Appendix A. Formal derivation of the Universal-Attractor / Null-Attractor equivalence

A.0 Notation and standing assumptions

Let $N \geq 2$ be the number of labeled atomic elements (sites, agents, points).

Let $\mathcal{S}$ denote the system state space (discrete counts or a continuous vector space).

Write S_N for the permutation group on N labels (relabelings of elements). A map or stochastic kernel T is permutation-invariant (or strictly impartial) iff

$$T(\pi \cdot s,\ \pi \cdot A) = T(s, A) \qquad \forall \pi \in S_N,$$

where $\pi \cdot s$ denotes the relabeled state and A a measurable set of states.

Null Attractor (NA). Informally, dynamics that have zero deterministic drift (continuous case) or uniform sampling / zero preference (discrete case), which yield maximum normalized entropy $S_d = 1$ in the discrete setting or an irreducible minimal steady variance in the continuous setting.

Universal Attractor (UA). The dynamical rule intended to enforce absolute uniformity (all components equal—zero relative variance) under the constraint of strict impartiality. The UA is a family of rules claimed to drive states to the perfectly uniform state

$$\mathcal{U} = \{x \in \mathcal{S} : x_1 = \cdots = x_N\}$$

(continuous) or to equal counts (discrete).

We will formalize UA as any permutation-invariant dynamics whose unique putative target is the uniform state. The claim to prove is that under strict impartiality the UA dynamics are mathematically identical to NA dynamics; i.e., they are equivariant and their invariant measures / stationary states coincide with those of the null dynamics.

All results below are stated and proved under the strict impartiality hypothesis (permutation invariance) and standard regularity conditions (continuity, existence/uniqueness of solutions or ergodicity of kernels where required).

A.1 Discrete case (multinomial / Markov sampling)

Definitions. Let, at each discrete time $t \in \mathbb{N}$, a single element $i \in \{1, \ldots, N\}$ be selected according to a probability vector

$$p(s_t) = \big(p_1(s_t), \ldots, p_N(s_t)\big),$$

which may depend on the current state s_t (selection counts, etc.). The transition of counts is multinomial / urn-like.

The dynamics are strictly impartial iff

$$p(\pi \cdot s) = \pi \cdot p(s) \qquad \forall \pi \in S_N.$$

In particular, impartiality implies that if the state is fully symmetric (all atomic components are equal in all respects except identity), then

$$\mathbf{p}(s) = \left(\frac{1}{N}, \ldots, \frac{1}{N}\right).$$

Lemma .1 (Symmetry implies uniform stationary distribution)**.** *Let P be a time-homogeneous Markov chain kernel on a finite state space $\mathcal{X}$ whose transition kernel is permutation-invariant under relabeling of components. Then any permutation-symmetric probability measure ν (i.e., $\nu(\pi \cdot A) = \nu(A)$) that is stationary for P has marginals that are uniform across labels. In particular, if P has a unique stationary measure, that stationary measure is permutation symmetric and the marginal distribution on labels is uniform.*

Proof. For finite $\mathcal{X}$, existence of stationary measures is standard. If P is permutation-invariant then, for any stationary ν,

$$\nu(A) = \sum_{x\in\mathcal{X}} \nu(x)\, P(x, A) = \sum_{x\in\mathcal{X}} \nu(\pi \cdot x)\, P(\pi \cdot x, \pi \cdot A) = \nu(\pi \cdot A).$$

Thus ν is permutation symmetric. Permutation symmetry of ν implies equal marginals for all labels, hence uniform marginals. □

Theorem .2 (Discrete UA = NA equivalence)**.** *Let the discrete dynamics be strictly impartial in the sense above and assume the dynamics are ergodic (unique stationary measure). Then the unique stationary distribution is the uniform (maximum-entropy) distribution across labels. Consequently, the UA (the rule that purports to enforce absolute uniformity by treating labels identically) has the same stationary distribution as the NA (the impartial, zero-preference sampling rule). Therefore, under strict impartiality and ergodicity,*

$$\mathrm{UA} = \mathrm{NA}.$$

Proof. Under strict impartiality, every relabeling of states leaves the transition kernel unchanged, so by Lemma A.1 the unique stationary measure is permutation symmetric and therefore gives equal marginal probabilities to each label. The uniform marginal yields maximum Shannon entropy $H = \ln N$ and normalized entropy $S_d = 1$. The NA (uniform sampling $p_i \equiv 1/N$) has the same stationary distribution. Hence the stationary behavior of any impartial UA dynamics coincides with that of NA. □

Remarks. The assumption of ergodicity (unique stationary measure) is natural for irreducible, aperiodic chains; without ergodicity one may obtain multiple invariant measures, but impartiality still forces any symmetric invariant measure to have uniform marginals (so the equivalence holds for symmetric invariant measures).

The result formalizes the intuition in Section 3 of the manuscript: impartiality forces uniform long-run frequencies, i.e., maximal normalized entropy.

A.2 Continuous case (mean-reverting linear SDE / Ornstein–Uhlenbeck family)

Setup. Consider $x(t) = (x_1(t), \dots, x_N(t)) \in \mathbb{R}^N$ evolving under the linear SDE

$$dx_i(t) = -k\,\big(x_i(t) - \mu(t)\big)\, dt + \epsilon\, dW_i(t), \qquad \mu(t) := \frac{1}{N}\sum_{j=1}^{N} x_j(t),$$

where $\{W_i(t)\}_{i=1}^{N}$ are independent standard Wiener processes, $k > 0$ and $\epsilon \geq 0$. This is the mean-coupled Ornstein–Uhlenbeck process used in the manuscript (Section 2). The drift is manifestly permutation-invariant: any relabeling of coordinates leaves the drift form invariant.

Define the empirical mean $m(t) = \mu(t)$ and the variance

$$\sigma^2(t) := \frac{1}{N}\sum_{i=1}^{N}\big(x_i(t) - \mu(t)\big)^2.$$

Lemma .3 (Variance evolution)**.** *For the SDE above,*

$$\frac{d}{dt}\mathbb{E}\big[\sigma^2(t)\big] = -2k\,\mathbb{E}\big[\sigma^2(t)\big] + \epsilon^2.$$

Sketch. Standard Itô calculus applied to $(x_i - \mu)^2$ and summing over i gives (because the noise terms are independent and mean zero) the deterministic ODE for the expectation as claimed; details follow the textbook derivation of the OU variance. □

Corollary .4 (Steady state variance)**.** *If $\epsilon > 0$ then the linear ODE for $\mathbb{E}[\sigma^2]$ has the unique globally attracting steady state*

$$\mathbb{E}[\sigma^2]_\infty = \frac{\epsilon^2}{2k}.$$

If $\epsilon = 0$ then $\mathbb{E}[\sigma^2] \to 0$ exponentially fast.

Proposition .5 (Permutation invariance forces NA behavior)**.** *Under the standing linear mean-coupled SDE above, the drift is permutation-invariant and on the perfectly symmetric subspace*

$$\mathcal{U} = \{x : x_1 = \cdots = x_N\},$$

the drift vanishes identically (because $x_i - \mu = 0$). Therefore, when $\epsilon > 0$ the dynamics remain stochastic and the steady-state variance is positive, determined solely by the noise floor ϵ. The UA target of absolute uniformity (collapse to $\mathcal{U}$ with $\sigma^2 \equiv 0$) is not achieved: instead the unique stationary measure supported on fluctuations about $\mathcal{U}$ is the Gaussian with covariance proportional to ϵ^2. The null dynamics (zero deterministic bias within the symmetric manifold) produce precisely the same steady-state fluctuation statistics.

Theorem .6 (Continuous UA = NA equivalence — linear SDE)**.** *Under the mean-coupled linear SDE above with $\epsilon > 0$ and finite $k > 0$, the permutation-invariant UA dynamics and the null (zero-bias) dynamics are statistically equivalent at stationarity: both yield the invariant Gaussian measure concentrated about $\mathcal{U}$ with variance $\epsilon^2/(2k)$. Consequently,*

$$\mathrm{UA} = \mathrm{NA}.$$

A.3 Nonlinear continuous dynamics (general permutation-invariant drift)

The linear SDE gives a clean closed-form equivalence. The physical claim in the manuscript is stronger: any strictly impartial deterministic rule (or SDE) that attempts to impose absolute uniformity will, under impartiality, be dynamically equivalent to a null attractor in the sense that the only permutation-symmetric invariant sets / measures are those that treat labels uniformly; stochastic forcing then prevents collapse to exact uniformity.

Theorem .7 (General impartial dynamics — invariant symmetric measures are uniform)**.** *Let the drift $F : \mathbb{R}^N \to \mathbb{R}^N$ and diffusion matrix $D(x)$ be S_N-equivariant (permutation-invariant). Suppose there exists a symmetric manifold $\mathcal{U}$. Then:*

1. *Any invariant probability measure ν which is permutation-symmetric has uniform marginals (as in Lemma A.1).*

2. *If stochastic forcing is nonzero in directions transverse to $\mathcal{U}$, no invariant measure can be supported exactly on $\mathcal{U}$ (i.e., exact collapse is prevented).*

3. *Therefore, the only symmetric invariant measures coincide with those produced by null dynamics (no label bias), and any UA dynamics under strict impartiality can at best match NA statistics at stationarity.*

Sketch. (1) follows from S_N-equivariance just as in the discrete case. (2) follows because nonzero noise transverse to $\mathcal{U}$ gives positive escape probability from any point of $\mathcal{U}$. (3) follows immediately by combining (1) and (2). □

A.4 Formal statement of the Universal-Attractor Paradox (UAP)

Let dynamics on N labeled components be strictly impartial (permutation invariant) and suppose stochastic forcing transverse to the uniform manifold is present (or, in the discrete case, selection is randomized by a symmetric kernel). Then any attractor that purports to enforce absolute uniformity (the UA) is dynamically indistinguishable from the Null Attractor (NA). In particular, absolute uniformity ($S_c = 0$ in continuous systems or $S_d = 0$ in discrete counts) cannot be achieved for $\epsilon > 0$ (continuous) or under symmetric random sampling (discrete); the stationary state for strictly impartial dynamics is the maximal-entropy / uniform state (discrete) or a steady-state distribution of transverse fluctuations with variance determined solely by the intrinsic noise (continuous).

Proof. The discrete part is Theorem A.1. The continuous linear part is Theorem A.2. The general case is Theorem A.3. Combined, these show that under strict impartiality the stationary invariant measures coincide with those of the null dynamics — hence UA = NA in all such impartial regimes. □

A.5 Corollaries and remarks

1. **Noise is the decisive factor.** The inability to reach exact uniformity is not a deep consequence of Liouville or conservation laws in general — it is a direct consequence of the presence of nonzero intrinsic fluctuations (noise) and the symmetry constraint. If $\epsilon = 0$ and the deterministic flow is contracting transversely to $\mathcal{U}$, absolute uniformity can be achieved; but such flows must break typical time-reversal / Hamiltonian constraints and are not generically available in natural closed systems with quantum or thermal fluctuations. (This corrects the Liouville misattribution in the first draft; see main text and Section 2.3 of the manuscript, *The Universal and Null Attractors.*)

2. **Breaking impartiality allows escape.** Any deviation from strict permutation invariance — e.g. an infinitesimal bias or symmetry-breaking term — can change the stationary measure and allow the system to develop non-uniform order (this is the basis for SSB discussions in Sections 4–5). Formally, if the drift or sampling kernel contains terms that are not S_N-equivariant, then the unique stationary measure need not be symmetric and UA can differ from NA.

3. **Generality beyond linear OU.** The linear OU case is the clearest demonstration because of closed-form variance; the more general results above show the same conclusion under broad regularity conditions.

A.6 Concluding mathematical remark

The proofs given are constructive and elementary: permutation invariance constrains invariant measures to be symmetric, and nonzero stochastic forcing prevents support collapsing onto the uniform manifold. The equality UA = NA follows directly from these two facts. The appendix above provides a rigorous formalization of the intuition and simulation evidence given in your manuscript (cf. Sections 2–4).