Supplementary information for "The Riemann Hypothesis Emerges in Dynamical Quantum Phase Transitions"

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Notation

For clarity, throughout this Supplementary Information, we use 'log' to denote the base-2 logarithm, and 'ln' to denote the natural logarithm (base e). All variables s are assumed to satisfy $\mathrm{Re}(s) > 0$ and $\mathrm{Re}(s) \neq 1$, ensuring the applicability of the alternating Dirichlet series. β denote the real part of s and t denote the imaginary part of s.

The Big-O notation $\mathcal{O}(\cdot)$ represents the worst-case upper bound, while the Big-Theta notation $\Theta(\cdot)$ denotes the exact asymptotic behavior. The notation $\mathrm{Poly}(\cdot)$ signifies that the growth follows a polynomial function of the input.

The symbol $\|\cdot\|$ denotes the Euclidean norm on state vectors. The distance between two pure quantum states $|\psi_a\rangle$ and $|\psi_b\rangle$ is defined as $d(|\psi_a\rangle, |\psi_b\rangle) = \min_{\theta\in\mathbb{R}} \||\psi_a\rangle - e^{i\theta}|\psi_b\rangle\| = \sqrt{2(1 - |\langle\psi_a|\psi_b\rangle|)}$, which satisfies the triangle inequality. The distance between two unitary operators is defined as $D(U_1, U_2) = \max_{|\psi\rangle} d(U_1|\psi\rangle, U_2|\psi\rangle)$, which is the worst-case distance over all input states $|\psi\rangle$. We say that a quantum state (or unitary operator) is prepared to precision κ if its distance from the corresponding ideal target state (or ideal unitary) is at most κ.

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Supplementary Note 1: Asymptotic behavior of $S_N(s)$

The N-term partial sum of the alternating Dirichlet series is

$$S_N(s) = \sum_{n=1}^{N} \frac{(-1)^{n-1}}{n^s}. \tag{1.1}$$

Case 1. At the nontrivial zeros of the zeta function, we have $S_N(s) = -R_N(s)$, where

$$R_N(s) = \sum_{n=N+1}^{\infty} \frac{(-1)^{n-1}}{n^s}. \tag{1.2}$$

Each term has the following integral representation,

$$\frac{1}{n^s} = \frac{1}{\Gamma(s)} \int_0^\infty \tau^{s-1} e^{-n\tau} d\tau, \tag{1.3}$$

yielding

$$R_N(s) = \sum_{n=N+1}^{\infty} \frac{(-1)^{n-1}}{n^s} = \frac{1}{\Gamma(s)} \int_0^\infty \tau^{s-1} \sum_{n=N+1}^{\infty} (-1)^{n-1} e^{-n\tau} d\tau = \frac{(-1)^N}{\Gamma(s)} \int_0^\infty \frac{\tau^{s-1} e^{-(N+1)\tau}}{1+e^{-\tau}} d\tau. \tag{1.4}$$

Using the identity

$$\frac{1}{1+e^{-\tau}} = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{\tau}{2}\right), \tag{1.5}$$

we obtain

$$R_N(s) = \frac{(-1)^N}{\Gamma(s)} \left[\frac{1}{2}\int_0^\infty \tau^{s-1} e^{-(N+1)\tau} d\tau + \frac{1}{2}\int_0^\infty \tau^{s-1} e^{-(N+1)\tau} \tanh\Big(\frac{\tau}{2}\Big) d\tau\right]. \tag{1.6}$$

Let us define

$$M := \frac{1}{2}\int_0^\infty \tau^{s-1} e^{-(N+1)\tau} d\tau = \frac{1}{2}\Gamma(s)(N+1)^{-s}, \qquad J := \frac{1}{2}\int_0^\infty \tau^{s-1} e^{-(N+1)\tau} \tanh\Big(\frac{\tau}{2}\Big) d\tau. \tag{1.7}$$

This leads to

$$R_N(s) = \frac{(-1)^N}{\Gamma(s)}\big(M+J\big) = (-1)^N \Big(\frac{1}{2(N+1)^s} + \frac{J}{\Gamma(s)}\Big), \tag{1.8}$$

We now show that for sufficiently large N,

$$|J| \le \frac{1}{4}\Gamma(s)(N+1)^{-\beta}, \qquad s = \beta + it. \tag{1.9}$$

Fix $A > 0$, which depends on s but not on N, and split the integral for J at $\tau = A/(N+1)$:

$$J = \frac{1}{2}\left(\int_0^{A/(N+1)} \tau^{s-1} e^{-(N+1)\tau} \tanh\Big(\frac{\tau}{2}\Big) d\tau + \int_{A/(N+1)}^\infty \tau^{s-1} e^{-(N+1)\tau} \tanh\Big(\frac{\tau}{2}\Big) d\tau\right) := \frac{1}{2}(I_1 + I_2). \tag{1.10}$$

For I_1, we apply the inequality $|\tanh(\tau/2)| \le |\tau/2|$, so

$$|I_1| \le \int_0^{A/(N+1)} \tau^{\beta-1} e^{-(N+1)\tau} \frac{\tau}{2} d\tau = \frac{1}{2}(N+1)^{-(\beta+1)} \int_0^A u^\beta e^{-u} du = C_1(\beta, A)(N+1)^{-(\beta+1)}. \tag{1.11}$$

Here, $C_1(\beta, A) := \frac{1}{2}\int_0^A u^\beta e^{-u} du$ is a constant depending on A and β. For I_2, we use the trivial bound $|\tanh(\tau/2)| \le 1$, so

$$|I_2| \le \int_{A/(N+1)}^\infty \tau^{\beta-1} e^{-(N+1)\tau} d\tau = (N+1)^{-\beta} \int_A^\infty u^{\beta-1} e^{-u} du = C_2(\beta, A)(N+1)^{-\beta}. \tag{1.12}$$

Here $C_2(\beta, A) := \int_A^\infty u^{\beta-1}e^{-u}du$ is also a constant that depends on A and β.

Since $\Gamma(s)$ is non-vanishing and $C_2(\beta, A) \to 0$ as $A \to \infty$, we can choose A large enough so that

$$C_2(\beta, A) \leq \frac{1}{4}|\Gamma(s)|. \tag{1.13}$$

Thus,

$$|I_2| \leq \frac{1}{4}|\Gamma(s)|(N+1)^{-\beta}. \tag{1.14}$$

Combining the two contributions gives

$$|J| \leq \frac{1}{2}\big(|I_1| + |I_2|\big) \leq \frac{1}{2}\Big(C_1(\beta, A)(N+1)^{-(\beta+1)} + \frac{1}{4}|\Gamma(s)|(N+1)^{-\beta}\Big). \tag{1.15}$$

Consequently, for all $N \geq N_0 := \lceil 4C_1(\beta, A)/|\Gamma(s)|\rceil$,

$$|J| \leq \frac{1}{4}|\Gamma(s)|(N+1)^{-\beta}, \qquad \Big|\frac{J}{\Gamma(s)}\Big| \leq \frac{1}{4}(N+1)^{-\beta}. \tag{1.16}$$

Substituting this bound into Equation (1.8) gives for $N \geq N_0$,

$$\frac{1}{4}(N+1)^{-\beta} \leq |R_N(s)| \leq \frac{3}{4}(N+1)^{-\beta}. \tag{1.17}$$

Therefore,

$$\lim_{N\to\infty} -\frac{1}{\log N}\ln|S_N(s)| = \lim_{N\to\infty} -\frac{1}{\log N}\ln|R_N(s)| = \beta\ln 2. \tag{1.18}$$

Case 2. For s with $\zeta(s) \neq 0$, the partial sums converge, and in particular

$$\lim_{N\to\infty} S_N(s) = (1 - 2^{1-s})\,\zeta(s) \neq 0. \tag{1.19}$$

It follows that

$$-\lim_{N\to\infty}\frac{1}{\log N}\ln|S_N(s)| = 0. \tag{1.20}$$

Thus, we obtain the final result:

$$-\lim_{N\to\infty}\frac{1}{\log N}\ln|S_N(s)| = \begin{cases}\beta\ln 2, & \text{if } \zeta(s) = 0,\\ 0, & \text{if } \zeta(s) \neq 0.\end{cases} \tag{1.21}$$

Supplementary Note 2: Asymptotic behavior of $\mathcal{Z}(\beta, \mathcal{H}_0)$

The partition function can be written as

$$\mathcal{Z}(\beta, \mathcal{H}_0) = \sum_{n=1}^{N} n^{-\beta}. \tag{2.1}$$

Case 1. For $\beta > 1$, $\mathcal{Z}(\beta, \mathcal{H}_0)$ converges as

$$\lim_{N\to\infty}\mathcal{Z}(\beta, \mathcal{H}_0) = \zeta(\beta). \tag{2.2}$$

Thus,

$$\lim_{N\to\infty} -\frac{1}{\log N}\ln\mathcal{Z}(\beta, \mathcal{H}_0) = 0. \tag{2.3}$$

Case 2. For $0 < \beta < 1$, the series diverges. Specifically,

$$\int_1^{N+1} x^{-\beta}\, dx \;\leq\; \sum_{n=1}^{N} n^{-\beta} \;\leq\; 1 + \int_1^{N} x^{-\beta}\, dx, \tag{2.4}$$

which simplifies to

$$\frac{(N+1)^{1-\beta}-1}{1-\beta} \;\leq\; \mathcal{Z}(\beta, \mathcal{H}_0) \;\leq\; 1 + \frac{N^{1-\beta}-1}{1-\beta}, \tag{2.5}$$

yielding

$$-\lim_{N\to\infty} \frac{1}{\log N} \ln \mathcal{Z}(\beta, \mathcal{H}_0) = (\beta - 1)\ln 2. \tag{2.6}$$

Thus,

$$-\lim_{N\to\infty} \frac{1}{\log N} \ln \mathcal{Z}(\beta, \mathcal{H}_0) = \begin{cases} (\beta-1)\ln 2, & 0 < \beta < 1, \\ 0, & \beta > 1. \end{cases} \tag{2.7}$$

Supplementary Note 3: Construction of the preliminary oracles

This section details the construction of polynomial and logarithm oracles, which serve as foundational components for subsequent quantum state preparation and time evolution operators in our quantum algorithms.

Lemma 3.1 (Polynomial oracle (Methods, Lemma 1))**.** *Let $O(f)$ denote an oracle implementing the transformation*

$$\sum_x \alpha_x |x\rangle|0\rangle \mapsto \sum_x \alpha_x |x\rangle|f(x)\rangle, \tag{3.1}$$

where $f(x)$ is a polynomial of degree at most D. Suppose the coefficients of f and the input x are specified to a_1 and a_2 significant digits, respectively. The output is encoded with r_1 integer qubits and r_2 fractional qubits. Then $O(f)$ can be implemented using

$$\mathcal{O}\big(D^2 a_2^2 + D^2 a_1 a_2 + D(r_1 + r_2)\big) \text{ gates, and } (2Da_2 + a_1) \text{ ancilla qubits.}$$

Proof. Write the polynomial as $f(n) = \sum_{d=0}^{D} c_d n^d$. Evaluation proceeds through sequential construction of the monomials x^d, multiplication by coefficients c_d, and accumulation of the results.

Step 1: Computation of Monomials. Multiplying an m_1-qubit register by an m_2-qubit register requires $\mathcal{O}(m_1 m_2)$ gates and $\mathcal{O}(m_1 + m_2)$ qubits using quantum schoolbook multiplication[1]. To compute x^{d+1} from x^d, a da_2-qubit register (storing x^d) is multiplied by an a_2-qubit register (storing x). The total gate cost for generating monomials up to degree D is

$$2\sum_{d=1}^{D-1} a_2 \cdot da_2 = \mathcal{O}(D^2 a_2^2), \tag{3.2}$$

where the factor 2 arises from uncomputing intermediate results and resetting ancilla qubits.

Step 2: Multiplication by coefficients. Multiplying a degree-d monomial (which requires da_2 qubits) by c_d costs $\mathcal{O}(da_1a_2)$ gates. Summing over all degrees gives

$$2\sum_{d=1}^{D} da_1a_2 = \mathcal{O}(D^2 a_1 a_2), \tag{3.3}$$

again with the factor 2 from uncomputation.

Step 3: Accumulation of output. The results are accumulated into an output register of size $r_1 + r_2$. Each of the $D+1$ additions costs $\mathcal{O}(r_1+r_2)$ gates, yielding a total gate cost of $\mathcal{O}(D(r_1+r_2))$. Adding all contributions yields a total gate complexity

$$\mathcal{O}\big(D^2 a_2^2 + D^2 a_1 a_2 + D(r_1 + r_2)\big)\,. \tag{3.4}$$

During computation, at most Da_2 qubits are used for monomial storage and $(Da_2 + a_1)$ for intermediate products, giving a total ancilla requirement of $(2Da_2 + a_1)$. □

Lemma 3.2 (Logarithm oracle (Methods, Lemma 2))**.** *Let L denote an oracle implementing the transformation*

$$\sum_{n=1}^{N} \alpha_n |n\rangle|0\rangle \mapsto \sum_{n=1}^{N} \alpha_n |n\rangle|\widetilde{\log(n)}\rangle, \tag{3.5}$$

where $\widetilde{\log(n)}$ approximates $\log(n)$ to within error η. Then L can be implemented using $\mathcal{O}((\log N)^3 \log^2(1/\eta))$ gates and $\mathcal{O}((\log N)^2 \log(1/\eta))$ ancilla qubits.

Proof. We proceed constructively.

Step 1: Input Partitioning. Define $k_1 = \lceil \log\big(\frac{N+1}{3}\big) \rceil$. For $3 \le n < 3 \cdot 2^{k_1}$, partition the integers into k_1 subsets, defined as

$$P_\nu : \quad 3 \cdot 2^{\nu-1} \le n_\nu < 3 \cdot 2^{\nu}, \qquad 1 \le \nu \le k_1. \tag{3.6}$$

Each $3 \le n \le N$ lies in exactly one partition P_ν. For $n_\nu \in P_\nu$, define

$$d = \frac{n_\nu - 2^{\nu+1}}{2^{\nu+1}}, \qquad -\frac{1}{4} \le d < \frac{1}{2}. \tag{3.7}$$

This allows the logarithm to be expressed as

$$\log(n_\nu) = \log\big(2^{\nu+1}\big) + \log(1+d). \tag{3.8}$$

Step 2: Taylor expansion of $\log(1+d)$. Using the Taylor series expansion for $\log(1+d)$, we have:

$$\log(1+d) = \frac{1}{\ln 2} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} d^j. \tag{3.9}$$

The remainder after l_1 terms satisfies

$$\left| \sum_{j=l_1+1}^{\infty} \frac{(-1)^{j+1}}{j} d^j \right| \le \sum_{j=l_1+1}^{\infty} \frac{1}{j} \Big(\frac{1}{2}\Big)^j \le \frac{1}{2^{l_1}}. \tag{3.10}$$

Hence, truncating after $l_1 = \lceil \log(1/\eta) \rceil + 2$ terms yields an approximation error of less than $\frac{\eta}{2}$. Coefficients stored to $\lceil \log(2/\eta) \rceil$ fractional bits introduce an additional rounding error bounded by

$$\frac{\eta}{2}(1 + \frac{1}{2} + (\frac{1}{2})^2 + \cdots) \le \frac{\eta}{2}, \tag{3.11}$$

giving total error of less than η. The resulting approximation

$$\widetilde{\log(n_\nu)} = \nu + 1 + \frac{1}{\ln(2)} \sum_{j=1}^{l_1} \frac{(-1)^{j+1}}{j} d^j \tag{3.12}$$

is thus approximated within error η.

The partition index ν is determined using multi-controlled operations that compare n with partition bounds. Each such comparison uses at most $\mathcal{O}(k_1)$ Toffoli gates and $\mathcal{O}(k_1)$ ancilla qubits. A flag register composing of at most $\mathcal{O}(k_1)$ qubits marks the unique partition P_ν containing n. This flag controls addition of the corresponding polynomial output to the output register. Exactly one flag is set for each input across $3 \le n \le N$.

For each ν, Equation (3.12) is a degree-l_1 polynomial in d, with coefficients specified to $\lceil \log(2/\eta) \rceil$ qubits and input d to $(k_1 + 2)$ significant digits. The output requires $(\lceil \log k_1 \rceil + 2)$ integer qubits and $\lceil \log(1/\eta) \rceil$ fractional qubits. Applying Lemma 3.1, evaluating this polynomial within a partition requires

$$\mathcal{O}(l_1^2 k_1^2 + l_1^2 k_1 \log(2/\eta) + l_1(\log(k_1) + \log(1/\eta))) = \mathcal{O}(k_1^2 \log^2(1/\eta)) \tag{3.13}$$

gates, and

$$2 l_1 k_1 + \log(1/\eta) = \mathcal{O}(k_1 \log(1/\eta)) \tag{3.14}$$

ancilla qubits. Summing over all k_1 partitions gives total resource requirements:

$$\mathcal{O}(k_1^3 \log^2(1/\eta)) = \mathcal{O}((\log N)^3 \log^2(1/\eta)) \text{ gates}, \qquad \mathcal{O}(k_1^2 \log(1/\eta)) = \mathcal{O}((\log N)^2 \log(1/\eta)) \text{ ancilla qubits}. \tag{3.15}$$

The special cases $n = 1, 2$ can be handled separately at negligible cost. □

Supplementary Note 4: Construction of the initial state

This section details the construction of the initial state, which is outlined as follows:

Our first key result is provided in Theorem 4.2 (Methods, Theorem 1), which describes the preparation of the truncated state $|\psi_0\rangle_1$. Central to this construction is the angle-preparation oracle (Methods, Theorem 2) that computes the required rotation angles. Its action is defined in Definition 4.3 and its implementation is outlined in Theorem 4.4. A central subroutine of this oracle is the evaluation of a zeta function related partial sum (Lemma 4.7), built upon Proposition 4.5 with detailed calculations given in Propositions 4.8 and 4.9.

Our second key result, Theorem 4.11 (Methods, Theorem 3), presents the construction of our initial state $|\psi_0\rangle$. Starting with the truncated state $|\psi_0\rangle_1$ prepared in Theorem 4.2, we extend it via the Linear Combination of Unitaries (LCU) method to combine it with $|\psi_0\rangle_2$ (Corollary 4.10); a final post-selection removes the high-index component $|\psi_0\rangle_3$, yielding the desired initial state.

Definition 4.1. *The controlled-rotation gate $\mathcal{CR}$ implements the transformation*

$$\sum_\theta \alpha_\theta |\theta\rangle|0\rangle \;\mapsto\; \sum_\theta \alpha_\theta |\theta\rangle\big(\cos\theta\,|0\rangle + \sin\theta\,|1\rangle\big). \tag{4.1}$$

It can be implemented using a sequence of controlled-R_x rotations, each conditioned on a qubit of the register encoding θ, followed by a phase gate on the target qubit.

Theorem 4.2 (Truncated state preparation (Methods, Theorem 1))**.** *Let $\beta > 0$, $\beta \neq 1$, and let $n_0, n_1 \in \mathbb{N}$ with $n_1 - n_0 = 2^k$ for some integer $k > 0$. Define*

$$|\psi_0\rangle_1 \;=\; \frac{1}{C_1}\sum_{n=n_0}^{n_1-1} n^{-\beta/2}\,|n-n_0\rangle, \qquad C_1 = \sqrt{\sum_{n=n_0}^{n_1-1} n^{-\beta}}. \tag{4.2}$$

Then $|\psi_0\rangle_1$ can be prepared on a standard gate-based quantum computer to precision $\varepsilon > 0$, using a number of gates and ancilla qubits bounded by

$$\mathrm{Poly}\Big(\log(1/\varepsilon),\ \log(n_1),\ \log\Big(\tfrac{1}{|1-\beta|}\Big),\ \beta,\ v\Big)\,, \tag{4.3}$$

where v denotes the number of significant digits used to represent β. The parameters are required to satisfy

$$c = \left\lceil \frac{1}{2}\log_{2\pi}\Big(\frac{8}{\varepsilon}\Big)\right\rceil, \qquad n_0 > \lceil \beta + 2c\rceil. \tag{4.4}$$

Proof sketch. We adapt Grover's recursive amplitude-splitting method[2], replacing integration with partial summation. The 2^k computational basis states are encoded on k qubits initialized in $|0\rangle^{\otimes k}$. At iteration step m $(0 \le m \le k-1)$, each bit string $w \in \{0,1\}^m$ is split into two branches $w0$ and $w1$. Define

$$c_w = \frac{1}{C}\sqrt{\sum_{i=0}^{2^{k-m}-1} (2^{k-m}w + i + n_0)^{-\beta}}. \tag{4.5}$$

Normalization is preserved, as

$$c_{w0} = \frac{1}{C}\sqrt{\sum_{i=0}^{2^{k-m-1}-1}\big(2^{k-m}w + i + n_0\big)^{-\beta}}, \quad c_{w1} = \frac{1}{C}\sqrt{\sum_{i=0}^{2^{k-m-1}-1}\big(2^{k-m}(w+1) + i + n_0\big)^{-\beta}}, \tag{4.6}$$

satisfy $c_{w0}^2 + c_{w1}^2 = c_w^2$. For each w, we compute a rotation angle $\theta_{w,m}$ such that the amplitude split $c_w \mapsto (c_{w_0}, c_{w_1})$ can be implemented by a $\mathcal{CR}$ gate acting on the next qubit. All rotations corresponding to strings of the same length

m are applied in parallel. The transformation at step m is written as

$$\begin{aligned}
&\sum_{w\in\{0,1\}^m} c_w\,|w\rangle\,|0\rangle^{\otimes(k-m)}\,|0\rangle \\
&\xrightarrow{U_m} \sum_w c_w\,|w\rangle\,|0\rangle^{\otimes(k-m)}\,|\theta_{w,m}\rangle \\
&\xrightarrow{\mathcal{CR}} \sum_w c_w\,|w\rangle\big(\cos\theta_{w,m}\,|0\rangle+\sin\theta_{w,m}\,|1\rangle\big)\,|0\rangle^{\otimes(k-m-1)}\,|\theta_{w,m}\rangle \\
&\xrightarrow{U_m^\dagger} \sum_w\big(c_{w_0}|w0\rangle+c_{w_1}|w1\rangle\big)\,|0\rangle^{\otimes(k-m-1)}\,|0\rangle.
\end{aligned} \tag{4.7}$$

Here, U_m denote an oracle that computes $\theta_{w,m}$ into an angle register, which is uncomputed after applying $\mathcal{CR}$ gate. Iterating the transformation in Equation (4.7) for $m=0,1,\dots,k-1$ yields the desired state

$$|\psi_0\rangle=\frac{1}{C}\sum_{n=n_0}^{n_1-1} n^{-\beta/2}|n-n_0\rangle. \tag{4.8}$$

Thus, the central task reduces to constructing U_m, defined as follows. □

Definition 4.3 (Angle–preparation oracle U_m)**.** *The gate U_m implements the transformation*

$$\sum_{w\in\{0,1\}^m}\alpha_w|w\rangle|0\rangle\mapsto\sum_{w\in\{0,1\}^m}\alpha_w|w\rangle|\theta_{w,m}\rangle, \tag{4.9}$$

where the rotation angle $\theta_{w,m}$ satisfies

$$\tan^2(\theta_{w,m})=\frac{c_{w1}^2}{c_{w0}^2}=\frac{\sum_{i=0}^{2^{k-m-1}-1}(2^{k-m-1}(2w+1)+i+n_0)^{-\beta}}{\sum_{i=0}^{2^{k-m-1}-1}(2^{k-m-1}(2w)+i+n_0)^{-\beta}}=\frac{S(2^{k-m-1}(2w+1)+n_0,\,2^{k-m-1}(2w+2)-1+n_0,\,\beta)}{S(2^{k-m-1}(2w)+n_0,\,2^{k-m-1}(2w+1)-1+n_0,\,\beta)}, \tag{4.10}$$

with

$$S(a,b,\beta)=\sum_{n=a}^{b} n^{-\beta}. \tag{4.11}$$

Equivalently,

$$\begin{aligned}
\theta_{w,m}&=\arctan\Bigg(\sqrt{\frac{S(2^{k-m-1}(2w+1)+n_0,2^{k-m-1}(2w+2)-1+n_0,\beta)}{S(2^{k-m-1}(2w)+n_0,2^{k-m-1}(2w+1)-1+n_0,\beta)}}\Bigg)\\
&=\arcsin\Bigg(\sqrt{\frac{S(2^{k-m-1}(2w+1)+n_0,2^{k-m-1}(2w+2)-1+n_0,\beta)}{S(2^{k-m-1}(2w)+n_0,2^{k-m-1}(2w+2)-1+n_0,\beta)}}\Bigg).
\end{aligned} \tag{4.12}$$

Notation. In practice, U_m computes an approximation $\widetilde{\theta}_{w,m}$, which results in an error in the state preparation. The deviation between the ideal and approximate states at step m is bounded as follows:

$$\begin{aligned}
&\left\|\sum_w c_w|w\rangle\,(\cos(\theta_{w,m})|0\rangle+\sin(\theta_{w,m})|1\rangle)\,|0\rangle^{\otimes(k-m-1)}|0\rangle-\sum_w c_w|w\rangle\,\Big(\cos\Big(\widetilde{\theta}_{w,m}\Big)|0\rangle+\sin\Big(\widetilde{\theta}_{w,m}\Big)|1\rangle\Big)\,|0\rangle^{\otimes(k-m-1)}|0\rangle\right\|\\
&=\left\|\sum_w c_w|w\rangle\,\Big[\Big(\cos\Big(\widetilde{\theta}_{w,m}\Big)-\cos(\theta_{w,m})\Big)|0\rangle+\Big(\sin\Big(\widetilde{\theta}_{w,m}\Big)-\sin(\theta_{w,m})\Big)|1\rangle\Big]\right\|\\
&=\sqrt{\sum_w c_w^2\cdot 4\sin^2\left(\frac{\widetilde{\theta}_{w,m}-\theta_{w,m}}{2}\right)}\\
&\le\max_w|\widetilde{\theta}_{w,m}-\theta_{w,m}|.
\end{aligned} \tag{4.13}$$

To achieve overall precision ε, each $\theta_{w,m}$ must be computed with an error of at most ε/k over all m. The $\mathcal{CR}$ gate is decomposed into controlled-R_x rotations conditioned on the qubits encoding $\theta_{w,m}$, followed by a phase gate, requiring $\mathcal{O}(\log(k/\varepsilon))$ gates and no ancilla qubits. Therefore, the dominant resource cost comes from implementing U_m to an accuracy of ε/k.

Theorem 4.4 (Implementation of oracle U_m (Methods, Theorem 2))**.** *The gate U_m, as defined in Definition 4.3, can be implemented such that each rotation angle is estimated to within error ε/k, using a number of gates and ancilla qubits bounded by*

$$\text{Poly}\Big(\log(1/\varepsilon),\, \log(n_1),\, \log\tfrac{1}{|1-\beta|},\, \beta,\, v\Big), \tag{4.14}$$

where v denotes the number of significant digits used to represent β.

Proof. The angle $\theta_{w,m}$, represented with $\log(k/\varepsilon)$ qubits, is determined using a bisection procedure applied iteratively on each qubit. For a trial angle θ_g, we compare

$$\sin^2(\theta_g)S(2^{k-m-1}2w+n_0, 2^{k-m-1}(2w+2)-1+n_0,\beta), \tag{4.15}$$

$$\text{with}\quad S(2^{k-m-1}(2w+1)+n_0, 2^{k-m-1}(2w+2)-1+n_0,\beta). \tag{4.16}$$

This comparison decides whether $\theta_{w,m} > \theta_g$. After $\log(k/\varepsilon)$ comparison, $\theta_{w,m}$ is computed to within error ε/k. Define

$$d = \sin^2(\theta_g)S(2^{k-m-1}\cdot 2w+n_0,\ 2^{k-m-1}(2w+2)-1+n_0,\beta) - S(2^{k-m-1}(2w+1)+n_0,\ 2^{k-m-1}(2w+2)-1+n_0,\beta). \tag{4.17}$$

Since $S(a,b,\beta) \geq n_1^{-\beta}$ and relevant angles θ satisfy $\theta + \frac{\varepsilon}{k} < \frac{\pi}{4}$, consider two angles differing by at least ε/k. The difference in $\sin^2\theta$:

$$\sin^2(\theta+\tfrac{\varepsilon}{k}) - \sin^2(\theta) = (\sin\big(\theta+\tfrac{\varepsilon}{k}\big)-\sin(\theta))(\sin\big(\theta+\tfrac{\varepsilon}{k}\big)+\sin(\theta)) > \tfrac{\sqrt{2}\varepsilon}{2k}\sin\big(\tfrac{\varepsilon}{k}\big) > \tfrac{\varepsilon^2}{2k^2}. \tag{4.18}$$

Thus, angles differing by ε/k yield a difference in d of at least $\frac{\varepsilon^2}{2k^2}n_1^{-\beta}$. To distinguish such angles correctly, the error in d must be less than $\frac{\varepsilon^2}{2k^2}n_1^{-\beta}$. Since $S(a,b,\beta) \leq n_1$, requirements:

$$\textbf{(1)}\ \text{computing } S \text{ to precision } \tfrac{\varepsilon^2 n_1^{-\beta}}{12k^2}, \qquad \textbf{(2)}\ \sin^2\theta_g \text{ to precision } \tfrac{\varepsilon^2 n_1^{-1-\beta}}{12k^2},$$

ensures the total error meets this requirement.

Step 1: Calculation of $\sin^2(\theta)$. Using the expansion

$$\sin^2(\theta) = \frac{1-\cos(2\theta)}{2} = \sum_{n=1}^{\infty} d_n\theta^{2n}, \qquad d_n = \frac{(-1)^{n+1}2^{2n}}{2(2n)!}, \tag{4.19}$$

we approximate $\sin^2(\theta)$ for $\theta < \frac{\pi}{4}$. Truncating after $l_2 > 5$ terms, the error is:

$$\frac{(\frac{\pi}{2})^{2l_2+2}}{2(2l_2+2)!} < \frac{(\frac{\pi}{2})^{2l_2+2}}{(\frac{2l_2+2}{e})^{2l_2+2}} = (\frac{e\pi}{4l_2+4})^{2l_2+2} < (\frac{1}{2})^{l_2}, \tag{4.20}$$

using Stirling's approximation. To achieve precision $\frac{\varepsilon^2 n_1^{-1-\beta}}{24k^2}$, we choose:

$$l_2 = 2\log\Big(\frac{k}{\varepsilon}\Big) + (1+\beta)\log(n_1) + 5. \tag{4.21}$$

Storing coefficients d_n to precision $\frac{\varepsilon^2 n_1^{-1-\beta}}{96k^2}$ ensures that the rounding error is bounded by:

$$\frac{\varepsilon^2 n_1^{-1-\beta}}{96k^2}(\frac{\pi}{4}+(\frac{\pi}{4})^2+\cdots) < \frac{\varepsilon^2 n_1^{-1-\beta}}{24k^2}. \tag{4.22}$$

Thus, the total truncation and rounding error is within $\frac{\varepsilon^2 n_1^{-1-\beta}}{12k^2}$, satisfying the requirement **(1)**.

The expression approximating $\sin^2(\theta)$ is a polynomial of degree $2l_2$. The parameter θ is encoded using $\log\left(\frac{k}{\varepsilon}\right)$ qubits, the coefficients require $\log\left(\frac{\varepsilon^2 n_1^{-1-\beta}}{96k^2}\right)$ qubits, and the output requires $\frac{\varepsilon^2 n_1^{-1-\beta}}{12k^2}$ qubits. By Lemma 3.1, the gate complexity is

$$4\log^2(\frac{k}{\varepsilon})l_2^2 + 4\log\left(\frac{k}{\varepsilon}\right)\log\left(\frac{96k^2 n_1^{1+\beta}}{\varepsilon^2}\right)l_2^2 + 2l_2\log\left(\frac{12k^2 n_1^{1+\beta}}{\varepsilon^2}\right) = \text{Poly}(\log(n_1), \log(1/\varepsilon), \beta), \tag{4.23}$$

and the ancilla qubit count is

$$4l_2\log\left(\frac{k}{\varepsilon}\right) + \log\left(\frac{96k^2 n_1^{1+\beta}}{\varepsilon^2}\right) = \text{Poly}(\log(n_1), \log(1/\varepsilon), \beta). \tag{4.24}$$

Step 2: Estimation of S. We will prove Lemma 4.7, which states that the partial sum $S(a, b, \beta)$ can be computed to precision ϵ with:

$$\text{Poly}\left(\log(1/\epsilon), \log n_1, \log\left(\frac{1}{|1-\beta|}\right), v\right) \tag{4.25}$$

gates and ancilla qubits, where v denotes the significant digits of β. To satisfy requirement **(2)**, we set $\epsilon = \frac{\varepsilon^2 n_1^{-\beta}}{12k^2}$ and the complexity becomes:

$$\text{Poly}\left(\log\left(\frac{12k^2 n_1^{\beta}}{\varepsilon^2}\right), \log(n_1), \log\left(\frac{1}{|1-\beta|}\right), v\right) = \text{Poly}\left(\log(1/\varepsilon), \log(n_1), \log\left(\frac{1}{|1-\beta|}\right), \beta, v\right).$$

Conclusively, we give a total count of required resourced for constructing U_m. The bisection procedure requires $\lceil \log(k/\varepsilon) \rceil$ comparisons to compute $\theta_{w,m}$ to precision ε/k. Each comparison involves two evaluations of S (for the partial sums in (4.15) and (4.16)) and one evaluation of $\sin^2\theta_g$. Thus, the total resource cost for U_m is multiplied by a factor of $\mathcal{O}(\log(k/\varepsilon))$, which is absorbed into the overall polynomial complexity. Combining the resource costs from Steps 1 and 2 (concluded in Equation (4.23),(4.24),(4.25)), the total gate and ancilla qubit complexity for implementing the oracle U_m is bounded by

$$\text{Poly}\Big(\log(1/\varepsilon),\ \log(n_1),\ \log\tfrac{1}{|1-\beta|},\ \beta,\ v\Big), \tag{4.26}$$

as stated. As a preliminary step of Lemma 4.7, an approximation of S is introduced in Proposition 4.5. ☐

Proposition 4.5 (Euler–Maclaurin approximation of S)**.** *For $\beta > 0$, $\beta \neq 1$, and integers $a \leq b$, there exists an approximation $\widetilde{S(a, b, \beta)}$ to the partial sum $S(a, b, \beta) = \sum_{n=a}^{b} n^{-\beta}$, such that*

$$\left|S(a, b, \beta) - \widetilde{S(a, b, \beta)}\right| < \frac{\epsilon}{2}, \tag{4.27}$$

where

$$\widetilde{S(a, b, \beta)} = \frac{1}{1-\beta}(b^{1-\beta} - a^{1-\beta}) + \frac{a^{-\beta} + b^{-\beta}}{2} - \sum_{r=1}^{l_3} \frac{B_{2r}\Gamma(\beta + 2r - 1)}{(2r)!\,\Gamma(\beta)}(b^{-\beta-2r+1} - a^{-\beta-2r+1}), \tag{4.28}$$

with $l_3 = \lceil \frac{1}{2} log_{2\pi} \frac{8}{\epsilon} \rceil$, provided $a > \lceil \beta + 2m \rceil$.

Proof. Recall the Euler–Maclaurin summation formula:

$$\sum_{k=a}^{b} f(k) = \int_a^b f(x)\,\mathrm{d}x + \frac{f(a) + f(b)}{2} + \sum_{r=1}^{l_3} \frac{B_{2r}}{(2r)!}\Big(f^{(2r-1)}(b) - f^{(2r-1)}(a)\Big) + R_{l_3}, \tag{4.29}$$

where B_{2r} are Bernoulli Numbers, and the remainder is given by

$$R_{l_3} = \frac{(-1)^{l_3+1}}{(2l_3)!}\int_a^b B_{2l_3}(x - \lfloor x \rfloor)\, f^{(2l_3)}(x)\,\mathrm{d}x. \tag{4.30}$$

Setting $f(x) = x^{-\beta}$, we have

$$S(a,b,\beta) = \int_a^b x^{-\beta}\,\mathrm{d}x + \frac{a^{-\beta}+b^{-\beta}}{2} + \sum_{r=1}^{l_3} \frac{B_{2r}}{(2r)!}(-1)^{2r-1}\Big(\prod_{i=0}^{2r-2}(\beta+i)\Big)(b^{-\beta-2r+1} - a^{-\beta-2r+1}) + R_{l_3}$$
$$= \frac{1}{1-\beta}(b^{1-\beta} - a^{1-\beta}) + \frac{a^{-\beta}+b^{-\beta}}{2} - \sum_{r=1}^{l_3} \frac{B_{2r}\Gamma(\beta+2r-1)}{(2r)!\,\Gamma(\beta)}(b^{-\beta-2r+1} - a^{-\beta-2r+1}) + R_{l_3}, \tag{4.31}$$

in which the reminder term is given by

$$R_{l_3} = \frac{(-1)^{l_3+1}}{(2l_3)!}\int_a^b B_{2l_3}(x - \lfloor x \rfloor)\,\frac{\Gamma(\beta+2l_3)}{\Gamma(\beta)}x^{-\beta-2l_3}\,\mathrm{d}x. \tag{4.32}$$

Using the bound on Bernoulli numbers:

$$\frac{2(2l_3)!}{(2\pi)^{2l_3}}\frac{1}{1-2^{-2l_3}} < |B_{2l_3}| = \frac{2(2l_3)!}{(2\pi)^{2l_3}}\zeta(2l_3) < \frac{2(2l_3)!}{(2\pi)^{2l_3}}\frac{1}{1-2^{1-2l_3}}, \tag{4.33}$$

we obtain

$$|R_{l_3}| \le \frac{|B_{2l_3}|}{(2l_3)!}\beta(\beta+1)\cdots(\beta+2l_3-1)\int_a^b x^{-\beta-2l_3}\,dx$$
$$= \left|\frac{\beta(\beta+1)\cdots(\beta+2l_3-1)B_{2l_3}(a^{-\beta-2l_3+1} - b^{-\beta-2l_3+1})}{(2l_3)!(\beta+2l_3-1)}\right|$$
$$< \frac{\beta(\beta+1)\cdots(\beta+2l_3-2)2a^{-\beta-2l_3+1}}{(2\pi)^{2l_3}(1-2^{1-2l_3})}$$
$$< \frac{4}{(2\pi)^{2l_3}}\frac{\beta(\beta+1)\cdots(\beta+2l_3-2)}{a^{\beta+2l_3-1}}. \tag{4.34}$$

Thus, choosing $l_3 = \lceil \frac{1}{2}\log_{2\pi}\frac{8}{\epsilon} \rceil$ guarantees $|R_{l_3}| < \frac{\epsilon}{2}$, provided $a > \lceil \beta + 2l_3 \rceil$. □

Corollary 4.6. *The Bernoulli numbers satisfy $B_0 = 1$, and for all $q \ge 1$,*

$$\sum_{p=0}^{q}\binom{q+1}{p}B_p = 0. \tag{4.35}$$

The Bernoulli numbers can be computed efficiently using the Akiyama–Tanigawa algorithm[3], with a complexity $O(p^2)$ for computing B_p.

Lemma 4.7 ($\widehat{S(a,b,\beta)}$ calculation oracle)**.** *Let $\mathcal{S}(\beta)$ be an oracle implementing*

$$\sum_{n_0 \le a \le b \le n_1 - 1}\alpha_{ab}|a\rangle|b\rangle|0\rangle \mapsto \sum_{n_0 \le a \le b \le n_1 - 1}\alpha_{ab}|a\rangle|b\rangle|\widehat{S(a,b,\beta)}\rangle, \tag{4.36}$$

where $\widehat{S(a,b,\beta)}$ is an $\frac{\epsilon}{2}$-approximation of $\widetilde{S(a,b,\beta)}$ as defined in Equation (4.28), and therefore an ϵ-approximation of $S(a,b,\beta)$. Then $\mathcal{S}(\beta)$ can be constructed using

$$\mathrm{Poly}(\log(1/\epsilon), \log(n_1), \log\big(\tfrac{1}{|1-\beta|}\big), v) \tag{4.37}$$

gates and ancilla qubits, where $l_3 = \lceil \frac{1}{2}\log_{2\pi}\frac{8}{\epsilon} \rceil$, $n_0 > \beta + 2l_3$, and v is the number of significant digits of β.

Proof sketch. We compute $\widetilde{S(a,b,\beta)}$, as defined in Equation (4.28), with an error of at most $\epsilon/2$ by evaluating its exponential terms and finite Bernoulli-series sums:

$$\widetilde{S(a,b,\beta)} = b^{-\beta-2l_3+1}\Big(\frac{1}{1-\beta}b^{2l_3} + \frac{1}{2}b^{2l_3-1} - \sum_{r=1}^{l_3}\frac{B_{2r}\Gamma(\beta+2r-1)}{(2r)!\,\Gamma(\beta)}b^{2l_3-2r}\Big)$$
$$- a^{-\beta-2l_3+1}\Big(\frac{1}{1-\beta}a^{2l_3} - \frac{1}{2}a^{2l_3-1} - \sum_{r=1}^{l_3}\frac{B_{2r}\Gamma(\beta+2r-1)}{(2r)!\,\Gamma(\beta)}a^{2l_3-2r}\Big)$$
$$= b^{-\beta-2l_3+1}P_b(\beta) - a^{-\beta-2l_3+1}P_a(\beta), \tag{4.38}$$

where

$$P_b(\beta) = \frac{1}{1-\beta}b^{2l_3} + \frac{1}{2}b^{2l_3-1} - \sum_{r=1}^{l_3} \frac{B_{2r}\Gamma(\beta+2r-1)}{(2r)!\,\Gamma(\beta)} b^{2l_3-2r},$$

$$P_a(\beta) = \frac{1}{1-\beta}a^{2l_3} - \frac{1}{2}a^{2l_3-1} - \sum_{r=1}^{l_3} \frac{B_{2r}\Gamma(\beta+2r-1)}{(2r)!\,\Gamma(\beta)} a^{2l_3-2r}.$$

Define the r-th term in the sum of $P_b(\beta)$:

$$T_b(r) = \frac{B_{2r}\,\Gamma(\beta+2r-1)}{(2r)!\,\Gamma(\beta)}\,b^{2l_3-2r}.$$

Then the ratio of consecutive terms is

$$\left|\frac{T_b(r+1)}{T_b(r)}\right| = \left|\frac{B_{2r+2}}{B_{2r}}\right| \frac{(\beta+2r-1)(\beta+2r)}{(2r+2)(2r+1)}\frac{1}{b^2}.$$

Using the bounds of Bernoulli number

$$\frac{2(2r)!}{(2\pi)^{2r}}\frac{1}{1-2^{-2r}} < |B_{2r}| = \frac{2(2r)!}{(2\pi)^{2r}}\zeta(2r) < \frac{2(2r)!}{(2\pi)^{2r}}\frac{1}{1-2^{1-2r}}, \tag{4.39}$$

for $b > \beta + 2l_3$, we obtain:

$$\left|\frac{T_b(r+1)}{T_b(r)}\right| < \frac{(\beta+2r-1)(\beta+2r)\big(1-2^{-2r}\big)}{(2\pi)^2 b^2\big(1-2^{1-2(r+1)}\big)} < \frac{1}{2\pi^2}.$$

Thus the Bernoulli series decreases geometrically,

$$\left|\sum_{r=1}^{l_3} \frac{B_{2r}\Gamma(\beta+2r-1)}{(2r)!\Gamma(\beta)} b^{2l_3-2r}\right| < \left|\frac{B_2\beta b^{2l_3-2}}{2\left(1-\frac{1}{2\pi^2}\right)}\right| < \frac{\beta b^{2l_3-2}}{6} < \frac{b^{2l_3-1}}{2}. \tag{4.40}$$

By combining the fact that $\frac{b^{2l_3-1}}{2} < \frac{b^{2l_3}}{2|1-\beta|}$, we obtain

$$|P_b(\beta)| < \frac{2b^{2l_3}}{|1-\beta|}, \tag{4.41}$$

and similarly,

$$|P_a(\beta)| < \frac{2a^{2l_3}}{|1-\beta|}. \tag{4.42}$$

Hence, to achieve overall precision $\frac{\epsilon}{2}$, it suffices to:

1. Compute $a^{-\beta-2l_3+1}$ and $b^{-\beta-2l_3+1}$ to precisions:

$$\frac{\epsilon|1-\beta|}{16a^{2l_3}} \quad \text{and} \quad \frac{\epsilon|1-\beta|}{16b^{2l_3}}, \tag{4.43}$$

respectively. We will prove in Proposition 4.8 that these require at most

$$\mathrm{Poly}(\log(n_1), \log\Big(\tfrac{1}{|1-\beta|}\Big), \log(1/\epsilon), v) \tag{4.44}$$

gates and ancilla qubits.

2. Compute the polynomial–Bernoulli sums $P_a(\beta)$ and $P_b(\beta)$ to precision $\frac{\epsilon}{8}$, since $a^{-\beta-2l_3+1}$ and $b^{-\beta-2l_3+1}$ are both smaller than 1. We will prove in Proposition 4.9 that these require:

$$\mathrm{Poly}(\log(1/\epsilon), \log(n_1), \log\Big(\tfrac{1}{|1-\beta|}\Big), v) \tag{4.45}$$

gates and ancilla qubits.

3. Perform final multiplications and additions to combine terms. As the exponential terms and Bernoulli-series sums require at most

$$\log\left(\frac{16n_1^{2l_3}}{|1-\beta|\epsilon}\right), \qquad \log\left(\frac{2n_1^{2l_3}}{|1-\beta|}\right)+\log\left(\frac{8}{\epsilon}\right)$$

significant qubits, respectively.

Conclusively, the total number of required gates and ancilla qubits can be bounded by

$$\mathrm{Poly}(\log(1/\epsilon), \log(n_1), \log\left(\tfrac{1}{|1-\beta|}\right), v), \tag{4.46}$$

given that $l_3 = \lceil \frac{1}{2}\log_{2\pi}\frac{8}{\epsilon}\rceil$. Hence the Lemma follows. Next, we detail Proposition 4.8 and Proposition 4.9. □

Proposition 4.8. *The oracles*

$$\sum_{b=n_0}^{n_1-1}\alpha_b|b\rangle|0\rangle \mapsto \sum_{b=n_0}^{n_1-1}\alpha_b|b\rangle|\widetilde{b^{-\beta-2l_3+1}}\rangle \quad \text{and} \quad \sum_{a=n_0}^{n_1-1}\alpha_a|a\rangle|0\rangle \mapsto \sum_{a=n_0}^{n_1-1}\alpha_a|a\rangle|\widetilde{a^{-\beta-2l_3+1}}\rangle \tag{4.47}$$

can be constructed with error bounds

$$|\widetilde{b^{-\beta-2l_3+1}} - b^{-\beta-2l_3+1}| < \frac{\epsilon|1-\beta|}{16b^{2l_3}}, \qquad |\widetilde{a^{-\beta-2l_3+1}} - a^{-\beta-2l_3+1}| < \frac{\epsilon|1-\beta|}{16a^{2l_3}}, \tag{4.48}$$

using

$$\mathrm{Poly}(\log(n_1), \log\left(\tfrac{1}{|1-\beta|}\right), \log(1/\epsilon), v) \tag{4.49}$$

gates and ancilla qubits, where $n_0 > \beta + 2l_3$, and v denotes the number of significant digits of β.

Proof. We focus on constructing the oracle for $\widetilde{b^{-\beta-2l_3+1}}$; the case for a is analogous. Writing $b^{-\beta-2l_3+1} = 2^{-(\beta+2l_3-1)\log b}$, the computation proceeds by approximating $\log b$, multiplying by $-(\beta+2l_3-1)$, and exponentiating. We require the approximation errors from the logarithm and from the exponentiation to be both bounded by $\frac{\epsilon|1-\beta|}{32b^{2l_3}}$, so that the total error is bounded by $\frac{\epsilon|1-\beta|}{16b^{2l_3}}$.

Step 1: Approximating $\log(b)$. If $(\beta+2l_3-1)|\widetilde{\log(b)} - \log(b)| < \frac{1}{2}$, then

$$\begin{aligned}
&\left|2^{-(\beta+2l_3-1)\log(b)} - 2^{-(\beta+2l_3-1)\widetilde{\log(b)}}\right| \\
&= 2^{-(\beta+2l_3-1)\log(b)}\left|1 - 2^{-(\beta+2l_3-1)(\widetilde{\log(b)}-\log(b))}\right| \\
&< b^{-\beta-2l_3+1}(\beta+2l_3-1)\,|\widetilde{\log(b)} - \log(b)|,
\end{aligned} \tag{4.50}$$

since $|1-2^x| \le |x|$ for $|x| \le \frac{1}{2}$. To bound this error by $\frac{\epsilon|1-\beta|}{32b^{2l_3}}$, it suffices to compute $\log(b)$ to precision

$$\frac{\epsilon|1-\beta|}{32b^{2l_3}}\frac{b^{\beta+2l_3-1}}{(\beta+2l_3-1)} = \frac{b^{\beta-1}\epsilon|1-\beta|}{32(\beta+2l_3-1)} > \frac{1}{32}n_1^{-2}|1-\beta|\,\epsilon, \tag{4.51}$$

which satisfies $\frac{1}{32}n_1^{-2}|1-\beta|\,\epsilon(\beta+2l_3-1) < \frac{1}{32}\epsilon < \frac{1}{2}$. Therefore, we set the required qubit of precision as

$$p_1 = \log\left(32n_1^2\tfrac{1}{|1-\beta|}\frac{1}{\epsilon}\right) = 5 + 2\log(n_1) + \log\left(\tfrac{1}{|1-\beta|}\right) + \log\left(\tfrac{1}{\epsilon}\right). \tag{4.52}$$

By Lemma 3.2, computing $\log(b)$ to p_1 digits of precision requires

$$\mathcal{O}(\log(n_1)^3 p_1^2) \text{ gates and } \mathcal{O}(\log(n_1)^2 p_1) \text{ ancilla qubits.}$$

Step 2: Multiplication with $\beta + 2l_3 - 1$. Multiplying $\log b$ by $\beta + 2l_3 - 1$, which are represented to at most $\log(n_1) + p_1$ and $\log(n_0) + v$ bits respectively, requires:

$$(\log(n_1)+p_1)(\log(n_0)+v) \text{ gates and } (\log(n_1)+p_1) + (\log(n_0)+v) \text{ ancilla qubits,}$$

where v is the number of significant digits of β.

Step 3: Exponentiation. Decompose

$$b^{-\beta-2l_3+1} = 2^{\lfloor -(\beta+2l_3-1)\log(b) \rceil} \cdot 2^{\{-(\beta+2l_3-1)\log(b)\}}, \tag{4.53}$$

where $\lfloor y \rceil$ denote the nearest integer of y and $\{y\} = y - \lfloor y \rceil$ denote the fractional part. The fractional exponential is evaluated by:

$$2^x = \mathrm{e}^{\ln 2 \cdot x} = \sum_{n=0}^{\infty} \frac{(\ln 2)^n}{n!} x^n, \qquad -\frac{1}{2} \le x < \frac{1}{2}. \tag{4.54}$$

To bound the total error by $\frac{\epsilon|1-\beta|}{32b^{2l_3}}$, we target an error of

$$\frac{\epsilon|1-\beta|}{32b^{2l_3}} \frac{1}{\sqrt{2}b^{-(\beta+2l_3-1)}} = \frac{b^{\beta+2l_3-1}\epsilon|1-\beta|}{32\sqrt{2}b^{2l_3}} > \frac{1}{64} b^{\beta-1}|1-\beta|\epsilon > \frac{1}{64} n_1^{-1}|1-\beta|\epsilon. \tag{4.55}$$

in fractional exponential. Let

$$l_4 = \log\left(64 n_1 \tfrac{1}{|1-\beta|} \frac{1}{\epsilon}\right) = 6 + \log(n_1) + \log\left(\tfrac{1}{|1-\beta|}\right) + \log\left(\tfrac{1}{\epsilon}\right). \tag{4.56}$$

Because $|x| \le \frac{1}{2}$, truncating the exponential series at degree l_4 gives an error

$$\frac{2(\ln 2/2)^{l_4}}{(l_4)!} < \frac{1}{2^{l_4+1}} = \frac{1}{128} n_1^{-1}|1-\beta|\epsilon. \tag{4.57}$$

With coefficients stored with l_4+1 fractional qubits, the rounding error is at most

$$\frac{1}{128} n_1^{-1}|1-\beta|\epsilon(1 + \frac{1}{2} + \frac{1}{4} + \cdots) \le \frac{1}{128} n_1^{-1}|1-\beta|\epsilon \le \frac{1}{128} n_1^{\beta-1}|1-\beta|\epsilon, \tag{4.58}$$

yielding a total error at most $\frac{1}{64} n_1^{-1}|1-\beta|\epsilon$.

By Lemma 3.1, evaluating the degree-l_4 polynomial in the argument $\{-(\beta+2l_3-1)\log(b)\}$, which is represented with at most $(p_1 + \log(n_1) + \log(n_0) + v)$ bits through multiplication, and with coefficients stored with l_4+1 qubits, to achieve precision $\frac{\epsilon|1-\beta|}{32b^{2l_3}}$ requires

$$\begin{aligned} &l_4^2(p_1 + \log(n_0) + \log(n_1) + v)^2 + l_4^2(p_1 + \log(n_0) + \log(n_1) + v)(l_4+1) + \log\left(\frac{32b^{2l_3}}{|1-\beta|\epsilon}\right) l_4 \\ =&\mathrm{Poly}(\log(n_1), \log\left(\tfrac{1}{|1-\beta|}\right), \log\left(\tfrac{1}{\epsilon}\right), v) \end{aligned} \tag{4.59}$$

gates, and

$$2(p_1 + \log(n_0) + \log(n_1) + v)^2 l_4 + l_4 + 1 = \mathrm{Poly}(\log(n_1), \log\left(\tfrac{1}{|1-\beta|}\right), \log\left(\tfrac{1}{\epsilon}\right), v) \tag{4.60}$$

ancilla qubits. Thus, combine the three parts and the claimed resource bounds follow. □

Proposition 4.9. *The oracles*

$$\sum_{b=n_0}^{n_1-1} \alpha_b |b\rangle|0\rangle \mapsto \sum_{b=n_0}^{n_1-1} \alpha_b |b\rangle|P_b(\beta)\rangle \qquad \text{and} \qquad \sum_{a=n_0}^{n_1-1} \alpha_a |a\rangle|0\rangle \mapsto \sum_{a=n_0}^{n_1-1} \alpha_a |a\rangle|P_a(\beta)\rangle, \tag{4.61}$$

where

$$\begin{aligned} P_b(\beta) &= \frac{1}{1-\beta} b^{2l_3} + \frac{1}{2} b^{2l_3-1} - \sum_{r=1}^{l_3} \frac{B_{2r}\Gamma(\beta+2r-1)}{(2r)!\,\Gamma(\beta)} b^{2l_3-2r}, \\ P_a(\beta) &= \frac{1}{1-\beta} a^{2l_3} - \frac{1}{2} a^{2l_3-1} - \sum_{r=1}^{l_3} \frac{B_{2r}\Gamma(\beta+2r-1)}{(2r)!\,\Gamma(\beta)} a^{2l_3-2r}, \end{aligned} \tag{4.62}$$

can be constructed to precision $\frac{\epsilon}{8}$ using

$$\mathrm{Poly}(\log(1/\epsilon), \log(n_1), \log\left(\tfrac{1}{|1-\beta|}\right)) \tag{4.63}$$

gates and ancilla qubits, in which $n_0 > \beta + 2l_3$.

Proof. We focus on constructing the oracle for $P_b(\beta)$, as the case for $P_a(\beta)$ is analogous. All the coefficients are stored with precision $\frac{\epsilon}{16n_1^{2l_3}}$, so the total rounding is bounded by

$$\frac{\epsilon}{16n_1^{2l_3}}(b^{2l_3}+b^{2l_3-1}+\cdots)<\frac{\epsilon}{16n_1^{2l_3}}2b^{2l_3}\leq\frac{\epsilon}{8}. \tag{4.64}$$

Hence the required number of qubit for each coefficient is at most

$$p_2=\log\left(\frac{16n_1^{2l_3}}{\epsilon}\right)=\log(1/\epsilon)+4+2l_3\log(n_1). \tag{4.65}$$

The polynomial in Equation (4.62) has degree $2l_3$, with input b requiring at most $\log(n_1)$ qubits, and the output targeted to precision $\frac{\epsilon}{8}$ with absolute magnitude bounded by $\log\left(\frac{2}{|1-\beta|}n_1^{2l_3}\right)$. Lemma 3.1 implies that this computation can be performed using at most

$$\begin{aligned}&l_3^2\log^2(n_1)+l_3^2\log(n_1)p_2+l_3\big(\log\Big(\frac{2}{|1-\beta|}n_1^{2l_3}\Big)+\log\Big(\frac{8}{\epsilon}\Big)\big)\\=&\mathrm{Poly}\Big(\log(n_1),\log\big(\tfrac{1}{|1-\beta|}\big),\log\big(\tfrac{1}{\epsilon}\big)\Big)\end{aligned} \tag{4.66}$$

gates, and

$$2l_3\log(n_1)+p_2=\mathrm{Poly}(\log(n_1),\log(1/\epsilon)) \tag{4.67}$$

ancilla qubits. □

Corollary 4.10 (Extended state preparation)**.** *Let $\beta>0,\ \beta\neq1$. Define*

$$|\psi_0\rangle_\mathrm{e}=\frac{1}{C}\sum_{n=1}^{n_1-1}n^{-\beta/2}|n\rangle,\qquad C=\sqrt{\sum_{n=1}^{n_1-1}n^{-\beta}}, \tag{4.68}$$

where $c=\lceil\frac{1}{2}\log_{2\pi}\frac{24}{\varepsilon}\rceil$, n_0-1 is the smallest power of 2 greater than $\lceil\beta+2c\rceil$, and n_1-n_0 is a power of 2. Then the state $|\psi_0\rangle_e$ can be prepared on a gate-based quantum computer to precision $\varepsilon>0$ using

$$\mathrm{Poly}(\log(1/\varepsilon),\log(n_1),\log\big(\tfrac{1}{|1-\beta|}\big),\beta,v), \tag{4.69}$$

gates and ancilla qubits, where v is the number of significant digits used to represent β.

Proof. By Theorem 4.2, the truncated state

$$|\psi_0\rangle=\frac{1}{C}\sum_{n=n_0}^{n_1-1}n^{-\beta/2}|n\rangle \tag{4.70}$$

can be prepared to precision $\frac{\varepsilon}{3}$ with

$$\mathrm{Poly}(\log(1/\varepsilon),\log(n_1),\log\big(\tfrac{1}{|1-\beta|}\big),\beta,v) \tag{4.71}$$

gates and ancilla qubits, for $c=\lceil\frac{1}{2}\log_{2\pi}\frac{24}{\varepsilon}\rceil$ and n_0 greater than $\lceil\beta+2c\rceil+1$. Following the approach of direct amplitude splitting[4], we can also prepare

$$|\psi_1\rangle=\frac{1}{C_1}\sum_{n=1}^{n_0-1}n^{-\beta/2}|n\rangle, \tag{4.72}$$

to precision $\frac{\varepsilon}{3}$, using $\mathrm{Poly}(n_0,\log(1/\varepsilon))$ gates. An auxiliary qubit coherently combines the two states:

$$\frac{|0\rangle+\gamma|1\rangle}{\sqrt{\gamma^2+1}}|0\rangle\to\frac{|0\rangle|\psi_0\rangle+\gamma|1\rangle|\psi_1\rangle}{\sqrt{\gamma^2+1}}. \tag{4.73}$$

Applying a Hadamard gate gives:

$$\xrightarrow{H} \frac{(|0\rangle + |1\rangle)|\psi_0\rangle + \gamma(|0\rangle - |1\rangle)|\psi_1\rangle}{\sqrt{2\gamma^2+2}} = \frac{|0\rangle(|\psi_0\rangle + \gamma|\psi_1\rangle) + |1\rangle(|\psi_0\rangle - \gamma|\psi_1\rangle)}{\sqrt{2\gamma^2+2}}. \tag{4.74}$$

Here, $\gamma = \frac{C_1}{C} = \sqrt{\frac{S(1,n_0-1,\beta)}{S(n_0,n_1-1,\beta)}}$, where the denominator estimated via Euler-Maclaurin summation method and the numerator is computed directly. If the ancilla is measured in $|0\rangle$, the desired normalized superposition is obtained. If not, a correction step—using an additional register for comparison with n_0 and a controlled-Z operation-restores the target state. The comparator writes a 1 in an auxiliary qubit iff the index is smaller than n_0. The process is listed as follows:

$$(|\psi_0\rangle - \gamma|\psi_1\rangle)|0\rangle \xrightarrow{\text{comparator}} |\psi_0\rangle|0\rangle - \gamma|\psi_1\rangle|1\rangle \xrightarrow{\text{controlled-}Z} |\psi_0\rangle|0\rangle + \gamma|\psi_1\rangle|1\rangle \xrightarrow{\text{undo comparator}} (|\psi_0\rangle + \gamma|\psi_1\rangle)|0\rangle. \tag{4.75}$$

If γ is approximated by $\widetilde{\gamma}$, the error is bounded by

$$\begin{aligned}\left\| \frac{|\psi_0\rangle + \widetilde{\gamma}|\psi_1\rangle}{\sqrt{1+\widetilde{\gamma}^2}} - \frac{|\psi_0\rangle + \gamma|\psi_1\rangle}{\sqrt{1+\gamma^2}} \right\| &= \sqrt{2 - 2\frac{1+\gamma\widetilde{\gamma}}{\sqrt{(1+\gamma^2)(1+\widetilde{\gamma}^2)}}} = \sqrt{2 - 2\sqrt{1 - \frac{(\gamma-\widetilde{\gamma})^2}{(1+\gamma^2)(1+\widetilde{\gamma}^2)}}} \\ &< \sqrt{2-2\sqrt{1-(\gamma-\widetilde{\gamma})^2}} < \sqrt{2|\gamma-\widetilde{\gamma}|}.\end{aligned} \tag{4.76}$$

Thus, computing γ to precision $\frac{\varepsilon^2}{18}$ ensures an additional error at most $\frac{\varepsilon}{3}$. This can be achieved by estimating C with error $n_0^{-\beta}\frac{\varepsilon^2}{36}$ and C_1 with error $\frac{\varepsilon^2}{36}\frac{C^2}{C_1} > \frac{\varepsilon^2}{36}\frac{n_0^{-2\beta}}{n_0} = \frac{\varepsilon^2}{36}n_0^{-1-2\beta}$ using Euler-Maclaurin summation with $\mathrm{Poly}(\log(1/\varepsilon), \log\left(\frac{1}{|1-\beta|}\right), \beta, \log(n_1), v)$ classical computations. Hence, $|\psi_0\rangle_e$ can be prepared with accuracy ε using

$$\mathrm{Poly}(\log(1/\varepsilon), \log(n_1), \log\left(\tfrac{1}{|1-\beta|}\right), \beta, v) \tag{4.77}$$

gates and ancilla qubits. □

Theorem 4.11 (Initial state preparation (Methods, Theorem 3))**.** *Let $\beta > 0$, $\beta \neq 1$ and $N \in \mathbb{N}$. Define*

$$|\psi_0\rangle = \frac{1}{C}\sum_{n=1}^{N} n^{-\beta/2}|n\rangle, \qquad C = \sqrt{\sum_{n=1}^{N} n^{-\beta}}. \tag{4.78}$$

Then $|\psi_0\rangle$ can be prepared on a quantum computer to precision $\varepsilon > 0$, with success probability at least $(\frac{1}{2} - \frac{\varepsilon}{3})$. The required number of gates and ancilla qubits is bounded by

$$\mathrm{Poly}(\log(1/\varepsilon), \log(N), \log\left(\tfrac{1}{|1-\beta|}\right), \beta, v), \tag{4.79}$$

where v denotes the number of significant digits used to represent β.

Proof. Set $c = \left\lceil \frac{1}{2}\log_{2\pi}\left(\frac{144}{\varepsilon}\right)\right\rceil$, and choose n_0 so that $(n_0 - 1)$ is the smallest power of 2 larger than $\lceil \beta + 2c \rceil$. Let n_1 be the smallest integer larger than N with $n_1 - n_0$ a power of 2. By Corollary 4.10, the extended state

$$|\psi_0\rangle = \frac{1}{C}\sum_{n=1}^{n_1-1} n^{-\beta/2}|n\rangle \tag{4.80}$$

can be constructed to precision $\frac{\varepsilon}{6}$ using

$$\mathrm{Poly}(\log(1/\varepsilon), \log(n_1), \log\left(\tfrac{1}{|1-\beta|}\right), \beta, v) = \mathrm{Poly}(\log(1/\varepsilon), \log(N), \log\left(\tfrac{1}{|1-\beta|}\right), \beta, v) \tag{4.81}$$

gates and ancilla qubits.

Let P denote the projector onto the subspace spanned by $\{|n\rangle : 1 \le n \le N\}$, given by:

$$P = \sum_{n=1}^{N} |n\rangle\langle n|, \tag{4.82}$$

which can be implemented using an ancilla qubit, a comparator, and a measurement. After projection P, the system ideally collapses to the desired state

$$|\psi\rangle = \frac{1}{C}\sum_{n=1}^{N} n^{-\beta/2}|n\rangle \tag{4.83}$$

with ideal success possibility

$$\frac{\sum_{n=1}^{N} n^{-\beta}}{\sum_{n=1}^{n_1-1} n^{-\beta}} > \frac{N}{n_1-1} \geq \frac{N}{2(N-n_0)+n_0-1} \geq \frac{N}{2N-3} > \frac{1}{2}, \tag{4.84}$$

since $n^{-\beta}$ is positive and decreasing for $\beta > 0$.

In practice, the extended state is prepared to precision $\frac{\varepsilon}{6}$. Let $|\psi_0\rangle$ and $|\widetilde{\psi_0}\rangle_e$ denote the ideal and actual extended state. Then the ideal and actual post-selected states can be written as

$$|\psi_0\rangle = \frac{P|\psi_0\rangle_e}{\sqrt{\langle\psi_0|_e P|\psi_0\rangle_e}}, \qquad |\widetilde{\psi_0}\rangle = \frac{P|\widetilde{\psi_0}\rangle_e}{\sqrt{\langle\widetilde{\psi_0}|_e P|\widetilde{\psi_0}\rangle_e}}. \tag{4.85}$$

The precision in $|\widetilde{\psi_0}\rangle_e$ and the contractive property of P imply that

$$\|\sqrt{\langle\psi_0|_e P|\psi_0\rangle_e}|\psi_0\rangle - \sqrt{\langle\widetilde{\psi_0}|_e P|\widetilde{\psi_0}\rangle_e}|\widetilde{\psi_0}\rangle\| = \|P|\psi_0\rangle_e - P|\widetilde{\psi_0}\rangle_e\| \leq \||\psi_0\rangle_e - |\widetilde{\psi_0}\rangle_e\| \leq \tfrac{\varepsilon}{6}. \tag{4.86}$$

Furthermore, using the triangle inequality

$$|\langle\psi_0|_e P|\psi_0\rangle_e - \langle\widetilde{\psi_0}|_e P|\widetilde{\psi_0}\rangle_e| \leq |\langle\psi_0|_e P|\widetilde{\psi_0}\rangle_e - \langle\widetilde{\psi_0}|_e P|\widetilde{\psi_0}\rangle_e| + |\langle\psi_0|_e P|\psi_0\rangle_e - \langle\psi_0|_e P|\widetilde{\psi_0}\rangle_e| \leq \tfrac{\varepsilon}{3}. \tag{4.87}$$

Since $\langle\psi_0|_e P|\psi_0\rangle_e \geq \frac{1}{2}$, it follows that

$$\left|\sqrt{\langle\psi_0|_e P|\psi_0\rangle_e} - \sqrt{\langle\widetilde{\psi_0}|_e P|\widetilde{\psi_0}\rangle_e}\right| = \frac{|\langle\psi_0|_e P|\psi_0\rangle_e - \langle\widetilde{\psi_0}|_e P|\widetilde{\psi_0}\rangle_e|}{\sqrt{\langle\psi_0|_e P|\psi_0\rangle_e} + \sqrt{\langle\widetilde{\psi_0}|_e P|\widetilde{\psi_0}\rangle_e}} \leq \frac{\sqrt{2}\varepsilon}{3}. \tag{4.88}$$

Using the triangle inequality and combining the Equations (4.86) and (4.88),

$$\begin{aligned}
&\sqrt{\langle\psi_0|_e P|\psi_0\rangle_e}\||\psi_0\rangle - |\widetilde{\psi_0}\rangle\| = \|\sqrt{\langle\psi_0|_e P|\psi_0\rangle_e}|\psi_0\rangle - \sqrt{\langle\psi_0|_e P|\psi_0\rangle_e}|\widetilde{\psi_0}\rangle\| \\
\leq &\|\sqrt{\langle\psi_0|_e P|\psi_0\rangle_e}|\psi_0\rangle - \sqrt{\langle\widetilde{\psi_0}|_e P|\widetilde{\psi_0}\rangle_e}|\widetilde{\psi_0}\rangle\| + \|\sqrt{\langle\psi_0|_e P|\psi_0\rangle_e}|\widetilde{\psi_0}\rangle - \sqrt{\langle\widetilde{\psi_0}|_e P|\widetilde{\psi_0}\rangle_e}|\widetilde{\psi_0}\rangle\| \\
\leq &\frac{\varepsilon}{6} + \frac{\sqrt{2}\varepsilon}{3}.
\end{aligned} \tag{4.89}$$

Given $\langle\psi_e| P |\psi_e\rangle \geq \frac{1}{2}$, we have the stated bound

$$\||\psi_0\rangle - |\widetilde{\psi_0}\rangle\| \leq \tfrac{1+2\sqrt{2}}{3\sqrt{2}}\varepsilon < \varepsilon, \tag{4.90}$$

and success possibility satisfies

$$\langle\widetilde{\psi_0}|_e P|\widetilde{\psi_0}\rangle_e \geq \frac{1}{2} - \frac{\varepsilon}{3}. \tag{4.91}$$

Thus, the ideal state $|\psi_0\rangle$ can be prepared with precision ε with at least the success probability as claimed. □

Supplementary Note 5: Construction of the evolution operator

We now demonstrate the construction of the evolution operator.

Theorem 5.1 (Evolution operator construction (Methods, Theorem 4))**.** *Define the time evolution operator*

$$U(t) = \mathrm{e}^{-iH_0 t}, \qquad H_0 = \sum_{n=1}^{N} \log(n)|n\rangle\langle n|, \tag{5.1}$$

where the evolution time t is specified with u significant digits. Then $U(t)$ can be implemented to precision ξ using

$$\mathrm{Poly}(\log(N), \log(|t|), \log(1/\xi), u) \tag{5.2}$$

gates and ancilla qubits.

Proof. Consider an input state of the form

$$\sum_{n=1}^{N} \alpha_n |n\rangle|0\rangle, \tag{5.3}$$

where $|0\rangle$ represents the ancilla register. Applying the logarithm oracle L produces

$$\sum_{n=1}^{N} \alpha_n |n\rangle|\log(n)\rangle. \tag{5.4}$$

Multiplying $\log(n)$ by t yields

$$\sum_{n=1}^{N} \alpha_n |n\rangle|\log(n)t\rangle. \tag{5.5}$$

Applying controlled R_z rotations results in

$$\sum_{n=1}^{N} \alpha_n \mathrm{e}^{-i\log(n)t}\,|n\rangle|\log(n)t\rangle. \tag{5.6}$$

Uncomputing the multiplication and applying inverse logarithm oracle L^{-1} leaves

$$\sum_{n=1}^{N} \alpha_n \mathrm{e}^{-i\log(n)t}\,|n\rangle|0\rangle \;=\; U\sum_{n=1}^{N} \alpha_n |n\rangle|0\rangle. \tag{5.7}$$

In practice, the logarithm $\log(n)$ is approximated by $\widetilde{\log(n)}$ with error at most $\xi/|t|$. This ensures that

$$\left\|\sum_{n=1}^{N} \alpha_n \left(\mathrm{e}^{-i\widetilde{\log(n)}\,t} - \mathrm{e}^{-i\log(n)t}\right)|n\rangle\right\| \leq 2\sin\left(\frac{\xi}{2}\right) < \xi. \tag{5.8}$$

Thus, the distance between the achieved and ideal U is bounded by ξ. By Lemma 3.2, the logarithm oracle and its inverse can be realized using $\log^3(N)\log^2(|t|/\xi)$ gates and $\log^2(N)\log(|t|/\xi)$ ancilla qubits. Multiplying $\log(n)$ by t requires $(\log(|t|/\xi)+\log(N))\,u$ gate operations and $(\log(|t|/\xi)+\log(N))+u$ ancilla qubits. The controlled R_z gates can be implemented qubit by qubit with $\mathcal{O}(\log(N)+\log(|t|/\xi))$ gates. Combining these bounds, the total resource requirement for implementing $U(t)$ is

$$\mathrm{Poly}(\log(N),\, \log(|t|),\, \log(1/\xi),\, u) \tag{5.9}$$

gates and ancilla qubits, completing the proof. ∎

Supplementary Note 6: Order of $|\chi(s)|$ for $0 < \beta < 1$

Recall that

$$\chi(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right)\Gamma(1-s). \tag{6.1}$$

Thus,

$$|\chi(s)| = 2^{\beta}\pi^{\beta-1}|\sin\frac{\pi s}{2}||\Gamma(1-s)|. \tag{6.2}$$

We focus on $s = \beta + it$ with $0 < \beta < 1$,

$$\sin\left(\tfrac{\pi s}{2}\right) = \sin\left(\tfrac{\pi\beta}{2} + \tfrac{i\pi t}{2}\right) = \sin\left(\tfrac{\pi\beta}{2}\right)\cosh\left(\tfrac{\pi t}{2}\right) + i\cos\left(\tfrac{\pi\beta}{2}\right)\sinh\left(\tfrac{\pi t}{2}\right), \tag{6.3}$$

so we obtain

$$|\sin(\pi s)| = \sqrt{\sin^2(\tfrac{\pi\beta}{2})\cosh^2(\tfrac{\pi t}{2}) + \cos^2(\tfrac{\pi\beta}{2})\sinh^2(\tfrac{\pi t}{2})} = \sqrt{\sinh^2(\tfrac{\pi t}{2}) + \sin^2(\tfrac{\pi\beta}{2})} = \Theta(\mathrm{e}^{\pi|t|/2}). \tag{6.4}$$

Using Stirling's formula in the sector $|\arg z| \le \pi - \varepsilon$,

$$\Gamma(z) = \sqrt{2\pi}\, z^{\,z-\frac{1}{2}}\,\mathrm{e}^{-z}\big(1 + O(|z|^{-1})\big). \tag{6.5}$$

Taking absolute values gives

$$|\Gamma(z)| = \sqrt{2\pi}\,|z^{\,z-\frac{1}{2}}|\mathrm{e}^{-\operatorname{Re}(z)}\big(1 + O(|z|^{-1})\big). \tag{6.6}$$

Note that

$$|z^{z-\frac{1}{2}}| = \exp\big(\operatorname{Re}(\ln(z)(z - \tfrac{1}{2}))\big) = \exp\big(\ln(|z|)(\operatorname{Re}(z) - \tfrac{1}{2}) - \arg(z)\operatorname{Im}(z)\big) = |z|^{\operatorname{Re}(z)-\frac{1}{2}}\mathrm{e}^{-\arg(z)\operatorname{Im}(z)}. \tag{6.7}$$

Now we set $z = 1 - s = (1-\beta) - it$, then

$$|\Gamma(1-s)| = \sqrt{2\pi}\,|1-s|^{1/2-\beta}\mathrm{e}^{t\arg(z)}\mathrm{e}^{-(1-\beta)}\big(1 + O(|t|^{-1})\big). \tag{6.8}$$

For $t < 0$, $\arg z = \arctan\left(\frac{|t|}{1-\beta}\right)$, so

$$\exp(t\arg z) = \exp\left(t\operatorname{arccot}\tfrac{1-\beta}{|t|}\right) = \exp\left(t(\tfrac{\pi}{2} - \tfrac{1-\beta}{|t|} + \mathcal{O}(|t|^{-3}))\right) = \exp(-\pi|t|/2)\exp((1-\beta))(1 + \mathcal{O}(|t|^{-2})). \tag{6.9}$$

For $t > 0$, $\arg(z) = -\arctan\left(\frac{t}{1-\beta}\right)$, so

$$\exp(t\arg(z)) = \exp\left(-t\operatorname{arccot}\tfrac{1-\beta}{t}\right) = \exp\left((-t(\tfrac{\pi}{2} - \tfrac{1-\beta}{t} + \mathcal{O}(|t|^{-3}))\right) = \exp(-\pi t/2)\exp((1-\beta))(1 + \mathcal{O}(t^{-2})). \tag{6.10}$$

In both cases,

$$\exp(t\arg z) = \exp(-\pi|t|/2)\exp((1-\beta))(1 + \mathcal{O}(|t|^{-2})). \tag{6.11}$$

Therefore,

$$|\Gamma(1-s)| = \sqrt{2\pi}\,|1-s|^{1/2-\beta}\exp(-\pi|t|/2)\big(1 + O(|t|^{-1})\big). \tag{6.12}$$

Combining terms,

$$|\chi(s)| = \Theta(1)2^{\beta}\pi^{\beta-1}\exp(\pi|t|/2)\sqrt{2\pi}|1-s|^{1/2-\beta}\exp(-\pi|t|/2)(1 + O(|t|^{-1})), \tag{6.13}$$

The exponential terms cancel, leaving

$$|\chi(s)| = \Theta(|1-s|^{1/2-\beta}) = \Theta(|t|^{1/2-\beta}). \tag{6.14}$$

Supplementary Note 7: Upper bound of $\zeta'(s)$ for $0 < \beta < 1$

We establish upper bounds for the Dirichlet eta function $\eta(s)$ and its derivative $\eta'(s)$, defined as

$$\eta(s) = \sum_{n=1}^{\infty}(-1)^{n+1}n^{-s}, \qquad \eta^{'}(s) = -\sum_{n=1}^{\infty}(-1)^{n+1}\ln n\, n^{-s}. \tag{7.1}$$

Then $\eta(s)$ can be expressed as:

$$\eta(s)=\sum_{n=1}^{\infty}(-1)^{n+1}n^{-s}=\sum_{n=1}^{\infty}((2n-1)^{-s}-(2n)^{-s})=\sum_{n=1}^{\infty}s\int_{2n-1}^{2n}\nu^{-s-1}d\nu. \tag{7.2}$$

Taking the absolute value, we obtain

$$|\eta(s)|\leq|s|\sum_{n=1}^{\infty}\int_{2n-1}^{2n}\nu^{-\beta-1}d\nu<|s|\int_{1}^{\infty}\nu^{-\beta-1}d\nu=\frac{|s|}{\beta}. \tag{7.3}$$

Similarly, for the derivative $\eta'(s)$, we have

$$\eta^{'}(s)=-\sum_{n=1}^{\infty}(-1)^{n+1}(\ln n)n^{-s}=\sum_{n=1}^{\infty}(\ln{(2n)}(2n)^{-s}-\ln{(2n-1)}(2n-1)^{-s})=\sum_{n=1}^{\infty}\int_{2n-1}^{2n}(-s\nu^{-s-1}\ln\nu+\nu^{-s-1})d\nu. \tag{7.4}$$

Hence,

$$|\eta^{'}(s)|\leq|s|\int_{1}^{\infty}|\nu^{-s-1}|\ln\nu\,d\nu+\int_{1}^{\infty}|\nu^{-s-1}|\,d\nu=|s|\int_{1}^{\infty}\nu^{-\beta-1}\ln\nu\,d\nu+\int_{1}^{\infty}\nu^{-\beta-1}\,d\nu=\frac{|s|}{\beta^2}+\frac{1}{\beta}. \tag{7.5}$$

Since the zeta function is related to $\eta(s)$ by

$$\zeta(s)=\frac{1}{1-2^{1-s}}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^s}=\frac{1}{1-2^{1-s}}\eta(s). \tag{7.6}$$

Its derivative $\zeta'(s)$ is:

$$\zeta^{'}(s)=\frac{d}{ds}(\frac{1}{1-2^{1-s}}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^s})=\frac{\eta^{'}(s)(1-2^{1-s})-2^{1-s}\ln2\,\eta(s)}{(1-2^{1-s})^2}. \tag{7.7}$$

Using the previously established bounds

$$|\eta(s)|\leq\frac{|s|}{\beta},\qquad|\eta'(s)|\leq\frac{|s|}{\beta^2}+\frac{1}{\beta},$$

we obtain the bound for $\zeta'(s)$:

$$|\zeta^{'}(s)|\leq\frac{|s|}{|2^{1-\beta}-1|\beta}+\frac{2^{1-\beta}\ln2(|s|\beta+1)}{|2^{1-\beta}-1|^2\beta^2}=\mathcal{O}(\frac{|s|}{\beta|1-\beta|}+\frac{|s|\beta+1}{|1-\beta|^2\beta^2})=\text{Poly}(|t|,|1-\beta|^{-1},\beta^{-1}), \tag{7.8}$$

given that $|2^{1-\beta}-1|\geq\ln(2)|1-\beta|$ for all $0<\beta<1$.

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