Supplementary Material:

S1. Experimental setup:

We describe here the experimental setup used to generate the optical supertopologies. To demonstrate the proof of concept, we generate the supertopologies through complex-amplitude modulation of two orthogonally polarised monochromatic beams using computer-generated holograms displayed on the two halves of the spatial light modulator (SLM). The diagonally polarised laser beam (wavelength $\lambda =$ 532 nm) is split into horizontal and vertical components by a polarising beam splitter (PBS1) and two mirrors (M1 and M2) arranged in a Sagnac interferometric configuration. To increase the modulation efficiency, we directly encode holograms for twisted lattice fields, rather than generating them through arrays of $2N$ Gaussian beamlets ($N = 3, 4,$ or 6). As the SLM modulates only horizontally polarised light beam, a small half-wave plate (HWP) is placed immediately before the second half of the SLM to rotate the polarisation appropriately, as shown in Fig S1(a) and S2.

After reflection from the SLM, the modulated beams are recombined using a second Sagnac interferometer formed by PBS2 and mirrors M4 and M5. The resulting beam is then converted into the right- and left-circular polarisation basis using a quarter-wave plate (QWP). For the reconstruction of the full polarisation field, the Stokes parameters are obtained from six intensity measurements corresponding to six analyser settings, as shown in Fig. S1(b).

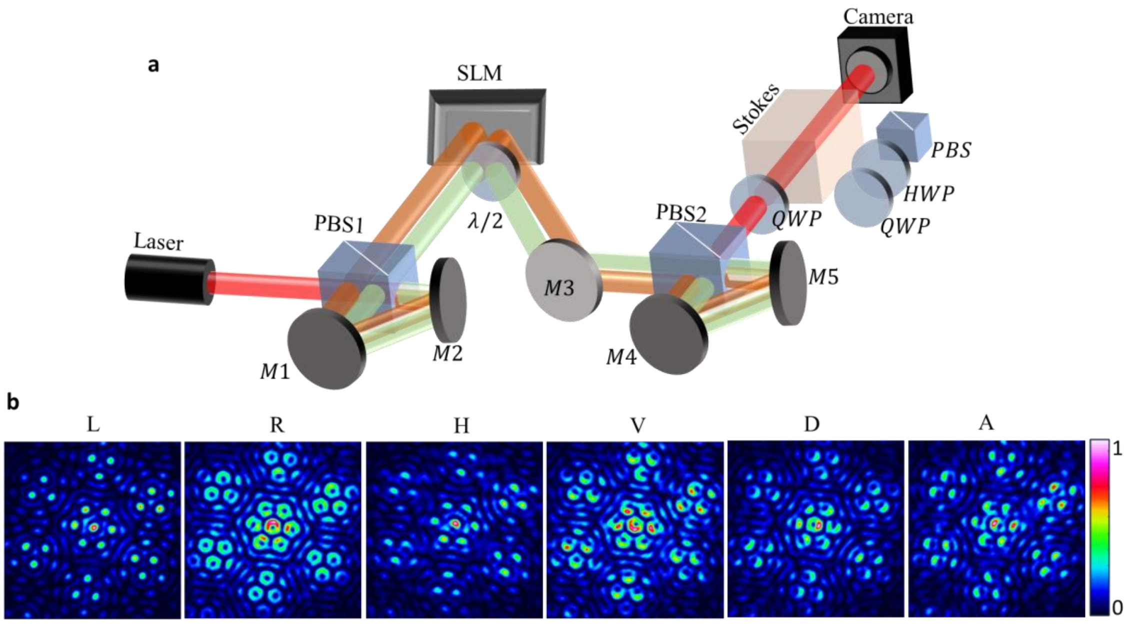

FIG. S1: (a) Experimental setup for the generation of supertopologies. (b) Measured intensities corresponding to six different polarisations for stokes vector reconstruction. PBS1, PBS2 polarisation beam splitters; $\lambda/2$, HWP, and QWP, half wave plate quarter wave plate for $\lambda = 532$ nm; M1-M5, mirrors; SLM, spatial light modulator.

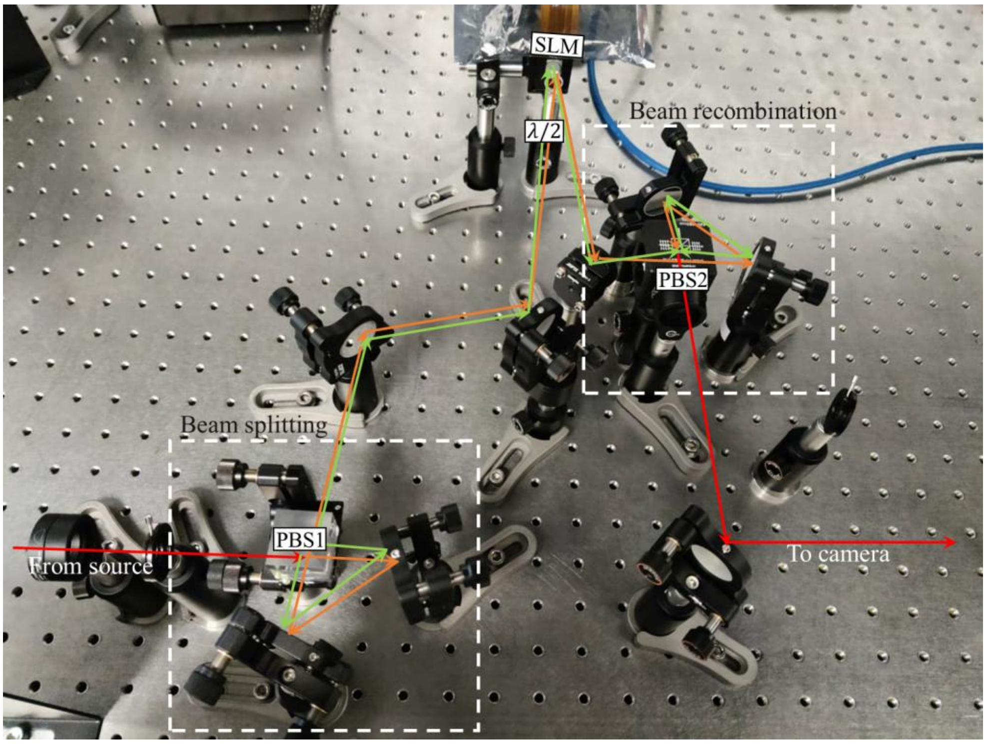

FIG. S2: Picture of experimental setup to generate supertopologies obtained from freespace twistronics.

S2. Hierarchical supertopology decomposition for C4:

Here we present the Fourier-space decomposition of a twisted C4 skyrmion superlattice. As an example, we consider the configuration when two C4 skyrmion lattices are twisted by an angle of $\theta = 16.26^{o}$. Owing to the periodicity of the Stokes vector, its Fourier transform consists of discrete spatial-frequency components. Figure S3(a) shows the Fourier transform of the in-plane Stokes vector, where the Fourier components are organised into four concentric rings with radial spatial frequencies k_1, k_2, k_3, and k_4. Each ring corresponds to a sublattice that contributes to the overall twisted skyrmion superlattice.

The sublattices associated with k_1 and k_3 each containing four Fourier components, reconstruct C4 meron–antimeron lattices with different lattice constants, as shown in Figs. S3(b) and S3(e). The ring at wavevector k_2 contains eight Fourier components, corresponds to a texture equivalent to two C4 meron–antimeron lattices twisted by $\theta = 16.26^{o}$, as shown in Fig. S3(c). The outer ring at wavevector k_4 consists of twelve Fourier components and represents a twisted trilayer texture in real space, as illustrated in Fig. S3(d).

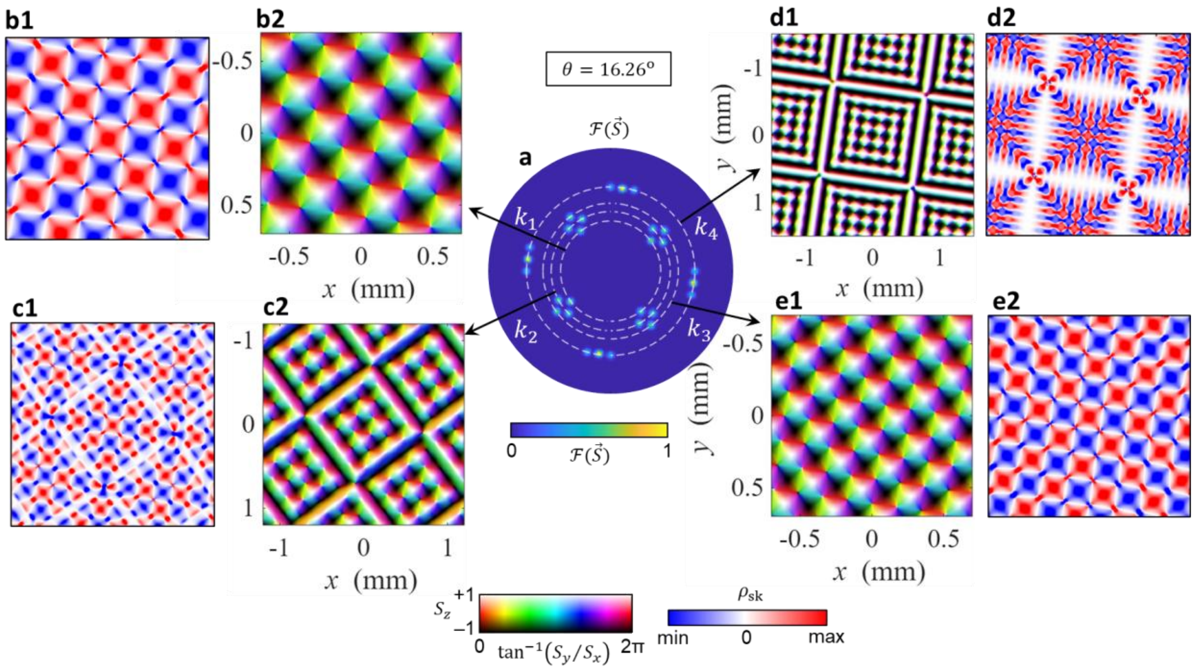

FIG. S3: Simulation results for hierarchical decomposition for twisted C4 lattices. (a) Fourier transform of in plane stokes vector of twisted C4 skyrmion lattices at $\theta = 16.26^{o}$, where Fourier components are arranged in four concentric rings of radial spatial frequencies $k_1 - k_4$. (b-e) Hierarchical real-space textures and skyrmion number density of reconstructed meron lattices and meron cluster in C4 geometry for frequencies $k_1 - k_4$.

S3. Twistronics in C3 meron-antimeron lattices:

Here we present the freespace twistronics results for C3 meron–antimeron topological lattices. For N = 3, the underlying topological texture forms a C3 meron–antimeron lattice with a triangular unit cell. When two such lattices are twisted, meron–antimeron cluster superlattices emerge at commensurate twist angles. Figures S4(a–d) show the topological textures obtained for twist angles $\theta = 13.2^o, 16.4^o, 21.8^o$, and 27.8^o. As evident, the size of the superunit cell decreases as the twist angle increases. The corresponding skyrmion-number density distributions, shown in Figs. S4(f–i), clearly reveal the meron–antimeron cluster superlattice structures. For a twist angle of $\theta = 60^o$, a distinct configuration appears that closely resembles a skyrmionium lattice, as shown in Figs. S4(e) and S4(j).

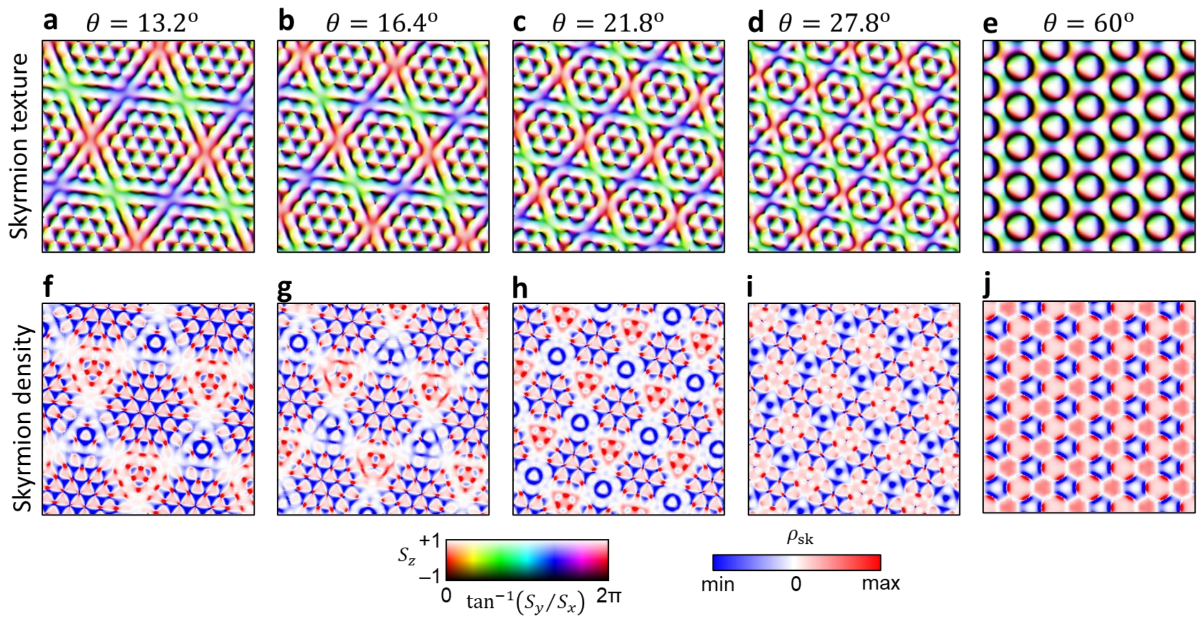

FIG. S4: Free space twistronics for C3 meron-antimeron lattices. (a-e) Topological texture, and (f-j) skyrmion number density obtained when two C3 meron-antimeron lattices are twisted at $\theta = 13.2^o$, 16.4^o, 21.8^o, 27.8^o, and 60^o.

S4. Hierarchical supertopology decomposition for C3:

Here we present the Fourier-space decomposition of twisted meron–antimeron cluster superlattices. As an example, we analyse the in-plane Stokes vector of a C3 meron–antimeron lattice twisted by $\theta = 21.8^{\circ}$. For an untwisted C3 meron–antimeron lattice, the Fourier transform of the Stokes vector contains six discrete frequencies with the same radial wavenumber, and therefore cannot be decomposed further into distinct sublattices. In contrast, for the twisted case we find that the Fourier spectrum consists of four distinct rings of spatial-frequency vectors, labelled $k_1 - k_4$, as shown in Fig. S5(a). Selective inverse Fourier transforms of these rings yield four meron–antimeron sublattices.

The frequency rings k_1, k_2, and k_4 each contain six equally spaced Fourier components, and the corresponding sublattices reconstruct meron–antimeron lattices with different lattice constants, as shown in Figs. S5(b–d). The ring at k_3 contains twelve Fourier components arranged in two sets of six, with one set rotated by $\theta = 21.8^{\circ}$ relative to the other. The resulting sublattice, shown in Fig. S5(e), exhibits a unique twisted topological texture that closely resembles a skyrmion-bag configuration.

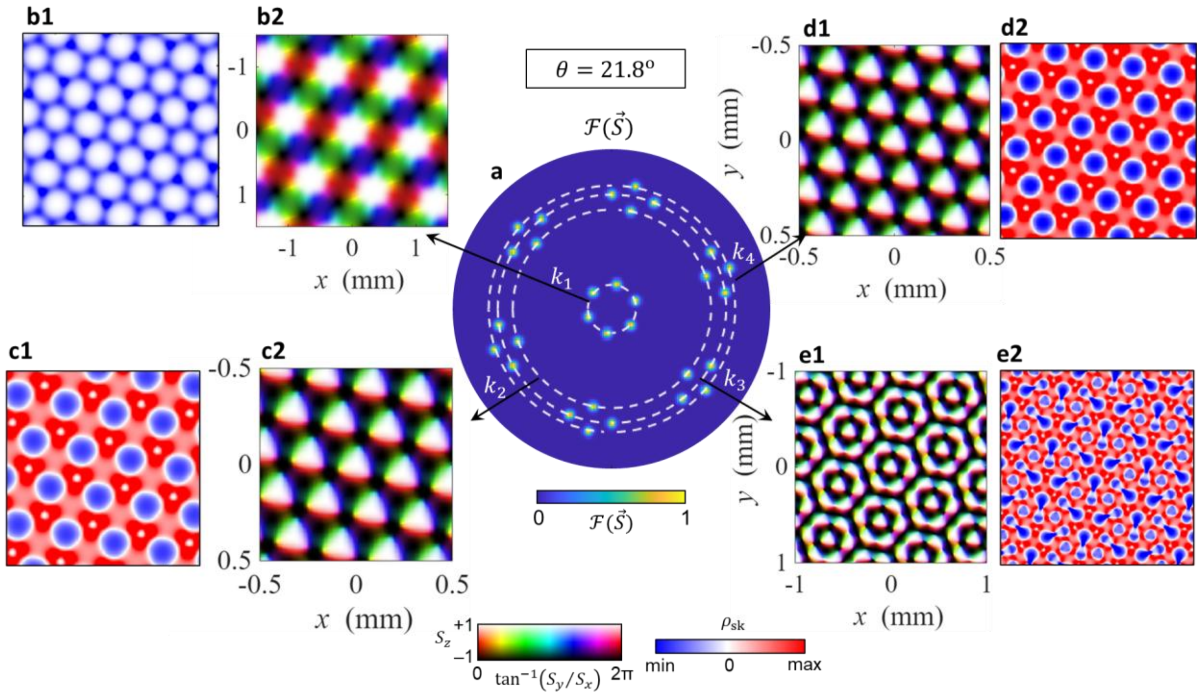

FIG. S5: Simulation results for hierarchical decomposition for twisted C3 lattices. (a) Fourier transform of in plane stokes vector of twisted C3 meron-antimeron lattices at $\theta = 21.8^{\circ}$, where Fourier components are arranged in four concentric rings of radial spatial frequencies $k_1 - k_4$. (b-e) Hierarchical real-space textures and skyrmion number density of reconstructed meron lattices and meron cluster in C3 geometry for frequencies $k_1 - k_4$.