Impact of gap anisotropy of Polar and Anderson-Brinkman-Morel p-wave superconductors on thermoelectric properties of quantum dot hybrids

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Green's function

The retarded Green's function G^r , represented as a 4×4 matrix in Nambu space using Zubarev notation, is given by:

$$\mathbf{G}^r = \left\langle \left\langle \begin{pmatrix} d_\uparrow & d_\downarrow^\dagger & d_\downarrow & d_\uparrow^\dagger \end{pmatrix}^T \mid \begin{pmatrix} d_\uparrow^\dagger & d_\downarrow & d_\downarrow^\dagger & d_\uparrow \end{pmatrix} \right\rangle \right\rangle. \tag{1}$$

We derive $\mathbf{G}^r$ using the equation-of-motion method. Given that the Coulomb interaction within the quantum dot is set to zero ($U = 0$), the Green's functions are exact. By applying equation-of-motion (EOM),

$$\varepsilon\langle\langle A : B\rangle\rangle = \langle\{A, B\}\rangle + \langle\langle[A, H] : B\rangle\rangle$$

to the elements of $\mathbf{G}^r$, we obtain following equations for the components of Green's function,

$$(\varepsilon - \varepsilon_\sigma - \Sigma^{(0)e}_{N\sigma})\langle\langle d_\sigma|d^\dagger_{\sigma'}\rangle\rangle = \delta_{\sigma\sigma'} + U\langle\langle d_\sigma n_{\bar{\sigma}}|d^\dagger_{\sigma'}\rangle\rangle + V^{TSC}_{k\sigma}\sum_k\langle\langle b_{k\sigma}|d^\dagger_{\sigma'}\rangle\rangle, \tag{2}$$

$$(\varepsilon - \varepsilon_\sigma - \Sigma^{(0)e}_{N\sigma})\langle\langle d_\sigma|d_{\sigma'}\rangle\rangle = U\langle\langle d_\sigma n_{\bar{\sigma}}|d_{\sigma'}\rangle\rangle + V^{TSC}_{k\sigma}\sum_k\langle\langle b_{k\sigma}|d_{\sigma'}\rangle\rangle. \tag{3}$$

Similarly,

$$(\varepsilon + \varepsilon_\sigma - \Sigma^{(0)h}_{N\sigma})\langle\langle d^\dagger_\sigma|d^\dagger_{\sigma'}\rangle\rangle = -U\langle\langle d^\dagger_\sigma n_{\bar{\sigma}}|d^\dagger_{\sigma'}\rangle\rangle - V^{TSC*}_{k\sigma}\sum_k\langle\langle b^\dagger_{-k\sigma}|d^\dagger_{\sigma'}\rangle\rangle, \tag{4}$$

$$(\varepsilon + \varepsilon_\sigma - \Sigma^{(0)h}_{N\sigma})\langle\langle d^\dagger_\sigma|d_{\sigma'}\rangle\rangle = \delta_{\sigma\sigma'} - U\langle\langle d^\dagger_\sigma n_{\bar{\sigma}}|d_{\sigma'}\rangle\rangle - V^{TSC*}_{k\sigma}\sum_k\langle\langle b^\dagger_{-k\sigma}|d_{\sigma'}\rangle\rangle. \tag{5}$$

Here $\Sigma^{(0)e(h)}_{N\sigma}$ denotes the electron (hole) component of the non-interacting self-energy due the ferromagnetic lead-quantum dot coupling. On the RHS, we made assumption of momentum-independent tunneling amplitudes i.e. $V^{TSC}_{k\sigma} = V^{TSC}$.

The triplet superconductor gap function can be represented as 2×2 complex matrix in spin space

$$\mathbf{\Delta}(k) = \begin{bmatrix} \Delta^k_{\sigma\sigma} & \Delta^k_{\sigma\bar{\sigma}} \\ \Delta^k_{\bar{\sigma}\sigma} & \Delta^k_{\bar{\sigma}\bar{\sigma}} \end{bmatrix} \tag{6}$$

with $\mathbf{\Delta}(k)$ obeying the relation $\mathbf{\Delta}(k) = -\mathbf{\Delta}(-k)^T$ i.e. $\mathbf{\Delta}(k)_{\sigma_1\sigma_2} = -\mathbf{\Delta}(-k)_{\sigma_2\sigma_1}$.
Applying the EOM to $\langle\langle b^{(\dagger)}_{k\sigma}|d^{(\dagger)}_{\sigma'}\rangle\rangle$ gives,

$$(\varepsilon - \varepsilon_{ks\sigma})A - (\Delta^k_{\sigma\sigma} - \Delta^{-k}_{\sigma\sigma})C - (\Delta^k_{\sigma\bar{\sigma}} - \Delta^{-k}_{\bar{\sigma}\sigma})D = V_{s\sigma}\langle\langle d_\sigma|d^\dagger_{\sigma'}\rangle\rangle, \tag{7a}$$

$$(\varepsilon-\varepsilon_{ks\bar{\sigma}})B-(\Delta^{k}_{\bar{\sigma}\sigma}-\Delta^{-k}_{\sigma\bar{\sigma}})C-(\Delta^{k}_{\bar{\sigma}\bar{\sigma}}-\Delta^{-k}_{\bar{\sigma}\bar{\sigma}})D=V_{s\bar{\sigma}}\langle\langle d_{\bar{\sigma}}|d^{\dagger}_{\sigma'}\rangle\rangle, \tag{7b}$$

$$-(\Delta^{k*}_{\sigma\sigma}-\Delta^{-k*}_{\sigma\sigma})A-(\Delta^{k*}_{\sigma\bar{\sigma}}-\Delta^{-k*}_{\bar{\sigma}\sigma})B+(\varepsilon+\varepsilon_{-ks\sigma})C=-V_{s\sigma}\langle\langle d^{\dagger}_{\sigma}|d^{\dagger}_{\sigma'}\rangle\rangle, \tag{7c}$$

$$-(\Delta^{k*}_{\bar{\sigma}\sigma}-\Delta^{-k*}_{\sigma\bar{\sigma}})A-(\Delta^{k*}_{\bar{\sigma}\bar{\sigma}}-\Delta^{-k*}_{\bar{\sigma}\bar{\sigma}})B+(\varepsilon+\varepsilon_{-ks\bar{\sigma}})D=-V_{s\bar{\sigma}}\langle\langle d^{\dagger}_{\bar{\sigma}}|d^{\dagger}_{\sigma'}\rangle\rangle. \tag{7d}$$

Here, $A=\langle\langle b_{k\sigma}|d^{\dagger}_{\sigma'}\rangle\rangle$, $B=\langle\langle b_{k\bar{\sigma}}|d^{\dagger}_{\sigma'}\rangle\rangle$, $C=\langle\langle b^{\dagger}_{-k\sigma}|d^{\dagger}_{\sigma'}\rangle\rangle$, $D=\langle\langle b^{\dagger}_{-k\bar{\sigma}}|d^{\dagger}_{\sigma'}\rangle\rangle$. By solving the coupled equations (7), four variables A,B,C,D can be obtained in term of elements of reduced Green's function given by Eq. (1).

For Polar and ABM state phases discussed in the main text, $\Delta_{\sigma\bar{\sigma}}=0=\Delta_{\bar{\sigma}\sigma}$. This simplifies the coupled system of Eq. (7) significantly, resulting in the following analytical form of the dot Green's function:

$$G^{r}_{11}=\frac{1}{\varepsilon-\varepsilon_{\sigma}-\Sigma^{(0)e}_{N\sigma}-\Sigma_{11}+\Sigma_{N}P}, \quad \text{with } P=\frac{\Sigma^{*}_{14}}{\varepsilon+\varepsilon_{\sigma}-\Sigma^{(0)h}_{N\sigma}-\Sigma_{14}}, \tag{8a}$$

$$G^{r}_{41}=P\,G^{r}_{11}, \tag{8b}$$

$$G^{r}_{12}=0, \tag{8c}$$

$$G^{r}_{13}=0. \tag{8d}$$

We further present the analytical expression for superconductor self-energy in Polar and ABM state.

Self-energy

The general form of self-energy elements in the Nambu space (see Eq. (1)) for the Polar or ABM state is given as,

$$\Sigma^{r}_{11}=\sum_{k}|V^{TSC}_{k\sigma}|^{2}\frac{\varepsilon+\varepsilon_{k\sigma}}{(\varepsilon+\varepsilon_{k\sigma})(\varepsilon-\varepsilon_{k\sigma})-\Delta^{2}_{k}} \tag{9a}$$

$$\Sigma^{r}_{14}=\sum_{k}|V^{TSC}_{k\sigma}|^{2}\frac{\Delta_{k}}{(\varepsilon+\varepsilon_{k\sigma})(\varepsilon-\varepsilon_{k\sigma})-\Delta^{2}_{k}} \tag{9b}$$

(9c)

A useful identity for converting the momentum sum into an integral is,

$$\sum_{k}\to V\int\frac{d^{3}k}{(2\pi)^{3}}=\int\frac{d\Omega_{k}}{4\pi}\int d\varepsilon_{k}N(\varepsilon_{k})\approx\frac{N_{0}}{4\pi}\int_{0}^{\pi}\sin\theta\,d\theta\int_{-\pi}^{\pi}d\phi\int_{-D}^{D}d\varepsilon_{k} \tag{10}$$

where $N(\varepsilon_k)\approx N(\varepsilon_F)=N_0$ is density of states of normal electrons at the Fermi level, $d\Omega_k=\sin\theta d\theta d\phi$ denotes the solid angle subtracted by the momentum vector in the momentum space of superconducting electrons and D represents the energy band width. Further, in the derivation of self-energy, we assume, $\varepsilon_{k\sigma}=\varepsilon_{k\bar{\sigma}}=\varepsilon_{-k\sigma}$ and placing $D\to\infty$. We further show the 'bare' self-energy, i.e., the self-energy obtained in the absence of spatial weighting of the coupling matrix elements between QD and TSC.

Self-energy for superconductor in Polar state

Gap function for Polar state is $\Delta_{pol}=\Delta_0\cos\theta$. From equation (9), the integration over momentum in polar coordinates leads to

$$\Sigma^{r}_{pol,11}=-\frac{i\Gamma_{TSC}}{2}\frac{\varepsilon}{\Delta_0}\left[\arcsin\left(\frac{\Delta_0}{\varepsilon}\right)\Theta(|\varepsilon|-\Delta_0)+\left(-i\ln\frac{\Delta_0+\sqrt{\Delta_0^2-\varepsilon^2}}{|\varepsilon|}+sign(\varepsilon)\frac{\pi}{2}\right)\Theta(\Delta_0-|\varepsilon|)\right], \tag{11a}$$

$$\Sigma^{r}_{pol,14}=0. \tag{11b}$$

Here, $\Theta(x)$ represents the Heaviside function and $\Gamma_{TSC} = 2\pi N_0 |V^{TSC}|^2$. The vanishing of $\Sigma^r_{pol,14}$ results due to effective integrand being an odd function of θ. Therefore, in the absence of weighting, Polar self-energy matrix is always diagonal. When weighting is introduced (as discussed in the main text), the non-diagonal (Σ^r_{14}) elements becomes finite.

Self-energy for superconductor in ABM state

Gap function for ABM state is $\Delta_{ABM} = \Delta_0 \sin\theta \exp(i\phi)$. In this case the self-energy acquires the form,

$$\begin{aligned}
\Sigma^r_{ABM,11} &= \frac{-i\Gamma_{TSC}}{4}\frac{\varepsilon}{\Delta_0}\ln\left|\frac{\varepsilon+\Delta_0}{\varepsilon-\Delta_0}\right| \\
&= \frac{-i\Gamma_{TSC}}{4}\frac{\varepsilon}{\Delta_0}\left[\ln\left(\frac{\varepsilon+\Delta_0}{\varepsilon-\Delta_0}\right)\Theta(|\varepsilon|-\Delta_0) + \ln\left(\frac{\varepsilon+\Delta_0}{\Delta_0-\varepsilon}\right)\Theta(\Delta_0-|\varepsilon|) - i\pi\Theta(\Delta_0-|\varepsilon|)\right] \\
\Sigma^r_{ABM,14} &= \left(\frac{iN_0|V_s|^2}{2}\right)\frac{\Delta_0}{\sqrt{\varepsilon^2-\Delta_0^2}}(-k)\int_0^{2\pi} d\phi e^{i\phi} I\left(\arcsin\frac{-1}{k}, k\right)
\end{aligned} \tag{12}$$

with

$$I\left(\arcsin\frac{-1}{k}, k\right) = \left[\int_0^{\arcsin\frac{-1}{k}} du\sqrt{1-k^2\sin^2 u}\right] \tag{13}$$

The parameters in $\Sigma^r_{ABM,14}$ are defined as $k = \frac{i}{a}$, $a = \frac{\Delta_0^2}{\varepsilon^2-\Delta_0^2}$, $\sin u = \tilde{x} = iax$. The integral $I\left(\arcsin\frac{-1}{k}, k\right)$ represents an elliptic function, such that its characteristic parameter $\cancel{|k|}$ can have value lesser or greater that unity. Because the ABM pairing contains the azimuthal phase factor $e^{i\phi}$, the angular integration $\int_0^{2\pi} e^{i\phi}\, d\phi = 0$. Therefore, the bare self-energy $\Sigma^r_{ABM,14}$ vanishes and off-diagonal elements satisfy $\Sigma^r_{ABM,14} = 0$. Nonzero off-diagonal elements appear only when direction dependent weighting is included, as discussed in the main text.

Tunneling Coefficients

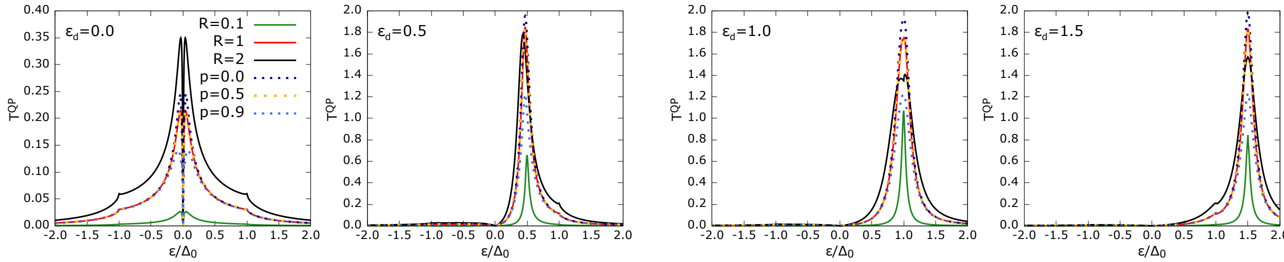
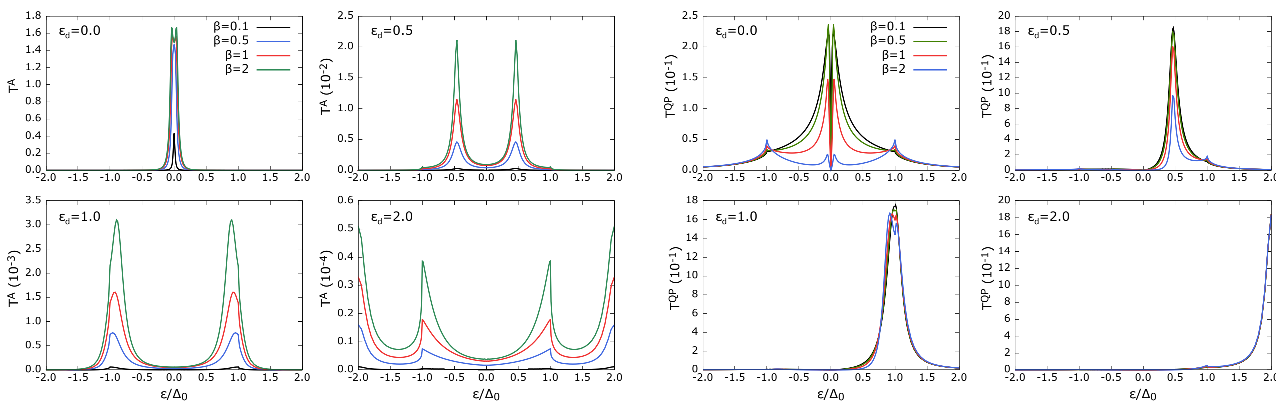


Figure S1. Tunneling coefficient (quasiparticle contribution) for the Polar state in the averaged self-energy case (without weighting), shown as a function of the tunneling electron energy ε. The Andreev reflection is absent, $T^A = 0$, due to the vanishing off-diagonal component of the self-energy, i.e., $\Sigma^r_{14} = 0$. The solid lines represent the variation of the QD–TSC coupling for fixed polarization $p = 0.5$. The dotted lines indicate the variation of the FM lead polarization for fixed coupling $R = 1$. Other parameters are set to: $U = 0$, $\Gamma_{FM} = 0.1$, $\Gamma_{SC} = 0.1$, and $k_B T = 0.1$.



(a) (b)

Figure S2. Tunneling coefficients for the Polar state in *parallel* configuration: (a) Andreev reflection and (b) quasiparticle contribution as functions of the tunneling energy ε, for the indicated values of the quantum-dot energy level ε_d. Other parameters are $U = 0$, $\Gamma_{FM} = 0.1$, $\Gamma_{SC} = 0.1$, $k_B T = 0.1$, and $p = 0.5$.

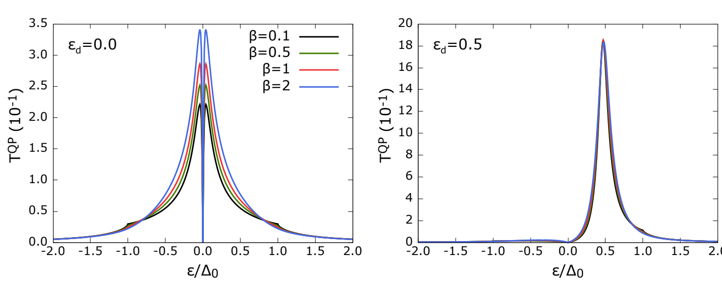


Figure S3. Tunneling coefficient for Polar state in *perpendicular* configuration as a function of energy ε of tunneling electron. The Andreev tunneling coefficients are not shown, since $T^A = 0$. Other Parameters U=0, $\Gamma_{FM} = 0.1$, $\Gamma_{SC} = 0.1$, $k_b T = 0.1$, $p = 0.5$.

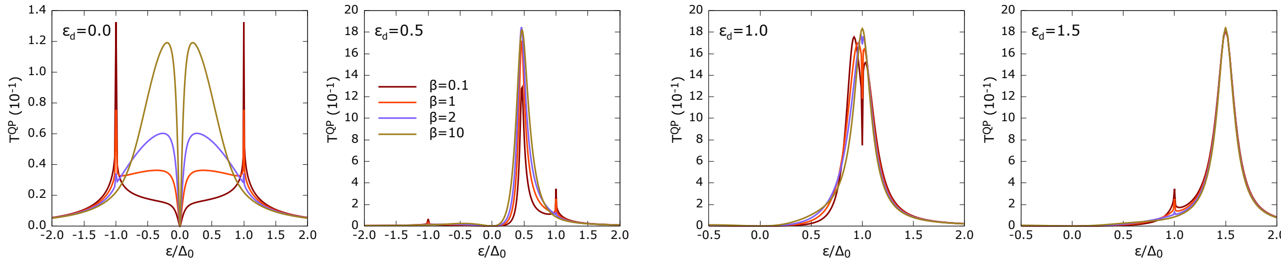
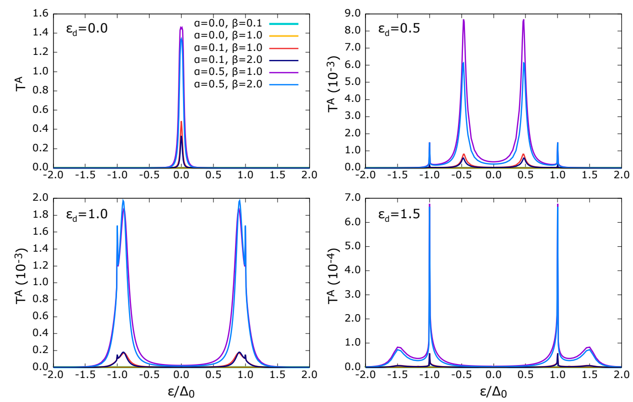
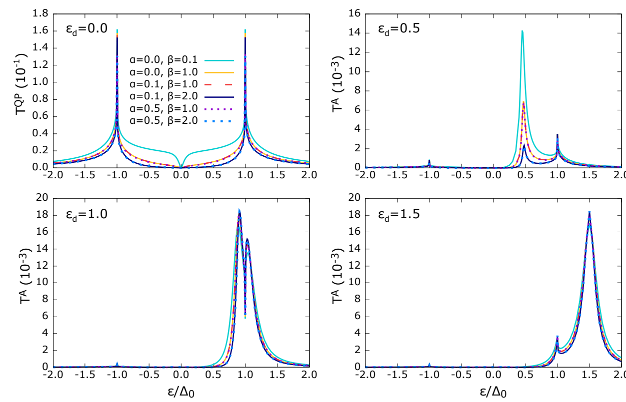


Figure S4. Tunneling coefficient for ABM state in *parallel* configuration as a function of energy ε of tunneling electron. The Andreev tunneling coefficients are not shown, since $T^A = 0$. Parameters: U=0, $\Gamma_{FM} = 0.1$, $\Gamma_{SC} = 0.1$, $k_b T = 0.1$, $p = 0.5$.



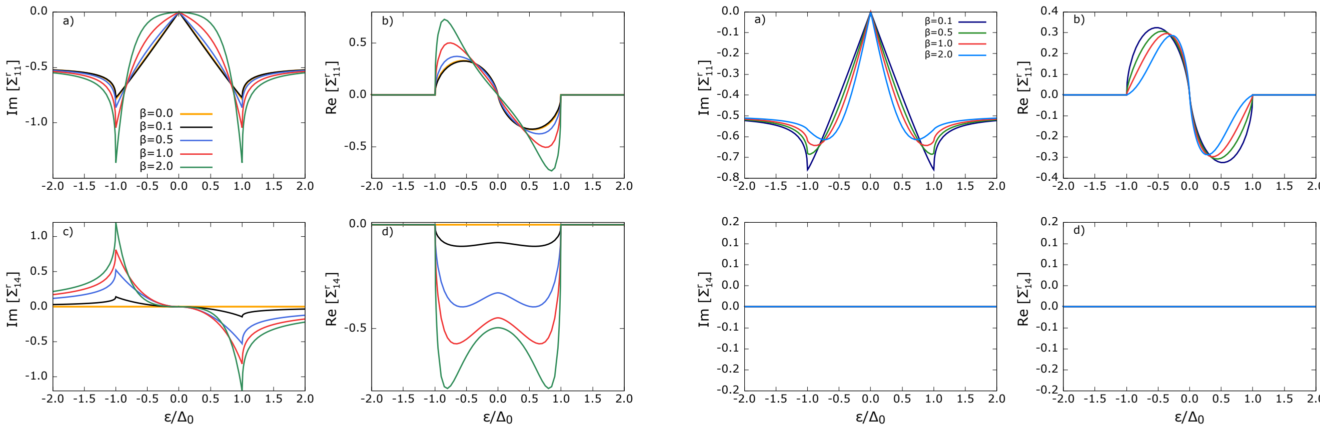
(a)



(b)

Figure S5. Tunneling coefficients for ABM state in *perpendicular* configuration for a) Andreev b) Quasiparticle tunneling as a function of energy ε of tunneling electron, for indicated values of quantum dot energy level ε_d. Other Parameters U=0, $\Gamma_{FM} = 0.1$, $\Gamma_{SC} = 0.1$, $k_b T = 0.1$, $p = 0.5$.

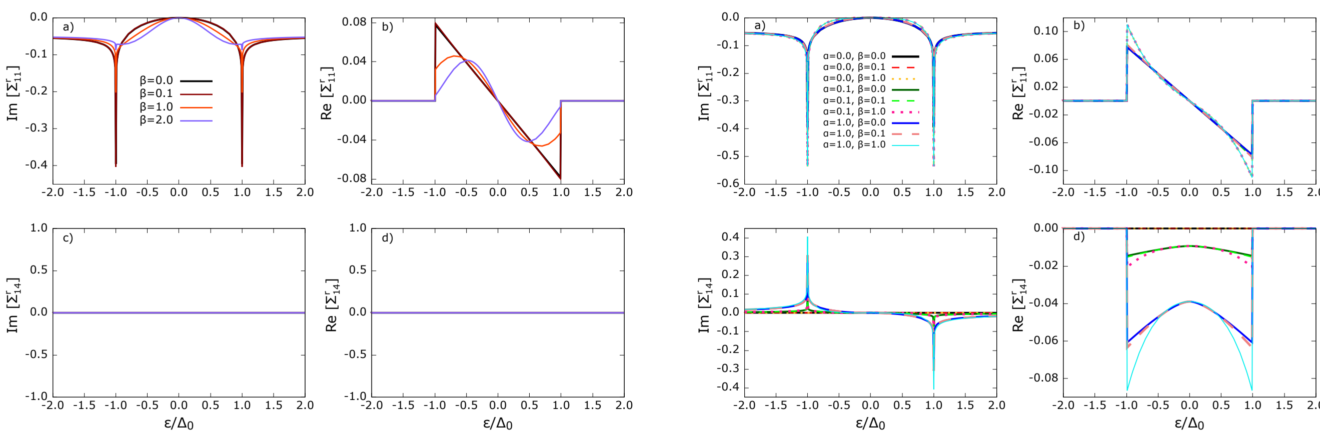
Weighted Self-Energy



Parallel

Perpendicular

Figure S6. Self-energy for *Polar* state in A) Parallel B) Perpendicular configuration for given β parameters. The α parameters are not shown for perpendicular case as the weight function cancels out with the normalization, therefore do not affect the self-energy. The 11^{th} and 14^{th} element of the self-energy in the Nambu space is shown. The finite spin-triplet Andreev reflection is associated with Σ^r_{14} being finite.



Parallel

Perpendicular

Figure S7. Self-energy for *ABM* state in A) Parallel B) Perpendicular configuration for given α, β parameters. The 11^{th} and 14^{th} element of the self-energy in the Nambu space is shown. The finite spin-triplet Andreev reflection is associated with Σ^r_{14} being finite.