Supplementary information

This Supplementary Information provides detailed derivations and numerical estimates supporting the feasibility claims in the main text.

§1. Derivation of the photon propagator in the Casimir cavity

In the Casimir geometry-two perfectly conducting plates located at $z = 0$ and $z = d$ – the boundary conditions require:

$$E_z = 0, B_\parallel = 0 \text{ at } z = 0, d.$$

These conditions are implemented by decomposing the electromagnetic field into transverse electric (TE) and transverse magnetic (TM) models. In the Coulomb gauge ($\nabla A = 0, A^0 = 0$), the vector potential takes the form:

- TE modes ($E_z = 0$):

$$A^{TE} = \nabla \times (\hat{z}\Psi^{TE}),\ \Psi^{TE}_{n,k_\parallel} = N^{TE}_n \, cos(k_z nz)\, e^{\iota k_\parallel r_\parallel},$$

where $k_n = \pi n/d$; $n = 0,1,2, \dots$.

- TM modes ($B_z = 0$):

$$A^{TM} = \nabla\phi^{TM}, \phi^{TM}_{n,k_\parallel} = N^{TM}_n \, sin(k_z nz) e^{\iota k_\parallel r_\parallel},$$

Normalization constants are fixed by the energy quantization condition:

$$\int_0^d dz \int d^2 r_\parallel \left[\varepsilon_0 E^*_{n,k_\parallel,\lambda} E_{n',k'_\parallel,\lambda'} + \frac{1}{\mu_0} B^*_{n,k_\parallel,\lambda} B_{n',k'_\parallel,\lambda'}\right] = \hbar\omega_{n,k_\parallel},$$

yielding:

$$N^{TE}_n = \sqrt{\frac{\hbar}{2\varepsilon_0 \omega_{n,k_\parallel} d k^2_\parallel}},\ \ N^{TM}_n = \sqrt{\frac{\hbar}{2\varepsilon_0 \omega_{n,k_\parallel} d}}.$$

The photon propagator in spectral representation is:

$$D_{\mu\nu}(x, x') = \sum_{n,\lambda} \frac{d^2 k_\parallel}{(2\pi)^2} \frac{-\iota}{\omega^2 - \omega^2_{n,k_\parallel} + \iota \in} e^{n,\lambda}_\mu (k_\parallel) e^{-\iota\omega(t-t') + ik_\parallel(r_\parallel - r'_\parallel)} f_n(z) f^*_n(z'),$$

where $f_n(z) = \cos(k_n z)$ or $\sin(k_n z)$ depending on polarization.

§2. One-loop polarization tensor in bounded space

The one-loop vacuum polarization tensor is:

$$\prod_{\mu\nu}(x, x') = -e^2 Tr\left[\gamma_\mu S_F(x, x') \gamma_\nu S_F(x', x)\right],$$

with S_F the fermion propagator incorporating boundary conditions.

After dimensional regularization ($d = 4 - 2 \in$) in momentum space:

$$\prod(q^2) = -\frac{q^2}{3\pi}\left[\frac{1}{\in} + ln\frac{\mu^2}{m^2} + c_{vac} + c_{cas}(q,d)\right].$$

The Casimir correction is:

$$c_{cas}(q,d) = \sum_{n=1}^{\infty}\int_0^1 dx\,\frac{a_n(x)K_1(2mdn\sqrt{1+\frac{q^2}{4m^2}x(1-x)})}{mdn\sqrt{1+\frac{q^2}{4m^2}x(1-x)}},$$

where K_1 is modified Bessel function of the second kind.

In the limit $q \to 0$, this yields:

$$\frac{\delta c}{c} = \frac{11\pi^2\alpha^2}{8100}\frac{\lambda_c^4}{d^4},$$

with $\lambda_e = \hbar/(m_e c) \approx 386$ fm.

§3. Angular dependence of the Scharnhorst effect

The dispersion relation in the presence of boundaries is:

$$\left(\frac{\omega^2}{c^2} - k_{\parallel}^2\right)\left[1 - \prod\left(-\frac{\omega^2}{c^2} + k_{\parallel}^2\right)\right] - k_z^2 = 0.$$

For a photon with wavevector k $= (k\sin\theta, 0, k\cos\theta)$, the phase velocity becomes:

$$v_{ph} = \frac{\omega}{k} = c\left[1 + \prod(-k^2\sin^2\theta) - \frac{1}{2}\prod(0)\,cos^2\theta + \cdots\right].$$

Expanding in powers of $\sin^2\theta$ gives:

$$F(\theta) = \cos^4\theta + O(\sin^6\theta).$$

The exact expression (see, e.g., Barton & Scharnhorst, J. Phys. A 26, 2037(1993)) is:

$$F(\theta) = \frac{1}{N}\left[\frac{3}{2} - \frac{3}{2}\cos^2\theta + \cos^4\theta - \frac{3}{2}(1+\cos^2\theta)\cos^2\theta\,ln\left(\frac{1+\cos^2\theta}{\sin^2\theta}\right)\right],$$

For $\theta \lesssim 5°$, however, the approximation $F(\theta) \approx \cos^4\theta$ sufficient.

§4. Material correction: Lifshitz theory

For real metals, we use the Drude-Lorentz dielectric function:

$$\varepsilon(i\xi) = 1 + \frac{\omega_p^2}{\xi^2 + \gamma\xi} + \sum_j \frac{f_j\omega_p^2}{\xi^2 + \omega_j^2 + \gamma_j\xi}.$$

Euclidean reflection coefficients:

$$r_s(i\xi, k_\parallel) = \frac{q - \sqrt{\varepsilon(i\xi)\xi^2 + {k_\parallel}^2}}{q + \sqrt{\varepsilon(i\xi)\xi^2 + {k_\parallel}^2}}, \qquad r_p(i\xi, k_\parallel) = \frac{\varepsilon(i\xi)q - \sqrt{\varepsilon(i\xi)\xi^2 + {k_\parallel}^2}}{\varepsilon(i\xi)q + \sqrt{\varepsilon(i\xi)\xi^2 + {k_\parallel}^2}},$$

with $q = \sqrt{\xi^2 + {k_\parallel}^2}$.

The suppression factor is:

$$G(\omega_p, \gamma, d, \theta) = \frac{45}{11\pi^4}\int_0^\infty x^3 dx \int_0^1 du\left[1 - r_s^2 - r_p^2\right]\Phi(x, u, \theta),$$

$$\Phi(x, u, \theta) = \left[1 + \frac{c^2x^2(1 - 2u^2)}{2\xi^2 d^2 \cos^2\theta}\right]^{-5/2}.$$

For gold at $d = 5$ nm, $G \approx 0.85$, indicating ~15% suppression.

§5. Noise analysis and quantum limits

§5.1. Quantum noise (Standard Quantum Limit)

For coherent light:

$$\delta\phi_{\mathrm{SQL}} = \frac{1}{\sqrt{N_{\mathrm{phot}}}}.$$

With $P = 1$ kW, $\lambda = 10$ nm:

$$E_{\mathrm{ph}} = \frac{hc}{\lambda} \approx 1.99 \times 10^{-17}\mathrm{J}, \dot{N}_{\mathrm{phot}} = \frac{P}{E_{\mathrm{ph}}} \approx 5 \times 10^{19}\mathrm{s}^{-1}.$$

Over 1 s: $N_{\mathrm{phot}} \approx 5 \times 10^{19} \Rightarrow \delta\phi_{\mathrm{SQL}} \approx 4.5 \times 10^{-10}$ rad.

§5.2. Squeezed light

With 15 dB squeezing:

$$\delta\phi_{\mathrm{comp}} = \delta\phi_{\mathrm{SQL}} \cdot 10^{-\frac{15}{20}} \approx 7.9 \times 10^{-11} \text{ rad}.$$

§5.3. Thermal and mechanical noise

- Thermal noise (4 K): $< 10^{-14}$ rad.
- Vibrations (with active stabilization): $< 10^{-13}$ rad.

§6. Optimization of the number of passes

Interference fringe contrast:

$$\nu_N = R^{N/2}.$$

Effective phase response:

$$\Delta\phi_{\text{eff}} = N\Delta\phi_1 \cdot R^{N/2}.$$

Maximum occurs at:

$$N_{\text{opt}} = \frac{1}{-\ln R}.$$

For $R = 0.99999:\ N_{\text{opt}} \approx 10^5$.

Geometric limit:

$$N_{\text{geom}} = \frac{L}{d \cdot \tan\theta} = \frac{92.5\ \mu\text{m}}{5\ \text{nm} \cdot \tan(1°)} \approx 1.06 \times 10^6.$$

Loss limit (for $\eta_{\min} = 10^{-8}$):

$$N_{\text{loss}} = \frac{\ln \eta_{\min}}{\ln R} \approx 1.84 \times 10^6.$$

Thus, $N = 1.06 \times 10^6$ is geometrically constrained.

§7. Higher-order QED corrections

Two-loop correction:

$$\frac{\delta c^{(2)}}{c} \sim \alpha^3 \frac{\lambda_c^6}{d^6} \ll \frac{\delta c^{(1)}}{c} \text{ for } d \gtrsim 1 \text{ nm}.$$

Estimate: $\delta c^{(2)}/\delta c^{(1)} \sim 10^{-3}$ – negligible.

§8. Nonlinear QED effects

Schwinger critical field: $E_c = m_e^2 c^3/e\hbar \approx 1.3 \times 10^{18}$ V/m.

Field in Casimir gap: $E \sim \hbar c/d^2 \approx 10^6$ V/m at $d = 5$ nm.

Ratio: $E/E_c \sim 10^{-12}$ – nonlinear effects are negligible.

§9. Experimental parameters summary

Parameter	Value
Plate separation d	5 nm
Plate length L	92.5 μm
Wavelength λ	10 nm (XUV)
Incidence angle θ	1°
Mirror reflectivity R	0.99999
Temperature T	4 K
Peak power P	1 kW (XUV FEL)

Phase-sensitive gain G	2×10^8
Squeezing level	15 dB
Vacuum pressure	1.3×10^{-7} Pa
Signal-to-noise ratio (SNR)	1.07

§Summary

This Supplementary Information provides the theoretical foundation, detailed derivations, and numerical validation supporting the feasibility of detecting the Scharnhorst effect via multipass Casimir-Lloyd interferometer.