Supplementary Information

Supplementary Note: Physical model and numerical algorithms

Stiff-amplified inflationary SGWB

The description of our physical model of the inflationary stochastic gravitational wave background (SGWB) is based on [1]. The primordial tensor power spectrum satisfies the power law:

$$\Delta_{\rm t}^2(f) = A_{\rm t}\left(\frac{f}{f_{\rm CMB}}\right)^{n_{\rm t}}, \qquad (1)$$

where the tensor amplitude $A_{\rm t}$ is related to the scalar amplitude $A_{\rm s}$ by the tensor-to-scalar ratio, $r \equiv A_{\rm t}/A_{\rm s}$, and $k_{\rm CMB} \equiv 2\pi f_{\rm CMB}/c = 0.05\,{\rm Mpc}^{-1}$ is the CMB pivot scale [2]. The parameters r, $n_{\rm t}$ (tensor spectral index), and $A_{\rm s}$ (scalar amplitude) are free parameters in our model.

On top of the primordial tensor spectrum, we consider the effect that $\Omega_{\rm GW}(f)$ can be additionally blue-tilted in the presence of a *kination* phase in the early expansion history, during which the universe is dominated by the kinetic energy of a scalar field [3–5]. Kination, also known as the "stiff phase," is characterized by an equation of state (EoS) of a stiff fluid, i.e., $w \equiv \bar{P}/\bar{\rho} = 1$ [6]. When kination is present, tensor modes that reenter the horizon during this phase contribute to the SGWB with $\Omega_{\rm GW}(f) \propto f^{n_{\rm t}+1}$ [7–10], as opposed to $\Omega_{\rm GW}(f) \propto f^{n_{\rm t}}$ for modes reentering during the radiation-dominated (RD) era ($w_{\rm RD} = 1/3$). This effect, termed kination or stiff amplification of the inflationary SGWB [11, 12], is further discussed in [13–15]. [1] demonstrate that, even when the inflationary consistency relation holds ($n_{\rm t} = -r/8$), the stiff-amplified SGWB can contribute several percent to the critical density during the RD era, offering a pathway to alleviate the Hubble tension via the $H_0 - N_{\rm eff}$ degeneracy [16].

The underlying physical model remains rooted in stiff-amplified inflationary GWs, as described in prior work, with an extended parameter set:

$$\mathbf{p} = \{r, n_{\rm t}, \kappa_{10}, T_{\rm re}, \Delta N_{\rm re}, \Omega_{\rm b}h^2, \Omega_{\rm c}h^2, H_0, A_{\rm s}\}, \qquad (2)$$

where the new parameters $\Omega_{\rm b}h^2$, $\Omega_{\rm c}h^2$, H_0, and $A_{\rm s}$ enable integration with cosmic microwave background (CMB) and large-scale structure (LSS) datasets.

Numerical algorithms

In the following, we review the formalism and numerical algorithm for calculating the stiff-amplified inflationary SGWB in our StiffGWpy code. The tensor transfer function is defined as $T(t,f) \equiv h(t,f)/h_{\rm ini}(f)$, where $h(t,f)$ is the amplitude of the tensor mode at frequency f and cosmic time t, and $h_{\rm ini}(f)$ is its initial superhorizon value. The late-time energy spectrum of the SGWB is given by:

$$\Omega_{\rm GW}(t,f) \equiv \frac{{\rm d}\Omega_{\rm GW}}{{\rm d}\ln f} = \Delta_{\rm t}^2(f)\frac{(2\pi f)^2 T^2(t,f)}{12 a^2 H^2}, \qquad (3)$$

where $a(t)$ is the scale factor, and $H(t) = \dot{a}/a$ is the Hubble parameter, with the overdot denoting the derivative with respect to cosmic time.

The time evolution of the tensor transfer function is governed by the wave equation [17]:

$$\ddot{T} + 3H\dot{T} + \left(\frac{2\pi f}{a}\right)^2 T = 0, \qquad (4)$$

where the Hubble rate H reflects the influence of the universe's expansion history and EoS on the SGWB. For modes reentering the horizon during a constant EoS era, the tensor transfer function follows a power law [9, 11], yielding:

$$\Omega_{\rm GW}(f) \propto \Delta_{\rm t}^2(f)\left(\frac{f a_f}{H_0}\right)^2 \propto f^{n_{\rm t}+\frac{2(3w_f-1)}{1+3w_f}}, \qquad (5)$$

where a_f is the scale factor at horizon reentry for frequency f, defined by $2\pi f \equiv a_f H(a_f)$, and w_f is the EoS parameter at that time [12].

However, the tensor wave equation (4) lacks analytical solutions in general, necessitating numerical methods. Following [1], we adopt a dynamical system approach, defining dimensionless variables for each frequency:

$$\zeta_f \equiv \ln\frac{2\pi f}{aH}, \quad x_f \equiv \frac{\dot{T}}{H}, \quad y_f \equiv \frac{2\pi f}{aH}T. \qquad (6)$$

Clearly, $T(t,f) = y_f/e^{\zeta_f}$. These variables transform Eq. (4) into the dynamical system:

$$\zeta_f' = \frac{3}{2}\sigma - 1, \qquad (7)$$

$$x_f' = -3x_f + \frac{3}{2}\sigma x_f - e^{\zeta_f} y_f, \qquad (8)$$

$$y_f' = -y_f + \frac{3}{2}\sigma y_f + e^{\zeta_f} x_f, \qquad (9)$$

where the prime denotes differentiation with respect to the number of e-folds, $N \equiv \ln a$ (${\rm d}N = H{\rm d}t$), and:

$$\sigma \equiv -\frac{2\dot{H}}{3H^2} = \frac{\bar{\rho}+\bar{P}}{\bar{\rho}} = 1 + w. \qquad (10)$$

The SGWB's contribution to the total EoS parameter w is encoded in σ via the density fraction $\Omega_{\rm GW}(N) \equiv \rho_{\rm GW}/\bar{\rho} = \int \Omega_{\rm GW}(N,f)\,{\rm d}\ln f$. Thus, Eqs. (7–10) form a set of coupled integro-differential equations, solved numerically by StiffGWpy over a range of adaptively chosen frequencies.

Cosmological model

Our cosmological model extends the base ΛCDM framework by incorporating kination and an extended parameter set. The thermal history may begin with kination, transitioning to the radiation-dominated (RD) era before Big Bang nucleosynthesis (BBN). Kination is modeled as a stiff fluid ($w_{\rm s} = 1$), with the kination-to-radiation transition parameterized by $\kappa_{10} \equiv (\rho_{\rm s}/\rho_\gamma)|_{T=10\,{\rm MeV}}$, the ratio of stiff-fluid to photon density at 10 MeV [18]. The model assumes the inflationary phase may end in a prolonged reheating epoch, dominated by coherent oscillations of the inflaton field ($w_{\rm re} = 0$), with the end of reheating at temperature $T_{\rm re}$. Unlike the original model, which fixed standard ΛCDM parameters, we now allow variation in the baryon density ($\Omega_{\rm b}h^2$), cold dark matter density ($\Omega_{\rm c}h^2$), Hubble constant (H_0), and scalar amplitude ($A_{\rm s}$),

Algorithm 1 Adaptive Frequency Sampling

Input: $\Omega_b h^2, \Omega_c h^2, H_0, A_s, r, n_t, T_{re}, \Delta N_{re}, \kappa_{10}$
Output: Frequency array f (in $\log_{10}(f/\mathrm{Hz})$)
Compute N_{re}, ρ_{re} from T_{re} via spline interpolation ▷ Reheating e-folds and density
$N_{inf} \leftarrow N_{re} + \Delta N_{re}$, $\rho_{rad,re} \leftarrow \rho_{re} \cdot (T_{CMB} \cdot e^{N_{10}})^4$
$\rho_{stiff,re} \leftarrow \kappa_{10} \cdot e^{2(N_{re}-N_{10})}$, $A_t \leftarrow r \cdot A_s$
$N_v \leftarrow [0, \Delta N, \ldots, N_{inf}]$, $(f_{hor}, f_{re}) \leftarrow \mathrm{ComputeExpansionHistory}(\cdot)$
$f_{max} \leftarrow f_{hor}[0]$, $f_{min} \leftarrow \min(f_{hor})$, $f_{CMB} \leftarrow \log_{10}(f_{piv})$
if $n_t > 0$ **then**
    $f_{max} \leftarrow \min(f_{max}, -\log_{10}(A_t)/n_t + \log_{10}(f_{piv}))$
end if
$f \leftarrow [f_{max}] \cup \mathrm{finelinearsampling} \cup \mathrm{logarithmicsampling}$ ▷ High-frequency
if $f_{max} \geq f_{re}$ **then**
    $f \leftarrow f \cup \mathrm{adaptivelinearsamplingaround} f_{re}$ ▷ Reheating
end if
if $\rho_{stiff,re} > \rho_{rad,re}$ **then**
    $f \leftarrow f \cup \mathrm{adaptivelinearsamplingaround} f_{sr}$ ▷ Kination
end if
$f \leftarrow f \cup \mathrm{coarselinearsamplingto} f_{CMB}, f_{min}$ ▷ Radiation and matter phases
$f \leftarrow f[f \geq f_{min}]$
return f

enabling a more comprehensive exploration of cosmological effects on the SGWB. In summary, our physical model includes nine free parameters:

$$\mathbf{p} = \{r, n_t, \kappa_{10}, T_{re}, \Delta N_{re}, \Omega_b h^2, \Omega_c h^2, H_0, A_s\}, \quad (11)$$

where ΔN_{re} is the number of e-folds during reheating.

The evolution of $\sigma(N)$ exhibits key cosmological transitions, modulated by the expanded parameter set, which affects the SGWB's spectral shape. These nontrivial evolutions necessitate exact numerical solutions to the tensor wave equation or the dynamical system (7–9) for accurate predictions of $\Omega_{GW}(f)$, as opposed to using fitting formulae. However, these solutions are computationally expensive, particularly due to the large number of modes required to capture the shape of $\Omega_{GW}(f)$ across the frequency range. An iterative run for a fixed set of model parameters may take $\mathcal{O}(10)$ seconds. Given this computational complexity, developing a fast solver, such as our Transformer-based emulator, is essential for efficient parameter sweeps and analysis of large GW and cosmological data sets.

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