Supplemental material – Parameter-free community detection by an operator on eigenvectors of the network adjacency matrix

Priya Bucha Jain,[1,2,3,*] Alexandra N. Busch,[1,2,3,*] Roberto C. Budzinski,[3,4] Christopher Steward,[2,3,5] Federico Pasini,[6] Ján Mináč,[1,3] and Lyle E. Muller[1,2,3,†]

[1] *Department of Mathematics, Western University, London ON, Canada*
[2] *Western Institute for Neuroscience, Western University, London ON, Canada*
[3] *Fields Lab for Network Computation, Fields Institute, Toronto ON, Canada*
[4] *Department of Neuroscience, University of Lethbridge, Lethbridge AB, Canada*
[5] *Department of Computer Science, Western University, London ON, Canada*
[6] *Huron University College, London ON, Canada*

I. PROOF OF THE METHOD

Given $(M+1)$ numbers $\{\alpha_i\}_{i=1}^{M}$ and β, we have a $N \times N$ block matrix

$$\mathcal{K} = \left(\begin{array}{c|c|c|c|c} \mathcal{A}_{11} & \mathcal{A}_{12} & \mathcal{A}_{13} & \cdots & \mathcal{A}_{1M} \\ \hline \mathcal{A}_{21} & \mathcal{A}_{22} & \mathcal{A}_{23} & \cdots & \mathcal{A}_{2M} \\ \hline \mathcal{A}_{31} & \mathcal{A}_{32} & \mathcal{A}_{33} & \cdots & \mathcal{A}_{3M} \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline \mathcal{A}_{M1} & \mathcal{A}_{M2} & \mathcal{A}_{M3} & \cdots & \mathcal{A}_{MM} \end{array}\right), \tag{I.1}$$

where the diagonal blocks

$$\mathcal{A}_{ii} = \begin{pmatrix} 0 & \alpha_i & \cdots & \alpha_i \\ \alpha_i & 0 & \cdots & \alpha_i \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_i & \alpha_i & \cdots & 0 \end{pmatrix} \tag{I.2}$$

are square matrices of size $N_i \times N_i$ and the off diagonal matrices

$$\mathcal{A}_{ij} = \beta \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} \tag{I.3}$$

for $i \neq j$ are matrices of dimension $N_i \times N_j$ and

$$\sum_{i=1}^{M} N_i = N.$$

We define a new $M \times M$ matrix whose (i,j) entry is the row sum of the block $\mathcal{A}_{ij}$:

$$\overline{\mathcal{K}} = \begin{pmatrix} (N_1-1)\alpha_1 & N_2\beta & \cdots & N_M\beta \\ N_1\beta & (N_2-1)\alpha_2 & \cdots & N_M\beta \\ \vdots & \vdots & \ddots & \vdots \\ N_1\beta & N_2\beta & \cdots & (N_M-1)\alpha_M \end{pmatrix} \tag{I.4}$$

---

* These authors contributed equally
† lmuller2@uwo.ca

Lemma I.1. *Eigenvalues of $\overline{\mathcal{K}}$ are also eigenvalues of $\mathcal{K}$ and if*

$$(x_1, x_2, x_3 \dots x_M)^t$$

is an eigenvector of $\overline{\mathcal{K}}$ with eigenvalue λ, then

$$(\underbrace{x_1, x_1, \dots, x_1}_{N_1}, \underbrace{x_2, x_2, \dots, x_2}_{N_2}, \dots, \underbrace{x_M, x_M, \dots, x_M}_{N_M})^t$$

is an eigenvector of $\mathcal{K}$ with eigenvalue λ.

Proof. The main theorem of [1, Theorem, Pg 2] gives us the statement about eigenvalues whereas [1, Corollary 16] proves the one on eigenvectors. □

Lemma I.2. *The eigenvalues of the matrix $\mathcal{K}$ are the eigenvalues of the matrix $\overline{\mathcal{K}}$ and the numbers $\{-\alpha_i\}_{i=1}^M$, where $-\alpha_i$ occur with multiplicity $N_i - 1$.*

Proof. By Circulant Diagonalisation and the the main theorem in [1]. □

Lemma I.3. *Under the assumption that for all $1 \leq k \leq M$*

$$|\alpha_k| > \frac{1}{N_k - 1} \max_{i \in [M]}(|\alpha_i|) + (M-1)\frac{N_k}{N_k - 1}|\beta| \tag{I.5}$$

where $[M]$ denote the natural numbers from 1 to M. The eigenvectors of $\mathcal{K}$ corresponding to the first M eigenvalues (sorted according to the absolute values) are of the form

$$(\underbrace{x_1, x_1, \dots, x_1}_{N_1}, \underbrace{x_2, x_2, \dots, x_2}_{N_2}, \dots, \underbrace{x_M, x_M, \dots, x_M}_{N_M})^t$$

Proof. Using [2, Corollary II.A.3], we have the following bound for any eigenvalue λ of $\overline{\mathcal{K}}$

$$|\lambda| \geq |(N_k - 1)\alpha_k - (M-1)N_k\beta|$$

for some k satisfying $1 \leq k \leq M$. Under the assumption, we have

$$|(N_k - 1)\alpha_k - (M-1)N_k\beta| \geq |(N_k - 1)\alpha_k| - |(M-1)N_k\beta| > \max_{i \in [M]}(|\alpha_i|). \tag{I.6}$$

Hence all eigenvalues of $\overline{\mathcal{K}}$ are bounded below by $|\alpha_i|$ for all i. Hence by Lemma I.2, the first M eigenvalues of $\mathcal{K}$ are the ones of $\overline{\mathcal{K}}$. We are done by Lemma I.1. □

Now we scramble the matrix $\mathcal{K}$ to form a new matrix $\mathcal{K}_\mathrm{s}$. This corresponds to choosing a permutation matrix P_s such that

$$\mathcal{K}_\mathrm{s} = P_s \mathcal{K} P_s^{-1}. \tag{I.7}$$

Lemma I.4. *For $\mathcal{K}_\mathrm{s}$ as defined in (I.7), the eigenvalues of $\mathcal{K}_\mathrm{s}$ are the same as eigenvalues of $\mathcal{K}$. Moreover, y is an eigenvector of $\mathcal{K}_\mathrm{s}$ with eigenvalue λ if and only if $y = P_s(x)$ for some eigenvector x of $\mathcal{K}$ with eigenvalue λ.*

Proof. The eigenvalues of $\mathcal{K}$ are the same as $\mathcal{K}_\mathrm{s}$ since they are similar. Moreover y is an eigenvector of $\mathcal{K}_\mathrm{s}$ with eigenvalue λ if and only if $\mathcal{K}_\mathrm{s}(y) = \lambda y$ if and only if $P_s \mathcal{K} P_s^{-1}(y) = \lambda y$ if and only if $\mathcal{K} P_s^{-1}(y) = \lambda P_s^{-1}(y)$. We can take x to be $P_s^{-1}(y)$ to finish off the proof. □

Notations I.5. Let σ be a permutation of n elements. Then we denote by P_σ the $n \times n$ permutation matrix whose elements are defined as

$$(P_\sigma)_{ij} = \begin{cases} 1 & \text{if } j = \sigma(i) \\ 0 & \text{otherwise.} \end{cases} \tag{I.8}$$

Any $n \times n$ permutation matrix is of the form P_σ for some permutation σ.

Lemma I.6. *If $\sigma(i) = j$, then the ith row of $P_\sigma K$ is the jth row of K and the the ith column of $K P_\sigma^t$ is the jth column of K for all matrices K.*

Lemma I.7. *Let P_σ and P_τ be permutation matrices corresponding to permutations σ and τ respectively, then $P_\sigma P_\tau$ is a permutation matrix corresponding to the permutation $\sigma \circ \tau$. The inverse of P_σ is also a permutation matrix corresponding to the permutation σ^{-1}.*

Lemma I.8. *Let y be a vector of the form*

$$(\underbrace{x_1, x_1, \ldots, x_1}_{N_1}, \underbrace{x_2, x_2, \ldots, x_2}_{N_2}, \ldots, \underbrace{x_M, x_M, \ldots, x_M}_{N_M})^t$$

where $x_i \neq x_j$ if $i \neq j$. Let Q be a $N \times N$ permutation matrix such that

$$Q(y) = y$$

Then Q is a block diagonal matrix of the form

$$Q = \left(\begin{array}{c|c|c|c|c} P_1 & 0 & 0 & \cdots & 0 \\ \hline 0 & P_2 & 0 & \cdots & 0 \\ \hline 0 & 0 & P_3 & \cdots & 0 \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & 0 & \cdots & P_M \end{array}\right), \tag{I.9}$$

where each of the P_i is a permutation matrix of size $N_i \times N_i$

Proof. Let us fix the notation

$$L_0 = 0$$

$$L_k = \sum_{i=1}^{k} N_k$$

Let σ be the N permutation such that $Q = P_\sigma$ and let y_l be the lth component of y for $1 \leq l \leq N$. In particular, this means $Q(y_l) = y_{\sigma(l)}$. Since $Q(y) = y$, $y_l = y_{\sigma(l)}$ for all l. Hence if $L_{k-1} < l \leq L_k$ for some k and $L_{m-1} < \sigma(l) \leq L_m$ for some m, $x_k = y_l = y_{\sigma(l)} = x_m$ which implies $k = m$ by hypothesis. Thus, σ permutes the set

$$\{L_{k-1} + 1, L_{k-1} + 2, L_{k-1} + 3, \ldots, L_k\}$$

into itself for k from 1 to M. This can be concisely restated as the equation

$$\sigma = \overline{\sigma_1} \circ \overline{\sigma_2} \circ \overline{\sigma_3} \ldots \overline{\sigma_M}$$

where each $\overline{\sigma_k}$ acts on the set

$$\{L_{k-1} + 1, L_{k-1} + 2, L_{k-1} + 3, \ldots, L_k\}$$

by a N_k permutation σ_k and leaves all other elements unchanged. Now, by definition

$$P_{\overline{\sigma_1}} = \left(\begin{array}{c|c|c|c|c} P_{\sigma_1} & 0 & 0 & \cdots & 0 \\ \hline 0 & Id_{N_2} & 0 & \cdots & 0 \\ \hline 0 & 0 & Id_{N_3} & \cdots & 0 \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & 0 & \cdots & Id_{N_M} \end{array}\right), \tag{I.10}$$

$$P_{\overline{\sigma_2}} = \left(\begin{array}{c|c|c|c|c} Id_{N_1} & 0 & 0 & \cdots & 0 \\ \hline 0 & P_{\sigma_2} & 0 & \cdots & 0 \\ \hline 0 & 0 & Id_{N_3} & \cdots & 0 \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & 0 & \cdots & Id_{N_M} \end{array}\right), \tag{I.11}$$

$$\vdots$$

$$P_{\overline{\sigma_M}} = \left(\begin{array}{c|c|c|c|c} Id_{N_1} & 0 & 0 & \cdots & 0 \\ \hline 0 & Id_{N_2} & 0 & \cdots & 0 \\ \hline 0 & 0 & Id_{N_3} & \cdots & 0 \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & 0 & \cdots & P_{\sigma_M} \end{array}\right). \tag{I.12}$$

So, we have

$$Q = P_{\overline{\sigma_1}} P_{\overline{\sigma_2}} P_{\overline{\sigma_3}} \dots P_{\overline{\sigma_M}} = \left(\begin{array}{c|c|c|c|c} P_{\sigma_1} & 0 & 0 & \cdots & 0 \\ \hline 0 & P_{\sigma_2} & 0 & \cdots & 0 \\ \hline 0 & 0 & P_{\sigma_3} & \cdots & 0 \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & 0 & \cdots & P_{\sigma_M} \end{array}\right) \tag{I.13}$$

by I.7. □

Given a M permutation σ, we define a new $N \times N$ block matrix

$$\mathcal{K}^\sigma = \left(\begin{array}{c|c|c|c|c} \mathcal{A}^\sigma_{11} & \mathcal{A}^\sigma_{12} & \mathcal{A}^\sigma_{13} & \cdots & \mathcal{A}^\sigma_{1M} \\ \hline \mathcal{A}^\sigma_{21} & \mathcal{A}^\sigma_{22} & \mathcal{A}^\sigma_{23} & \cdots & \mathcal{A}^\sigma_{2M} \\ \hline \mathcal{A}^\sigma_{31} & \mathcal{A}^\sigma_{32} & \mathcal{A}^\sigma_{33} & \cdots & \mathcal{A}^\sigma_{3M} \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline \mathcal{A}^\sigma_{M1} & \mathcal{A}^\sigma_{M2} & \mathcal{A}^\sigma_{M3} & \cdots & \mathcal{A}^\sigma_{MM} \end{array}\right), \tag{I.14}$$

where the diagonal blocks

$$\mathcal{A}^\sigma_{ii} = \begin{pmatrix} 0 & \alpha_{\sigma(i)} & \cdots & \alpha_{\sigma(i)} \\ \alpha_{\sigma(i)} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{\sigma(i)} & \alpha_{\sigma(i)} & \cdots & 0 \end{pmatrix} \tag{I.15}$$

are square matrices of size $N_{\sigma(i)} \times N_{\sigma(i)}$ and the off diagonal matrices

$$\mathcal{A}^\sigma_{ij} = \beta \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} \tag{I.16}$$

for $i \neq j$ are matrices of dimension $N_{\sigma(i)} \times N_{\sigma(j)}$.

Lemma I.9. *Let*

$$x = (\underbrace{x_1, x_1, \dots, x_1}_{N_1}, \underbrace{x_2, x_2, \dots, x_2}_{N_2}, \dots, \underbrace{x_M, x_M, \dots, x_M}_{N_M})^t$$

and

$$y = (\underbrace{y_1, y_1, \dots, y_1}_{N'_1}, \underbrace{y_2, y_2, \dots, y_2}_{N'_2}, \dots, \underbrace{y_M, y_M, \dots, y_M}_{N'_M})^t$$

be permutations of each other. Then there exists a M permutation σ and a $N \times N$ permutation matrix $\mathcal{P}$ such that

1. $\mathcal{P}(x) = y$
2. $\mathcal{P}\,\mathcal{K}\,\mathcal{P}^{-1} = \mathcal{K}^\sigma$

Proof. Since x and y are permutations of each other, we can choose a M permutation σ such that

1. $N'_i = N_{\sigma(i)}$
2. $y_i = x_{\sigma(i)}$

Then we define a $N \times N$ block matrix $\mathcal{P}$

$$\mathcal{P} = \left(\begin{array}{c|c|c|c|c} P_{11} & P_{12} & P_{13} & \cdots & P_{1M} \\ \hline P_{21} & P_{22} & P_{23} & \cdots & P_{2M} \\ \hline P_{31} & P_{32} & P_{33} & \cdots & P_{3M} \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline P_{M1} & P_{M2} & P_{M3} & \cdots & P_{MM} \end{array}\right), \tag{I.17}$$

where each block P_{ij} is a matrix of dimension $N_{\sigma(i)} \times N_j$ and is defined by the equation

$$P_{ij} = \begin{cases} Id_j & \text{if } \sigma(i) = j \\ 0 & \text{otherwise} \end{cases} \tag{I.18}$$

Then, we have

$$\begin{aligned} \mathcal{P}\,x &= \left(\begin{array}{c|c|c|c|c} P_{11} & P_{12} & P_{13} & \cdots & P_{1M} \\ \hline P_{21} & P_{22} & P_{23} & \cdots & P_{2M} \\ \hline P_{31} & P_{32} & P_{33} & \cdots & P_{3M} \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline P_{M1} & P_{M2} & P_{M3} & \cdots & P_{MM} \end{array}\right) (\underbrace{x_1, x_1, \ldots, x_1}_{N_1}, \underbrace{x_2, x_2, \ldots, x_2}_{N_2}, \ldots, \underbrace{x_M, x_M, \ldots, x_M}_{N_M})^t \\ &= (\underbrace{x_{\sigma(1)}, x_{\sigma(1)}, \ldots, x_{\sigma(1)}}_{N_{\sigma(1)}}, \underbrace{x_{\sigma(2)}, x_{\sigma(2)}, \ldots, x_{\sigma(2)}}_{N_{\sigma(2)}}, \ldots, \underbrace{x_{\sigma(M)}, x_{\sigma(M)}, \ldots, x_{\sigma(M)}}_{N_{\sigma(M)}})^t \\ &= (\underbrace{y_1, y_1, \ldots, y_1}_{N'_1}, \underbrace{y_2, y_2, \ldots, y_2}_{N'_2}, \ldots, \underbrace{y_M, y_M, \ldots, y_M}_{N'_M})^t \\ &= y. \end{aligned}$$

where the second last equality follows from our assumption on σ. We now prove that

$$\mathcal{P}\,\mathcal{K}\,\mathcal{P}^{-1} = \mathcal{K}^{\sigma} \tag{I.19}$$

We first note that, since $\mathcal{P}$ is a permutation matrix, $\mathcal{P}^{-1}$ is just the transpose $\mathcal{P}^t$. Thus it is a block matrix of the form

$$\mathcal{P}^{-1} = \left(\begin{array}{c|c|c|c|c} P'_{11} & P'_{12} & P'_{13} & \cdots & P'_{1M} \\ \hline P'_{21} & P'_{22} & P'_{23} & \cdots & P'_{2M} \\ \hline P'_{31} & P'_{32} & P'_{33} & \cdots & P'_{3M} \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline P'_{M1} & P'_{M2} & P'_{M3} & \cdots & P'_{MM} \end{array}\right), \tag{I.20}$$

where each block P'_{ij} is a $N_i \times N_{\sigma(j)}$ matrix defined by the equation

$$P'_{ij} = \begin{cases} Id_i & \text{if } \sigma(j) = i \\ 0 & \text{otherwise} \end{cases} \tag{I.21}$$

Hence, we have that $\mathcal{P}\,\mathcal{K}\,\mathcal{P}^{-1}$ is a $N \times N$ block matrix

$$\mathcal{P}\,\mathcal{K}\,\mathcal{P}^{-1} = \left(\begin{array}{c|c|c|c|c} \mathcal{A}'_{11} & \mathcal{A}'_{12} & \mathcal{A}'_{13} & \cdots & \mathcal{A}'_{1M} \\ \hline \mathcal{A}'_{21} & \mathcal{A}'_{22} & \mathcal{A}'_{23} & \cdots & \mathcal{A}'_{2M} \\ \hline \mathcal{A}'_{31} & \mathcal{A}'_{32} & \mathcal{A}'_{33} & \cdots & \mathcal{A}'_{3M} \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline \mathcal{A}'_{M1} & \mathcal{A}'_{M2} & \mathcal{A}'_{M3} & \cdots & \mathcal{A}'_{MM} \end{array}\right), \tag{I.22}$$

whose diagonal blocks $\mathcal{A}'_{ii}$ are matrices of dimension $N_{\sigma(i)} \times N_{\sigma(i)}$ defined by the equation

$$\begin{aligned}\mathcal{A}'_{ii} &= \sum_{j,l=1}^{M} \mathcal{P}_{ij}\, \mathcal{A}_{jl}\, \mathcal{P}^{-1}_{li} \\ &= \mathcal{A}_{\sigma(i)\sigma(i)} \\ &= \mathcal{A}^{\sigma}_{ii}\,.\end{aligned}$$

and the off diagonal blocks $\mathcal{A}'_{ij}$ are matrices of dimension $N_{\sigma(i)} \times N_{\sigma(j)}$ defined by the equation

$$\begin{aligned}\mathcal{A}'_{ij} &= \sum_{l,k=1}^{M} \mathcal{P}_{il}\, \mathcal{A}_{lk}\, \mathcal{P}^{-1}_{kj} \\ &= \mathcal{A}_{\sigma(i)\sigma(j)} \\ &= \mathcal{A}^{\sigma}_{ij}\end{aligned}$$

for $i \neq j$. This proves Equation (I.19). □

Lemma I.10. *Let P_i be an $N_i \times N_i$ permutation matrix. Let $\mathcal{A}_{ii}$ be a matrix as in Equation (I.2). Then*

$$P_i\, \mathcal{A}_{ii}\, P_i^{-1} = \mathcal{A}_{ii}\,.$$

Proof. Let σ be the N_i permutation such that $P_i = P_\sigma$ (refer Notation I.5). Then we have that

$$\begin{aligned}(P_i\, \mathcal{A}_{ii}\, P_i^{-1})_{jk} &= (P_\sigma\, \mathcal{A}_{ii}\, P_\sigma^{-1})_{jk} \\ &= \sum_{l,m} (P_\sigma)_{jl} (\mathcal{A}_{ii})_{lm} (P_\sigma^{-1})_{mk} \\ &= (P_\sigma)_{j\sigma(j)} (\mathcal{A}_{ii})_{\sigma(j)\sigma(k)} (P_\sigma^{-1})_{\sigma(k)k} \\ &= (\mathcal{A}_{ii})_{\sigma(j)\sigma(k)}.\end{aligned}$$

For $j \neq k$, $(P_i\, \mathcal{A}_{ii}\, P_i^{-1})_{jk} = (\mathcal{A}_{ii})_{\sigma(j)\sigma(k)} = \alpha = (\mathcal{A}_{ii})_{jk}$. On the other hand, $(P_i\, \mathcal{A}_{ii}\, P_i^{-1})_{jj} = (\mathcal{A}_{ii})_{\sigma(j)\sigma(j)} = 0 = (\mathcal{A}_{ii})_{jj}$.

□

Lemma I.11. *Let Q be a block diagonal matrix of the form*

$$Q = \left(\begin{array}{c|c|c|c|c} P_1 & 0 & 0 & \cdots & 0 \\ \hline 0 & P_2 & 0 & \cdots & 0 \\ \hline 0 & 0 & P_3 & \cdots & 0 \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & 0 & \cdots & P_M \end{array}\right), \tag{I.23}$$

where each of the P_i is a $N_i \times N_i$ permutation matrix. Then for a matrix $\mathcal{K}$ as in Eq. (I.1),

$$Q\, \mathcal{K}\, Q^{-1} = \mathcal{K}$$

Proof. By product of block matrices, we have that $Q\, \mathcal{K}\, Q^{-1}$ is of the form

$$\mathcal{K} = \left(\begin{array}{c|c|c|c|c} P_1\, \mathcal{A}_{11}\, P_1^{-1} & P_1\, \mathcal{A}_{12}\, P_2^{-1} & P_1\, \mathcal{A}_{13}\, P_3^{-1} & \cdots & P_1\, \mathcal{A}_{1M}\, P_M^{-1} \\ \hline P_2\, \mathcal{A}_{21}\, P_1^{-1} & P_2\, \mathcal{A}_{22}\, P_2^{-1} & P_2\, \mathcal{A}_{23}\, P_3^{-1} & \cdots & P_2\, \mathcal{A}_{2M}\, P_M^{-1} \\ \hline P_3\, \mathcal{A}_{31}\, P_1^{-1} & P_3\, \mathcal{A}_{32}\, P_2^{-1} & P_3\, \mathcal{A}_{33}\, P_3^{-1} & \cdots & P_3\, \mathcal{A}_{3M}\, P_M^{-1} \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline P_M\, \mathcal{A}_{M1}\, P_1^{-1} & P_M\, \mathcal{A}_{M2}\, P_2^{-1} & P_M\, \mathcal{A}_{M3}\, P_3^{-1} & \cdots & P_M\, \mathcal{A}_{MM}\, P_M^{-1} \end{array}\right), \tag{I.24}$$

The diagonal blocks are $P_i\, \mathcal{A}_{ii}\, P_i^{-1}$ and hence $\mathcal{A}_{ii}$ by I.10. Moreover, for $i \neq j$, all elements of the off diagonal blocks $\mathcal{A}_{ij}$ are the same, implying that any permutations leave $\mathcal{A}_{ij}$ invariant. Hence the off-diagonal matrices are $P_i\, \mathcal{A}_{ij}\, P_j^{-1}$ $= \mathcal{A}_{ij}$. □

Theorem I.12. *Let y' be a linear combination of 1st M eigenvectors (eigenvectors corresponding to 1st M eigenvalues) of $\mathcal{K}_s$. Then by I.4, there exists x such that $P_s(x) = y'$ and x is a linear combination of leading M eigenvectors (eigenvectors corresponding to leading M eigenvalues sorted according to their absolute values) of $\mathcal{K}$. Under the assumption*

$$|\alpha_k| > \frac{1}{N_k - 1} \max_{i \in [M]}(|\alpha_i|) + (M-1)\frac{N_k}{N_k - 1}|\beta|,$$

x is of the form

$$(\underbrace{x_1, x_1, \ldots, x_1}_{N_1}, \underbrace{x_2, x_2, \ldots, x_2}_{N_2}, \ldots, \underbrace{x_M, x_M, \ldots, x_M}_{N_M})^t$$

by Lemma I.3. Furthermore, we assume we have chosen y' such that $x_i \neq x_j$ for $i \neq j$. Now let T be a permutation matrix such that the entries $y = Ty'$ is sorted in an ascending order. Then

$$T\,\mathcal{K}_s\,T^{-1} = \mathcal{K}^\sigma$$

for some M permutation σ.

Proof. We have $y = Ty' = TP_s(x)$ and y is of the form

$$(\underbrace{y_1, y_1, \ldots, y_1}_{N'_1}, \underbrace{y_2, y_2, \ldots, y_2}_{N'_2}, \ldots, \underbrace{y_M, y_M, \ldots, y_M}_{N'_M})^t.$$

By Lemma I.9, we know the existence of a M permutation σ and a permutation matrix $\mathcal{P}$ such that $\mathcal{P}(x) = y$. Hence, we have

$$\mathcal{P}^{-1}\,TP_s(x) = x$$

Now by Lemma I.8, we have

$$\mathcal{P}^{-1}\,TP_s = Q$$

where Q is of the form (I.9). That is

$$TP_s = \mathcal{P}\,Q \tag{I.25}$$

We have

$$\mathcal{P}\,Q\,\mathcal{K}(\mathcal{P}\,Q)^{-1} = \mathcal{P}\,Q\,\mathcal{K}\,Q^{-1}\,\mathcal{P}^{-1} = \mathcal{K}^\sigma \tag{I.26}$$

by Lemma I.11 and property of $\mathcal{P}$. Thus, we get

$$T\,\mathcal{K}_s\,T^{-1} = TP_s\,\mathcal{K}(TP_s)^{-1} = \mathcal{P}\,Q\,\mathcal{K}\,Q^{-1}\,\mathcal{P}^{-1} = \mathcal{K}^\sigma \tag{I.27}$$

by (I.25) and (I.26). □

II. ESTIMATING THE NUMBER OF COMMUNITIES

Let $\mathcal{K}$ be as in (I.1) with $N_i = N$ for all i and $\alpha_i = \alpha$ for all i. Let $\{\lambda_1, \lambda_2, \ldots \lambda_{NM}\}$ be the eigenvalues of $\mathcal{K}$ sorted in a descending order by their absolute values. We introduce a new $NM \times NM$ matrix L whose $(i,j)th$ entry is defined by

$$L_{ij} = \frac{1}{|\lambda_i - \lambda_j| + 1}. \tag{II.1}$$

We also introduce the *row sum* vector R and *difference* vector D. R is a $NM \times 1$ vector defined by

$$R_i = \sum_{j=1}^{NM} T_{ij} \tag{II.2}$$

whereas D is a $(NM-1) \times 1$ vector defined by

$$D_i = R_i - R_{i+1} \tag{II.3}$$

With the above notations in mind, we have the following

Lemma II.1. *Assuming that (I.5) holds, the index corresponding to the largest (in absolute value) entry of D_i is M.*

Proof. By Lemma I.2 and Lemma I.3, the eigenvalues λ_i for i from $M+1$ to MN are all $-\alpha$. The eigenvalues λ_i for i from 1 to M on the other hand are eigenvalues of the matrix $\overline{\mathcal{K}}$ which again is a circulant matrix in this case. It follows from the Circulant Diagonalisation Theorem that $\lambda_1 = (M-1)\alpha + N(M-1)\beta$ and $\lambda_i = (M-1)\alpha + (NM-N)\beta$ for i from 2 through to M. Hence the only nonzero entries of D are at index 1 and index M. Furthermore, we have

$$\begin{aligned} D_M = (M-1) &+ \frac{1}{|\lambda_1 - \lambda_2| + 1} + \frac{NM-N}{|\lambda_2 - \lambda_{M+1}| + 1} \\ &- \frac{M-1}{|\lambda_2 - \lambda_{M+1}|} - \frac{1}{|\lambda_1 - \lambda_M| + 1} - (NM-N) \\ &> (N-1)(M-1) \end{aligned}$$

and

$$\begin{aligned} D_1 = 1 &+ \frac{M-11}{|\lambda_1 - \lambda_2| + 1} + \frac{NM-N}{|\lambda_1 - \lambda_{M+1}| + 1} \\ &- (M-1) - \frac{1}{|\lambda_1 - \lambda_2| + 1} - \frac{NM-N}{|\lambda_2 - \lambda_{M+1}| + 1} \\ &< (N-1)(M-1). \end{aligned}$$

This finishes the proof. □

III. NOISE AND ERROR

Consider a network with N nodes and apparent community structure, as discussed in the previous sections. We now make the adjacency matrix K noisy by adding to each element some noise drawn from a standard normal distribution, η. Formally,

$$\mathcal{K}_\delta = \mathcal{K} + \delta F,$$

where $\mathcal{K}_\delta$ is the adjacency matrix after adding noise, δ is a non-negative real number and F is a matrix of order N with elements drawn from a standard normal distribution.

We quantify the amount of added noise with respect to the difference between the inter-community and intra-community connection strengths. Noise levels that are small compared to this difference largely preserve the community structure, but noise on the order of this difference would disrupt the structure completely. Here, we add increasing amounts of noise, ranging from very low ($\delta = 0.05$) to high ($\delta = 0.5$), while remaining in the regime where community structure is preserved so that the approach is still applicable.

We then randomly relabel the nodes, so that the adjacency matrix loses its distinct block structure. The method (Section II of that main text) is then applied to these matrices; i.e. we sort the sum of eigenvectors corresponding to the largest M eigenvalues of these scrambled matrices where M is obtained by using the method in section III of the main text and apply the same sorting operator to the matrices.

Importantly, the choice of noise regime was not arbitrary. It was guided by a threshold in the overlap score, which as its name suggests, quantifies the extent of overlap between inter- and intra-connection strengths (Fig. S1). We first define a common range to identify the global minimum and maximum values across all inter- and intra-samples. This common range is then discretized into bins of fixed width (10^{-6}). The overlap was calculated for each noise level as the sum of counts in bins that contained contributions from both inter- and intra-distributions, and normalized by the total number of samples. This yielded a score between 0 and 1, where higher values indicate greater overlap and hence lower separability of inter- and intra-connections.

Having established how the noise regime is determined, we now turn to quantifying the performance of the method under these conditions. To do so, we formalize the notion of error in order to better understand its relationship with noise. The most natural definition of error, denoted by ϵ, is the fraction of nodes that are misplaced:

$$\epsilon = \frac{\text{Number of misplaced nodes}}{\text{Total number of nodes}}.$$

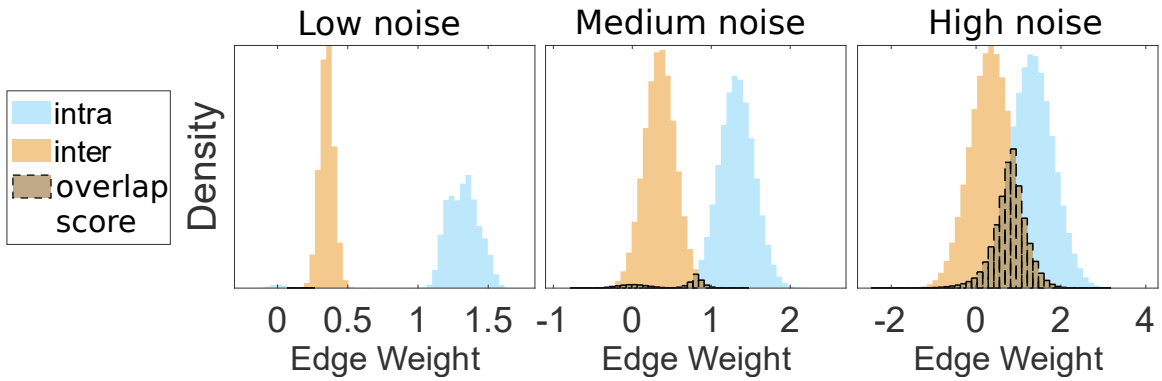

Figure S1. **Visual representation of overlap score at various noise levels** In the network of Figure 4 of the main text, an overlap score is computed to quantify the extent of overlap between intra and inter-connection strengths.

IV. LFR BENCHMARK

The LFR (Lancichinetti–Fortunato–Radicchi) Benchmark, introduced by Andrea Lancichinetti, Santo Fortunato, and Francesco Radicchi in their 2008 paper [3], is one of the most widely used synthetic benchmarks for evaluating community detection algorithms. The LFR benchmark extends earlier models like the Girvan–Newman benchmark [4] by introducing heterogeneity in the distributions of node degree and community size to reflect the structure of real-world complex networks more realistically. In particular, both the degree distribution and the community size distribution follow a power law, capturing the heavy-tailed nature commonly observed in empirical networks.

The mixing parameter for topology μ_t controls the fraction of each node's edges that connect to nodes outside its community. Lower values of μ_t indicate well-separated communities, while higher values correspond to overlapping communities. The minus exponent for the degree sequence, t_1, determines the shape of the power-law degree distribution. Smaller values of t_1 result in a heavier tail, which means the network will contain more high-degree nodes or hubs. Similarly, the minus exponent for the community size distribution t_2 governs the skewness in the distribution of community sizes, with smaller values producing more large communities and a broader range of sizes. The mixing parameter for weights μ_w controls how the total edge weight of a node is distributed between its own community and the rest of the network.

We applied our method to a wide range of parameters of the LFR Benchmark and the model performed well in all the cases. (Fig. S2)

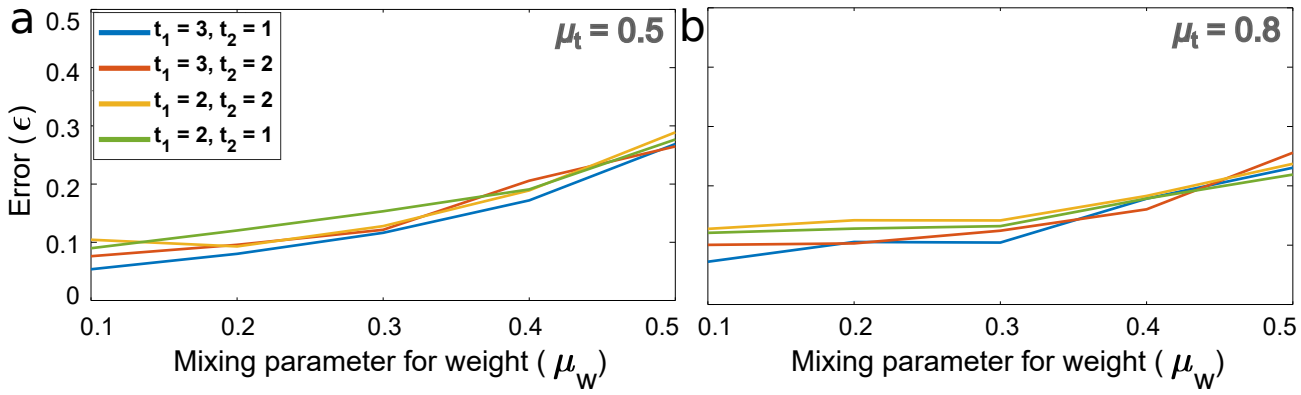

Figure S2. The method does well on applicable cases of the LFR benchmark. Specifically, changing parameters: Mixing parameter for topology (**(a)** $\mu_t = 0.5$, **(b)** $\mu_t = 0.8$), minus exponent for the degree sequence (t_1), minus exponent for the community size distribution (t_2) has little effect on the error when plotted against mixing parameter for weights (μ_w).

V. SCANS

Next, we quantified the effect of each parameter on the accuracy of the approach (Fig. S3). To do so, we followed a similar simulation procedure as outlined above, this time varying one parameter at a time, while fixing all others. Fig. S3(a) demonstrates how error changes with intra-connection strength, α. We start with a network of size, $N = 600$ with $M = 6$ communities of sizes 100 each. The inter-connection strength, β is 0.2 and the intra-connection strength, α is same throughout different communities. We add a noise level, $\delta = 0.1$. As expected, the error decrease with increase in α. Larger α means intra-connections weights are much stronger than inter-connections weights and so the community structure is much stronger. Mathematically, larger α implies larger difference between α and β and less noise percentage with respect to this difference.

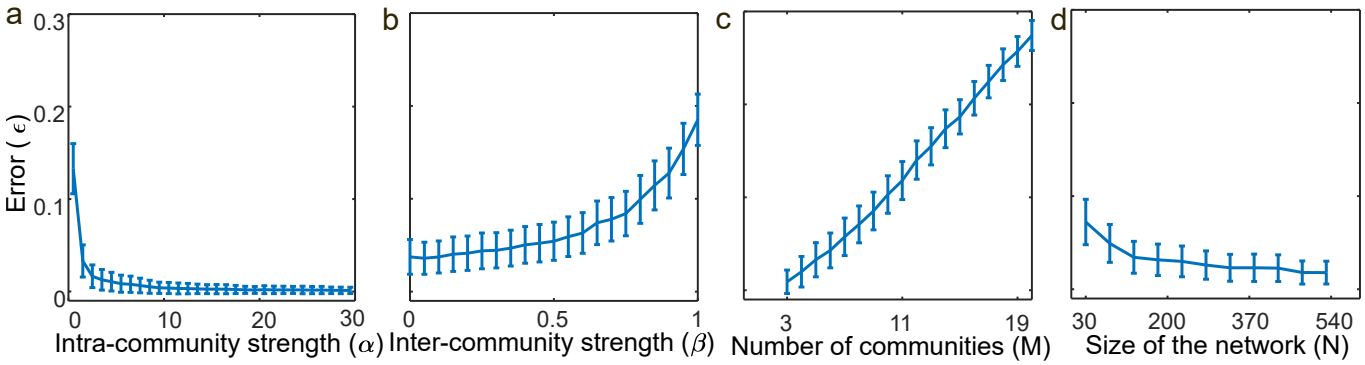

Figure S3. **Relation between various parameters and error**: A network with 6 communities of size 100 each with intra-community connection strength 1.2, inter-community connection strength 0.2 and noise level of 0.1 is used with varying one parameter at a time. Error is plotted against y-axis with. **(a)** Along the x-axis are the various intra-community strengths α. The error decreases as α increases, since a larger gap between α and β makes the communities more distinct. **(b)** Along the x-axis are the various inter-community strengths β. The error increases as β increases, since a larger gap between α and β makes the communities less distinct.**(c)** Along the x-axis are the various number of communities. The error increases as the number of communities increases since the inequality (I.5) is violated. **(d)** Along the x-axis are the various community sizes. Error decreases as community size increases.

In Fig. S3(b), we see how error changes with inter-connection strength, β. We start with a network of size, $N = 600$ with $M = 6$ communities of sizes 100 each. The intra-connection strength, α is 1.2 and the intra-connection strength, β is uniform throughout the network. We add a noise level, $\delta = 0.1$. The error increases with increase in β. Increase in β means inter-connections weights are getting stronger but the intra-connections weights remains the same and so the community structure is getting weaker. In other words, larger β implies smaller difference between α and β and high noise percentage with respect to this difference. In Fig. S3(c), we see how error changes with number of communities, M. We start with a network of size, $N = 600$ with M communities of sizes 100 each. The intra-connection strength, α is 1.2 and the intra-connection strength, β is 0.2. We add a noise level, $\delta = 0.1$. The error increases with increase in M. This can be easily explained by the inequality (I.5) as increasing M while keeping α and β constant does not satisfy the bounds in this condition.

In Fig. S3(d), we see how error changes with community size, N. We start with a network of size, N with $M = 6$ communities of sizes 100 each. The intra-connection strength, α is 1.2 and the intra-connection strength, β is 0.2. We add a noise level, $\delta = 0.1$. The error decreases with increase in N. Increasing N while keeping M constant increases the community sizes N_k. This decreases the quantity $\frac{1}{N_k - 1} \max_{i \in [M]}(|\alpha_i|)$. This, in turn implies that the inequality (I.5) is satisfied more often, thus improving the efficacy of the approach.

VI. RANDOM GRAPHS

When applied on random graph obtained from LFR benchmark graphs(Fig. S4a), the method detects only one community in all the 100 iterations (Fig. S4d). Hence, only the first eigenvector is used for sorting and there are no sharp jumps in the eigenvector (Fig. S4b). This shows that our approach is robust and doesn't detect communities where there are none.

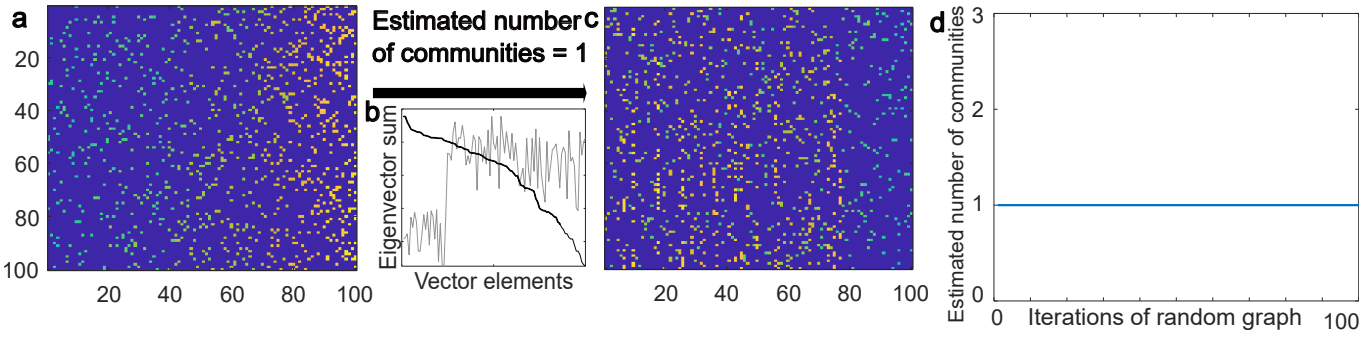

Figure S4. **Community detection on random graphs (a)** We begin with a random graph of 100 nodes generated using the LFR benchmark **(b)** The estimated number of communities in this case is 1**(c)** Along the x-axis are the various number of communities. The error increases as the number of communities increases since the inequality (I.5) is violated. **(d)** In all 100 iterations, only a single community is detected, demonstrating the robustness of our approach.

VII. "STEP 2"

Our method reorders nodes according to the sorted sum of M leading eigenvectors, where nodes belonging to the same community are positioned next to one another and community boundaries are revealed by "jumps" in the value of the sorted sum. The intuition behind this approach is that, when nodes are ordered in this way, each of the leading eigenvectors displays block structure that distinguishes one or more of the communities (see for example Fig. 6a,b of the main text). Single eigenvectors rarely describe the full community structure, which instead is revealed more robustly through the combination of M leading eigenvectors. However, in cases where this sum is ambiguous (for example, due to the presence of noise) (Figs. S5a, b), individual eigenvectors may help to refine the ordering of nodes (Figs. S5c, d). We demonstrate this idea by introducing a simple "Step 2", which uses targeted resorting to further align nodes belonging to the same community (Section 5 of the main text).

"Step 2" inspects the ordering at a finer scale by analysing small spectral "gaps" i.e., differences between consecutive values of the sorting variable: small differences correspond to internally homogeneous portions of a community, while larger jumps signal potential boundaries between communities. Using a percentile threshold, we identify these tentative boundaries and partition the sorted node list into segments. Within each segment, we measure the standard deviation of the first eigenvector's entries, which acts as a good marker of whether nodes in that segment belong to the same underlying community. Segments with unusually large variation compared to others are flagged as potentially containing inter-community mixing. For these flagged segments, we perform a local re-sorting using only the first eigenvector, which provides a finer separation of nodes (Fig. S5e). Segments that show no such irregularity are left unchanged. The output of this refined procedure is a permutation that improves the block-structure alignment of the adjacency matrix beyond what is achieved in the initial method (Fig. S5f). In our current implementation, we consider only the leading eigenvector for simplicity, however, inspecting each of the M eigenvectors contributing to the sum could further refine this procedure.

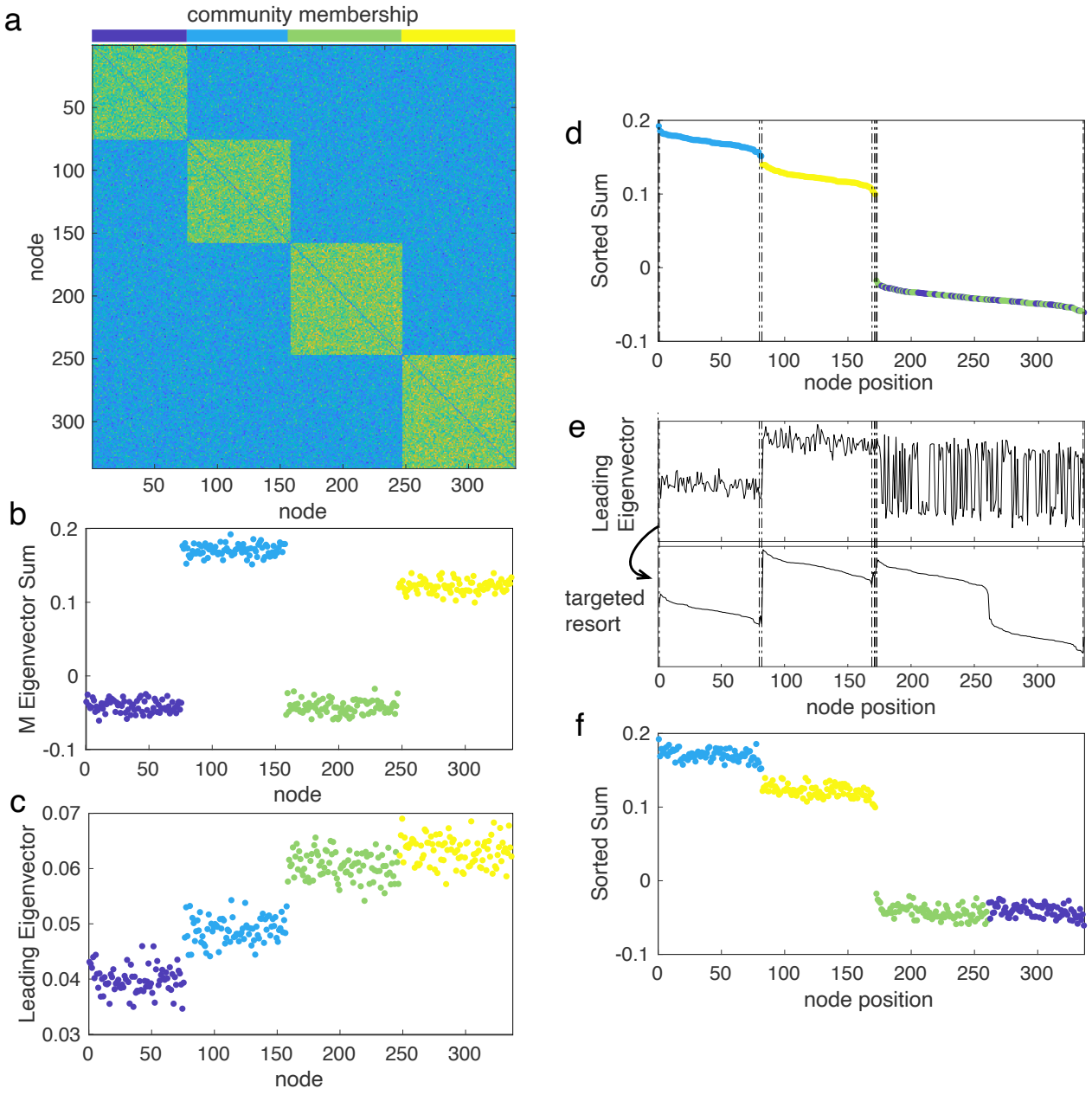

Figure S5. **Step 2: targeted resorting. (a)** The adjacency matrix of an example network with four communities (where the community membership of each node is indicated in colour code below the plot). **(b)** The sum of the 4 leading eigenvectors, plotted with each entry coloured according to the corresponding node's community membership. Here, the purple and green communities have similar values in the eigenvector sum: sorting thus obscures these two communities (as in panel d). **(c)** However, these two communities are well-separated in the first eigenvector. Here, the leading eigenvector is plotted with entries coloured by the corresponding node's community membership. We can leverage this separation to re-sort the section of the eigenvector sum corresponding to the purple and green communities, thus distinguishing the two communities. **(d)** The sorted sum of the 4 leading eigenvectors (from panel b), plotted with entries coloured by the corresponding node's community membership. Here, the blue and yellow communities are clustered as desired, but the purple and green communities are mixed together, as expected from panel b. Dashed lines indicate large jumps between consecutive entries of the sorted sum (exceeding a percentile threshold). These tentative boundaries are used to identify regions where a targeted resorting may be beneficial. **(e)** The leading eigenvector displays large variance in the segment corresponding to the green and purple nodes (as expected from panel c). Resorting within each segment (dashed lines) according to the leading eigenvector reveals a clear jump in this segment that was previously obscured. **(f)** The re-sorted sum of the 4 leading eigenvectors, now plotted with node positions determined by the targeted resorting procedure in panel e. The green and purple communities are now distinguished.

VIII. NETWORKS OF KURAMOTO OSCILLATORS

The dynamics of a network of Kuramoto oscillators is defined by

$$\dot{\theta}_i(t) = \omega + \epsilon \sum_{j=0}^{N-1} A_{ij} \sin\left(\theta_j(t) - \theta_i(t) - \phi_{ij}\right), \tag{VIII.1}$$

where $\theta_i(t)$ is the phase of the i^{th} oscillator at time t, ω is the natural frequency of oscillation, N is the number of oscillators in the network, ϵ is the coupling strength, A_{ij} represents the elements of the adjacency matrix that defines the connectivity in the network, and ϕ_{ij} is the phase-lag in the interaction between oscillator i and j. We consider a network of identical oscillators, and we set $\omega = 0$ without loss of generality.

For the analyses in Fig. 6 of the main text, we integrate Eq. (VIII.1) using Euler method with timestep $dt = 0.001$, where we start the system with random initial conditions uniformly distributed in $[-\pi, \pi]$. For the original network, Fig. 6a of the main text represents the adjacency matrix and no phase-lag is considered in the coupling, i.e. $\phi_{ij} = 0$. Using the insights from our recent work [5–7], we obtain a rewired network, such that the argument of the leading eigenvector displays three communities, so that the dynamics will display synchronized clusters. The adjacency matrix in this case is represented in Fig. 6f of the main text, and there is also phase-lag in the coupling which follow a similar pattern as the adjacency matrix.

[1] J. Đoàn, J. Mináč, L. Muller, T. T. Nguyen, and F. W. Pasini, Joins of circulant matrices, Linear Algebra and its Applications **650**, 190 (2022).

[2] K. R. Garren, Bounds for the eigenvalues of a matrix, NASA Technical Note **4373** (1968).

[3] A. Lancichinetti, S. Fortunato, and F. Radicchi, Benchmark graphs for testing community detection algorithms, Phys. Rev. E **78**, 046110 (2008).

[4] M. Girvan and M. E. J. Newman, Community structure in social and biological networks, Proceedings of the National Academy of Sciences **99**, 7821 (2002).

[5] R. C. Budzinski, T. T. Nguyen, J. Đoàn, J. Mináč, T. J. Sejnowski, and L. E. Muller, Geometry unites synchrony, chimeras, and waves in nonlinear oscillator networks, Chaos: An Interdisciplinary Journal of Nonlinear Science **32** (2022).

[6] R. C. Budzinski, T. T. Nguyen, G. B. Benigno, J. Đoàn, J. Mináč, T. J. Sejnowski, and L. E. Muller, Analytical prediction of specific spatiotemporal patterns in nonlinear oscillator networks with distance-dependent time delays, Physical Review Research **5**, 013159 (2023).

[7] T. T. Nguyen, R. C. Budzinski, J. Doàn, F. W. Pasini, J. Mináč, and L. E. Muller, Equilibria in kuramoto oscillator networks: An algebraic approach, SIAM Journal on Applied Dynamical Systems **22**, 802 (2023).