}{\sigma^{n-1}\sqrt{n!}}R\beta\|A_S\|^2\right)\end{aligned}$$

Summing all terms still gives

$$\left\|\int_{-\infty}^{\infty}\gamma(\omega)\left(\rho_\beta^{-1}\widetilde{\mathcal{G}}_{A^i,f}(\omega)\rho_\beta - \widetilde{\mathcal{G}}_{A^i,f}(\omega)\right)\mathrm{d}\omega\right\| = \mathcal{O}\left(R\beta\|A_S\|^2\right)$$

Similar result can be proved for $\left(\widetilde{\mathcal{G}}_{A_S,f}(\omega)\right)^\dagger$. We conclude the proof. □

Proof of Lemma 14. Let $\mathcal{B}$ be a Lindbladian, define

$$\mathcal{K}(\rho_\beta,\mathcal{B}) = \rho_\beta^{-1/4}\mathcal{B}[\rho_\beta^{1/4}\cdot\rho_\beta^{1/4}]\rho_\beta^{-1/4}$$

with

$$(\mathcal{K}(\rho_\beta,\mathcal{B}))^\dagger = \rho_\beta^{1/4}\mathcal{B}^\dagger[\rho_\beta^{-1/4}\cdot\rho_\beta^{-1/4}]\rho_\beta^{1/4}\,.$$

We note that if $\mathcal{K}(\rho_\beta,\mathcal{B}) = (\mathcal{K}(\rho_\beta,\mathcal{B}))^\dagger$, we have

$$\rho_\beta^{-1/4}\mathcal{B}[\rho_\beta]\rho_\beta^{-1/4} = \mathcal{K}(\rho_\beta,\mathcal{B})[\sqrt{\rho_\beta}] = (\mathcal{K}(\rho_\beta,\mathcal{B}))^\dagger[\sqrt{\rho_\beta}] = \rho_\beta^{1/4}\mathcal{B}^\dagger[I]\rho_\beta^{1/4} = 0\,.$$

This implies $\mathcal{B}$ fixes the thermal state ρ_β. Furthermore,

$$\begin{aligned}&\left\|\mathcal{K}(\rho_\beta,\mathcal{B}) - (\mathcal{K}(\rho_\beta,\mathcal{B}))^\dagger\right\|_{2\leftrightarrow 2} = \left\|\mathcal{K}(\rho_\beta,\mathcal{B}) - (\mathcal{K}(\rho_\beta,\mathcal{B}))^\dagger\right\|_{2\leftrightarrow 2}\|\sqrt{\rho_\beta}\|_2 \\ \geq& \left\|\mathcal{K}(\rho_\beta,\mathcal{B})[\sqrt{\rho_\beta}] - (\mathcal{K}(\rho_\beta,\mathcal{B}))^\dagger[\sqrt{\rho_\beta}]\right\|_2 = \left\|\rho_\beta^{-1/4}\mathcal{B}[\rho_\beta]\rho_\beta^{-1/4}\right\|_2 \\ =& \left\|\rho_\beta^{-1/4}\mathcal{B}[\rho_\beta]\rho_\beta^{-1/4}\right\|_2\|\rho_\beta^{1/4}\|_4^2 \geq \|\mathcal{B}[\rho_\beta]\|_1\end{aligned}. \tag{E9}$$

Here $\|\cdot\|_p$ is the Schatten-p norm defined in Section A. In the last inequality, we use Hölder's inequality $\|BAB\|_1 \leq \|B\|_4^2\|A\|_2$. This inequality implies that, if $\mathcal{K}(\rho_\beta,\mathcal{B})$ is approximately self-adjoint, $\mathcal{B}$ can also approximately preserve the thermal state.

The rest of the proof follows a similar procedure as the proof of [17, Theorem I.3] to show that $\|\mathcal{D}[\rho_\beta]\|_1$ is small. Let $\mathcal{D}=\mathbb{E}_{A_S}\left(\int_{-\infty}^{\infty}\gamma(\omega)\mathcal{D}_{\widetilde{V}_{A_S,f}(-\omega)}(\rho_\beta)\mathrm{d}\omega\right)$. In the proof of [17, Theorem I.3], the authors first approximate $\mathcal{D}$ with secular version $\mathcal{D}_{sec}$ (See [17, Lemma A.2]). The secular approximation is an artificial cutoff in frequency space on the transition energies induced by Lindblad jump operators, which causes only a small error when σ is sufficiently large. Following the proof of [17, Theorem I.3],we have

$$\|\mathcal{D}_{sec}-\mathcal{D}\|_{1\leftrightarrow 1}+\left\|\mathcal{K}(\rho_\beta,\mathcal{D}_{sec})(\rho_\beta)-(\mathcal{K}(\rho_\beta,\mathcal{D}_{sec})(\rho_\beta))^\dagger\right\|_{2\leftrightarrow 2}=\mathcal{O}\left(\|\gamma(\omega)\|_\infty\mathbb{E}\left(\|A_S\|^2\right)\frac{\beta}{\sigma}\sqrt{\log(\sigma/\beta)}\right).$$

This implies that

$$\begin{aligned}&\|\mathcal{D}(\rho_\beta)\|_1\leq\|\mathcal{D}_{sec}-\mathcal{D}\|_{1\leftrightarrow 1}+\|\mathcal{D}_{sec}(\rho_\beta)\|_1\\ \leq&\|\mathcal{D}_{sec}-\mathcal{D}\|_{1\leftrightarrow 1}+\left\|\mathcal{K}(\rho_\beta,\mathcal{D}_{sec})(\rho_\beta)-(\mathcal{K}(\rho_\beta,\mathcal{D}_{sec})(\rho_\beta))^\dagger\right\|_{2\leftrightarrow 2}=\mathcal{O}\left(\|\gamma(\omega)\|_\infty\mathbb{E}\left(\|A_S\|^2\right)\frac{\beta}{\sigma}\sqrt{\log(\sigma/\beta)}\right).\end{aligned}$$

In the second inequality, we use (E9). □

2. Approximate fixed point – Ground state

In this section, we provide a rigorous version of Theorem 10 in Theorem 16 and provide the proof. We consider (2) with $\beta=\infty$ and $f(t)=\frac{1}{(2\pi)^{1/4}\sigma^{1/2}}\exp\left(-\frac{t^2}{4\sigma^2}\right)$. Similar to the thermal state case, we first consider a simplified CPTP map by removing the error terms in Theorem 4 and take the limit $T\to\infty$, as mentioned in Section A,

$$\widetilde{\Phi}=\mathcal{U}_S(T)\circ\exp\left(\widetilde{\mathcal{L}}\alpha^2\right)\circ\mathcal{U}_S(T)\,. \tag{E10}$$

Here

$$\widetilde{\mathcal{L}}(\rho)=\mathbb{E}_{A_S}\left(-i\left[\widetilde{H}_{\mathrm{LS},A_S},\rho\right]+\int_{-\infty}^{0}(g(\omega)+g(-\omega))\mathcal{D}_{\widetilde{V}_{A_S,f}(\omega)}(\rho)\mathrm{d}\omega\right), \tag{E11}$$

where

$$\widetilde{H}_{\mathrm{LS},A_S}=-\mathrm{Im}\left(\int_{-\infty}^{0}g(\omega)\widetilde{\mathcal{G}}_{A_S^\dagger,f}(\omega)\mathrm{d}\omega+\int_{0}^{\infty}g(\omega)\widetilde{\mathcal{G}}_{A_S,f}(-\omega)\mathrm{d}\omega\right),\quad \widetilde{V}_{A_S,f}(\omega)=\int_{-\infty}^{\infty}f(t)A_S(t)\exp(-i\omega t)\mathrm{d}t\,,$$

with

$$\widetilde{\mathcal{G}}_{A_S,f}(\omega)=\int_{-\infty}^{\infty}\int_{-\infty}^{s_1}f(s_2)f(s_1)A_S^\dagger(s_2)A_S(s_1)\exp(i\omega(s_2-s_1))\mathrm{d}s_2\mathrm{d}s_1\,.$$

Same as Lemma 11, the error between Φ and $\widetilde{\Phi}$ can be controlled in the following lemma:

Lemma 15. *When $T>\sigma$, We have*

$$\left\|\Phi-\widetilde{\Phi}\right\|_{1\leftrightarrow 1}=\mathcal{O}\left(\alpha^2\sigma\exp\left(-T^2/(4\sigma^2)\right)\mathbb{E}(\|A_S\|^2)+\alpha^4T^4\sigma^{-2}\mathbb{E}\left(\|A_S\|^4\right)\right)$$

The proof of Lemma 15 is almost the same as the proof of Lemma 11. Thus, we omit it. Using $\widetilde{\Phi}$, we are ready to state the rigorous version of Theorem 10 and provide the proof:

Theorem 16. *Assume H has a spectral gap Δ and $T>\sigma$. Then, for any $\epsilon>0$,*

$$\begin{aligned}&\|\rho_{\mathrm{fix}}(\Phi)-|\psi_0\rangle\langle\psi_0|\|_1\\ =&\mathcal{O}\left(\left(\|H\|^{1/2}\sigma^{3/2}\exp\left(-\sigma^2\Delta^2/8\right)\mathbb{E}(\|A_S\|^2)+\sigma\exp\left(-T^2/(4\sigma^2)\right)\mathbb{E}(\|A_S\|^2)+\alpha^2T^4\sigma^{-2}\mathbb{E}\left(\|A_S\|^4\right)\right)\alpha^2\tau_{\mathrm{mix},\Phi}(\epsilon)+\epsilon\right)\end{aligned}$$

The proof of this theorem follows a similar approach to that of Theorem 12, where we demonstrate that both the Lamb shift term and the dissipative term approximately preserve the ground state. Although the overall proof strategy is similar, we adopt a different technique in the proof below. Specifically, using the spectral gap Δ, we directly establish a small fixed-point error when the number of Bohr frequencies is constant. In the general case, where the number of Bohr frequencies cannot be bounded, we approximate the Hamiltonian by a rounded version with a controllable number of eigenvalues, inspired by the secular approximation idea in [18]. The errors introduced in the Lamb shift and dissipative terms due to this rounding can also be controlled by exploiting the Gaussian structure of f. Furthermore, when handling the Lamb shift term, the rounding technique and the spectral gap assumption allow us to establish a uniform fixed-point error bound for $\left\|\left[|\psi_0\rangle\langle\psi_0|,\widetilde{\mathcal{G}}_{A_S,f}(-\omega)\right]\right\|_1+\left\|\left[|\psi_0\rangle\langle\psi_0|,\left(\widetilde{\mathcal{G}}_{A_S,f}(-\omega)\right)^\dagger\right]\right\|_1+\left\|\mathcal{D}_{\widetilde{V}_{A_S,f}(\omega)}(|\psi_0\rangle\langle\psi_0|)\right\|_1$ in ω prior to taking the expectation over ω. This enables the use of an arbitrary distribution g in the theorem above.

Proof of Theorem 16. Similar to the proof of Theorem 12, it suffices to show $\left\|\widetilde{\Phi}(|\psi_0\rangle\langle\psi_0|)-|\psi_0\rangle\langle\psi_0|\right\|_1$ is small. Because the unitary evolution $\mathcal{U}_S(T)$ preserves the ground state, we have

$$\left\|\widetilde{\Phi}\left(|\psi_0\rangle\langle\psi_0|\right)-|\psi_0\rangle\langle\psi_0|\right\|_1\leq\alpha^2\left\|\widetilde{\mathcal{L}}\left(|\psi_0\rangle\langle\psi_0|\right)\right\|_1, \tag{E12}$$

where $\widetilde{\mathcal{L}}$ is defined in Eq. (E11).

Now, we consider the Lamb shift term and dissipative term separately. For simplicity, we consider a fixed A_S in the following calculation. Recall that

$$A_S(\nu)=\sum_{\lambda_j-\lambda_i=\nu}|\psi_j\rangle\langle\psi_j|A_S|\psi_i\rangle\langle\psi_i|,\quad A_S^\dagger(\nu)=\sum_{\lambda_j-\lambda_i=\nu}|\psi_j\rangle\langle\psi_j|A_S^\dagger|\psi_i\rangle\langle\psi_i|.$$

- Lamb shift term: Recall the definition of $\widetilde{\mathcal{G}}_{A_S,f}$:

$$\widetilde{\mathcal{G}}_{A_S,f}(-\omega)=\int_{-\infty}^{\infty}\int_{-\infty}^{s_1}f(s_2)f(s_1)A_S^\dagger(s_2)A_S(s_1)\exp(-i\omega(s_2-s_1))\mathrm{d}s_2\mathrm{d}s_1\,.$$

Using change of variable $p=(s_1+s_2)/\sigma$ and $q=(s_1-s_2)/\sigma$,

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{-\infty}^{s_1}f(s_2)f(s_1)A_S^\dagger(s_2)A_S(s_1)\exp(-i\omega(s_2-s_1))\mathrm{d}s_2\mathrm{d}s_1\\=&\sum_{\nu_1,\nu_2\in B(H)}A_S^\dagger(\nu_2)A_S(\nu_1)\int_{-\infty}^{\infty}\int_{-\infty}^{s_1}f(s_2)f(s_1)\exp(i\nu_2 s_2)\exp(i\nu_1 s_1)\exp(-i\omega(s_2-s_1))\mathrm{d}s_2\mathrm{d}s_1\\=&\frac{\sigma}{2\sqrt{2\pi}}\sum_{\nu_1,\nu_2\in B(H)}A_S^\dagger(\nu_2)A_S(\nu_1)\\&\cdot\underbrace{\int_{-\infty}^{\infty}\exp\left(i\frac{\sigma p}{2}(\nu_1+\nu_2)\right)\exp\left(-\frac{p^2}{8}\right)\mathrm{d}p}_{=\mathcal{O}(\exp(-\sigma^2(\nu_1+\nu_2)^2/2))}\underbrace{\int_0^{\infty}\exp\left(-\frac{q^2}{8}\right)\exp\left(i\frac{\sigma q}{2}(\nu_1-\nu_2)\right)\exp(i\sigma\omega q)\mathrm{d}q}_{=\mathcal{O}(1)}\end{aligned}$$

where $B(H)$ is the set of Bohr frequencies.

We note that

$$\left[|\psi_0\rangle\langle\psi_0|,A_S^\dagger(\nu_2)A_S(\nu_1)\right]=0$$

when $|\nu_2+\nu_1|<\Delta$. We show this using the proof by contradiction: When $\left[|\psi_0\rangle\langle\psi_0|,A_S^\dagger(\nu_2)A_S(\nu_1)\right]\neq0$, we must have $|\psi_0\rangle\langle\psi_0|A_S^\dagger(\nu_2)A_S(\nu_1)\neq0$ or $A_S^\dagger(\nu_2)A_S(\nu_1)|\psi_0\rangle\langle\psi_0|\neq0$. We consider these two cases separately:

  - In the first case, we have $\left(A_S^\dagger(\nu_2)\right)^\dagger|\psi_0\rangle=A_S(-\nu_2)|\psi_0\rangle\neq0$, which implies $\nu_2\leq0$. Now, since $|\nu_2+\nu_1|<\Delta$ and $|\psi_0\rangle\langle\psi_0|A_S^\dagger(\nu_2)A_S(\nu_1)\neq0$, we have $\nu_1=-\nu_2$. This implies $\left[|\psi_0\rangle\langle\psi_0|,A_S^\dagger(\nu_2)A_S(\nu_1)\right]=0$.

– In the second case, we have $A_S(\nu_1)\left|\psi_0\right\rangle \neq 0$, which implies $\nu_1 \geq 0$. Now, since $|\nu_2+\nu_1| < \Delta$ and $A_S^\dagger(\nu_2)A_S(\nu_1)\left|\psi_0\right\rangle \neq 0$, we have $\nu_1 = -\nu_2$. This implies $\left[\left|\psi_0\right\rangle\left\langle\psi_0\right|, A_S^\dagger(\nu_2)A_S(\nu_1)\right] = 0$.

These two cases give a contradiction. This implies

$$\left[\left|\psi_0\right\rangle\left\langle\psi_0\right|, \widetilde{\mathcal{G}}_{A_S,f}(-\omega)\right] = \sum_{|\nu_2+\nu_1|\geq\Delta} \underbrace{F(\nu_1,\nu_2)}_{|F(\nu_1,\nu_2)|=\mathcal{O}(\sigma\exp(-\sigma^2\Delta^2/2))} \left[\left|\psi_0\right\rangle\left\langle\psi_0\right|, A_S^\dagger(\nu_2)A_S(\nu_1)\right]. \tag{E13}$$

Now, we are ready to show (E13) is small. First, let us assume H has discrete eigenvalues in $[-\|H\|,\|H\|]$ with uniform gap η, meaning $|\lambda_i-\lambda_j|\geq\eta$ if $\lambda_i\neq\lambda_j$. This implies $|B(H)| = \mathcal{O}(\|H\|/\eta)$, where $|B(H)|$ means the number of elements in $B(H)$. Then,

$$\begin{aligned}\left\|\sum_{\nu_2\leq 0,\nu_1\geq 0}(\dots)\right\| &= \mathcal{O}\left(\|A_S\|^2|B(H)|\sigma\exp\left(-\sigma^2\Delta^2/2\right)\right)\\ =&\mathcal{O}\left(\|A_S\|^2\|H\|\sigma\exp\left(-\sigma^2\Delta^2/2\right)/\eta\right).\end{aligned}$$

Because every Hamiltonian can be approximated by a rounding Hamiltonian H_η such that: 1. $\|H-H_\eta\|\leq\eta$; 2. H_η has the same ground state; 3. H_η has discrete eigenvalues in $[-\|H\|,\|H\|]$ with uniform gap η. We conclude that

$$\begin{aligned}&\left\|\sum_{\nu_2\leq 0,\nu_1\geq 0}(\dots)\right\|\\ =&\mathcal{O}\left(\|A_S\|^2\|H\|\sigma\exp\left(-\sigma^2\Delta^2/2\right)/\eta\right)+\mathcal{O}\left(\|A_S\|^2\int_{-\infty}^{\infty}\int_{-\infty}^{s_1} f(s_1)f(s_2)(|s_1|+|s_2|)\eta\mathrm{d}s_1\mathrm{d}s_2\right)\\ =&\mathcal{O}\left(\|A_S\|^2\min_\eta\left(\sigma\exp\left(-\sigma^2\Delta^2/2\right)\|H\|/\eta+\eta\sigma^2\right)\right)\\ =&\mathcal{O}\left(\|A_S\|^2\|H\|^{1/2}\sigma^{3/2}\exp\left(-\left(\sigma^2\Delta^2/4\right)\right)\right)\end{aligned} \tag{E14}$$

Here, the second term arises from approximating $\left[\left|\psi_0\right\rangle\left\langle\psi_0\right|, \widetilde{\mathcal{G}}_{A_S,f}(-\omega)\right]$ by replacing H with H_η. This concludes the calculation for the Lamb shift term.

- Dissipative term: When $f(t) = \frac{1}{(2\pi)^{1/4}\sigma^{1/2}}\exp(-t^2/(4\sigma^2))$,

$$\widetilde{V}_{A_S,f}(\omega) = \int_{-\infty}^{\infty} f(t)A_S(t)\exp(-i\omega t)\mathrm{d}t = 2^{3/4}\pi^{1/4}\sqrt{\sigma}\sum_{\nu\in B(H)}\exp(-(\nu-\omega)^2\sigma^2)A_S(\nu).$$

Define the component of V that preserves the ground state as V^+:

$$V^+_{A_S,f}(\omega) = \begin{cases} 2^{3/4}\pi^{1/4}\sqrt{\sigma}\displaystyle\sum_{\nu\in B(H),\nu<0}\exp(-(\nu-\omega)^2\sigma^2)A(\nu), & \omega < -\dfrac{\Delta}{2}\\ 2^{3/4}\pi^{1/4}\sqrt{\sigma}\displaystyle\sum_{\text{If } i=0 \text{ or } j=0, \text{ then } i+j=0}\exp(-(\lambda_i-\lambda_j-\omega)^2\sigma^2)\left\langle\psi_i\right|A\left|\psi_j\right\rangle\left|\psi_i\right\rangle\left\langle\psi_j\right|, & -\dfrac{\Delta}{2}\leq\omega<0\end{cases}$$

Here, $|\psi_i\rangle$ is the eigenvector of H with eigenvalue λ_i with $\lambda_0, |\psi_0\rangle$ being the ground state energy and ground state. Recall (E11):

$$\widetilde{\mathcal{L}}(\rho) = \mathbb{E}_{A_S}\left(-i\left[\widetilde{H}_{\mathrm{LS},A_S},\rho\right]+\int_{-\infty}^{0}(g(\omega)+g(-\omega))\mathcal{D}_{\widetilde{V}_{A_S,f}(\omega)}(\rho)\mathrm{d}\omega\right).$$

We will show that the choice of $V^+_{A_S,f}(\omega)$ ensures that: 1. $\left|\psi_0\right\rangle\left\langle\psi_0\right| \in \mathrm{Ker}\left(\mathcal{D}_{V^+_{A_S,f}(\omega)}\right)$ for any $\omega<0$; 2. $\widetilde{V}_{A_S,f}(\omega)$ is close to $V^+_{A_S,f}(\omega)$. Consider two cases:

– When $\omega < -\Delta/2$, $|\psi_0\rangle\langle\psi_0| \in \mathrm{Ker}\left(\mathcal{D}_{V^+_{A_S,f}(\omega)}\right)$ is straightforward because $V^+_{A_S,f}(\omega)|\psi_0\rangle = 0$. To show that $\widetilde{V}_{A_S,f}(\omega)$ is close to $V^+_{A_S,f}(\omega)$, we use the rounding Hamiltonian technique similar to the calculation for the Lamb shift term. First, let us assume H has discrete eigenvalues in $[-\|H\|, \|H\|]$ with uniform gap η, meaning $|\lambda_i - \lambda_j| = \eta$ if $\lambda_i \neq \lambda_j$. This implies $|B(H)| = \mathcal{O}(\|H\|/\eta)$. Then, for $\omega \geq 0$,

$$\left\|\widetilde{V}_{A_S,f}(\omega) - V^+_{A_S,f}(\omega)\right\| = \mathcal{O}\left(\|A_S\||B(H)|\sqrt{\sigma}\exp\left(-\sigma^2\Delta^2/4\right)\right) = \mathcal{O}\left(\|A_S\|\|H\|\sqrt{\sigma}\exp\left(-\sigma^2\Delta^2/4\right)/\eta\right) . \tag{E15}$$

Similar to the Lamb shift term, we approximated the Hamiltonian by the rounding Hamiltonian H_η such that $\|H - H_\eta\| \leq \eta$ and H_η has discrete eigenvalues in $[-\|H\|, \|H\|]$ with uniform gap η. We conclude that, for general H,

$$\begin{aligned}\left\|\widetilde{V}_{A_S,f}(\omega) - V^+_{A_S,f}(\omega)\right\| &= \mathcal{O}\left(\min_\eta\left(\|A_S\|\|H\|\sqrt{\sigma}\exp\left(-\sigma^2\Delta^2/4\right)/\eta + \|A_S\|\eta\underbrace{\|tf(t)\|_{L^1}}_{=\mathcal{O}(\sigma^{3/2})}\right)\right) . \\ &=\mathcal{O}\left(\|A_S\|\|H\|^{1/2}\sigma\exp(-\sigma^2\Delta^2/8)\right)\end{aligned} \tag{E16}$$

– When $-\Delta/2 < \omega \leq 0$, we can rewrite

$$V^+_{A_S,f}(\omega) = (\dots)|\psi_0\rangle\langle\psi_0| + \sum_{i,j\neq 0}(\dots)|\psi_i\rangle\langle\psi_j| .$$

This ensures that $[V^+_{A_S,f}(\omega), |\psi_0\rangle\langle\psi_0|]$ and thus $|\psi_0\rangle\langle\psi_0| \in \mathrm{Ker}\left(\mathcal{D}_{V^+_{A_S,f}(\omega)}\right)$. Next, to show $\widetilde{V}_{A_S,f}(\omega)$ is close to $V^+_{A_S,f}(\omega)$, we note that

$$\begin{aligned}V_{A_S,f}(\omega) =& V^+_{A_S,f}(\omega) + 2^{3/4}\pi^{1/4}\sigma^{1/2}\sum_{i\neq 0}\exp(-(\lambda_i - \lambda_0 - \omega)^2\sigma^2)\langle\psi_i|A|\psi_0\rangle|\psi_i\rangle\langle\psi_0| \\ &+ 2^{3/4}\pi^{1/4}\sigma^{1/2}\sum_{i\neq 0}\exp(-(\lambda_0 - \lambda_i - \omega)^2\sigma^2)\langle\psi_0|A|\psi_i\rangle|\psi_0\rangle\langle\psi_i| .\end{aligned}$$

In the above summation, since $i \neq 0$ and H has spectral gap Δ, we have $|\lambda_i - \lambda_0| \geq \Delta$ and $|\lambda_i - \lambda_0 - \omega| \geq \Delta/2$ when $-\Delta/2 < \omega \leq 0$. This guarantees that each term in the summation can be upper bounded, meaning

$$\left\|2^{3/4}\pi^{1/4}\sigma^{1/2}\sum_{\lambda_i=\lambda}\exp(-(\lambda_0 - \lambda_i - \omega)^2\sigma^2)\langle\psi_0|A|\psi_i\rangle|\psi_0\rangle\langle\psi_i|\right\| = \mathcal{O}\left(\|A_S\|\sqrt{\sigma}\exp\left(-\sigma^2\Delta^2/4\right)\right)$$

for each eigenvalue λ. Thus, similar to the first case, we also have (E15) and (E16).

Because both cases satisfy (E16), we have

$$\begin{aligned}\left\|\mathcal{L}_{\widetilde{V}_{A_S,f}(\omega)}(|\psi_0\rangle\langle\psi_0|)\right\|_1 &= \mathcal{O}\left(\left\|\mathcal{L}_{\widetilde{V}_{A_S,f}(\omega)} - \mathcal{L}_{V^+_{A_S,f}(\omega)}\right\|_{1\leftrightarrow 1}\right) = \mathcal{O}\left(\left\|\widetilde{V}_{A_S,f}(\omega) - V^+_{A_S,f}(\omega)\right\|\left\|\widetilde{V}_{A_S,f}(\omega)\right\|\right) \\ &=\mathcal{O}\left(\|A_S\|^2\|H\|^{1/2}\sigma^{3/2}\exp(-\sigma^2\Delta^2/8)\right) .\end{aligned} \tag{E17}$$

Combining (E14) and (E17), we have

$$\left\|\widetilde{\mathcal{L}}(|\psi_0\rangle\langle\psi_0|)\right\|_1 = \mathcal{O}\left(\|A_S\|^2\|H\|^{1/2}\sigma^{3/2}\exp(-\sigma^2\Delta^2/8)\right)$$

Plugging this into (E12),

$$\left\|\widetilde{\Phi}(|\psi_0\rangle\langle\psi_0|) - |\psi_0\rangle\langle\psi_0|\right\|_1 = \mathcal{O}\left(\alpha^2\sigma^{3/2}\exp(-\sigma^2\Delta^2/8)\|A_S\|^2\|H\|^{1/2}\right)$$

This concludes the proof. □

Appendix F: Mixing time and End-to-end efficiency analysis

As concrete examples to guarantee fast mixing, in this section, we choose $\mathcal{A}$ to be the set of all single-qubit Pauli operators (and their negatives) for qubit systems, and the set of creation and annihilation operators (and their negatives) for fermionic systems. For the functions g and f, we set

$$g(\omega)=\frac{1}{\omega_{\max}}\mathbf{1}_{[0,\omega_{\max}]},\ f(t)=\frac{1}{(2\pi)^{1/4}\sigma^{1/2}}\exp\left(-\frac{t^2}{4\sigma^2}\right).$$

The parameters $\omega_{\max}$ are selected so that the system-bath interaction can induce energy transitions effectively. The choice of $\omega_{\max}$ can be system-dependent and should generally be at least as large as the largest eigenvalue gap, and typically does not grow with system size. The parameter σ in the filter function $f(t)$ is typically chosen to be sufficiently large to ensure that the Lamb shift term in Theorem 4 approximately commutes with the thermal or ground state, as discussed in Section E.

According to Theorem 9 and Theorem 10, to establish end-to-end efficiency, it suffices to provide an upper bound on $t_{\mathrm{mix},\Phi}$. However, we emphasize that in Theorem 9 and Theorem 10, the mixing time $t_{\mathrm{mix},\Phi}$ and the parameter σ are *not* independent of each other. The bath Hamiltonian H_B, the coupling operator B_E, and the filter function $f(t)$ must be carefully designed to ensure that the conditions required for the theorems are meaningfully satisfied.

In this section, we provide the result of upper bounding the mixing time of the map Φ defined in Eq. (2) and a complete end-to-end efficiency analysis for preparing both the thermal state and the ground state. Specifically, we consider three examples of physical systems: a single qubit example (as a toy model), free fermionic systems, and commuting local Hamiltonians. In all three cases, we show that the mixing time of Φ can be upper bounded by a constant independent of σ, provided that σ is sufficiently large. This enables us to achieve an arbitrarily small fixed-point error by appropriately choosing a large σ and a small α. For clarity, we first state the results, and defer all proofs to later sections.

1. Single qubit example

We first consider a toy model to illustrate the key ideas. Assume the system Hamiltonian $H=-Z$. In Eq. (2), we set $\mathcal{A}=\{X,-X\}$ and $g(\omega)=\frac{1}{3}\mathbf{1}_{[0,3]}(\omega)$ ($\omega_{\max}=3$). Then, we have the following result:

Theorem 17. *For thermal state preparation, given any $\beta,\epsilon>0$, there exists a constant $C=\mathrm{poly}(\beta,1/\epsilon)$ such that if $\sigma>C$, $T=\widetilde{\Omega}(\sigma)$, and $\alpha<\sigma^{-1}C^{-1}$, we have*

$$t_{\mathrm{mix},\Phi}(\epsilon)=\mathcal{O}(\log(1/\epsilon)).$$

For ground state preparation ($\beta=\infty$), given $\epsilon>0$, there exists a constant $C=\mathrm{polylog}(1/\epsilon)$ such that if $\sigma>C$, $T=\widetilde{\Omega}(\sigma)$, and $\alpha<\sigma^{-1}\epsilon^{1/2}C^{-1}$, we have

$$t_{\mathrm{mix},\Phi}(\epsilon)=\mathcal{O}(\log(1/\epsilon)).$$

Although this is a toy model, it highlights a key mechanism underlying the efficiency of our protocol when $\sigma\gg 1$: the design of the jump operator $V_{A_S,f,T}(\omega)$ should support a wide range of nondegenerate energy transitions. In the present setting, it suffices to have nondegenerate jumps between $|0\rangle$ and $|1\rangle$; see Eqs. (G1) and (G3). Even for this simplified model, achieving this property requires a careful choice of both the function $f(t)$ and the bath. A contrasting example that fails to meet this condition is discussed in Section G, Theorem 22.

The proof of Theorem 17 is given in Section G. We emphasize that, in this theorem, when σ is sufficiently large, the mixing time $t_{\mathrm{mix},\Phi}(\epsilon)$ becomes independent of σ. Plugging this bound into Theorem 9 and Theorem 10 yields a result demonstrating the end-to-end efficiency of our protocol; see Theorem 21.

2. Free fermionic systems

Consider a local fermionic Hamiltonian H defined on a D-dimensional lattice of fermionic systems, $\Lambda=[0,L]^D$, given by

$$H=\sum_{i,j=1}^{N}h_{i,j}c_i^\dagger c_j\,. \tag{F1}$$

where $N=(L+1)^D$ is the number of fermionic modes, $(h_{i,j})$ is a Hermitian matrix, and $c_j^\dagger$ and c_j are the creation and annihilation operators at site j. We also assume that the coefficient matrix h satisfies $\|h\|=\mathcal{O}(1)$. Note that the operator norm of the Hamiltonian $\|H\|$ can still increase with respect to the system size N. We choose A_S to be uniformly sampled from the set of all single fermionic operators $\{\pm c_i^\dagger, \pm c_i\}_{i=1}^n$.

The mixing time analysis for ground state preparation is simpler, so we present it first. The rigorous version of Theorem 18 appears in Section H as Theorem 23.

Theorem 18 (Ground state of quadratic fermionic Hamiltonian, informal)**.** *Assume H has a spectral gap Δ. Let $g(\omega)=\frac{1}{\omega_{\max}}\mathbf{1}_{[0,\omega_{\max}]}$ with $\omega_{\max}=2\|h\|$. Given any $\epsilon>0$, if $\sigma=\widetilde{\Theta}(\Delta^{-1})$, $T=\widetilde{\Theta}(\Delta^{-1})$, and $\alpha=\widetilde{\mathcal{O}}(\epsilon^{1/2}\Delta N^{-1/2})$, we have*

$$t_{\mathrm{mix},\Phi}(\epsilon)=\mathcal{O}\left(N\log(N/\epsilon)\right)\,.$$

Here, $\widetilde{\Theta}$ suppresses logarithmic dependencies on Δ^{-1}, $1/\epsilon$, and N.

To prove this result, we adopt the strategy from [22, Section IV], which analyzes the Heisenberg evolution of the number operator. Following the argument in [22, Section IV], the convergence of the Lindblad dynamics to the ground state can be established by showing that the expectation of the number operator converges to zero. Moreover, since the unitary evolution commutes with the number operator, it does not affect this convergence. Finally, the convergence of the number operator can be directly related to the trace distance between the current state and the ground state using the Fuchs–van de Graaf inequality; see Section H for details.

For thermal state preparation, we have an analogous result.

Theorem 19 (Thermal state of quadratic fermionic Hamiltonian at constant temperature, informal)**.** *For any constant temperature β^{-1}, with a proper choice of $g(\omega)$, let $\sigma=\widetilde{\Theta}(\epsilon^{-1}N^2), T=\widetilde{\Theta}(\epsilon^{-1}N^2), \alpha=\widetilde{\Theta}(\epsilon^{3/2}N^{-3})$, we have*

$$t_{\mathrm{mix},\Phi}(\epsilon)=\mathcal{O}\left(N^2\log(N/\epsilon)\right)\,.$$

Here, the notation $\widetilde{\Theta}$ suppresses logarithmic dependencies on $1/\epsilon$, and N.

The rigorous version of Theorem 19 is presented in Section J as Theorem 29. Compared to the ground state result, the additional N factor in t_{mix} mainly arises from the initial dependence of the norm $\|\rho_\beta^{-1/4}[\cdot]\rho_\beta^{-1/4}\|_2$; see the detailed discussion at the end of Section J. It is worth noting that the choice of $g(\omega)$ in Theorem 29 is chosen to simplify the analysis, and can be suboptimal at large β [41, Section VII].

In Theorem 19, it may be possible to further reduce the dependence of $t_{\mathrm{mix},\Phi}$ to linear in N by employing advanced mixing time analysis techniques, such as the modified logarithmic Sobolev inequality or the oscillator norm method [22, 34, 36, 37, 39, 40]. However, due to the additional analytical challenges introduced by the Lamb-shift term, pursuing this improvement lies beyond the current scope of this work. On the other hand, we believe that the linear N dependence of $t_{\mathrm{mix},\Phi}$ in Theorems 18 and 19 is intrinsic, since the algorithm samples only one jump operator per iteration. This situation closely parallels that of Lindbladian-dynamics-based algorithms: while rapid mixing can, in principle, be achieved when employing $O(N)$ jump operators, the total end-to-end simulation cost still scales linearly with N [43–45].

3. Commuting local Hamiltonians

Let $H=\sum_i h_i$ be a commuting local Hamiltonian defined on a D-dimensional lattice, where each local term h_i commutes with all others and is supported on a ball of constant radius. Furthermore, each qubit j is acted upon by only a constant number of terms h_i. Let I_j denote the set of indices i such that h_i acts non-trivially on qubit j, and define $H_j=\sum_{i\in I_j}h_i$. Let $\Delta_\lambda=\max_{j,k}\left(\lambda_{k+1}(H_j)-\lambda_k(H_j)\right)$ be the maximal nearby eigenvalue difference among all H_j. We note that for local commuting Hamiltonians, Δ_λ is often a constant independent of the system size.

We choose A_S to be randomly sampled from all local Pauli operators $\{\pm X_i, \pm Y_i, \pm Z_i\}_{i=1}^n$. We have the following result:

Theorem 20 (Commuting local Hamiltonian at high temperature, informal)**.** *Let H be a commuting local Hamiltonian defined on a D-dimensional lattice and $g(\omega)=\frac{1}{\omega_{\max}}\mathbf{1}_{[0,\omega_{\max}]}$ with $\omega_{\max}=2\Delta_\lambda$. There exists a constant β_c dependent on the Hamiltonian H such that for every $\beta\le\beta_c$ and any $\epsilon>0$, if $\sigma=\widetilde{\Theta}(\epsilon^{-1}N^2), T=\widetilde{\Theta}(\epsilon^{-1}N^2), \alpha=\widetilde{\Theta}(\epsilon^{3/2}N^{-3})$, we have*

$$t_{\mathrm{mix},\Phi}(\epsilon)=\mathcal{O}\left(N^2\log(1/\epsilon)\right)\,.$$

Here, $\widetilde{\Theta}$ suppresses logarithmic dependencies on $1/\epsilon$, and N.

A more general version of Theorem 20 is given in Section K Theorem 31. Here we use the result of [35] stating that for commuting local Hamiltonians, there exists a critical inverse temperature β_c such that, when $\beta \leq \beta_c$, the spectral gap of the Davies generator is bounded below; see Theorem 32.

Although the mixing time bounds in Theorem 19 and Theorem 20 appear similar, their proof strategies differ substantially. For the thermal state case, the main idea is to show that the dissipative part of the Lindbladian approximately satisfies the detailed balance condition, while the Lamb shift term approximately commutes with the thermal state when $\sigma \gg 1$. However, the Lindbladian with the Lamb shift term does not satisfy the quantum detailed balance condition. Therefore existing techniques using the contraction of χ^2-distance, relative entropy [34], or local oscillator norm [22, 40] are not directly applicable.

Instead, we follow the approach of [17, Appendix E.3.a, Proposition II.2], which analyzes the spectral gap of the dissipative part of the generator after a similarity transformation, as introduced in [17, Appendix E.2]. In particular, we prove contraction under the weighted Hilbert-Schmidt norm $\|\rho_\beta^{-1/4}[\cdot]\rho_\beta^{-1/4}\|_2$. This contraction still holds in the presence of the unitary evolution in Eq. (7), and therefore also holds for the map Φ. Further details are given in Section I, in particular Theorem 25 and Theorem 26.

4. End-to-end efficiency analysis

In the previous section, we have established that the fixed-point approximation error and the upper bound on the mixing time are independent of σ, when σ is sufficiently large. This property is crucial for ensuring the validity of the fixed-point error bound in Section E.

Combining the result in Section E, we obtain the following corollary:

Corollary 21. *For the single-qubit, free-fermion, and (high-temperature) local commuting Hamiltonian problems above, for any $\epsilon > 0$, it suffices to choose $\sigma, T, \alpha^{-1} = \mathrm{poly}(N, 1/\epsilon)$ to ensure that*

$$\|\rho_{\mathrm{fix}}(\Phi) - \rho_\beta\|_1 < \epsilon,$$
$$\tau_{\mathrm{mix},\Phi}(\epsilon) = \frac{t_{\mathrm{mix},\Phi}(\epsilon)}{\alpha^2} = \mathrm{poly}(N, 1/\epsilon)\,.$$

For the single qubit and gapped free fermionic systems above, for any $\epsilon > 0$, it suffices to choose $\sigma, T, \alpha^{-1} = \mathrm{poly}(N, 1/\epsilon)$ to ensure that

$$\|\rho_{\mathrm{fix}}(\Phi) - |\psi_0\rangle\langle\psi_0|\,\|_1 < \epsilon,$$
$$\tau_{\mathrm{mix},\Phi}(\epsilon) = \frac{t_{\mathrm{mix},\Phi}(\epsilon)}{\alpha^2} = \mathrm{poly}(N, 1/\epsilon)\,.$$

In the above corollary, $\tau_{\mathrm{mix},\Phi}(\epsilon)$ denotes the number of times the map Φ defined in (2) should be applied to achieve ϵ-mixing.

To establish end-to-end efficiency, it remains to analyze the simulation complexity of Φ, which follows from a standard analysis of Trotter errors (see e.g. [61]). Recall the quantum channel $\Phi_\alpha^{\mathrm{approx}}$ in (4). We have $\|\Phi_\alpha^{\mathrm{approx}} - \Phi\|_{1\leftrightarrow 1} = \mathcal{O}\left(\alpha T(\|H\| + \omega_{\max})^2 \|A_S\| \tau^2/\sigma^{1/2}\right)$, where τ is the Trotter step size. Since $\|A_S\| \leq 1$, to achieve η-accuracy in each application of Φ, the number of Trotter steps per iteration is $M = \Theta\left(\alpha^{1/2} T^{3/2} (\|H\| + \omega_{\max})\, \eta^{-1/2} \sigma^{-1/4}\right)$. Given a mixing time of $\tau_{\mathrm{mix},\Phi}(\epsilon)$, we set $\eta = \epsilon/\tau_{\mathrm{mix},\Phi}$ to ensure the total quantum channel error is bounded by ϵ in $1 \leftrightarrow 1$ norm. This leads to the total number of steps is $M_{\mathrm{total}} = M \cdot \tau_{\mathrm{mix},\Phi}(\epsilon) = \mathrm{poly}(N, 1/\epsilon)$. Each step involves a short-time (τ) simulation of the system Hamiltonian, a single Z rotation, and one simulation step for the system-bath interaction term whose gate complexity depends on the choice of A_S. We note that, in Theorem 21, the dependence of $\tau_{\mathrm{mix},\Phi}$ and M_{total} on N, β, and $1/\epsilon$ could potentially be further improved, not only by establishing a tighter upper bound on the mixing time, but also by allowing more relaxed choices of α and σ. For instance, although this work focuses on the weak-interaction regime—where $\alpha\sqrt{\sigma}$ is small—there is currently no evidence that this is the only regime that is valid (see [54] for example). Relaxing this assumption represents an interesting direction for future research.

Appendix G: Mixing analysis of thermal and ground state preparation for the single qubit example

In this section, we consider a toy model $H = -Z$. In Eq. (2), we set $\mathcal{A} = \{X, -X\}$ and $g(\omega) = \frac{1}{3}\mathbf{1}_{[0,3]}(\omega)$ ($\omega_{\max} = 3$). To prove Theorem 17, it suffices to show that, for both thermal state and ground state preparation, the mixing time of Φ is independent of σ when σ is sufficiently large.

Similar to Section E 1 and Section E 2, we first consider a simplified CPTP map defined as follows:

$$\widetilde{\Phi} = \mathcal{U}_S(T) \circ \exp\left(\widetilde{\mathcal{L}}\alpha^2\right) \circ \mathcal{U}_S(T)\,.$$

Here $\widetilde{\mathcal{L}}$ omits the error in Theorem 4 and take the limit $T \to \infty$. Specifically,

$$\widetilde{\mathcal{L}}(\rho) = -i\left[\widetilde{H}_{\mathrm{LS}}, \rho\right] + \int_{-\infty}^{\infty} \gamma(\omega)\mathcal{D}_{\widetilde{V}_{X,f}(\omega)}(\rho)\mathrm{d}\omega\,,$$

where

$$\widetilde{H}_{\mathrm{LS}} = -\mathrm{Im}\left(\int_{-\infty}^{\infty}\gamma(\omega)\widetilde{\mathcal{G}}_{X,f}(-\omega)\mathrm{d}\omega\right), \quad \widetilde{V}_{X,f}(\omega) = \int_{-\infty}^{\infty} f(t)X(t)\exp(-i\omega t)\mathrm{d}t\,,$$

with

$$\widetilde{\mathcal{G}}_{X,f}(\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{s_1} f(s_2)f(s_1)X(s_2)X(s_1)\exp(i\omega(s_2-s_1))\mathrm{d}s_2\mathrm{d}s_1\,.$$

According to Lemma 11 or Lemma 15, for thermal and ground state preparation, respectively, we first have

$$\left\|\Phi - \widetilde{\Phi}\right\|_{1\leftrightarrow 1} = \mathcal{O}\left(\alpha^2\sigma\exp\left(-T^2/(4\sigma^2)\right) + \alpha^4T^4\sigma^{-2}\right)\,.$$

According to Theorem 8, when α is sufficiently small and T is sufficiently large, it suffices to consider the mixing time of $\widetilde{\Phi}$. In this case, we can compute $\widetilde{V}$ and $\widetilde{H}_{\mathrm{LS}}$ explicitly. Noticing,

$$\widetilde{V}_{X,f}(\omega) = \int_{-\infty}^{\infty} f(t)X(t)e^{-i\omega t}\,\mathrm{d}t = 2^{3/4}\sigma^{1/2}\pi^{1/4}\left(\exp\left(-\sigma^2(\omega-2)^2\right)|1\rangle\langle 0| + \exp\left(-\sigma^2(\omega+2)^2\right)|0\rangle\langle 1|\right)\,,$$

and

$$\begin{aligned}\widetilde{\mathcal{G}}_{X,f}(\omega) =& \sum_{\nu_1,\nu_2\in B(H)} X(\nu_2)X(\nu_1)\int_{-\infty}^{\infty}\int_{-\infty}^{s_1} f(s_2)f(s_1)\exp(i\nu_2 s_2)\exp(i\nu_1 s_1)\exp(i\omega(s_2-s_1))\mathrm{d}u\mathrm{d}v\\ =& C_{0,\sigma}(\omega)|0\rangle\langle 0| + C_{1,\sigma}(\omega)|1\rangle\langle 1|\end{aligned}$$

where $C_{0,\sigma}(\omega)$ and $C_{1,\sigma}(\omega)$ are functions of ω that depend on σ. Because the Lamb shift term does not effect the proof later, we do not specify the form of C_0 and C_1.

Now, we consider the thermal state and ground state separately:

- Thermal state: We notice that

$$\begin{aligned}&\int_{-\infty}^{\infty}\gamma(\omega)\,\mathcal{D}_{V_{X,f,T}(\omega)}(\rho)\,\mathrm{d}\omega\\ =&2^{3/2}\pi^{1/2}\int_{-\infty}^{\infty}\gamma(\omega)\sigma\exp(-2\sigma^2(\omega-2)^2)\mathrm{d}\omega\mathcal{D}_{|1\rangle\langle 0|}(\rho) + 2^{3/2}\pi^{1/2}\int_{-\infty}^{\infty}\gamma(\omega)\sigma\exp(-2\sigma^2(\omega+2)^2)\mathrm{d}\omega\mathcal{D}_{|0\rangle\langle 1|}(\rho) + \mathcal{O}\left(\exp(-8\sigma^2)\right)\\ =&2\pi\left(\gamma(2)\mathcal{D}_{|1\rangle\langle 0|}(\rho) + \gamma(-2)\mathcal{D}_{|0\rangle\langle 1|}(\rho)\right) + \mathcal{O}\left(\frac{\beta}{\sigma}\right)\,,\end{aligned}$$

  and

$$\widetilde{H}_{\mathrm{LS}} = C_{0,\beta,\sigma}|0\rangle\langle 0| + C_{1,\beta,\sigma}|1\rangle\langle 1|\,,$$

  where $C_{0,\beta,\sigma}$ and $C_{1,\beta,\sigma}$ are constants that depend on σ. Define

$$\widehat{\mathcal{L}}_\beta = -i\left[\widetilde{H}_{\mathrm{LS}}, \rho\right] + 2\pi\left(\gamma(2)\mathcal{D}_{|1\rangle\langle 0|} + \gamma(-2)\mathcal{D}_{|0\rangle\langle 1|}\right) \tag{G1}$$

  and $\widehat{\Phi}_\beta = \mathcal{U}_S(T) \circ \exp\left(\widehat{\mathcal{L}}_\beta\alpha^2\right) \circ \mathcal{U}_S(T)$. Then, we have

$$\left\|\Phi - \widehat{\Phi}_\beta\right\|_{1\leftrightarrow 1} = \mathcal{O}\left(\alpha^2\sigma\exp\left(-T^2/(4\sigma^2)\right) + \alpha^4T^4\sigma^{-2} + \frac{\beta}{\sigma}\right)\,. \tag{G2}$$

In the case when σ is sufficiently large, according to Theorem 8, we only need to consider the mixing time of $\widehat{\Phi}_\beta$. Since this result follows from a more general theorem in Theorem 31, it is sufficient to demonstrate that the mixing time is independent of σ.

We express ρ_n in the computational basis as $\rho_n = \sum_{a,b=0}^{1} c_{a,b,n} |a\rangle \langle b|$, where $c_{a,b,n}$ are the corresponding coefficients. To show convergence, it suffices to verify that $c_{0,0,n}$ and $c_{1,1,n}$ converge to $\frac{\exp(2\beta)}{1+\exp(2\beta)}$ and $\frac{1}{1+\exp(2\beta)}$, respectively, while $|c_{0,1,n}|^2$ and $|c_{1,0,n}|^2$ converge to zero. It is straightforward to check that the Lamb shift Hamiltonian $\widetilde{H}_{\mathrm{LS}}$ and the unitary dynamics do not affect the evolution of $c_{0,0,n}$, $c_{1,1,n}$, or $|c_{0,1,n}|^2$, $|c_{1,0,n}|^2$. After plugging ρ_n into (G1), an ordinary differential equation (ODE) is obtained for the evolution of $c_{0,0,n}$, $c_{1,1,n}$, $c_{0,1,n}$, and $c_{1,0,n}$. A direct calculation verifies that the solution converges to the desired fixed point. Since the evolution is independent of σ, it follows that the mixing time of $\widehat{\Phi}_\beta$ is also independent of σ. Combing this mixing time and (G2) with Theorem 8, we conclude the proof for the thermal state part.

- Ground state:

$$\int_{-\infty}^{\infty} \gamma(\omega)\, \mathcal{D}_{V_{X,f,T}(\omega)}(\rho)\, \mathrm{d}\omega = C_{\infty,\sigma}\mathcal{D}_{|0\rangle\langle 1|} + \mathcal{O}\left(\sigma \exp(-4\sigma^2)\right)\,,$$

where $C_{\infty,\sigma}$ is a constant that depends on σ. We note that, there exists a uniform constant C_∞ such that $C_{\infty,\sigma} \geq C_\infty$ for $\sigma \geq 1$.

$$\widetilde{H}_{\mathrm{LS}} = C_{0,\infty,\sigma} |0\rangle \langle 0| + C_{1,\infty,\sigma} |1\rangle \langle 1|\,,$$

where $C_{0,\infty,\sigma}$ and $C_{1,\infty,\sigma}$ that only depends on σ.

Define

$$\widehat{\mathcal{L}}_\infty = -i\left[\widetilde{H}_{\mathrm{LS}}, \rho\right] + C_{\infty,\sigma}\mathcal{D}_{|0\rangle\langle 1|} \tag{G3}$$

and $\widehat{\Phi}_\infty = \mathcal{U}_S(T) \circ \exp\left(\widehat{\mathcal{L}}_\infty \alpha^2\right) \circ \mathcal{U}_S(T)$. Then, we have

$$\left\|\Phi - \widehat{\Phi}_\infty\right\|_{1\leftrightarrow 1} = \mathcal{O}\left(\alpha^2\sigma \exp\left(-T^2/(4\sigma^2)\right) + \alpha^4 T^4 \sigma^{-2} + \sigma\exp(-4\sigma^2)\right)\,. \tag{G4}$$

Finally, we consider the mixing time of $\widehat{\Phi}_\infty$. Similar to the thermal state case, we express ρ_n in the computational basis as $\rho_n = \sum_{a,b=0}^{1} c_{a,b,n} |a\rangle \langle b|$, where $c_{a,b,n}$ are the corresponding coefficients. To show convergence, it suffices to verify that $c_{0,0,n}$ and $c_{1,1,n}$ converge to 1 and 0, respectively, while $|c_{0,1,n}|^2$ and $|c_{1,0,n}|^2$ converge to zero. Same as before, the Lamb shift Hamiltonian $\widetilde{H}_{\mathrm{LS}}$ and the unitary dynamics do not affect the evolution of $c_{0,0,n}$, $c_{1,1,n}$, or $|c_{0,1,n}|^2$, $|c_{1,0,n}|^2$. Similar to before, a direct calculation verifies that the dissipative part of $\widehat{\mathcal{L}}_\infty$ converges and is independent of σ, which implies that the mixing time of $\widehat{\Phi}_\infty$ is independent of σ. Combining this mixing time and (G4) with Theorem 8, we conclude the proof for the ground state part.

Remark 22. *Different from our setting, [47, Section III] considers the filter function*

$$f(t) = \sqrt{\frac{2}{\pi\sigma^2}}\,\exp\left(-\frac{2}{\sigma^2}\left(t - \frac{i\beta}{4}\right)^2\right) \tag{G5}$$

Under this choice, the corresponding jump operator in the Lindblad dynamics is

$$L_{A_S} = \sum_{\omega\in B(H)} \exp(-\beta\omega/4)\,\exp\left(-\frac{(\sigma\omega)^2}{8}\right) A_S(\omega)\,.$$

As $\sigma \to \infty$, the support of L_{A_S} effectively shrinks to a narrow energy window of width $\mathcal{O}(1/\sigma)$:

$$L_{A_S} = \underbrace{\sum_{|\omega|\leq 1/2} \exp(-\beta\omega/4)\,\exp\left(-\frac{(\sigma\omega)^2}{8}\right) A_S(\omega)}_{=A_S(0)} + \underbrace{\sum_{|\omega|\geq 1/2} \exp(-\beta\omega/4)\,\exp\left(-\frac{(\sigma\omega)^2}{8}\right) A_S(\omega)}_{=\mathcal{O}(\exp(-\sigma^2))}\,,$$

where we use $B(H)=\{2,0,-2\}$ in the above equality. This implies that when $\sigma\gg 1$, transitions between eigenvectors corresponding to different eigenvalues are strongly suppressed. For instance, when $A_S=X$, we have $A_S(0)=0$ and $\|L_{A_S}\|=\mathcal{O}(\exp(-\sigma^2))$, so the dissipative term becomes exponentially weak. This leads to a mixing time scaling as $t_{\mathrm{mix}}=\Omega(\exp(\sigma^2))$. Substituting this into the fixed-point error bound in Theorem 9 yields a vacuous upper bound on the error.

In our algorithm, since the bath is initialized in the thermal state of H_E, we do not need to choose an interaction function f that simultaneously depends on both β and σ to satisfy an approximate detailed balance condition. This avoids the restriction—present in Eq. (G5)—that energy transitions must remain near 0 when σ is large. This key distinction prevents the mixing time from degrading in the large-σ regime and enables substantially faster mixing.

Appendix H: Rigorous version of Theorem 18

Recall $H=\sum_{j,k}^N h_{j,k}c_j^\dagger c_k$, where $c_j^\dagger$, c_k are creation and annihilation operators, respectively. Because h is a Hermitian matrix, there exists an unitary matrix U such that $\Lambda=U^\dagger hU$ is diagonal. Specifically,

$$H=\sum_{k=1}^N\lambda_k\left(\sum_j (U^\dagger)_{k,j}c_j\right)^\dagger\left(\sum_j (U^\dagger)_{k,j}c_j\right):=\sum_{k=1}^N\lambda_k b_k^\dagger b_k\,,$$

where $b_k^\dagger=\left(\sum_j (U^\dagger)_{k,j}c_j\right)^\dagger$, $b_k=\sum_j (U^\dagger)_{k,j}c_j$ formulate a new set of creation and annihilation operators after the unitary transformation. Then, the spectral gap $\Delta=\min_i|\lambda_i|$.

Now, we are ready to introduce the rigorous version of Theorem 18:

Theorem 23. *Let $g(\omega)=\frac{1}{\omega_{\max}}\mathbf{1}_{[0,\omega_{\max}]}$ with $\omega_{\max}=2\|h\|$. Given any $\epsilon>0$, when*

$$\left(\frac{N\|h\|}{\alpha^2}\log(N/\epsilon)\right)\left(\alpha^2\sigma\exp\left(-T^2/(4\sigma^2)\right)+\alpha^2\sigma\sqrt{N}\exp(-\Delta^2\sigma^2)+\alpha^4T^4\sigma^{-2}\right)=\mathcal{O}(\epsilon)\,,$$

we have

$$t_{\mathrm{mix},\Phi}(\epsilon)=\mathcal{O}\left(\|h\|N\log(N/\epsilon)\right)\,.$$

Proof of Theorem 23. First, according to Theorem 8 Eq. (D3) and Lemma 15, it suffices to prove the mixing time of $\widetilde{\Phi}$ defined in (E10). Recall

$$\widetilde{\Phi}=\mathcal{U}_S(T)\circ\exp\left(\widetilde{\mathcal{L}}\alpha^2\right)\circ\mathcal{U}_S(T)\,.$$

Here

$$\widetilde{\mathcal{L}}(\rho)=\mathbb{E}_{A_S}\left(-i\left[\widetilde{H}_{\mathrm{LS},A_S},\rho\right]+\int_{-\infty}^0(g(\omega)+g(-\omega))\mathcal{D}_{\widetilde{V}_{A_S,f}(\omega)}(\rho)\mathrm{d}\omega\right)\,,$$

where

$$\widetilde{H}_{\mathrm{LS},A_S}=-\mathrm{Im}\left(\int_{-\infty}^0 g(\omega)\widetilde{\mathcal{G}}_{A_S^\dagger,f}(\omega)\mathrm{d}\omega+\int_0^\infty g(\omega)\widetilde{\mathcal{G}}_{A_S,f}(-\omega)\mathrm{d}\omega\right),\quad \widetilde{V}_{A_S,f}(\omega)=\int_{-\infty}^\infty f(t)A_S(t)\exp(-i\omega t)\mathrm{d}t\,,$$

with

$$\widetilde{\mathcal{G}}_{A_S,f}(\omega)=\int_{-\infty}^\infty\int_{-\infty}^{s_1}f(s_2)f(s_1)A_S^\dagger(s_2)A_S(s_1)\exp(i\omega(s_2-s_1))\mathrm{d}s_2\mathrm{d}s_1\,.$$

Define the number operator:

$$\hat{\mathsf{N}}=\sum_{\lambda_k>0}b_k^\dagger b_k+\sum_{\lambda_k<0}b_kb_k^\dagger\,.$$

Let H has eigendecomposition $\{(\lambda_i, |\psi_i\rangle)\}_{i=0}^{d-1}$ with $\lambda_0 \leq \lambda_1 \leq \ldots, \lambda_{d-1}$. We note that $1 - \langle\psi_0|\,\rho\,|\psi_0\rangle \leq \mathrm{Tr}(\rho\hat{\mathsf{N}})$. Using the Fuchs-van de Graaf inequality, we obtain

$$\|\rho - |\psi_0\rangle\langle\psi_0|\,\|_1 \leq 2\sqrt{(1-\langle\psi_0|\,\rho\,|\psi_0\rangle)} \leq 2\sqrt{\mathrm{Tr}(\rho\hat{\mathsf{N}})}.$$

we can show the decaying of $\mathrm{Tr}\left(\widetilde{\Phi}[\rho]\hat{\mathsf{N}}\right)$, compared with $\mathrm{Tr}\left(\rho\hat{\mathsf{N}}\right)$. Furthermore, because $\hat{\mathsf{N}}$ commutes with H, the unitary evolution part $\mathcal{U}_S$ does not effect the expectation. Thus, if we can show

$$\mathrm{Tr}\left(\exp\left(\widetilde{\mathcal{L}}\alpha^2\right)[\rho]\hat{\mathsf{N}}\right) \leq (1-\delta)\mathrm{Tr}(\rho\hat{\mathsf{N}})$$

for some $0 < \delta < 1$ and any ρ, then we have

$$\mathrm{Tr}\left(\widetilde{\Phi}_\alpha^K[\rho]\hat{\mathsf{N}}\right) = \mathrm{Tr}\left(\exp\left(\widetilde{\mathcal{L}}\alpha^2\right)\circ\widetilde{\Phi}_\alpha^{K-1}[\rho]\hat{\mathsf{N}}\right) \leq (1-\delta)\mathrm{Tr}\left(\widetilde{\Phi}_\alpha^{K-1}[\rho]\hat{\mathsf{N}}\right) \leq (1-\delta)^K\mathrm{Tr}(\rho\hat{\mathsf{N}})\,.$$

for any $K > 0$ and ρ. This implies the fast decaying of $\mathrm{Tr}(\rho_K\hat{\mathsf{N}})$.

However, because $\widetilde{\mathcal{L}}$ does not exactly preserve the ground state, it is difficult to directly show the exponential decay of $\mathrm{Tr}\left(\exp\left(\widetilde{\mathcal{L}}\alpha^2\right)[\rho]\hat{\mathsf{N}}\right)$ in the above form. Instead, we will construct a new Lindbladian operator $\widehat{\mathcal{L}}$ in the proof so that such that $\left\|\widehat{\mathcal{L}}-\widetilde{\mathcal{L}}\right\|_{1\leftrightarrow 1}$ is bounded, $\widehat{\mathcal{L}}$ exactly fixes the ground state, and $\widehat{\mathcal{L}}$ satisfies a decay property, namely,

$$\mathrm{Tr}\left(\exp\left(\widehat{\mathcal{L}}\alpha^2\right)[\rho]\hat{\mathsf{N}}\right) \leq (1-\delta)\,\mathrm{Tr}(\rho\hat{\mathsf{N}}). \tag{H1}$$

Recall that A_S is uniformly sampled from $\{\pm c_k^\dagger, \pm c_k\}_{k=1}^N$. We first deal with the dissipative part and define $\widehat{\mathcal{L}}$. We note that

$$\exp(iHt)b_j\exp(-iHt) = \exp(-i\lambda_j t)b_j, \quad \exp(iHt)b_j^\dagger\exp(-iHt) = \exp(i\lambda_j t)b_j^\dagger\,. \tag{H2}$$

When $A_S = c_k^\dagger = \left(\sum_j U_{k,j}b_j\right)^\dagger$, we have

$$A_S(t) = \exp(iHt)A_S\exp(-iHt) = \sum_j \overline{U_{k,j}}\exp(i\lambda_j t)b_j^\dagger\,.$$

Because the integral in $\widetilde{V}$ is restricted to the regime $\omega \leq 0$, we have

$$\begin{aligned}\widetilde{V}_k(\omega) := \widetilde{V}_{A_S,f}(\omega) &= \int_{-\infty}^{\infty} f(t)A_S(t)\exp(-i\omega t)\mathrm{d}t = \sum_j \overline{U_{k,j}}\hat{f}(-\omega+\lambda_j)b_j^\dagger \\ &= \sum_{\lambda_j<0}\overline{U_{k,j}}\hat{f}(-\omega+\lambda_j)b_j^\dagger + \underbrace{\mathcal{O}\left(\sqrt{\sigma N}\exp(-\Delta^2\sigma^2)\right)}_{\text{contains the part with }\lambda_j\geq 0}\end{aligned}$$

when $\omega \leq 0$. We note that the Lindbladian with jump operator $\widehat{V}_k = \sum_{\lambda_j<0}\overline{U_{k,j}}\hat{f}(-\omega+\lambda_j)b_j^\dagger$ preserves the ground state, since $\widehat{V}_k\,|\psi_0\rangle = 0$. In addition, for $\omega \leq 0$, we note

$$\left\|\widetilde{V}_k(\omega) - \widehat{V}_k(\omega)\right\| = \mathcal{O}\left(\sqrt{\sigma N}\exp(-\Delta^2\sigma^2)\right)\,. \tag{H3}$$

Now, defining $\widehat{\mathcal{L}}$ with $\widehat{V}_k$ and the same Lamb shift term $\widetilde{H}_{\mathrm{LS}}$, we obtain

$$\left\|\widehat{\mathcal{L}}-\widetilde{\mathcal{L}}\right\|_{1\leftrightarrow 1} = \mathcal{O}\left(\sup_k\left\|\widetilde{V}_k(\omega)-\widehat{V}_k(\omega)\right\|\left\|\widetilde{V}_k(\omega)\right\|\right) = \mathcal{O}\left(\sigma\sqrt{N}\exp(-\Delta^2\sigma^2)\right) \tag{H4}$$

Define $\widehat{\Phi}$ with $\widehat{\mathcal{L}}$ similar to Eq. (E10). We have

$$\left\|\widehat{\Phi}-\widetilde{\Phi}\right\|_{1\leftrightarrow 1} = \mathcal{O}\left(\alpha^2\sigma\sqrt{N}\exp(-\Delta^2\sigma^2)\right)\,. \tag{H5}$$

Combining (H5) and Lemma 15, we have

$$\left\|\Phi-\widetilde{\Phi}\right\|_{1\leftrightarrow 1}=\mathcal{O}\left(\alpha^2\sigma\exp\left(-T^2/(4\sigma^2)\right)+\alpha^2\sigma\sqrt{N}\exp(-\Delta^2\sigma^2)+\alpha^4T^4\sigma^{-2}\right)\,. \tag{H6}$$

Now, given an observable $O=b_i^\dagger b_i$ with $\lambda_i>0$, we notice $[b_j,O]=\delta_{ij}b_i,\quad [b_j^\dagger,O]=-\delta_{ij}b_j^\dagger$. Then, we have

$$\mathcal{L}^\dagger_{\widehat{V}_k}(O)=\frac{1}{2}\left([\widehat{V}_k^\dagger,O]\widehat{V}_k-\widehat{V}_k^\dagger[\widehat{V}_k,O]\right)=0$$

Given an observable $O=b_ib_i^\dagger$ with $\lambda_i<0$, we notice $[b_j,O]=-\delta_{ij}b_i,\quad [b_j^\dagger,O]=\delta_{ij}b_i^\dagger$. Then, we have

$$\begin{aligned}&\mathcal{L}^\dagger_{\widehat{V}_k}(O)=\frac{1}{2}\left([\widehat{V}_k^\dagger,O]\widehat{V}_k-\widehat{V}_k^\dagger[\widehat{V}_k,O]\right)\\ =&\frac{1}{2}\left(\left(-U_{k,i}\hat{f}(-\omega+\lambda_i)b_i\right)\widehat{V}_k-\widehat{V}_k^\dagger\left(\overline{U_{k,i}}\hat{f}(-\omega+\lambda_i)b_i^\dagger\right)\right)\\ =&-\frac{1}{2}\sum_{\lambda_j<0}U_{k,i}\overline{U_{k,j}}\hat{f}(-\omega+\lambda_i)\hat{f}(-\omega+\lambda_j)b_ib_j^\dagger-\frac{1}{2}\sum_{\lambda_j<0}\overline{U_{k,i}}U_{k,j}\hat{f}(-\omega+\lambda_i)\hat{f}(-\omega+\lambda_j)b_jb_i^\dagger\end{aligned}\,.$$

Because $\sum_k U_{k,i}\overline{U_{k,j}}=\delta_{i,j}$, this implies

$$\sum_k\left(\sum_{\lambda_i>0}\mathcal{L}^\dagger_{\widehat{V}_k}(b_i^\dagger b_i)+\sum_{\lambda_i<0}\mathcal{L}^\dagger_{\widehat{V}_k}(b_ib_i^\dagger)\right)=-\sum_{\lambda_i<0}\left|\hat{f}(-\omega+\lambda_i)\right|^2b_ib_i^\dagger \tag{H7}$$

Similarly, when $A_S=c_k$, we can also define $\hat{V}_k$ that preserves the ground state and satisfies (H3) to (H6). Further more, similar to (H7), we have

$$\sum_k\left(\sum_{\lambda_i>0}\mathcal{L}^\dagger_{\widehat{V}_k}(b_i^\dagger b_i)+\sum_{\lambda_i<0}\mathcal{L}^\dagger_{\widehat{V}_k}(b_ib_i^\dagger)\right)=-\sum_{\lambda_i>0}\left|\hat{f}(-\omega-\lambda_i)\right|^2b_i^\dagger b_i$$

Because $g(\omega)=\frac{1}{2\|h\|}\mathbf{1}_{[0,2\|h\|]}$, we have $\mathbb{E}_\omega|\hat{f}(-\omega-\mathrm{sign}(\lambda_i)\lambda_i)|^2=\Omega(\|h\|^{-1})$. Thus,

$$\widehat{\mathcal{L}}^\dagger(\hat{\mathsf{N}})\leq-\frac{C}{\|h\|N}\hat{\mathsf{N}}\,, \tag{H8}$$

with a uniform constant C. Here, N comes from the expectation of V_k, which gives an $\frac{1}{N}$ factor before the summation of k.

Next, for the Lamb shift term, when $A_S=c_k^\dagger=\left(\sum_j U_{k,j}b_j\right)^\dagger$, we have

$$\begin{aligned}&\widetilde{\mathcal{G}}_{A_S,f}(-\omega)=\int_{-\infty}^{\infty}\int_{-\infty}^{s_1}f(s_2)f(s_1)A_S^\dagger(s_2)A_S(s_1)\exp(-i\omega(s_2-s_1))\mathrm{d}s_2\mathrm{d}s_1\\ =&\sum_{\nu_1,\nu_2=1}^{N}U_{k,\nu_2}\overline{U_{k,\nu_1}}b_{\nu_2}b_{\nu_1}^\dagger\int_{-\infty}^{\infty}\int_{-\infty}^{s_1}f(s_2)f(s_1)\exp(-i\lambda_{\nu_2}s_2)\exp(i\lambda_{\nu_1}s_1)\exp(-i\omega(s_2-s_1))\mathrm{d}u\mathrm{d}v\end{aligned}\,.$$

After summing in k, the remaining terms commute with $\widehat{\mathsf{N}}$ and thus, does not change $\mathrm{Tr}(\rho(t)\widehat{\mathsf{N}})$. Similarly, when A_S is chosen to be c_k, we have the same commuting properties.

In conclusion, using Eq. (H8) and the commuting property of $\widetilde{\mathcal{G}}_{A_S,f}$, we have

$$\mathrm{Tr}\left(\exp\left(\widehat{\mathcal{L}}\alpha^2\right)[\rho]\hat{\mathsf{N}}\right)\leq\left(1-\frac{C\alpha^2}{\|h\|N}\right)\mathrm{Tr}(\rho\hat{\mathsf{N}}).$$

This implies that

$$\left\|\widehat{\Phi}^k_\alpha[\rho]-|\psi_0\rangle\langle\psi_0|\right\|_1\leq\mathrm{Tr}\left(\widehat{\Phi}^k_\alpha[\rho]\hat{\mathsf{N}}\right)\leq\left(1-\frac{C\alpha^2}{\|h\|N}\right)^k\mathrm{Tr}(\rho\hat{\mathsf{N}})\leq N\left(1-\frac{C\alpha^2}{\|h\|N}\right)^k\,.$$

Thus, given $\epsilon>0$, we have

$$\tau_{\mathrm{mix},\widehat{\Phi}}(\epsilon)=\mathcal{O}\left(\frac{N\|h\|}{\alpha^2}\log(N/\epsilon)\right)$$

Combining this, (H6), and Theorem 8, we conclude the proof. □

Appendix I: Mixing time of Φ for thermal state preparation

Before showing Theorem 19 and Theorem 20, we provide a framework for studying the mixing time of the CPTP maps that take the form of

$$\Phi = \mathcal{U}_S(T) \circ \exp\left(\mathcal{M}\alpha^2\right) \circ \mathcal{U}_S(T)\,,$$

where $\mathcal{M}$ is an arbitrary Lindbladian that preserves the thermal state ρ_β. This framework is inspired by [17, 18].

To start, we first introduce the detailed balance condition that allows coherent term:

Definition 24 (Detailed balance condition with unitary drift [18, 62])**.** *For any Lindbladian $\mathcal{M}$ and full-rank state ρ_β, take a similarity transformation and decompose into the Hermitian and the anti-Hermitian parts*

$$\mathcal{K}(\rho_\beta,\mathcal{M}) = \rho_\beta^{-1/4}\mathcal{M}\left[\rho_\beta^{1/4}\cdot\rho_\beta^{1/4}\right]\rho_\beta^{-1/4} = \mathcal{H}(\rho_\beta,\mathcal{M}) + \mathcal{A}(\rho_\beta,\mathcal{M})$$
$$\mathcal{K}(\rho_\beta,\mathcal{M})^\dagger = \rho_\beta^{1/4}\mathcal{M}^\dagger\left[\rho_\beta^{-1/4}\cdot\rho_\beta^{-1/4}\right]\rho_\beta^{1/4} = \mathcal{H}(\rho_\beta,\mathcal{M}) - \mathcal{A}(\rho_\beta,\mathcal{M})$$

We say the Lindbladian $\mathcal{M}$ satisfies the detailed balance with unitary drift if there exists a Hermitian operator H_C such that

$$\mathcal{A}(\rho_\beta,\mathcal{M}) = -i\rho_\beta^{1/4}[H_C,\rho_\beta^{-1/4}(\cdot)\rho_\beta^{-1/4}]\rho_\beta^{1/4}\,.$$

We note that the above detailed balance condition allows a coherent term that commutes with ρ_β in $\mathcal{M}$. It is straightforward to check that if $\mathcal{M}$ satisfies the detailed balance with unitary drift, then $\mathcal{H}(\rho_\beta,\mathcal{M})(\sqrt{\rho_\beta}) = 0$ and $\mathcal{M}(\rho_\beta) = 0$. Furthermore, if $\mathcal{M}$ approximately satisfies the detailed balance with unitary drift, we have the following result to quantify the mixing time of Φ:

Theorem 25. *Assume $\mathcal{H}(\rho_\beta,\mathcal{M}) = \mathcal{H}_1(\rho_\beta,\mathcal{M}) + \mathcal{H}_2(\rho_\beta,\mathcal{M})$. If $\mathcal{H}_1$ is a self-adjoint operator under Hilbert-Schmidt such that $\mathcal{H}_1(\rho_\beta,\mathcal{M})(\sqrt{\rho_\beta}) = 0$ and $\mathcal{H}_1(\rho_\beta,\mathcal{M})$ has a spectral gap $\lambda_{\mathrm{gap}}(\mathcal{H}_1) > \|\mathcal{H}_2\|_{2\leftrightarrow 2}$. Given any ρ_1,ρ_2, we have*

$$\left\|\Phi^k(\rho_1-\rho_2)\right\|_1 \le 2\exp\left(\left(-\lambda_{\mathrm{gap}}(\mathcal{H}_1) + \|\mathcal{H}_2\|_{2\leftrightarrow 2}\right)k\alpha^2\right)\left\|\rho_\beta^{-1/2}\right\|\|\rho_1-\rho_2\|_1\,.$$

Specifically, for any $\epsilon > 0$, we have

$$t_{\mathrm{mix},\Phi}(\epsilon) \le \frac{1}{\lambda_{\mathrm{gap}}(\mathcal{H}_1) - \|\mathcal{H}_2\|_{2\leftrightarrow 2}}\log\left(\frac{2\left\|\rho_\beta^{-1/2}\right\|}{\epsilon}\right) + 1$$

We emphasize that Theorem 25 does not guarantee the correctness of the fixed point. However, it still provides an upper bound on the mixing time of Φ. In the regime where $\mathcal{H}_2 \ll 1$, it is possible to establish a small fixed-point error.

Proof of Theorem 25. Given any density operator ρ_1,ρ_2, we define $\mathcal{E} = \rho_1 - \rho_2$. We consider the change of $\left\|\rho_\beta^{-1/4}\mathcal{E}\rho_\beta^{-1/4}\right\|_2$ after applying Φ, where $\|\cdot\|_2$ is the Schatten-2 norm (Hilbert-Schmidt norm). First, because $\mathcal{U}_S$ commutes with $\rho_\beta^{-1/4}(\cdot)\rho_\beta^{-1/4}$, we have

$$\left\|\rho_\beta^{-1/4}\mathcal{U}_S(\mathcal{E})\rho_\beta^{-1/4}\right\|_2 = \left\|\mathcal{U}_S\left(\rho_\beta^{-1/4}\mathcal{E}\rho_\beta^{-1/4}\right)\right\|_2 = \left\|\rho_\beta^{-1/4}\mathcal{E}\rho_\beta^{-1/4}\right\|_2$$

Thus,

$$\left\|\rho_\beta^{-1/4}\Phi(\mathcal{E})\rho_\beta^{-1/4}\right\|_2 = \left\|\rho_\beta^{-1/4}\exp(\mathcal{M}\alpha^2)(\mathcal{E})\rho_\beta^{-1/4}\right\|_2 = \left\|\exp(\mathcal{K}(\rho_\beta,\mathcal{M})\alpha^2)\left[\rho_\beta^{-1/4}\mathcal{E}\rho_\beta^{-1/4}\right]\right\|_2$$

Let $\mathcal{E}(t) = \exp(\mathcal{M}t)\mathcal{E}$. Because $\mathcal{E}(t)$ is traceless, we have $\rho_\beta^{-1/4}\mathcal{E}(t)\rho_\beta^{-1/4}$ is orthogonal to $\sqrt{\rho_\beta}$ under Hilbert Schemitz inner product. This implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\rho_\beta^{-1/4}\exp(\mathcal{M}t)(\mathcal{E})\rho_\beta^{-1/4}\right\|_2^2 = \frac{\mathrm{d}}{\mathrm{d}t}\left\|\exp(\mathcal{K}(\rho_\beta,\mathcal{M})t)\left[\rho_\beta^{-1/4}\mathcal{E}\rho_\beta^{-1/4}\right]\right\|_2^2$$
$$=2\left\langle\rho_\beta^{-1/4}\mathcal{E}\rho_\beta^{-1/4},(\mathcal{H}_1+\mathcal{H}_2)\left[\rho_\beta^{-1/4}\mathcal{E}(t)\rho_\beta^{-1/4}\right]\right\rangle_2 \le 2\left(-\lambda_{\mathrm{gap}}(\mathcal{H}_1) + \|\mathcal{H}_2\|_{2\leftrightarrow 2}\right)\left\|\rho_\beta^{-1/4}\mathcal{E}(t)\rho_\beta^{-1/4}\right\|_2^2$$

This implies that

$$\left\|\rho_\beta^{-1/4}\Phi(\mathcal{E})\rho_\beta^{-1/4}\right\|_2 = \left\|\exp(\mathcal{K}(\rho_\beta,\mathcal{M})\alpha^2)\left[\rho_\beta^{-1/4}\mathcal{E}\rho_\beta^{-1/4}\right]\right\|_2 \leq \exp\left(\left(-\lambda_{\mathrm{gap}}(\mathcal{H}_1)+\|\mathcal{H}_2\|_{2\leftrightarrow 2}\right)\alpha^2\right)\left\|\rho_\beta^{-1/4}\mathcal{E}\rho_\beta^{-1/4}\right\|_2 .$$

In summary, we have

$$\left\|\rho_\beta^{-1/4}\Phi^k(\mathcal{E})\rho_\beta^{-1/4}\right\|_2 \leq \exp\left(\left(-\lambda_{\mathrm{gap}}(\mathcal{H}_1)+\|\mathcal{H}_2\|_{2\leftrightarrow 2}\right)k\alpha^2\right)\left\|\rho_\beta^{-1/4}\mathcal{E}\rho_\beta^{-1/4}\right\|_2 .$$

Finally, using $\|BAB\|_1 \leq \|B\|_4^2\|A\|_2$, we have

$$\|\mathcal{E}\|_1 \leq \left\|\rho_\beta^{1/4}\right\|_4^2\left\|\rho_\beta^{-1/4}\mathcal{E}\rho_\beta^{-1/4}\right\|_2 = \left\|\rho_\beta^{-1/4}\mathcal{E}\rho_\beta^{-1/4}\right\|_2 \leq \left\|\rho_\beta^{-1/4}\right\|^2\|\mathcal{E}\|_2 \leq \left\|\rho_\beta^{-1/2}\right\|\|\mathcal{E}\|_1 .$$

This implies

$$\left\|\Phi^k(\mathcal{E})\right\|_1 \leq 2\exp\left(\left(-\lambda_{\mathrm{gap}}(\mathcal{H}_1)+\|\mathcal{H}_2\|_{2\leftrightarrow 2}\right)k\alpha^2\right)\|\rho_\beta^{-1/2}\|\,\|\mathcal{E}\|_1 .$$

This concludes the proof. □

Let H has eigendecomposition $\{(\lambda_i,|\psi_i\rangle)\}_{i=0}^{2^N-1}$ with $\lambda_0 \leq \lambda_1 \leq \ldots, \lambda_{2^N-1}$. Given a coupling operator A, for any $\omega > 0$, define $A(\omega) = \sum_{\lambda_i-\lambda_j=\omega}|\psi_i\rangle\langle\psi_j|\langle\psi_i|A|\psi_j\rangle$. The Davies generator of a set of coupling operator $\mathcal{A}$ is defined as

$$\mathcal{L}_{D,\mathcal{A}}[\rho] = \sum_{A\in\mathcal{A}}\sum_{\omega}A(\omega)\rho A(\omega)^\dagger - \frac{1}{2}\left\{A^\dagger(\omega)A(\omega),\rho\right\} .$$

A direct corollary of Theorem 25 is in the following:

Corollary 26. *For $\mathcal{M} = -i[H_C,\cdot]+\mathcal{L}_D(\cdot)$, where $\mathcal{L}_D$ is a generator that satisfies GNS detailed balance condition or KMS detailed balance condition and has a gap $\lambda_{\mathrm{gap}}(\mathcal{L}_D)$. If*

$$\left\|\rho_\beta^{-1/4}H_C\rho_\beta^{1/4} - \rho_\beta^{1/4}H_C\rho_\beta^{-1/4}\right\| \leq \delta < \lambda_{\mathrm{gap}}(\mathcal{L}_D) .$$

Then, the mixing time of $\mathcal{M}$ is

$$t_{\mathrm{mix},\Phi}(\epsilon) \leq \frac{1}{\lambda_{\mathrm{gap}}(\mathcal{L}_D)-\delta}\log\left(\frac{2\left\|\rho_\beta^{-1/2}\right\|}{\epsilon}\right)+1$$

Proof of Corollary 26. Because $\mathcal{L}$ satisfies GNS/KMS detailed balance condition, we have

$$\mathcal{K}(\rho_\beta,\mathcal{L}_D) = \mathcal{K}(\rho_\beta,\mathcal{L}_D)^\dagger = \mathcal{H}(\rho_\beta,\mathcal{L}_D), \quad \lambda_{\mathrm{gap}}(\mathcal{H}(\rho_\beta,\mathcal{L}_D)) = \lambda_{\mathrm{gap}}(\mathcal{L}_D) .$$

Thus,

$$\mathcal{H}(\rho_\beta,\mathcal{M}) = \mathcal{H}(\rho_\beta,\mathcal{L}_D) - \frac{i}{2}\left\{\rho_\beta^{-1/4}H_C\rho_\beta^{1/4} - \rho_\beta^{1/4}H_C\rho_\beta^{-1/4},\rho\right\} .$$

Noticing

$$\left\|\frac{i}{2}\left\{\rho_\beta^{-1/4}H_C\rho_\beta^{1/4} - \rho_\beta^{1/4}H_C\rho_\beta^{-1/4},[\cdot]\right\}\right\|_{2\leftrightarrow 2} \leq \left\|\rho_\beta^{-1/4}H_C\rho_\beta^{1/4} - \rho_\beta^{1/4}H_C\rho_\beta^{-1/4}\right\| \leq \delta ,$$

we conclude the proof using Theorem 25 with $\mathcal{H}_1 = \mathcal{H}(\rho_\beta,\mathcal{L}_D)$. □

Next, we show that, with a proper choice of $g(\omega)$, the dissipative part of (8) approximates a Lindbladian dynamics satisfying the KMS detailed balance condition when σ and T are sufficiently large. This can be used to show the mixing time of the free fermions in Section J.

Theorem 27. *Given $x = \Omega(\frac{\beta}{\sigma^2})$ such that $\frac{\beta^2}{\sigma^2}\frac{1+x/\sqrt{2x/\beta-1/(4\sigma^2)}}{x-\beta/(8\sigma^2)} = \mathcal{O}(1)$, we set*

$$g_x(\omega) = \frac{1}{Z_x}\exp\left(-\frac{(\omega+x)^2}{2\left(\frac{2x}{\beta}-\frac{1}{4\sigma^2}\right)}\right), \quad Z_x = \sqrt{2\pi\left(\frac{2x}{\beta}-\frac{1}{4\sigma^2}\right)}.$$

Then, there exists a Lindbladian $\widehat{\mathcal{L}}_{KMS,x}$ that satisfies KMS detailed balance condition and a Hermitian operator H_x such that

$$\left\|\mathcal{L}-\left(-i[H_x,\rho]+\widehat{\mathcal{L}}_{KMS,x}\right)\right\|_{1\leftrightarrow 1} = \mathcal{O}\left(\sigma\exp\left(-T^2/(4\sigma^2)\right)+\frac{1}{Z_x}\left(\frac{\beta^2}{\sigma^2}\frac{x+\sqrt{2x/\beta-1/(4\sigma^2)}}{x-\beta/(8\sigma^2)}+\frac{\beta}{\sigma}\right)\right),$$

and

$$\left\|\sigma_\beta^{-1/4}H_x\sigma_\beta^{1/4}-\sigma_\beta^{1/4}H_x\sigma_\beta^{-1/4}\right\| = \mathcal{O}\left(\frac{\beta}{\sigma}\sqrt{\frac{\beta}{x-\frac{\beta}{8\sigma^2}}}+\frac{\beta^3}{\sigma^2}\frac{x+\sqrt{2x/\beta-1/(4\sigma^2)}}{x-\beta/(8\sigma^2)}\right).$$

Here, $\widehat{\mathcal{L}}_{KMS,x}$ takes the form of

$$\widehat{\mathcal{L}}_{KMS,x}[\rho] = \mathbb{E}_{A_S}\left(-i\left[\frac{B_{A_S}}{Z_x},\rho\right]+\int_{-\infty}^{\infty}\widehat{\gamma}_x(\omega)\mathcal{D}_{V_{A_S,f,\infty}(\omega)}(\rho)\mathrm{d}\omega\right), \tag{I1}$$

with $\widehat{\gamma}_x(\omega) = g_x(\omega)$ and

$$B_{A_S} = -\int_{-\infty}^{\infty}h_1(t_1)e^{-iHt_1}\left(\int_{-\infty}^{\infty}h_2(t_2)A_S(t_2)A_S(-t_2)\mathrm{d}t_2\right)e^{iHt_1}\mathrm{d}t_1,$$

where

$$h_1(t) = \frac{1}{2\sigma\pi\beta}\exp\left(\frac{\beta^2}{32\sigma^2}\right)\left(\frac{1}{\cosh 2\pi t/\beta}*_t\sin(-\beta t/(4\sigma^2))\exp\left(-t^2/(2\sigma^2)\right)\right),$$

and

$$h_2(t) = 2\sqrt{\frac{2x}{\beta}-\frac{1}{4\sigma^2}}\exp\left(\left(-\frac{4t^2}{\beta}-2it\right)x\right).$$

Furthermore, when $x = \frac{\beta}{8\sigma^2}+\omega_{\max}$ with $\omega_{\max} = \Omega(\beta)$, we have error bounds

$$\left\|\mathcal{L}-\left(-i[H_x,\rho]+\widehat{\mathcal{L}}_{KMS,x}\right)\right\|_{1\leftrightarrow 1} = \mathcal{O}\left(\sigma\exp\left(-T^2/(4\sigma^2)\right)+\frac{\beta}{\sigma}\right),$$

and

$$\left\|\sigma_\beta^{-1/4}H_x\sigma_\beta^{1/4}-\sigma_\beta^{1/4}H_x\sigma_\beta^{-1/4}\right\| = \mathcal{O}\left(\frac{\beta}{\sigma}\right).$$

Remark 28. *We notice that $\widehat{\gamma}_x$ satisfies [18, Eqn. (1.4)] up to a normalization factor. According to [18, Lemma II.2] with $\sigma_E = 1/(2\sigma)$, $\sigma_r = \sqrt{2x/\beta-\sigma_E^2}$, and $\omega_r = x$, and the above choice of g_x, the transition part of the Lindbladian $\widehat{\mathcal{L}}_{KMS,x}$ can be written as*

$$\mathcal{T}(\rho) = \sum_{\nu_1,\nu_2\in B(H)}\gamma_{\nu_1,\nu_2}A_{\nu_1}\rho A_{\nu_2}^\dagger,$$

where

$$\gamma_{\nu_1,\nu_2} = \frac{1}{2\sqrt{4\pi x/\beta}}\exp\left(-\frac{(\nu_1+\nu_2+2x)^2}{16x/\beta}\right)\exp\left(-\frac{(\nu_1-\nu_2)^2\sigma^2}{2}\right).$$

This implies that, when σ changes, it only affects the $\nu_1-\nu_2$ term.

Proof of Theorem 27. The formula of g gives

$$\gamma_x(\omega) = \frac{g_x(\omega)+g_x(-\omega)}{1+\exp(\beta\omega)} = g_x(\omega)\frac{1+g_x(-\omega)/g_x(\omega)}{1+\exp(\beta\omega)} = g_x(\omega)\frac{1+\exp(\beta\omega)\exp\left(\frac{\beta^2\omega}{8x\sigma^2-\beta}\right)}{1+\exp(\beta\omega)}\,.$$

Let $\widehat{\gamma}_x(\omega) = g_x(\omega)$, we then have

$$\begin{aligned}
\|\gamma_x - \hat{\gamma}_x\|_{L^\infty} \leq & \left\|g_x(\omega)\left|1-\exp\left(\frac{\beta^2\omega}{8x\sigma^2-\beta}\right)\right|\right\|_{L^\infty_\omega} \\
= & \left\|\frac{1}{Z_x}\exp\left(-\frac{(\omega+x)^2}{2\left(\frac{2x}{\beta}-\frac{1}{4\sigma^2}\right)}\right)\left|1-\exp\left(\frac{\beta^2\omega}{8x\sigma^2-\beta}\right)\right|\right\|_{L^\infty_\omega} \\
= & \left\|\frac{1}{Z_x}\exp\left(-\frac{u^2}{2}\right)\left|1-\exp\left(\frac{\beta^2\left(u\sqrt{2x/\beta-1/(4\sigma^2)}-x\right)}{8x\sigma^2-\beta}\right)\right|\right\|_{L^\infty_u} \\
= & \mathcal{O}\left(\frac{\beta^2}{\sigma^2}\frac{1}{\sqrt{2x/\beta-1/(4\sigma^2)}}\frac{\sqrt{2x/\beta-1/(4\sigma^2)}+x}{x-\beta/(8\sigma^2)}\right) \\
= & \mathcal{O}\left(\frac{\beta^2}{\sigma^2}\frac{1+x/\sqrt{2x/\beta-1/(4\sigma^2)}}{x-\beta/(8\sigma^2)}\right)\,.
\end{aligned} \tag{I2}$$

when $\frac{\beta^2}{\sigma^2}\frac{1+x/\sqrt{2x/\beta-1/(4\sigma^2)}}{x-\beta/(8\sigma^2)} = \mathcal{O}(1)$. In the second equality, we let $u = \omega + x/\sqrt{2x/\beta-1/(4\sigma^2)}$. Similarly,

$$\begin{aligned}
\|\gamma_x - \hat{\gamma}_x\|_{L^1} \leq & \int_{-\infty}^{\infty} g_x(\omega)\frac{\exp(\beta\omega)}{1+\exp(\beta\omega)}\left|1-\exp\left(\frac{\beta^2\omega}{8x\sigma^2-\beta}\right)\right|\mathrm{d}\omega \\
\leq & \frac{1}{Z_x}\int_{-\infty}^{\infty}\exp\left(-\frac{(\omega+x)^2}{2\left(\frac{2x}{\beta}-\frac{1}{4\sigma^2}\right)}\right)\left|1-\exp\left(\frac{\beta^2\omega}{8x\sigma^2-\beta}\right)\right|\mathrm{d}\omega \\
= & \frac{1}{\sqrt{2\pi\left(\frac{2x}{\beta}-\frac{1}{4\sigma^2}\right)}}\int_{-\infty}^{\infty}\exp\left(-\frac{(\omega+x)^2}{2\left(\frac{2x}{\beta}-\frac{1}{4\sigma^2}\right)}\right)\left|1-\exp\left(\frac{\beta^2\omega}{8x\sigma^2-\beta}\right)\right|\mathrm{d}\omega \\
= & \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(-u^2/2\right)\left|1-\exp\left(\frac{\beta^2\left(u\sqrt{2x/\beta-1/(4\sigma^2)}-x\right)}{8x\sigma^2-\beta}\right)\right|\mathrm{d}\omega \\
= & \mathcal{O}\left(\frac{\beta^2}{\sigma^2}\frac{x+\sqrt{2x/\beta-1/(4\sigma^2)}}{x-\beta/(8\sigma^2)}\right)\,.
\end{aligned} \tag{I3}$$

Define

$$\widehat{\mathcal{D}}_{A_S,x}(\rho) = \int_{-\infty}^{\infty}\widehat{\gamma}_x(\omega)\mathcal{D}_{V_{A_S,f,\infty}(\omega)}(\rho)\mathrm{d}\omega\,.$$

Using the above estimation, we have

$$\left\|\widehat{\mathcal{D}}_{A_S,x}(\rho) - \int_{-\infty}^{\infty}\gamma_x(\omega)\mathcal{D}_{V_{A_S,f,\infty}(\omega)}(\rho)\mathrm{d}\omega\right\|_{1\leftrightarrow 1} = \mathcal{O}\left(\|\gamma_x-\hat{\gamma}_x\|_{L^\infty}\|f(t)\|_{L^2}\right) = \mathcal{O}\left(\frac{\beta^2}{\sigma^2}\frac{1+x/\sqrt{2x/\beta-1/(4\sigma^2)}}{x-\beta/(8\sigma^2)}\right),$$

where the second term is the original dissipative part with $\gamma_x(\omega)$. Here, the first equality is a result of [17, Proposition A.1] and [17, Lemma A.1].

Next, it is straightforward to check that, $\widehat{\gamma}_x$ satisfies [18, Eqn. (1.4)] up to a normalization factor. According to [18, Appendix A Corollaries A.1,A.2] with $\sigma_E = 1/(2\sigma)$, $\sigma_r = \sqrt{2x/\beta-\sigma_E^2}$, $g(\omega) = \delta(\omega - x)$, (I1) satisfies the KMS detailed balance. Furthermore, we have

$$\|h_1\|_{L^1} = \mathcal{O}\left(\left\|\frac{1}{\cosh 2\pi t/\beta}\right\|_{L^1} \left\|(\sigma\beta)^{-1}\sin(-t\beta/(4\sigma^2))\exp\left(-t^2/(2\sigma^2)\right)\right\|_{L^1}\right) = \mathcal{O}(\beta/\sigma), \quad \|h_2\|_{L^1} = \mathcal{O}\left(1\right),$$

which implies $\|B_{A_S}\| \leq \|h_1\|_{L^1}\|h_2\|_{L^1} = \mathcal{O}(\beta/\sigma)$. In summary, we have

$$\left\|\underbrace{-i\left[\frac{B_{A_S}}{Z_x},\rho\right] + \widehat{\mathcal{D}}_{A_S,x}(\rho)}_{\widehat{\mathcal{L}}_{\mathrm{KMS},x}} - \int_{-\infty}^{\infty}\gamma_x(\omega)\mathcal{D}_{V_{A_S,f,\infty}(\omega)}(\rho)\mathrm{d}\omega\right\|_{1\leftrightarrow 1} = \mathcal{O}\left(\frac{\beta^2}{\sigma^2}\frac{1+x/\sqrt{2x/\beta-1/(4\sigma^2)}}{x-\beta/(8\sigma^2)} + \frac{\beta}{\sigma Z_x}\right).$$

Furthermore, we can verify that

$$\widehat{R} := \int_0^\infty \left|\int_{-\infty}^\infty \widehat{\gamma}_x(\omega)\exp(i\omega\sigma q)\mathrm{d}\omega\right|\exp(-q^2/8)\mathrm{d}q = \mathcal{O}\left(\frac{\sqrt{\beta}}{\sigma\sqrt{x-\frac{\beta}{8\sigma^2}}}\right),$$

which implies

$$R := \int_0^\infty \left|\int_{-\infty}^\infty \gamma_x(\omega)\exp(i\omega\sigma q)\mathrm{d}\omega\right|\exp(-q^2/8)\mathrm{d}q = \mathcal{O}\left(\frac{\sqrt{\beta}}{\sigma\sqrt{x-\frac{\beta}{8\sigma^2}}} + \frac{\beta^2}{\sigma^2}\frac{x+\sqrt{2x/\beta-1/(4\sigma^2)}}{x-\beta/(8\sigma^2)}\right). \tag{I4}$$

Here, $\hat{R}, R$ are defined according to Lemma 13 (E5). According to the proof of Lemma 11 and Lemma 13, there exists a Hermitian matrix H_x such that

$$\left\|\mathcal{L} - \left(-i[H_x,\rho] + \widehat{\mathcal{L}}_{\mathrm{KMS},x}\right)\right\|_{1\leftrightarrow 1} = \mathcal{O}\left(\sigma\exp\left(-T^2/(4\sigma^2)\right) + \frac{1}{Z_x}\left(\frac{\beta^2}{\sigma^2}\frac{x+\sqrt{2x/\beta-1/(4\sigma^2)}}{x-\beta/(8\sigma^2)} + \frac{\beta}{\sigma}\right) + \frac{\beta}{\sigma}\right),$$

and

$$\left\|\sigma_\beta^{-1/4}H_x\sigma_\beta^{1/4} - \sigma_\beta^{1/4}H_x\sigma_\beta^{-1/4}\right\| = \mathcal{O}\left(\frac{\beta}{\sigma}\sqrt{\frac{\beta}{x-\frac{\beta}{8\sigma^2}}} + \frac{\beta^3}{\sigma^2}\frac{x+\sqrt{2x/\beta-1/(4\sigma^2)}}{x-\beta/(8\sigma^2)}\right).$$

This concludes the proof of the first part of Theorem 27.

Now, let $x = \frac{\beta}{8\sigma^2} + \omega_{\max}$ with $\omega_{\max} = \Omega(\beta)$. We can provide a better estimation for (I3). Let $c = 2\omega_{\max}/\beta$. Then $c = \Omega(1)$ and

$$\begin{aligned}
\|\gamma_x - \hat{\gamma}_x\|_{L^1} \leq & \int_{-\infty}^{\infty} g_x(\omega)\frac{\exp(\beta\omega)}{1+\exp(\beta\omega)}\left|1-\exp\left(\frac{\beta^2\omega}{8x\sigma^2-\beta}\right)\right|\mathrm{d}\omega \\
\leq & \frac{1}{\sqrt{2\pi c}}\int_{-\infty}^{\infty}\exp\left(-\frac{(\omega+x)^2}{2c}\right)\frac{\exp(\beta\omega)}{1+\exp(\beta\omega)}\left|1-\exp\left(\frac{\beta\omega}{4\sigma^2 c}\right)\right|\mathrm{d}\omega \\
= & \frac{1}{\sqrt{2\pi c}}\int_{-\infty}^{0}\exp\left(-\frac{(\omega+x)^2}{2c}\right)\frac{\exp(\beta\omega)}{1+\exp(\beta\omega)}\left|1-\exp\left(\frac{\beta\omega}{4\sigma^2 c}\right)\right|\mathrm{d}\omega \\
& + \frac{1}{\sqrt{2\pi c}}\int_{0}^{\infty}\exp\left(-\frac{(\omega+x)^2}{2c}\right)\frac{\exp(\beta\omega)}{1+\exp(\beta\omega)}\left|1-\exp\left(\frac{\beta\omega}{4\sigma^2 c}\right)\right|\mathrm{d}\omega
\end{aligned}$$

For the first term, we have

$$\begin{aligned}
& \frac{1}{\sqrt{2\pi c}}\int_{-\infty}^{0}\exp\left(-\frac{(\omega+x)^2}{2c}\right)\frac{\exp(\beta\omega)}{1+\exp(\beta\omega)}\left|1-\exp\left(\frac{\beta\omega}{4\sigma^2 c}\right)\right|\mathrm{d}\omega \\
\leq & \max_{\omega\in(-\infty,0)}\exp(\beta\omega)\left|1-\exp\left(\frac{\beta\omega}{4\sigma^2 c}\right)\right| = \max_{\omega\in(-\infty,0)}\exp(\omega)\left|1-\exp\left(\frac{\omega}{4\sigma^2 c}\right)\right| = \mathcal{O}(1/(c\sigma^2)).
\end{aligned}$$

For the second term, we have

$$
\begin{aligned}
&\frac{1}{\sqrt{2\pi c}}\int_0^\infty \exp\left(-\frac{(\omega+x)^2}{2c}\right)\frac{\exp(\beta\omega)}{1+\exp(\beta\omega)}\left|1-\exp\left(\frac{\beta\omega}{4\sigma^2 c}\right)\right|\mathrm{d}\omega\\
=&\frac{1}{\sqrt{2\pi c}}\int_0^\infty \exp\left(-\frac{(\omega+x)^2}{2c}\right)\left|1-\exp\left(\frac{\beta\omega}{4\sigma^2 c}\right)\right|\mathrm{d}\omega\\
\leq&\frac{1}{\sqrt{2\pi}}\int_0^\infty \exp\left(-\frac{(u+x/\sqrt{c})^2}{2}\right)\left|1-\exp\left(\frac{\beta u}{4\sigma^2 \sqrt{c}}\right)\right|\mathrm{d}u\\
=&\frac{1}{\sqrt{2\pi}}\left(\int_0^{\log(\sigma)} \exp\left(-\frac{(u+x/\sqrt{c})^2}{2}\right)\left|1-\exp\left(\frac{\beta u}{4\sigma^2 \sqrt{c}}\right)\right|\mathrm{d}u+\int_{\log(\sigma)}^\infty \exp\left(-\frac{(u+x/\sqrt{c})^2}{2}\right)\left|1-\exp\left(\frac{\beta u}{4\sigma^2 \sqrt{c}}\right)\right|\mathrm{d}u\right)
\end{aligned}
$$

For the first part, we have

$$
\begin{aligned}
&\int_0^{\log(\sigma)} \exp\left(-\frac{(u+x/\sqrt{c})^2}{2}\right)\left|1-\exp\left(\frac{\beta u}{4\sigma^2 \sqrt{c}}\right)\right|\mathrm{d}u\\
\leq&\log(\sigma)\max_{u\in[0,\log(\sigma)]}\exp\left(-\frac{(u+x/\sqrt{c})^2}{2}\right)\left|1-\exp\left(\frac{\beta u}{4\sigma^2 \sqrt{c}}\right)\right|\\
=&\mathcal{O}\left(\frac{\log(\sigma)}{\sqrt{c}}\max_{u\in[0,\log(\sigma)]}\frac{\beta u}{\sigma^2}\exp(-(u+\beta\sqrt{c}/2)^2/2)\right)\\
=&\mathcal{O}\left(\frac{\log^2(\sigma)}{\sqrt{c}}\max_{u\in[0,\log(\sigma)]}\frac{\beta}{\sigma^2}\exp(-(u+\beta\sqrt{c}/2)^2/2)\right)=\mathcal{O}\left(\sigma^{-2}\log^2(\sigma)/\sqrt{c}\right)
\end{aligned}
$$

For the second part, we have

$$
\begin{aligned}
&\int_{\log(\sigma)}^\infty \exp\left(-\frac{(u+x/\sqrt{c})^2}{2}\right)\left|1-\exp\left(\frac{\beta u}{4\sigma^2 \sqrt{c}}\right)\right|\mathrm{d}u\\
\leq&2\int_{\log(\sigma)}^\infty \exp\left(-\frac{(u+x/\sqrt{c})^2-\beta u/(2\sigma\sqrt{c})}{2}\right)\mathrm{d}u\\
\leq&2\int_{\log(\sigma)}^\infty \exp\left(-\frac{(u+x/\sqrt{c})^2-u/(2\sqrt{c})}{2}\right)\mathrm{d}u\leq 2\int_{\log(\sigma)}^\infty \exp\left(-\frac{u^2/2}{2}\right)\mathrm{d}u=\mathcal{O}\left(\exp(-\log(\sigma)^2/4)\right)
\end{aligned}
$$

where we use $u>1/\sqrt{c}$ when $u>\log(\sigma)$. Plugging this back, we have

$$
\|\gamma_x-\hat{\gamma}_x\|_{L^1}=\mathcal{O}\left(\sigma^{-2}\log^2(\sigma)\right)\,.
$$

when $\sigma=\Omega(\beta)$. The remaining part of the calculation is very similar to the first part of the proof. Thus, we omitted. □

In the following section, we will upper bound the mixing time of free fermion in Section J using KMS DBC and the mixing time of commuting local Hamiltonian in Section K using GNS DBC.

Appendix J: Mixing time for thermal state preparation of free fermions

In this section, we give a rigorous version of Theorem 19 and provide the proof.

Theorem 29. *Assume $\beta=\Theta(1)$ and $\|h\|=\mathcal{O}(1)$. We set*

$$
g(\omega)=\frac{1}{Z_x}\exp\left(-\frac{(\omega+\beta/(8\sigma^2)+\omega_{\max})^2}{4\omega_{\max}/\beta}\right),\quad Z_x=\sqrt{4\pi\omega_{\max}/\beta}\,, \tag{J1}
$$

where $\omega_{\max}=\Theta(1)$. Then, if

$$
\sigma=\widetilde{\Omega}(N^2\epsilon^{-1}),\quad T=\widetilde{\Omega}(\sigma),\quad \alpha=\widetilde{\mathcal{O}}(\sigma^{-1}N^{-1}\epsilon^{1/2})
$$

we have

$$
t_{\mathrm{mix},\Phi}(\epsilon)=\mathcal{O}\left(N\log\left(\frac{2\left\|\sigma_\beta^{-1/2}\right\|}{\epsilon}\right)\right)=\mathcal{O}(N(N+\log(1/\epsilon)))\,.
$$

Remark 30. *We note that since g is not a uniform distribution, we cannot directly apply Theorem 9 to control the fixed-point error. However, using* (I4) *and the assumption* $\beta = \Theta(1)$*, we obtain*

$$R = \int_0^\infty \left| \int_{-\infty}^\infty \gamma(\omega) \exp(i\omega\sigma q) \mathrm{d}\omega \right| \exp(-q^2/8) \mathrm{d}q = \mathcal{O}\left(\frac{1}{\sigma}\right) .$$

Plugging this into Theorem 12, we obtain a fixed-point error bound that is essentially the same as in Theorem 9. Specifically, for any $\epsilon > 0$*, if*

$$\sigma = \widetilde{\Omega}\left(\epsilon^{-1} t_{\mathrm{mix},\Phi}(\epsilon)\right), \quad T = \Omega(\sigma \log(\sigma/\epsilon)),$$

and $\alpha = \mathcal{O}\left(\sigma T^{-2} \epsilon^{1/2} t_{\mathrm{mix},\Phi}^{-1/2}(\epsilon)\right)$*, then*

$$\|\rho_{\mathrm{fix}}(\Phi) - \rho_\beta\|_1 < \epsilon .$$

The upper bound on $t_{\mathrm{mix},\Phi}(\epsilon)$ *established in Theorem 29 then implies that Theorem 21 also holds in this setting.*

Proof of Theorem 29. The proof of the theorem is based on Corollary 26 and Theorem 27. Specifically, in Theorem 27, we choose $x = \frac{\beta}{8\sigma^2} + \omega_{\max}$ and define $\widehat{\mathcal{L}}_{\mathrm{KMS,c}} = \widehat{\mathcal{L}}_{\mathrm{KMS},x}$ and $H_c = H_x$. According to Theorem 27, we have

$$\left\| \mathcal{L} - \left(-i[H_c, \rho] + \widehat{\mathcal{L}}_{\mathrm{KMS,c}} \right) \right\|_{1\leftrightarrow 1} = \mathcal{O}\left(\sigma \exp\left(-T^2/(4\sigma^2)\right) + \frac{\beta}{\sigma} \right) ,$$

and

$$\left\| \sigma_\beta^{-1/4} H_c \sigma_\beta^{1/4} - \sigma_\beta^{1/4} H_c \sigma_\beta^{-1/4} \right\| = \mathcal{O}(\beta/\sigma) . \tag{J2}$$

Thus, it suffices to study the spectral gap of $\widehat{\mathcal{L}}_{\mathrm{KMS,c}}$ defined in the above lemma. For this part, we mainly follow the approach in [42, Section III.A].

First, given a set of creation and annihilation operators $\{c_k, c_k^\dagger\}_{k=1}^N$, we define the Majorana operators as

$$m_{2j-1} = c_j + c_j^\dagger, \quad m_{2j} = i(c_j - c_j^\dagger), \quad j = 1, \dots, N ,$$

which satisfies $\{m_i, m_j\} = 2\delta_{i,j}$. Let $\vec{m} = [m_1, \dots, m_{2N}]^T$. Then, we have

$$H = \sum_{i,j} h_{i,j} c_i^\dagger c_j = \sum_{i,j=1}^{2N} h_{i,j}^m m_i m_j - C I_{2^N \times 2^N} = \vec{m}^T \cdot h^m \cdot \vec{m} - C I_{2^N \times 2^N} .$$

We note that the eigenvalues of h^m is a Hermitian and antisymmetric matrix with eigenvalues $\{\lambda_k(h)/4, -\lambda_k(h)/4\}_{k=1}^N$, and C is a constant.

Next, for each creation and annihilation operator pair $(c_j, c_j^\dagger)$, we have

$$\begin{bmatrix} c_j^\dagger \\ c_j \end{bmatrix} = \begin{bmatrix} 1/2 & i/2 \\ 1/2 & -i/2 \end{bmatrix} \begin{bmatrix} m_{2j-1} \\ m_{2j} \end{bmatrix}$$

Define the coupling operator vector $\vec{A} = [c_1^\dagger, c_1, c_2^\dagger, c_2, \dots, c_N^\dagger, c_N]^T$. Then,

$$\vec{A} = \frac{M}{\sqrt{2}} \vec{m} , \tag{J3}$$

where M is a unitary matrix.

In $\hat{L}_{\mathrm{KMS,c}}$, we first evaluate the coherent component B. By (J3), the coupling operator $\sqrt{2}, \vec{A}$ can be written as a unitary transformation of $\vec{m}$. As shown in [42, Lemma III.1], the coherent part under KMS detailed balance satisfies $B = 0$, meaning that the coherent contribution vanishes.

Next, we follow the proof of [42, Lemma III.2] to calculate the spectral gap of the dissipative term. We first formulate

$$\mathcal{H}_0[\rho] = \sigma_\beta^{-1/4} \cdot \widehat{\mathcal{L}}_{\mathrm{KMS,c}}^\dagger [\sigma_\beta^{1/4} (\rho) \sigma_\beta^{1/4}] \sigma_\beta^{-1/4} .$$

Because $\widehat{\mathcal{L}}_{\mathrm{KMS,c}}$ satisfies the KMS detailed balance condition, $\mathcal{H}_0$ is a self-adjoint operator with respect to the HilbertSchmidt inner product. Furthermore, $\mathcal{H}_0$ is a similarity transformation of the Lindbladian $\widehat{\mathcal{L}}^\dagger_{\mathrm{KMS,c}}$, which means that the spectral gap of $\mathcal{H}_0$ is the same as the spectral gap of $\widehat{\mathcal{L}}^\dagger_{\mathrm{KMS,c}}$.

To calculate each term in the parent Hamiltonian $\mathcal{H}_0[\rho]$, we first note that, for each ω,

$$\begin{bmatrix} \sigma_\beta^{-1/4} V_{c_1^\dagger}(\omega)\sigma_\beta^{1/4} \\ \sigma_\beta^{-1/4} V_{c_1}(\omega)\sigma_\beta^{1/4} \\ \vdots \\ \sigma_\beta^{-1/4} V_{c_N^\dagger}(\omega)\sigma_\beta^{1/4} \\ \sigma_\beta^{-1/4} V_{c_N}(\omega)\sigma_\beta^{1/4} \end{bmatrix} = \frac{M}{\sqrt{2}} \cdot \hat{f}(-4h^m-\omega)e^{-\beta h^m} \cdot \vec{m}\,,$$

and

$$\sum_j \sigma_\beta^{-1/4} V^\dagger_{c_j^\dagger}(\omega)V_{c_j^\dagger}(\omega)\sigma_\beta^{1/4} + \sigma_\beta^{-1/4} V^\dagger_{c_j}(\omega)V_{c_j}(\omega)\sigma_\beta^{1/4} = \frac{1}{2}\vec{m}^\dagger \cdot \left|\hat{f}(-4h^m-\omega)\right|^2 \cdot \vec{m}\,.$$

Plugging this equality into the expression for $\mathcal{H}_0[\rho]$,

$$\begin{aligned} &\mathcal{H}_0[\rho] \\ =&\frac{1}{2N}\int \frac{\widehat{\gamma}(\omega)}{2}\Big(\vec{m}^\dagger \cdot \hat{f}(-4h^m-\omega)e^{-\beta h^m}\cdot M^\dagger \cdot \rho \cdot M \cdot \hat{f}(-4h^m-\omega)e^{-\beta h^m}\cdot \vec{m} \\ &-\frac{1}{2}\vec{m}^\dagger \cdot \left|\hat{f}(-4h^m-\omega)\right|^2 \cdot \vec{m}\cdot\rho - \rho\cdot \frac{1}{2}\vec{m}^\dagger \cdot \left|\hat{f}(-4h^m-\omega)\right|^2 \cdot \vec{m}\Big)\,\mathrm{d}\omega\,. \end{aligned}$$

Because $\beta=\Theta(1)$, $\|h\|=\mathcal{O}(1)$, and $\omega_{\max}=\Theta(1)$, it is straightforward to check that

$$\left(\int \widehat{\gamma}(\omega)\left|\hat{f}(-4h^m-\omega)\right|^2\mathrm{d}\omega\right)\exp(-\beta h^m) \geq C\,,$$

where C is a constant independent of σ, meaning the coefficients of the above quadratic expansion does not generate when σ approaches to zero. Following the proof of [42, Proposition III.2], the spectral gap of $\mathcal{H}_0$ is lower bounded by a constant over N independent of σ.

Let

$$\widehat{\Phi} = \mathcal{U}_S(T)\circ \exp\left(\mathcal{M}\alpha^2\right)\circ \mathcal{U}_S(T), \quad \mathcal{M} = -i[H_c,\rho] + \widehat{\mathcal{L}}_{\mathrm{KMS,c}}\,.$$

Combining Lemma 11 and Theorem 27, we first have

$$\left\|\widehat{\Phi}-\Phi_\alpha\right\|_{1\leftrightarrow 1} = \mathcal{O}\left(\alpha^2\left(\sigma\exp\left(-T^2/(4\sigma^2)\right)+\alpha^2T^4\sigma^{-2}+\frac{\beta}{\sigma}\right)\right)\,.$$

In addition, according to Corollary 26 and (J2), when $\beta/\sigma=\mathcal{O}(1)$, we have

$$t_{\mathrm{mix},\widehat{\Phi}}(\epsilon) = \mathcal{O}\left(N\log\left(\frac{2\left\|\sigma_\beta^{-1/2}\right\|}{\epsilon}\right)\right)\,.$$

We note that $\log\left(\left\|\sigma_\beta^{-1/2}\right\|\right) = \mathcal{O}(\beta\|H\|+N) = \mathcal{O}(\beta N) = \mathcal{O}(N)$ in our case. Applying Theorem 8 to Φ and $\widehat{\Phi}$, we obtain that, if

$$\sigma\exp\left(-T^2/(4\sigma^2)\right)+\alpha^2T^4\sigma^{-2}+\frac{1}{\sigma} = \mathcal{O}(\epsilon N^{-1}(N+\log(1/\epsilon))^{-1})\,,$$

then

$$t_{\mathrm{mix},\Phi_\alpha}(\epsilon) = \mathcal{O}\left(N\log\left(\frac{2\left\|\sigma_\beta^{-1/2}\right\|}{\epsilon}\right)\right)\,.$$

This concludes the proof.

□

Appendix K: Mixing time for commuting local Hamiltonians

In this section, we prove a general version of Theorem 20. Unlike the previous section, here we show that, the dissipative part of $\mathcal{L}$ can converge to the Davies generator. The Davies generator satisfies the GNS detailed balance condition (defined in Section A), and therefore also satisfies the KMS detailed balance condition.

In the general case, establishing rigorous convergence to the Davies generator requires σ to scale exponentially with the system size, since one may need to resolve exponentially close Bohr frequencies ω_1 and ω_2 in order to generate distinct jump operators $A_S(\omega_1)$ and $A_S(\omega_2)$ that appear in the Davies generator. However, in cases where the effective Bohr frequencies are not exponentially close, such as for local commuting Hamiltonians, σ does not need to be exponentially large, allowing for an efficient approximation. This property is mainly summarized in the following theorem.

Theorem 31. *Given a coupling set $\mathcal{A}$. Assume*

- *The Davies generator $\mathcal{L}_{D,\mathcal{A}}[\rho]$ has a spectral gap $\lambda_{\mathrm{gap}} > 0$.*
- *There exists a constant $\delta > 0$ such that: for any $A \in \mathcal{A}$, $\omega_1 \neq \omega_2$, if $A(\omega_1) \neq 0$ and $A(\omega_2) \neq 0$, we must have $|\omega_1 - \omega_2| \geq \delta$.*
- *There exists a constant M such that $\sup_{A\in\mathcal{A}} |\{\omega | A(\omega) \neq 0\}| \leq M$.*

Given any $\epsilon > 0$, if

$$\sigma = \Omega\left(\beta|\mathcal{A}|\lambda_{\mathrm{gap}}^{-1}\delta^{-1}\epsilon^{-1}M\log\left(\left\|\sigma_\beta^{-1/2}\right\|/\epsilon\right)\mathbb{E}(\|A_S\|^2)\right),\ T = \widetilde{\Omega}(\sigma)\,,$$

and

$$\alpha = \mathcal{O}\left(\epsilon^{1/2}\sigma^{-1}|\mathcal{A}|^{-1/2}\lambda_{\mathrm{gap}}^{1/2}\log^{-1/2}\left(\left\|\sigma_\beta^{-1/2}\right\|/\epsilon\right)\mathbb{E}^{-1/2}(\|A_S\|^4)\right)\,,$$

then

$$t_{\mathrm{mix},\widehat{\Phi}}(\epsilon) \leq \frac{|\mathcal{A}|}{\lambda_{\mathrm{gap}}}\log\left(\frac{8\left\|\sigma_\beta^{-1/2}\right\|}{\epsilon}\right) + 1$$

In Theorem 31, we note that the second condition is a technical assumption used to bound the difference between $\widetilde{\mathcal{L}}$ and the Davies generator. When δ is not exponentially small, it is not necessary to fully resolve all Bohr frequencies to ensure that the Lindbladian dynamics converges to the Davies generator. This condition is easily satisfied in the case of local commuting Hamiltonians: $A(\omega)$ is determined by the interaction between A and a constant number of local Hamiltonians. Hence, it suffices to choose δ to be a constant to guarantee that different components $A(\omega)$ are well separated.

Remark 32. *According to [35, Section VIII], when H is a local commuting Hamiltonian as stated in Theorem 20, there exists a constant β_c dependent on the Hamiltonian H such that for every $\beta \leq \beta_c$, the spectral gap of the Davies generator can be lower bounded by a constant, meaning $\lambda_{\mathrm{gap}} = \Theta(1)$. Furthermore, we also have $\delta = \Omega(1)$, $M = \mathcal{O}(1)$, $|\mathcal{A}| = \mathcal{O}(N)$, $\|H\| = \mathcal{O}(N)$. Noticing $\log\left(\left\|\sigma_\beta^{-1/2}\right\|\right) = \mathcal{O}(\beta\|H\| + N) = \mathcal{O}((\beta+1)N)$, we can choose*

$$\sigma = \widetilde{\Omega}\left(\epsilon^{-1}(\beta+1)^2N^2\right), \quad T = \widetilde{\Omega}(\sigma), \quad \text{and} \quad \alpha = \widetilde{\mathcal{O}}\left(\epsilon^{3/2}(\beta+1)^{-5/2}N^{-3}\right),$$

to obtain

$$t_{\mathrm{mix},\Phi}(\epsilon) = \mathcal{O}\left(N(\|H\|\beta + N)\log(1/\epsilon)\right) = \mathcal{O}\left(N^2(\beta+1)\log(1/\epsilon)\right).$$

This gives Theorem 20.

Proof. Recall $\widetilde{\Phi}$ defined in Eq. (E2):

$$\widetilde{\Phi} = \mathcal{U}_S(T) \circ \exp\left(\widetilde{\mathcal{L}}\alpha^2\right) \circ \mathcal{U}_S(T)\,. \tag{K1}$$

Here

$$\widetilde{\mathcal{L}}(\rho) = -i\left[\widetilde{H}_{\rm LS}, \rho\right] + \mathbb{E}_{A_S}\left(\int_{-\infty}^{\infty} \gamma(\omega)\mathcal{D}_{\widetilde{V}_{A_S,f(\omega)}}(\rho)\mathrm{d}\omega\right), \tag{K2}$$

where

$$\widetilde{H}_{\rm LS} = -\mathbb{E}_{A_S}\left(\mathrm{Im}\left(\int_{-\infty}^{\infty}\gamma(\omega)\widetilde{\mathcal{G}}_{A_S,f}(-\omega)\mathrm{d}\omega\right)\right), \quad \widetilde{V}_{A_S,f}(\omega) = \int_{-\infty}^{\infty} f(t)A_S(t)\exp(-i\omega t)\mathrm{d}t\,,$$

with

$$\widetilde{\mathcal{G}}_{A,f}(\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{s_1} f(s_2)f(s_1)A^\dagger(s_2)A(s_1)\exp(i\omega(s_2-s_1))\mathrm{d}s_2\mathrm{d}s_1\,. \tag{K3}$$

Similar to the proof of Theorem 23, we will construct

$$\widehat{\Phi} = \mathcal{U}_S(T)\circ\exp\left(\widehat{\mathcal{L}}\alpha^2\right)\circ\mathcal{U}_S(T) \tag{K4}$$

with

$$\widehat{\mathcal{L}}(\rho) = -i\left[\widehat{H}_{\rm LS},\rho\right] + \mathbb{E}_{A_S}\left(\sum_{\omega}\mathcal{L}_{D,A_S}\right), \tag{K5}$$

such that $\left\|\widehat{\Phi}-\widetilde{\Phi}\right\|_{1\leftrightarrow 1}$ is small. Here $\mathcal{L}_{D,A_S}$ is the Davies generator associated with the coupling operator A_S.

We first deal with the Lamb shift term. For each ω, we have

$$\begin{aligned}
\widetilde{\mathcal{G}}_{A_S,f}(\omega) =& \int_{-\infty}^{\infty}\int_{-\infty}^{s_1} f(s_2)f(s_1)A_S(s_2)A_S(s_1)\exp(-i\omega(s_2-s_1))\mathrm{d}s_2\mathrm{d}s_1\\
=& \sum_{\nu_1,\nu_2\in B(H_S)} A_S^\dagger(\nu_2)A_S(\nu_1)\int_{-\infty}^{\infty}\int_{-\infty}^{s_1} f(s_2)f(s_1)\exp(i\nu_2 s_2)\exp(i\nu_1 s_1)\exp(-i\omega(s_2-s_1))\mathrm{d}u\mathrm{d}v\\
=& \frac{\sigma}{2\sqrt{2\pi}}\sum_{\nu_1,\nu_2\in B(H_S)} A_S^\dagger(\nu_2)A_S(\nu_1)\\
&\cdot\underbrace{\int_{-\infty}^{\infty}\exp\left(i\frac{\sigma p}{2}(\nu_1+\nu_2)\right)\exp\left(-\frac{p^2}{8}\right)\mathrm{d}p}_{=\mathcal{O}(\exp(-\sigma^2(\nu_1+\nu_2)^2/2))}\underbrace{\int_0^{\infty}\exp\left(-\frac{q^2}{8}\right)\exp\left(i\frac{\sigma q}{2}(\nu_1-\nu_2)\right)\exp(i\sigma\omega q)\mathrm{d}q}_{=\mathcal{O}(1)}
\end{aligned}$$

Define the commuting part as $\widehat{\mathcal{G}}_{A_S,f}(\omega)$:

$$\begin{aligned}
\widehat{\mathcal{G}}_{A_S,f}(\omega) =& \frac{\sigma}{2\sqrt{2\pi}}\sum_{\nu_1+\nu_2=0} A_S^\dagger(\nu_2)A_S(\nu_1)\\
&\cdot\int_{-\infty}^{\infty}\exp\left(i\frac{\sigma p}{2}(\nu_1+\nu_2)\right)\exp\left(-\frac{p^2}{8}\right)\mathrm{d}p\int_0^{\infty}\exp\left(-\frac{q^2}{8}\right)\exp\left(i\frac{\sigma q}{2}(\nu_1-\nu_2)\right)\exp(i\sigma\omega q)\mathrm{d}q
\end{aligned}$$

According to the assumption of A_S, we have $|\nu_1+\nu_2|\geq\delta$ in the summation of $\widetilde{\mathcal{G}}_{A_S,f}(\omega)$. Thus,

$$\left\|\widetilde{\mathcal{G}}_{A_S,f}(\omega)-\widehat{\mathcal{G}}_{A_S,f}(\omega)\right\| = \mathcal{O}\left(\sigma\exp(-\sigma^2\delta^2/2)\|A_S\|^2M^2\right),$$

where M comes from the total number of terms in the summation.

Define $\widehat{H}_{\rm LS} = -\mathbb{E}_{A_S}\left(\mathrm{Im}\left(\int_{-\infty}^{\infty}\gamma(\omega)\widehat{\mathcal{G}}_{A_S,f}(-\omega)\mathrm{d}\omega\right)\right)$. We have

$$\left\|\widetilde{H}_{\rm LS}-\widehat{H}_{\rm LS}\right\| = \mathcal{O}\left(\sigma\exp(-\sigma^2\delta^2/2)M^2\mathbb{E}(\|A_S\|^2)\right).$$

Next, for the dissipative operator, we have

$$\widetilde{V}_{A,f}(\omega) = \int_{-\infty}^{\infty} f(t)A(t)\exp(-i\omega t)\mathrm{d}t = 2^{3/4}\sigma^{1/2}\pi^{1/4}\sum_{\xi\in B(H)}\exp(-(\xi-\omega)^2\sigma^2)A(\xi)$$

Then

$$\int_{-\infty}^{\infty}\gamma(\omega)\mathcal{D}_{\widetilde{V}_{A_S,f(\omega)}}(\rho)\mathrm{d}\omega-\mathcal{L}_{D,A_S}$$

$$=\sum_{\xi\in B(H)}\left(\underbrace{\int_{-\infty}^{\infty}2^{3/2}\pi^{1/2}\sigma\exp(-2(\xi-\omega)^2\sigma^2)\gamma(\omega)\mathrm{d}\omega-2\pi\gamma(\xi)}_{|\cdot|=\mathcal{O}(\beta\sigma^{-1})}\right)\mathcal{L}_{A_S(\xi)}+\underbrace{\int_{-\infty}^{\infty}\gamma(\omega)\left(\sum_{\xi_1\neq\xi_2}\cdots\right)\mathrm{d}\omega}_{\|\cdot\|_{1\leftrightarrow 1}=\mathcal{O}(\sigma\exp(-\sigma^2\delta^2/2)M^2\|A_S\|^2)}\quad.$$

This implies that

$$\left\|\int_{-\infty}^{\infty}\gamma(\omega)\mathcal{D}_{\widetilde{V}_{A_S,f(\omega)}}(\rho)\mathrm{d}\omega-\mathcal{L}_{D,A_S}\right\|_{1\leftrightarrow 1}=\mathcal{O}\left(\left(\beta\sigma^{-1}M+\sigma\exp(-\sigma^2\delta^2/2)M^2\right)\|A_S\|^2\right)\,.$$

In conclusion,

$$\left\|\widehat{\Phi}-\widetilde{\Phi}\right\|_{1\leftrightarrow 1}=\mathcal{O}\left(\alpha^2\left(\beta\sigma^{-1}M+\sigma\exp(-\sigma^2\delta^2/2)M^2\right)\mathbb{E}(\|A_S\|^2)\right)\,.$$

Combining this and Lemma 11, we have

$$\left\|\widehat{\Phi}-\Phi\right\|_{1\leftrightarrow 1}=\mathcal{O}\left(\alpha^2\left(\beta\sigma^{-1}M+\sigma\exp(-\sigma^2\delta^2/2)M^2+\sigma\exp\left(-T^2/(4\sigma^2)\right)\right)\mathbb{E}(\|A_S\|^2)+\alpha^4T^4\sigma^{-2}\mathbb{E}\left(\|A_S\|^4\right)\right)\,.$$

Furthermore, according to Corollary 26, we have

$$t_{\mathrm{mix},\widehat{\Phi}}(\epsilon)\leq\frac{|\mathcal{A}|}{\lambda_{\mathrm{gap}}}\log\left(\frac{2\left\|\sigma_\beta^{-1/2}\right\|}{\epsilon}\right)+1\,.$$

Finally, when

$$\sigma=\Omega\left(|\mathcal{A}|\lambda_{\mathrm{gap}}^{-1}\delta^{-1}\epsilon^{-1}\beta M\log\left(\left\|\sigma_\beta^{-1/2}\right\|/\epsilon\right)\mathbb{E}\left(\|A_S\|^2\right)\right),\ T=\widetilde{\Omega}(\sigma)\,,$$

and

$$\alpha=\mathcal{O}\left(\epsilon^{1/2}\sigma^{-1}|\mathcal{A}|^{-1/2}\lambda_{\mathrm{gap}}^{1/2}\log^{-1/2}\left(\left\|\sigma_\beta^{-1/2}\right\|/\epsilon\right)\mathbb{E}^{-1/2}\left(\|A_S\|^4\right)\right)\,,$$

we have

$$t_{\mathrm{mix},\widehat{\Phi}}(\epsilon)\left\|\widehat{\Phi}-\Phi\right\|_{1\leftrightarrow 1}\leq\epsilon\,.$$

Applying Theorem 8, we conclude the proof. □