Supplemental Material

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A. Dynamic Modulus of a Generalized Rouse Network

We generalize the Rouse model from a linear chain to a connected network of N beads with arbitrary configurations [1–3]. The position of bead i follows the Langevin equation:

$$\zeta \dot{\boldsymbol{R}}_i = -kL_{ij}\boldsymbol{R}_j + \boldsymbol{\xi}_i \tag{S1}$$

Here, ζ is the friction coefficient and k is the spring constant, which is related to the Kuhn length b by $k = 3k_BT/b^2$ where k_B is the Boltzmann constant and T is the temperature. The random force $\boldsymbol{\xi}_i$ is generated by thermal fluctuation and satisfies the fluctuation-dissipation theorem (FDT) [4],

$$\langle \xi_{i,\alpha}(t)\xi_{j,\beta}(t')\rangle = 2\zeta k_BT\delta_{ij}\delta_{\alpha\beta}\delta(t-t'). \tag{S2}$$

Here δ_{ij} and $\delta_{\alpha\beta}$ are Kronecker delta functions, and $\delta(t-t')$ is Dirac delta function. The Greek letters represent the directions in the Cartesian coordinate system. The connectivity matrix L_{ij} is a $N \times N$ symmetric matrix defined as $L_{ij} = \delta_{ij}\sum_k A_{ik} - A_{ij}$, where A_{ij} is the adjacency matrix with $A_{ij} = 1$ if beads i and j are connected and 0 otherwise. The connectivity matrix describes the connectivity between beads with nonzero eigenvalues $\lambda_p \geq 0$ and orthonormal real eigenvectors $|u_p\rangle$. We project the $N \times 3$ position vector $|\boldsymbol{R}\rangle$ onto the eigenmodes such that $\boldsymbol{c}_p = \langle u_p|\boldsymbol{R}\rangle$, a 1×3 vector that satisfies

$$\zeta \dot{\boldsymbol{c}}_p = -k\lambda_p \boldsymbol{c}_p + \boldsymbol{f}_p, \tag{S3}$$

where $\boldsymbol{f}_p = \langle u_p|\boldsymbol{\xi}\rangle$, which also satisfies the FDT,

$$\langle f_{p,\alpha}(t)f_{q,\beta}(t')\rangle = 2\zeta k_BT\delta_{pq}\delta_{\alpha\beta}\delta(t-t'). \tag{S4}$$

We introduce a step shear with strain γ at $t = 0$. The shear deformation is applied along the x-direction with a gradient along the y-direction, transforming the bead positions as:

$$x_i(0^+) = x_i(0^-) + \gamma y_i(0^-), \quad y_i(0^+) = y_i(0^-), \quad z_i(0^+) = z_i(0^-). \tag{S5}$$

The shear stress σ_{xy}, defined as force per unit area, can be written as [5]

$$\sigma_{xy} = -\frac{1}{V}\sum_i F_{i,x}y_i = \frac{k}{V}\langle x|L|y\rangle, \tag{S6}$$

where $F_{i,x}$ is the total force on monomer i in the x-direction and V is the system volume. Decomposing the position vectors using the eigenvectors of L, where $|x\rangle = \sum_p c_{px}|u_p\rangle$ and $|y\rangle = \sum_q c_{qy}|u_q\rangle$, we obtain

$$\sigma_{xy} = \frac{k}{V}\sum_{p,q} c_{px}c_{qy}\langle u_p|L|u_q\rangle = \frac{k}{V}\sum_p \lambda_p c_{px}c_{py}, \tag{S7}$$

where we used the orthogonality $\langle u_p|u_q\rangle = \delta_{pq}$ and the eigenvalue equation $L|u_p\rangle = \lambda_p|u_p\rangle$. Under the step strain at $t = 0$, the initial conditions can be rewritten as

$$\begin{aligned} c_{py}(0^+) &= c_{py}(0^-), \\ c_{px}(0^+) &= c_{px}(0^-) + \gamma c_{py}(0^-). \end{aligned} \tag{S8}$$

This leads to the initial cross-correlation:

$$\langle c_{px}(0^+)c_{py}(0^+)\rangle = \gamma\langle c_{py}^2\rangle = \gamma\frac{k_BT}{k\lambda_p}, \tag{S9}$$

where the equilibrium variance $\langle c_{py}^2 \rangle = k_B T/(k\lambda_p)$ follows from equipartition theorem. Taking the time derivative of $\langle c_{px} c_{py} \rangle$ and using Eq. (S3):

$$\frac{d}{dt}\langle c_{px} c_{py} \rangle = \langle \dot{c}_{px} c_{py} \rangle + \langle c_{px} \dot{c}_{py} \rangle = -\frac{2k\lambda_p}{\zeta}\langle c_{px} c_{py} \rangle, \tag{S10}$$

leading to

$$\langle c_{px}(t) c_{py}(t) \rangle = \gamma \frac{k_B T}{k\lambda_p} e^{-(2k\lambda_p/\zeta)t}, \tag{S11}$$

and the time-dependent shear stress

$$\langle \sigma_{xy}(t) \rangle = \frac{k}{V} \sum_{p,\, \lambda_p \neq 0} \lambda_p \left(\gamma \frac{k_B T}{k\lambda_p} e^{-(2k\lambda_p/\zeta)t} \right) = \frac{\gamma k_B T}{V} \sum_{p,\, \lambda_p \neq 0} e^{-t/\tau_p}, \tag{S12}$$

where $\tau_p \equiv \zeta/(2k\lambda_p)$ is the relaxation time of mode p. The shear relaxation modulus $G(t)$ is then:

$$G(t) \equiv \frac{\langle \sigma_{xy}(t) \rangle}{\gamma} = \frac{k_B T}{V} \sum_{p,\, \lambda_p \neq 0} e^{-t/\tau_p}. \tag{S13}$$

The frequency-dependent complex shear modulus $G^*(\omega) = G'(\omega) + iG''(\omega)$ is obtained from the Fourier transform of the relaxation modulus:

$$G^*(\omega) = i\omega \int_0^\infty G(t) e^{-i\omega t} dt. \tag{S14}$$

Substituting Eq. (S13) into the transform yields:

$$G^*(\omega) = i\omega \frac{k_B T}{V} \sum_{p,\, \lambda_p \neq 0} \int_0^\infty e^{-t/\tau_p} e^{-i\omega t} dt = \frac{k_B T}{V} \sum_{p,\, \lambda_p \neq 0} \frac{i\omega\tau_p}{1 + i\omega\tau_p}. \tag{S15}$$

Separating the real and imaginary parts of Eq. (S15) gives the storage modulus $G'(\omega)$ and the loss modulus $G''(\omega)$:

$$\begin{aligned} G'(\omega) &= \frac{k_B T}{V} \sum_{p,\, \lambda_p \neq 0} \frac{\omega^2 \tau_p^2}{1 + \omega^2 \tau_p^2}, \\ G''(\omega) &= \frac{k_B T}{V} \sum_{p,\, \lambda_p \neq 0} \frac{\omega\tau_p}{1 + \omega^2 \tau_p^2}. \end{aligned} \tag{S16}$$

B. Relationship Between Viscosity and Radius of Gyration

The viscosity η is defined as the time integral of relaxation modulus after a step strain: $\eta = \int_0^\infty \langle \sigma_{xy}(t) \rangle dt/\gamma$. Using Eq. (S12), the viscosity can be expressed as the sum of the inverse of all nonzero eigenvalues:

$$\eta = \frac{\zeta k_B T}{2kV} \sum_p \frac{1}{\lambda_p}. \tag{S17}$$

In the following, we introduce the dimensionless viscosity as $\tilde{\eta} = \sum_p \lambda_p^{-1}$ to simplify the notation.

The mean-square radius of gyration quantifies the spatial size of a macromolecule, which can be computed as $\langle R_g^2 \rangle = \sum_{i,j} \langle (\boldsymbol{R}_i - \boldsymbol{R}_j)^2 \rangle / 2N^2$ [6]. Here, the average is over all possible configurations of the molecule at thermal equilibrium. We rewrite $\boldsymbol{R}_i - \boldsymbol{R}_j = \sum_p \boldsymbol{c}_p (u_{pi} - u_{pj})$, and it is convenient to consider the fluctuation in one direction:

$$\begin{aligned} \sum_{i,j} \langle (x_i - x_j)^2 \rangle &= \langle \sum_{p,q} c_{px} c_{qx} \sum_{i,j} (u_{pi} - u_{pj})(u_{qi} - u_{qj}) \rangle \\ &= \sum_p \langle c_{px}^2 \rangle \sum_{i,j} (u_{pi}^2 + u_{pj}^2 - 2u_{pi}u_{pj}) \\ &= \sum_p \frac{2k_B T}{k\lambda_p} [N \sum_i u_{pi}^2 - (\sum_i u_{pi})^2] \\ &= \sum_p \frac{2Nk_B T}{k\lambda_p}. \end{aligned} \tag{S18}$$

In the above derivation, we use the fact that the fluctuation of each eigenmode is independent, as well as the equipartition theorem for $\langle c_{px}^2 \rangle$. We also use the normalization condition: $\sum_i u_{pi}^2 = 1$, and the condition, $\sum_i u_{pi} = 0$ since rigid-body motion is not included. Since the three directions are equivalent, we obtain

$$\langle R_g^2 \rangle = \frac{3k_BT}{Nk} \sum_p \frac{1}{\lambda_p} = \frac{b^2}{N} \sum_p \frac{1}{\lambda_p}. \tag{S19}$$

Combining (S17) and (S19), we obtain one of our key results:

$$\eta = \frac{N\zeta}{6V} \langle R_g^2 \rangle. \tag{S20}$$

Converting the viscosity and mean-square radius of gyration to dimensionless form, we obtain the equation, $\eta = N\langle R_g^2 \rangle$, which we show in the main text.

C. Calculating the Viscosity of a Tree-like Network Using its Coarse-grained Graph

The advantage of Eq. (S20) is that one can convert the problem of finding viscosity (i.e., finding all the eigenvalues of the connectivity matrix L) to the problem of calculating the mean-square radius of gyration. In particular, for tree-like networks, the Kramers theorem significantly simplifies the calculation of the radius of gyration: $\langle R_g^2 \rangle = b^2 \sum_e \pi_e / N^2$, where π_e is the number of all strands between different beads that pass through a given edge [6]. Therefore, one can compute the dimensionless viscosity as

$$\tilde{\eta} = \sum_e \pi_e / N. \tag{S21}$$

In this section, we use $\tilde{\eta}$ to represent the dimensionless viscosity, which is related to the physical viscosity via $\eta = \tilde{\eta}\zeta b^2/(6V)$.

In the following, we apply the relationship between viscosity and radius of gyration to a tree-like network. Consider a connected tree-like network of n linear precursor chains, each containing m monomers. The network topology is defined by the coarse-grained tree graph G where each node represents a linear chain, and edges represent crosslinks between chains, formed by randomly selecting one monomer from each of two distinct chains to create a connection. We compute the dimensionless viscosity $\tilde{\eta}$ of the complete network and separate the edges into two categories $nm\tilde{\eta} = \sum_{e_c} \pi_{e_c} + \sum_{e_b} \pi_{e_b}$ where e_c are crosslinks between polymers and e_b denotes backbone bonds within each linear chain. We calculate these two terms separately as follows.

Crosslink contribution—For crosslinks, each e_c corresponds to a coarse-grained edge e_c' in graph G with $\pi_{e_c} = m^2 \pi_{e_c'}^G$ (see Figure 1b in the main text), leading to the total contribution $\sum_{e_c} \pi_{e_c} = m^2 N_p \tilde{\eta}_G$ where $\tilde{\eta}_G$ is the coarse-grained graph G's dimensionless viscosity.

Backbone contribution—For each chain s, the backbone contribution $\sum_{e_b \in s} \pi_{e_b}$ is equal to the sum, over all pairs of beads (i, j), of the length of the path segment between i and j that lies within chain s (see the yellow paths in Figure S1). This total path length within s can be decomposed into three categories based on the locations of the endpoints i and j (Figure S1): (1) paths with both endpoints in s, (2) paths with one endpoint in s and the other outside s, (3) paths with both endpoints outside s.

In the following, we compute the three categories of backbone contribution separately:

(1) The sum of path distances over all pairs with both endpoints in s is simply $\sum_{i=0}^{i=m} i(m-i) = m\tilde{\eta}_p$ (Figure S1a), where $\tilde{\eta}_p = (m^2-1)/6$ is the dimensionless viscosity of an isolated linear chain with m beads. Summing over all n chains gives the contribution $nm\tilde{\eta}_p$.

(2) The second term represents the sum of path lengths traversing bonds in s for all pairs with one endpoint in s and the other outside ($i \in s, j \notin s$) (Figure S1b). The total number of such bead pairs is $m \times (n-1)m$. For each pair (i, j), the average path distance on chain s (connecting j to the branch attachment point j') is given by: $\bar{d} = \frac{1}{m^2} \sum_{i=1}^{m} \sum_{j'=1}^{m} |i - j'| = \frac{2\tilde{\eta}_p}{m}$. Summing up all n chains gives the contribution $m^2(n-1) \times \bar{d} \times n = 2nm(n-1)\tilde{\eta}_p$.

(3) To calculate the third term, we introduce a branch size function $[s, l]$ as follows: for two linear chains s and l connected by a crosslink, $[s, l]$ is the number of chains of the branch containing s when the crosslink is removed. By definition, the branch size function satisfies $[s, l] + [l, s] = n$, and $\sum_l [l, s] = n - 1$. For each pair of chains $l_1 \neq l_2$ directly connected to s, the number of path pairs with end points (i, j) (i, j do not have to be in chain l_1 or l_2) is determined by the size of their respective branches (Figure S1c). There are $m[l_1, s]$ beads in branch l_1 and $m[l_2, s]$

beads in branch l_2, yielding $m^2[l_1,s][l_2,s]$ possible bead pairs. Thus, the expectation of the sum of path lengths traversing bonds in s for the third term is $\bar{d}\sum_{l_1<l_2} m^2[l_1,s][l_2,s] = 2\tilde{\eta}_p m \sum_{l_1<l_2}[l_1,s][l_2,s]$. We compute the double summation using the branch size properties and viscosity definitions:

$$\begin{aligned}
&\sum_s \sum_{l_1<l_2} [l_1,s][l_2,s] = \frac{1}{2}\sum_s \sum_{l_1}[l_1,s]\left(\sum_{l_2\neq l_1}[l_2,s]\right)\\
&= \frac{1}{2}\sum_s \sum_{l_1}[l_1,s]\left(\sum_{l_2}[l_2,s]-[l_1,s]\right)\\
&= \frac{1}{2}\sum_s \sum_{l_1}[l_1,s]\left((n-1)-(n-[s,l_1])\right)\\
&= \frac{1}{2}\sum_s \sum_{l_1}[l_1,s]\left([s,l_1]-1\right)\\
&= \frac{1}{2}\sum_s \sum_{l_1}[l_1,s][s,l_1] - \frac{1}{2}\sum_s (n-1)\\
&= \frac{1}{2}\sum_s \sum_{l_1}[l_1,s][s,l_1] - \frac{n(n-1)}{2}\\
&= \tilde{\eta}_G n - \frac{n(n-1)}{2}.
\end{aligned} \tag{S22}$$

In the final step, we use the formula Eq. (S21) to compute the dimensionless viscosity for the coarse-grained graph G. Hence the total contribution for the third term is: $2\tilde{\eta}_p m[\tilde{\eta}_G n - n(n-1)/2]$. Combining crosslink contribution and the three terms in backbone contribution, we obtain one of our key results, the exact formula of the added dimensionless viscosity of a tree-like network relative to n isolated linear chains due to crosslinking,

$$\Delta\tilde{\eta} = \frac{m^2+3m-1}{3}\tilde{\eta}_G. \tag{S23}$$

The above formula shows that to calculate the viscosity of the complete network, one needs to compute the viscosity of the coarse-grained network, which we calculate in the following for a randomly crosslinked network. For a polymer solution with multiple disconnected tree-like networks, one only needs to sum the contributions from all networks.

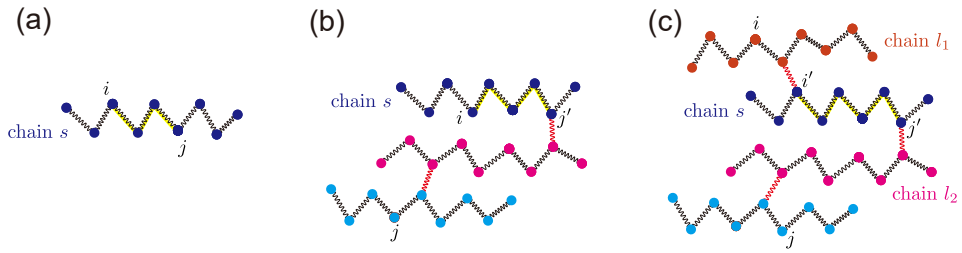

FIG. S1. Illustration of the three categories of paths due to backbone contribution. (a) Case 1: Paths with both endpoints (i,j) in chain s. (b) Case 2: Paths with one endpoint (i) in chain s and the other (j) outside; j' is the connection point in s. (c) Case 3: Paths with both endpoints (i,j) outside chain s. Their path traverses s, entering at point i' and exiting at point j'. In all cases, the yellow-highlighted segment on chain s (dark blue) represents the portion of the path that lies within s and contributes to the backbone viscosity.

D. Derivation of the Branch Size Distribution in Erdős-Rényi Random Graphs

We consider an undirected random graph $G(N_p, c)$ with $N_p \to \infty$ nodes and mean degree c. In the subcritical regime ($c < 1$), the graph decomposes almost surely into small, tree-like components without cycles, where the degree distribution of a randomly chosen node follows a Poisson distribution:

$$P(\deg = k_1) = \frac{e^{-c}c^{k_1}}{k_1!}. \tag{S24}$$

If we randomly select an edge and randomly choose one of its connected nodes, the distribution of the remaining degree k_2 (i.e., the number of edges incident to the node excluding the initially selected edge) is:

$$P(\text{remaining deg} = k_2) = \frac{P(\deg = k_2 + 1)(k_2 + 1)}{\sum_k P(\deg = k)k}. \tag{S25}$$

Since $\sum_k P(k)k = c$, this simplifies to:

$$P(\text{remaining deg} = k_2) = \frac{e^{-c}c^{k_2}}{k_2!}, \tag{S26}$$

which is identical to the original Poisson distribution. We introduce the generating function for Eq. (S24):

$$G_p(x) = \sum_k P(k)x^k = e^{c(x-1)}. \tag{S27}$$

To analyze branch sizes, we define a branch selection process: (1) randomly select an edge, (2) cut it to split the component into two branches, and (3) choose one branch randomly. Let $P(\text{size} = n)$ be the size distribution of the selected branch and $G(x)$ be its generating function. Starting from the root node of the chosen branch, let k_2 denote its remaining degree (excluding the cut edge). The root connects to k_2 sub-branches with sizes $n_1, n_2, \ldots, n_{k_2}$, giving total branch size $n = 1 + \sum_{i=1}^{k_2} n_i$. For fixed k_2, the conditional generating function is $G(x|k_2) = xG(x)^{k_2}$. Averaging over all possible k_2 and using the generating function $G_p(z) = e^{c(z-1)}$ of the remaining degree distribution, we obtain the recursive relation:

$$G(x) = \mathbb{E}_{k_2}[xG(x)^{k_2}] = xG_p(G(x)) = xe^{c(G(x)-1)}. \tag{S28}$$

We set $u = G(x)$, so $x(u) = ue^{c(1-u)}$. The Lagrange inversion formula gives the coefficient of x^n in $G(x)$ [7]:

$$[x^n]G(x) = \frac{1}{n}[u^{n-1}]\left(\frac{u}{x(u)}\right)^n. \tag{S29}$$

Therefore,

$$[x^n]G(x) = \frac{1}{n}[u^{n-1}]\left(\frac{u}{ue^{c(1-u)}}\right)^n = \frac{1}{n}[u^{n-1}]e^{nc(u-1)}. \tag{S30}$$

We expand the exponential:

$$e^{ncu} = \sum_{k=0}^{\infty} \frac{(ncu)^k}{k!} \tag{S31}$$

so the coefficient of u^{n-1} in e^{ncu} is $\frac{(nc)^{n-1}}{(n-1)!}$. Therefore,

$$[u^{n-1}]e^{ncu} = \frac{(nc)^{n-1}}{(n-1)!} \tag{S32}$$

and

$$[x^n]G(x) = \frac{1}{n}e^{-nc}\frac{(nc)^{n-1}}{(n-1)!} = \frac{e^{-nc}(nc)^{n-1}}{n!}. \tag{S33}$$

Thus, the branch size distribution is:

$$P(n) = \frac{e^{-nc}(nc)^{n-1}}{n!}. \tag{S34}$$

The asymptotic behavior for large n is found using Stirling's approximation $n! \approx \sqrt{2\pi n}(n/e)^n$:

$$P(n) \approx \frac{e^{-nc}(nc)^{n-1}}{\sqrt{2\pi n}(n/e)^n} = \frac{c^{n-1}e^{n(1-c)}}{\sqrt{2\pi}n^{3/2}}. \tag{S35}$$

E. Viscosity of Fixed-Size Trees

We consider a labeled tree with n nodes. For any edge connecting nodes a and b, we divide the tree into two components, each with i and j nodes, respectively ($i+j=n-2$), excluding nodes a and b. We then use Eq. (S21) and get the number of paths that pass through the edge, $\pi_e=(i+1)(j+1)$. The number of ways to choose which i nodes of the $n-2$ nodes are in the left component is $(i+j)!/(i!j!)$. For each such partition, the number of distinct labeled trees with this specific edge division is $(i+1)^{i-1}(j+1)^{j-1}$, using the fact that the number of labeled trees of size n is n^{n-2} by Cayley's formula [8], where all nodes are distinguishable. Summing over all possible partitions and all $n(n-1)/2$ possible edges, the total sum of π_e values for all trees with size n is:

$$\sum_{\text{all trees}}\sum_{e}\pi_e=\frac{n(n-1)}{2}\sum_{i+j=n-2}\frac{(i+j)!}{i!j!}(i+1)^i(j+1)^j=\frac{n!}{2}\sum_{i+j=n-2}\frac{(i+1)^i(j+1)^j}{i!j!}. \tag{S36}$$

Since there are n^{n-2} distinct labeled trees, the average viscosity becomes:

$$\langle\tilde{\eta}(n)\rangle=\frac{1}{n^{n-2}}\cdot\frac{1}{n}\sum_{\text{all trees}}\sum_{e}\pi_e=\frac{n!}{2n^{n-1}}\sum_{i+j=n-2}\frac{(i+1)^{(i+1)}(j+1)^{(j+1)}}{(i+1)!(j+1)!}. \tag{S37}$$

Applying Stirling's approximation $m!\approx\sqrt{2\pi m}(m/e)^m$ leads to

$$\frac{(i+1)^{(i+1)}(j+1)^{(j+1)}}{(i+1)!(j+1)!}\approx\frac{e^n}{2\pi\sqrt{(i+1)(j+1)}}. \tag{S38}$$

The sum then becomes:

$$\sum_{i+j=n-2}\frac{e^n}{2\pi\sqrt{(i+1)(j+1)}}\approx\frac{e^n}{2\pi}\sum_{k=1}^{n-1}\frac{1}{\sqrt{k(n-k)}}\approx\frac{e^n}{2}. \tag{S39}$$

Substituting back and applying Stirling's approximation to $n!$, we get the dimensionless viscosity:

$$\langle\tilde{\eta}(n)\rangle\approx\frac{n!}{2n^{n-1}}\cdot\frac{e^n}{2}\approx\frac{\sqrt{2\pi n}(n/e)^n}{2n^{n-1}}\cdot\frac{e^n}{2}=\sqrt{\frac{\pi}{8}}n^{3/2}. \tag{S40}$$

F. Viscosity of a Random Graph

Using Eq. (S21), we compute the dimensionless viscosity of the coarse-grained polymer solution where each linear chain is coarse-grained as a monomer. We sum over all edges, with each edge's contribution equal to the number of paths traversing it:

$$\begin{aligned}\tilde{\eta}_G&=\frac{N_pc}{2}\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}P(n_1)P(n_2)\frac{n_1n_2}{n_1+n_2}\\&=\sum_{N\geq2}\sum_{n=1}^{N-1}\frac{N_pe^{-cN}c^{N-1}n^n(N-n)^{N-n}}{2Nn!(N-n)!}\\&=\sum_{N\geq2}\tilde{\eta}_{G,N}(c).\end{aligned} \tag{S41}$$

Here, N_p is the total number of nodes, $P(n)=e^{-nc}(nc)^{n-1}/n!$ is the distribution of branch size by randomly choosing an edge, and $\tilde{\eta}_{G,N}(c)$ is defined as the viscosity contributions of all clusters of size N.

We further write $\tilde{\eta}_G$ as the sum of the contributions of clusters of size N: $\tilde{\eta}_G=N_p\sum_{N\geq2}f_N(c)$ with

$$f_N(c)=\sum_{n=1}^{N-1}\frac{e^{-cN}c^{N-1}n^n(N-n)^{N-n}}{2Nn!(N-n)!}. \tag{S42}$$

Applying Stirling's approximation $n^n/n! \approx e^n/\sqrt{2\pi n}$ to (S41) yields

$$\begin{aligned}\tilde{\eta}_{G,N}(c)/N_p &\approx \frac{e^{(1-c)N}c^{N-1}}{4\pi N}\sum_{n_1=1}^{N-1}\frac{1}{\sqrt{n_1(N-n_1)}} \\ &\approx \frac{e^{(1-c)N}c^{N-1}}{4\pi N}\int_0^1 \frac{dx}{\sqrt{x(1-x)}} \\ &\approx \frac{e^{(1-c+\ln c)N}}{4cN}.\end{aligned} \tag{S43}$$

In the weakly crosslinked limit ($c \ll 1$), the dimensionless viscosity of the coarse-grained network is dominated by the smallest cluster size ($N = 2$), yielding $\tilde{\eta}_G = N_p c/4$ using Eq. (S42). As expected, the added viscosity is proportional to c in the weakly crosslinked limit.

Near the gelation point ($\epsilon = 1 - c \ll 1$), $\tilde{\eta}_{G,N}(c)$ can be further approximated as $\tilde{\eta}_{G,N}(c) \approx N_p e^{-\epsilon^2 N/2}/4N$, from which we find a logarithmic divergence of the dimensionless viscosity of the coarse-grained network near the gelation point:$\tilde{\eta}_G \approx N_p \sum_{N\geq 2} e^{-\epsilon^2 N/2}/4N \approx -(N_p/2)\ln\epsilon$.

It is noteworthy that near the gelation point, the cluster-size probability distribution of the random graph follows a power law $P(N) \propto N^{-5/2}$, and each cluster exhibits a fractal dimension of 4 (i.e., $\langle R_g^2\rangle \propto N^{1/2}$) [6]. According to Eq. (S20), the total viscosity of clusters of size N is: $\tilde{\eta}_{G,N} \propto P(N)\cdot N\langle R_g^2\rangle = N^{-1}$. This summation elucidates the logarithmic divergence of viscosity near the gelation point, i.e., $\tilde{\eta}_G \sim -\ln\epsilon$. The scaling behavior reveals a universal feature wherein clusters across all scales contribute comparably to the divergence, as the integrand N^{-1} maintains comparable weight for different cluster sizes.

Using the exact formula for the viscosity increment derived in Eq. (S23), we summarize the behavior of the viscosity for the entire solution. For long precursor chains ($m \gg 1$), the viscosity increment scales as $\Delta\tilde{\eta} \sim \frac{m^2}{3}\tilde{\eta}_G$, where $\tilde{\eta}_G$ is the viscosity of the coarse-grained network. As discussed, $\tilde{\eta}_G$ varies with the crosslinking density c: in the weakly crosslinked limit ($c \ll 1$), $\tilde{\eta}_G \sim c$, leading to $\Delta\tilde{\eta} \propto m^2 N_p c$; near the gelation point ($c \to 1$), $\tilde{\eta}_G \sim -\ln(1-c)$, resulting in $\Delta\tilde{\eta} \propto m^2 N_p(-\ln(1-c))$. This demonstrates how the viscosity evolves with crosslinking density across the entire range below the gelation point. We illustrate this behavior in Figure S2, where the dimensionless viscosity of the coarse-grained solution is plotted as a function of the crosslinking density c.

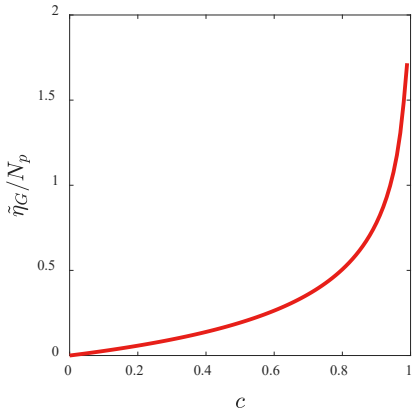

FIG. S2. Dimensionless viscosity divided by the total number of nodes $\tilde{\eta}_G/N_p$ as a function of crosslinking density c for the coarse-grained solution corresponding to a randomly crosslinked polymer solution. The viscosity increases with crosslinking density, exhibiting a rapid rise as c approaches the gelation point at $c = 1$.

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