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Article

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Universal spectrum and scaling laws for dark matter halo mass functions and density profiles

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ABSTRACT

We identify a universal linear-to-nonlinear transition centred on a characteristic halo mass $m_h^* \propto t$, near which gravity organizes the matter field into a quasi-universal power spectrum $P(k) \propto k^{-1}$, independent of the primordial spectrum. Five critical scales, defined operationally from N -body simulations, reorder at a crossover a_* , when the cutoff mass M^* of the mass function overtakes m_h^* ; only afterwards does the universal range form. The two halo masses are fronts of random walks on two diffusion clocks: m_h^* follows a cascade clock that slows at a_* , and M^* a linear clock that does not. A generalized cascade that redistributes mass and energy across scales then sets scaling laws at two levels. In mass space, haloes random-walk with waiting time $\tau_g \propto m_h^{-\lambda}$, giving a double- λ mass function whose exponential cutoff carries the gravitational value $\lambda_2 = 2/3$ and whose effective small-mass slope, $\lambda_1 = 5/6$, is fixed by small-scale permanence and the ratio of the two clocks. Within haloes, a radial cascade at rate $\varepsilon = \varepsilon_u (m_h/m_h^*)^{2/3}$, time-invariant at the injection scale m_h^* and with a value set by the energy-containing scale M^* , gives $\rho \propto r^{-4/3}$ inside the scale radius r_s ; the front of a particle random walk sets the cutoff radius R_h , and the mass budget fixes r_s , so that the concentration grows only slowly. Reading the two fronts through three density laws, the fixed contrast, the virial density and the cascade law, gives six halo radii whose scalings agree with Illustris-1-Dark. The two levels are dual: energy fixes the cutoff in mass space but the density slope in haloes, and permanence fixes the slope in mass space but the scale radius in haloes. Both levels reach a non-equilibrium steady state with scale-independent rates, a mass cascade rate $\varepsilon_m \propto a^{-1}$ and a constant energy cascade rate $|\varepsilon_u| \approx (2-5) \times 10^{-7} \text{ m}^2 \text{ s}^{-3}$, and both show small-scale permanence: the halo group mass and the $-4/3$ inner profile of haloes at $m_h^*(z)$ are time-invariant. Kinetic energy cascades *inversely* at ε_u and potential energy *directly* at $-1.4 \varepsilon_u$, so the total energy, injected near m_h^* and contained below M^* , is "dissipated" by cosmic expansion at $-0.4 \varepsilon_u$.

Key words: Dark matter; N-body simulations; Theoretical models; Universal spectrum; Scaling laws;

1 INTRODUCTION

Collisionless systems governed by self-gravity exhibit remarkably ordered collective behaviour, pointing to universal principles underlying their dynamics. Self-gravitating collisionless fluid dynamics (SG-CFD) seeks to identify these principles [1, 2, 3]. A central example is the gravitational collapse of cold dark matter (CDM), a non-equilibrium problem governed by the coupled Vlasov–Poisson equations [4]. Because CDM fluctuations have larger amplitudes on smaller scales [5], structure forms hierarchically: small haloes collapse first and merge into larger ones.

Since the evolution from Gaussian initial conditions is fully determined by the Vlasov–Poisson system, most analytic frameworks express halo statistics through the initial spectrum. Virialized haloes are the building blocks of large-scale structure [6, 7]. Their abundance follows from Press–Schechter (PS) theory [8], which assigns collapse to regions whose smoothed linear contrast exceeds the spherical-collapse threshold [9, 10], and from its excursion-set extensions [11, 12, 13, 14]. Their internal structure is described by density

profiles [15]: self-similar collapse predicts power laws [16, 17], and simulations reveal a nearly universal cuspy profile [18, 19, 20, 21].

This view is exact in the linear regime, where the power in every Fourier mode grows as a^2 and the spectral shape is preserved. In the nonlinear regime it is incomplete. Low-mass haloes assemble early and retain the primordial imprint, whereas massive haloes are built by repeated mergers and accretion that progressively erase it. Two effects drive this memory loss: the equality scale k_{eq} breaks scale-free self-similarity, and hierarchical merging redistributes mass and energy across scales, a process we call a *cascade*. The consequences are visible in both observables. PS-type mass functions underpredict massive and overpredict low-mass haloes relative to simulations [22]. Simulated inner density slopes range from shallower than -1 to about -1.3 [18, 23, 24, 25, 26, 27] without a first-principles prediction, while observations favour lower central densities [28, 29, 30, 31, 32], the cusp–core problem.

The memory loss is also directly visible in the spectrum. Figure 1 shows the compensated spectrum $P(k)k$ for three simulations with different cosmologies and resolutions. A horizontal plateau, $P(k) \propto k^{-1}$, appears in all of them between the linear and deeply nonlinear regimes, widens with time, and is absent at early times. We argue that this plateau is a fixed point of gravitational dynamics, and that the

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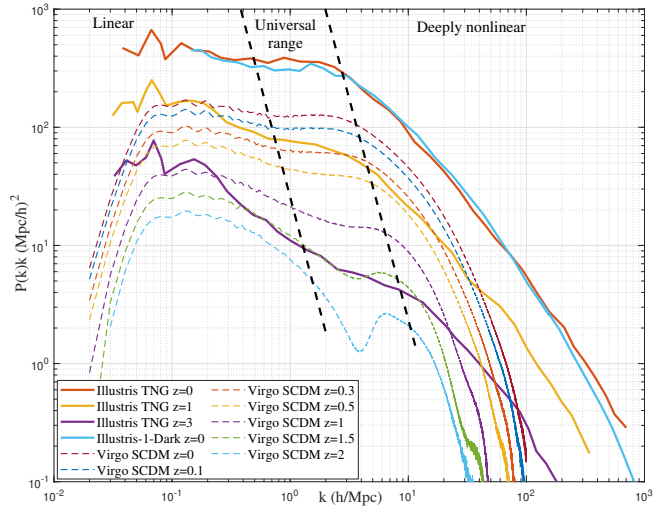


Figure 1. Compensated density power spectrum $P(k)k$ for Illustris TNG [33], Illustris-1-Dark [34] at $z = 0, 1, 3$ (Appendix A), and the Virgo SCDM simulation (Fig. 2). A horizontal plateau (roughly between the two dashed lines) signals $n_{\text{eff}} = -1$, the gravity-dominated fixed point of Section 4. It appears only after the crossover a_* (Section 3) and extends to lower k with time, while the spectrum at large k retains the primordial imprint.

same universal spectrum sets a gravity-dominated range in both the mass function and the density profile, coexisting with a primordial-dominated range.

Analogy with turbulence. In fully developed turbulence, eddies generated at an integral scale break up and pass kinetic energy to smaller eddies through an inertial range at a scale-independent rate ε , which balances viscous dissipation at the smallest scales [35]; in two dimensions the cascade is inverse [36]. Richardson’s verse [37] captures the hierarchy:

"Big whirls have little whirls, That feed on their velocity;
And little whirls have lesser whirls, And so on to viscosity."

SG-CFD has neither viscosity nor eddies, yet two correspondences hold. First, haloes play the role of whirls: a broad spectrum of halo masses maximizes entropy [38], and merging transfers mass from small to large scales, an *inverse* cascade:

"Little haloes have big haloes, That feed on their mass;
And big haloes have greater haloes, And so on to growth."

Second, cosmic expansion plays the role of viscosity. The expanding background breaks time-translation symmetry, the total energy of the dark matter decreases (Appendix B), and this decrease acts as the sink that sustains a nonzero cascade rate. The analogy is functional only: no energy is converted to radiation.

This work. We first define five critical scales operationally from N -body data (Section 2) and show that they reorder at a crossover a_* , after which the universal range forms (Section 3). We then derive the shape $P(k) \propto k^{-1}$ and growth $\propto a^3$ of the universal spectrum and the scalings of the five scales (Section 4). The universal spectrum fixes the gravitational exponent $\lambda = 2/3$ of the halo random walk in mass space, which gives the halo mass function and the mass and energy cascades (Section 5). Within haloes we define three radii through three density laws, show that a radial cascade, whose rate is fixed at the cutoff mass M^* , gives $\rho \propto r^{-4/3}$ inside the scale radius, and obtain the full profile from a particle random walk, in direct parallel with the mass function (Section 6). Section 7 concludes; supporting derivations and a summary are collected in the appendices.

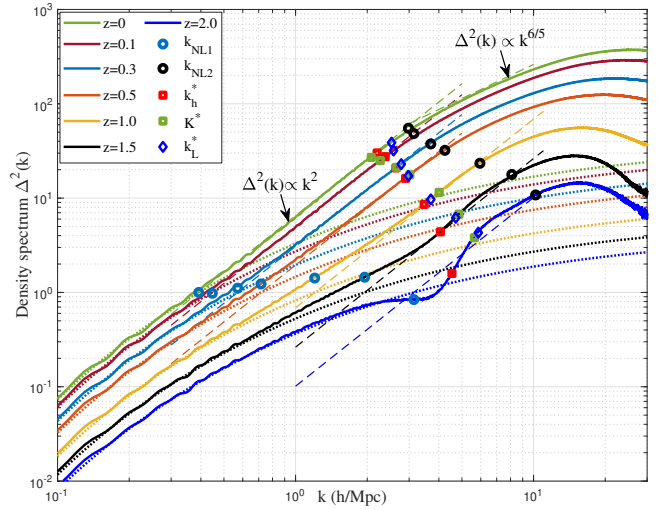


Figure 2. Dimensionless power spectrum $\Delta^2(k, a)$ from the Virgo SCDM simulation. *Dotted*: linear spectrum (Eq. (1)); *solid*: measured nonlinear spectrum. The five critical scales of Table 1 are placed on the measured spectrum: k_{NL1} (blue circles), k_{NL2} (black circles; thin dashed lines show the two asymptotes), k_h^* (red squares), K^* (green squares) and k_L^* (diamonds). The k^2 (universal) range exists only after the crossover $a_* \approx 0.77$.

Table 1. The five critical scales. All masses use Eq. (8).

Scale	Defining condition	Measured from
$k_{\text{NL1}}, M_{\text{NL1}}^*$	$\Delta_L^2(k_{\text{NL1}}) = 1$	linear spectrum
$k_{\text{NL2}}, M_{\text{NL2}}^*$	k^2 and $k^{6/5}$ asymptotes meet	nonlinear spectrum
k_h^*, m_h^*	$\sigma_v^2(m_h^*) = \sigma_v^2(m_h^*)$	halo velocities
K^*, M^*	$\sigma_v^2(M^*) = u^2$	halo velocities
k_L^*, M_L^*	$\Delta^2(k_L^*) = \Delta_L^2(\infty)$	both spectra

2 FIVE CRITICAL SCALES

Each scale below is defined by a single condition measured in the simulation. All five scales (Table 1) are marked in Fig. 2 at the measured nonlinear amplitude $\Delta^2(k)$, and their evolution is shown in Figs. 4 and 5. Interpretation is deferred to Sections 3 and 4, and all scales and exponents are summarized in Appendix I.

2.0.0.1 Spectral boundaries k_{NL1} and k_{NL2} . The linear spectrum is described by the Bond & Efstathiou approximation [39],

$$\Delta_L^2(k, a) = \frac{a^2 A k^4}{[1 + Q(q)^{\nu_{\text{BE}}}]^{2/\nu_{\text{BE}}}}, \quad q \equiv \frac{k}{\Gamma}, \quad (1)$$

$$Q(q) = a_{\text{BE}} q + (b_{\text{BE}} q)^{3/2} + (c_{\text{BE}} q)^2,$$

with $a_{\text{BE}} = 6.4 h^{-1} \text{Mpc}$, $b_{\text{BE}} = 3 h^{-1} \text{Mpc}$, $c_{\text{BE}} = 1.7 h^{-1} \text{Mpc}$, $\nu_{\text{BE}} = 1.13$, and $A = 5172.4$ fixed by σ_8 at $a = 1$. The *linear boundary* is

$$\Delta_L^2(k_{\text{NL1}}, a) \equiv 1. \quad (2)$$

Beyond k_{NL1} the measured spectrum after a_* first follows $\Delta^2 \propto k^2$, i.e. the $P(k) \propto k^{-1}$ plateau of Fig. 1 (universal range), and then a shallower slope (deeply nonlinear range). For the latter we use the stable-clustering asymptote $\Delta^2 \propto k^{3+n_a}$ with $n_a = -6/(5+n_i)$ [40], where n_i is the linear index; for the effective CDM index $n_i = -5/3$ at k_{NL1} this gives $n_a = -9/5$ and $\Delta^2 \propto k^{6/5}$. Both slopes are derived in Section 4 (Eqs. (17) and (29)); here they serve only to define the boundary. The *nonlinear boundary* k_{NL2} is the intersection of these

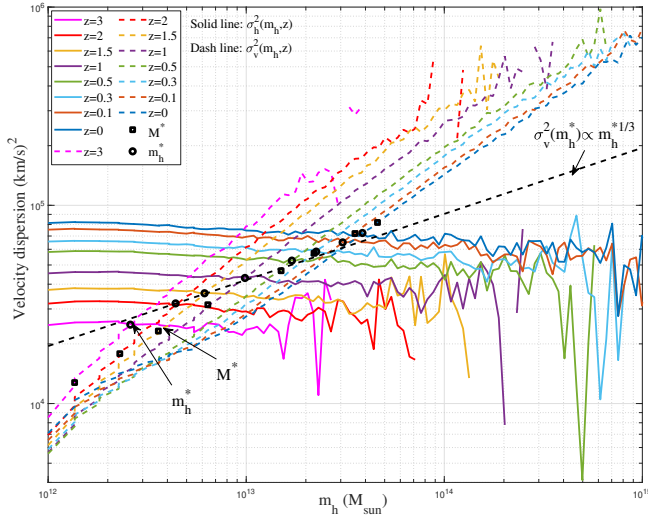


Figure 3. Virial dispersion $\sigma_v^2(m_h, a)$ (dashed) and bulk dispersion $\sigma_h^2(m_h, a)$ (solid) versus halo mass at several redshifts from the Virgo SCDM simulation. Their intersection defines m_h^* (circles); the condition $\sigma_v^2(M^*) = u^2$ defines M^* (squares). The crossover values follow $\sigma_v^2(m_h^*) \propto m_h^{1/3}$ (dashed black line). $M^* < m_h^*$ before a_* and $M^* > m_h^*$ after a_* .

two asymptotes,

$$(k_{\text{NL2}}/k_{\text{NL1}})^2 = C_{\text{NL}}(a) k_{\text{NL2}}^{6/5}, \quad (3)$$

where the universal asymptote is anchored at $\Delta^2 = 1$ at k_{NL1} and $C_{\text{NL}}(a)$ is fitted to the measured spectrum with the slope fixed at $6/5$. The nonlinear boundary k_{NL2} is plotted at the measured $\Delta^2(k_{\text{NL2}})$.

2.0.0.2 Kinematic scale m_h^* . For haloes of mass m_h , let $\sigma_v^2(m_h, a)$ be the one-dimensional dispersion of particle velocities about the halo mean (virial motion) and $\sigma_h^2(m_h, a)$ the one-dimensional dispersion of halo mean velocities (bulk motion). The characteristic mass is defined by the crossover (Fig. 3)

$$\sigma_v^2(m_h^*, a) \equiv \sigma_h^2(m_h^*, a). \quad (4)$$

σ_h^2 is nearly independent of mass, while $\sigma_v^2 \propto m_h^{2/3}$, so the two cross once. Below m_h^* , $\sigma_v^2 < \sigma_h^2$: the bulk kinetic energy exceeds the internal virial energy, and haloes are carried by the coherent flow as wholes. Above m_h^* , $\sigma_v^2 > \sigma_h^2$: internal motions dominate, and haloes grow by merging with collapsed neighbours. At m_h^* the two energies are equal, so a halo of this mass has just decoupled kinematically from the surrounding flow; m_h^* is thus the peculiar-flow analogue of turnaround from the Hubble flow in spherical collapse model.

2.0.0.3 Cutoff scale M^* . Let $M_h(a)$ be the total mass in haloes and $u^2(a) \equiv \langle \sigma_v^2(m_h, a) \rangle$ the one-dimensional virial dispersion of all halo particles in M_h , i.e. the mass-weighted mean of σ_v^2 over all haloes. The cutoff mass is defined by (Fig. 3)

$$\sigma_v^2(M^*, a) \equiv u^2(a). \quad (5)$$

Because the halo mass fraction rises as $m_h^{1-\lambda_1}$ with $\lambda_1 < 1$ up to the exponential cutoff of the mass function (Section 5), haloes near the cutoff dominate the mean virial energy, $\langle K_{\text{pv}} \rangle = \frac{3}{2}u^2 \simeq \frac{3}{2}\sigma_v^2(M^*)$. M^* is therefore the cutoff of the mass function: it separates collapsed scales from linear ones, and haloes above it are exponentially rare.

2.0.0.4 Linear-extrapolation scale k_L^* . As $k \rightarrow \infty$ the $(c_{\text{BE}}q)^2$ term in Q dominates the denominator of Eq. (1), and the linear spectrum saturates at

$$\Delta_L^2(\infty, a) = a^2 A (\Gamma/c_{\text{BE}})^4 \approx 38.7 a^2, \quad (6)$$

the largest amplitude linear theory assigns to any mode at epoch a . The plateau is a property of the Bond & Efstathiou form, which is exactly the spectrum used to generate the Virgo initial conditions. An exact CDM transfer function falls as $T(k) \propto \ln k/k^2$ at large k , so Δ_L^2 keeps growing slowly as $\ln^2 k$; over the simulated range $k \lesssim 30 h \text{ Mpc}^{-1}$ the two differ by less than about 15%. We define k_L^* as the scale where the measured spectrum reaches this value,

$$\Delta^2(k_L^*, a) \equiv \Delta_L^2(\infty, a). \quad (7)$$

Modes beyond k_L^* carry more power than linear extrapolation can produce. In spherical collapse, an overdensity passes through turnaround (linear contrast $\delta_L \approx 1.06$, actual overdensity $1 + \delta_{\text{ta}} = (3\pi/4)^2$) before collapse and virialization ($\delta_c \approx 1.686$, $\Delta_{\text{vir}} = 18\pi^2$). Linear theory cannot follow it past turnaround, so k_L^* marks the last scale on which linear extrapolation retains meaning.

2.0.0.5 Mass–wavenumber conversion. To compare all five scales on one footing, every wavenumber is converted with the same virialized extrapolation,

$$M = \frac{4\pi}{3} \bar{\rho}_0 \Delta_{\text{vir}} k^{-3}, \quad \Delta_{\text{vir}} = 18\pi^2, \quad (8)$$

where $\bar{\rho}_0 = \Omega_m \rho_{c,0}$ is the comoving mean matter density. This gives M_{NL1}^* , M_{NL2}^* , M_L^* , and, inversely, k_h^* from m_h^* and K^* from M^* . The same conversion in physical units defines the fixed-contrast radius of a halo of mass m_h ,

$$r_h(m_h, a) = \left(\frac{3m_h}{4\pi \Delta_{\text{vir}} \bar{\rho}(a)} \right)^{1/3} \propto m_h^{1/3} a, \quad (9)$$

used for haloes in Section 6. The corresponding Lagrangian radius at mean density is

$$R_L(M) = \left(\frac{3M}{4\pi \bar{\rho}_0} \right)^{1/3}. \quad (10)$$

Linear-theory masses such as the Press–Schechter mass M_{PS}^* , defined by $\sigma_\delta^L(R_L)D(a) = \delta_c$, are Lagrangian; at fixed radius they are smaller than masses from Eq. (8) by $\Delta_{\text{vir}} \approx 178$. The mass M_{NL1}^* plotted here is therefore not M_{PS}^* .

2.0.0.6 Measured evolution. Figures 4 and 5 summarize what the theory must explain. (i) $k_{\text{NL1}} \propto a^{-3/2}$ at late times ($a \gtrsim 0.6$) and evolves faster earlier. (ii) k_{NL2} is the largest of the five wavenumbers at all epochs and decreases with time, more slowly than $a^{-3/2}$ (roughly as $a^{-0.9}$ for $z \leq 0.5$). (iii) k_h^* , K^* and k_L^* lie between the two boundaries. At late times k_h^* and k_L^* converge on $k \propto a^{-1/2}$, while K^* falls faster, as a^{-1} , so that it crosses k_h^* at a_* . (iv) In mass, m_h^* grows as a^3 at early times and as $a^{3/2}$ at late times, with the break clearest in Illustris-1-Dark (Fig. 5); the cutoff M^* grows faster, as $a^{15/4}$ before a_* and close to a^3 after in Illustris-1-Dark, whereas in Virgo its early growth is slower. (v) The two halo scales cross at $a_* \approx 0.77$ in Virgo and $a_* \approx 0.5$ in Illustris-1-Dark. (vi) The total halo mass grows as $M_h \propto a^{1/2}$ once the steady-state cascade is established after a_* .

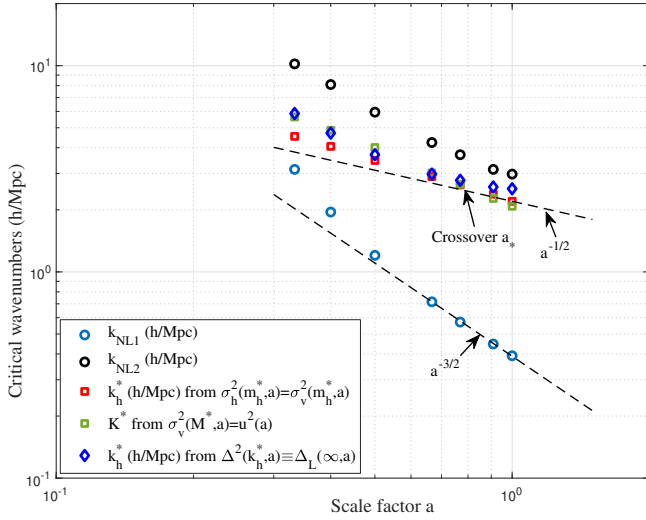


Figure 4. Evolution of the critical wavenumbers in the Virgo SCDM simulation: k_{NL1} (blue circles), k_{NL2} (black circles), k_h^* from $\sigma_v^2(m_h^*) = \sigma_h^2(m_h^*)$ (red squares), K^* from $\sigma_v^2(M^*) = u^2$ (green squares), and k_L^* from $\Delta^2(k_L^*) = \Delta_L^2(\infty)$ (diamonds). Dashed lines: $a^{-3/2}$ and $a^{-1/2}$. K^* falls faster than k_h^* , so $K^* > k_h^*$ before the crossover a_* and $K^* < k_h^*$ after; k_L^* follows K^* before a_* and k_h^* after.

3 THE LINEAR-NONLINEAR CROSSOVER AT a_*

3.0.0.1 Definition. We define the crossover a_* by the equality of the two halo mass scales,

$$M^*(a_*) = m_h^*(a_*) \iff K^*(a_*) = k_h^*(a_*). \quad (11)$$

In Virgo SCDM, $a_* \approx 0.77$ ($z_* \approx 0.3$; Figs. 3 and 4). In Illustris-1-Dark the corresponding break in m_h^* occurs earlier, near $a \approx 0.5$ (Fig. 5), consistent with the onset of the steady mass cascade there (Section 5); the two simulations differ in cosmology and resolution.

3.0.0.2 What changes at a_* . Four measured features change together at a_* : (i) the mass ordering reverses, from $M^* < m_h^*$ to $M^* > m_h^*$; (ii) the linear-extrapolation scale follows K^* before a_* and k_h^* after (Fig. 4), so that $M_L^* \approx \min(M^*, m_h^*)$ is the largest fully collapsed mass; (iii) the nonlinear virial motion overtakes the linear bulk flow (Fig. 6); and (iv) the $n = -1$ plateau appears in the compensated spectrum (Fig. 1), when about half of the dark matter is in haloes ($f_h \approx 0.5$; Fig. 5). All four mark the same event, the transition of the matter field from the linear to the nonlinear regime, which shapes the mass function and the density profile on either side of a_* .

3.0.0.3 Physical picture. A *multistream* region is one where shell crossing has occurred, so that matter from different Lagrangian positions overlaps at the same Eulerian point. Walls, filaments and haloes are multistream; voids remain single-stream [41, 42]. The coherent, curl-free large-scale flow displaces matter by an rms amount per dimension $\Psi_{\parallel}$, with $\Psi_{\parallel}^2 = (6\pi^2)^{-1} \int P_L(k) dk$, at velocity $V_{\parallel} = aHf\Psi_{\parallel}$.

Before a_* , multistream regions are isolated islands in a single-stream flow, and the coherent flow carries each island as a whole. Since $m_h^* > M^*$, haloes near m_h^* lie above the cutoff and are rare. Each forms by the near-spherical collapse of an isolated high peak from cold matter close to the mean density, so these haloes are freshly virialized at a fixed density contrast Δ_{vir} .

At a_* , collapse reaches scales large enough for walls and fila-

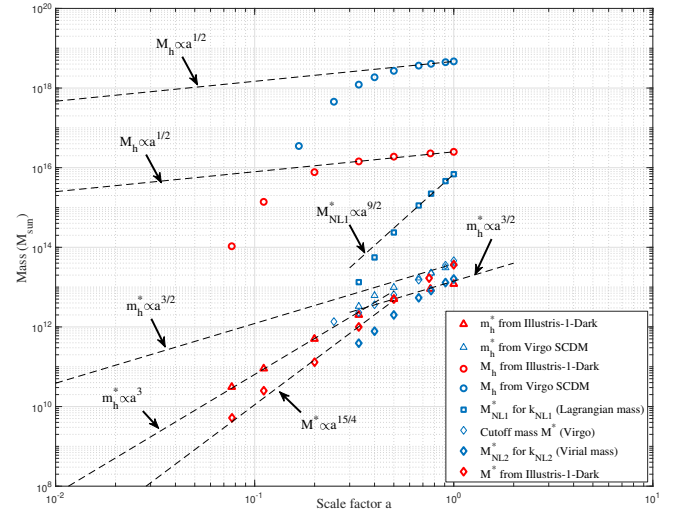


Figure 5. Evolution of the mass scales in Illustris-1-Dark (red) and Virgo SCDM (blue): characteristic mass m_h^* (triangles), $\propto a^3$ before and $\propto a^{3/2}$ after the crossover; total halo mass M_h (circles), $\propto a^{1/2}$ after a_* ; cutoff mass M^* (red diamonds, Illustris-1-Dark; open blue diamonds, Virgo), $\propto a^{15/4}$ before a_* in Illustris-1-Dark; and the spectral masses M_{NL1}^* (squares), $\propto a^{9/2}$ at late times, and M_{NL2}^* (filled blue diamonds), both from Eq. (8). Dashed lines show the predicted scalings. At a_* about half of the dark matter is in haloes ($f_h \approx 0.5$) in both simulations.

ments to join the islands into a single network spanning the volume, with haloes at filament crossings. The multistream regions become *connected*: one can pass from one halo to another without leaving shell-crossed matter. A natural indicator of this overlap is the ratio of the Lagrangian radius of cutoff haloes to the coherent displacement, $R_L(M^*)/\Psi_{\parallel}$. We therefore identify a_* by $M^* = m_h^*$ (Eq. (11)), where the bulk flow enters through σ_h via m_h^* (Eq. (4)).

After a_* , $m_h^* < M^*$, and haloes near m_h^* are typical members of larger structures that are still collapsing on scales up to M^* . They still move with the large-scale flow, but they accrete from their local web. This matter is *pre-processed*: having already crossed shells in walls, filaments and the outskirts of larger haloes, it arrives warm, at densities well above the mean, and partly bound in smaller haloes. Once decoupled, the haloes built from it keep a fixed comoving size, so their physical density falls as $\rho_h^* \propto a^{-3/2}$ and their contrast grows as $\Delta_h^* \propto a^{3/2}$ (Section 4).

3.0.0.4 Velocity-crossover relations. At fixed time $\sigma_v^2 \propto m_h^{2/3}$, while the crossover values follow $\sigma_v^2(m_h^*) \propto m_h^{*1/3}$ (Fig. 3). Writing $\sigma_h^2(m_h^*) \propto a^{\theta_h}$ and $\sigma_v^2 \propto m_h^{2/3} a^{-\theta_v}$ at fixed mass, the crossover gives $m_h^* \propto a^{3(\theta_h + \theta_v)/2}$, and the $m_h^{*1/3}$ law requires

$$\theta_h = \theta_v \equiv \theta, \quad m_h^* \propto a^{3\theta}, \quad \rho_h^* m_h^* = \text{const}, \quad (12)$$

where ρ_h^* is the physical (dynamical) density of haloes at m_h^* , since $\sigma_v^2 \propto G m_h^{2/3} \rho_h^{*1/3}$. The specific kinetic energy of all particles grows linearly in time, $K_p = -\epsilon_u t$ (Appendix B), and its virial part, averaged over haloes, also grows linearly, $\langle K_{pv} \rangle \propto t$, in both Illustris-1-Dark and Virgo SCDM (Fig. 6). With $\langle K_{pv} \rangle \approx \frac{3}{2} \sigma_v^2(M^*) \propto M^{*2/3} a^{-\theta}$ (Section 2) and $M^* \propto a^{6/(n+3)}$, where n is the effective spectral index on the scale M^* ,

$$M^* \propto (a^{\theta} t)^{3/2} \propto t^{3/2} m_h^{*1/2}, \quad \theta = \frac{4}{n+3} - \frac{3}{2}. \quad (13)$$

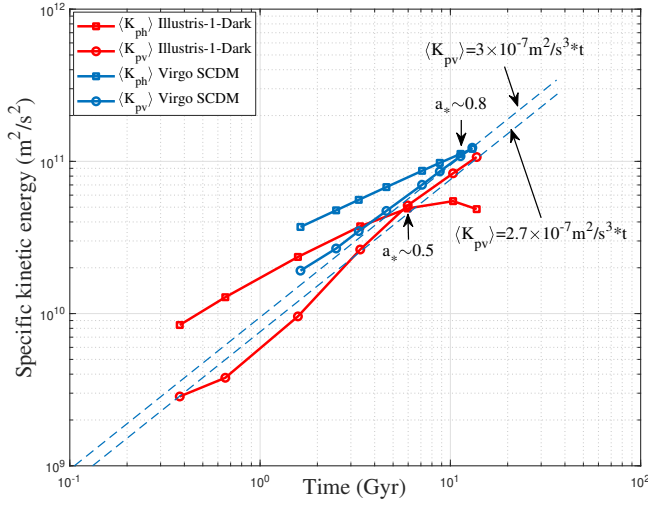


Figure 6. Mean specific virial ($\langle K_{pv} \rangle$, circles) and bulk ($\langle K_{ph} \rangle$, squares) kinetic energy of halo particles versus time in Illustris-1-Dark (red) and Virgo SCDM (blue). Dashed lines: $\langle K_{pv} \rangle \propto t$ with slopes 2.7 and $3 \times 10^{-7} \text{ m}^2 \text{ s}^{-3}$, below the all-particle rate $|\varepsilon_u| \approx 4.6 \times 10^{-7} \text{ m}^2 \text{ s}^{-3}$ (Eq. (B5)), which includes bulk motion and unbound particles. $\langle K_{pv} \rangle$ overtakes $\langle K_{ph} \rangle$ at $a_* \approx 0.5$ (Illustris-1-Dark) and ≈ 0.8 (Virgo SCDM), as Eq. (14) requires. The same $\langle K_{pv} \rangle = \frac{3}{2} u^2$ defines the cutoff M^* in Fig. 3.

Both regimes are compatible with $\langle K_{pv} \rangle \propto t$; $\theta = 1$ for $n = -7/5$, and $\theta = 1/2$ for the universal $n = -1$ (Section 4). The value $n = -7/5$ is the effective index implied by $\theta = 1$ at M^* ; it is not derived independently here, but the corresponding $M^* \propto a^{15/4}$ is measured in Illustris-1-Dark (Fig. 5). Because bulk energy is set by $\sigma_h^2(m_h^*) = \sigma_v^2(m_h^*)$, the ratio of virial to bulk energy of all haloes is

$$\frac{\langle K_{pv} \rangle}{\langle K_{ph} \rangle} \simeq \left(\frac{M^*}{m_h^*} \right)^{2/3}, \quad (14)$$

so the crossover is equivalently $\langle K_{pv} \rangle = \langle K_{ph} \rangle$: linear bulk energy dominates before a_* , and nonlinear virial energy after (Fig. 6).

3.0.0.5 Before and after a_* . Before a_* , the rare haloes found at m_h^* have a fixed contrast, $\sigma_v \sim aHR_L$, and move with the linear bulk flow, $\sigma_h^2(m_h^*) \simeq V_{\parallel}^2$, so $\theta = 1$. Equations (12) and (13) give $m_h^* \propto t^2$ and $M^* \propto t^{5/2} \propto a^{15/4}$, as measured in Illustris-1-Dark (Fig. 5; in Virgo the early growth is slower, roughly $a^{2.5}-a^{2.7}$ for $1 \leq z \leq 3$), and $\langle K_{pv} \rangle / \langle K_{ph} \rangle \propto t^{1/3}$ rises to unity at a_* . After a_* the contrast grows, $\theta = 1/2$ (derived in Section 4), and

$$m_h^* \propto a^{3/2} \propto t, \quad M^* \propto a^3 \propto t^2, \quad (15)$$

so the range below the cutoff widens as $M^*/m_h^* \propto t$. Below M^* the halo distribution becomes statistically steady (*small-scale permanence*; [43]) and mass passes through each scale at a scale-independent rate ε_m ; this mass cascade is developed in Section 5. The switch and its consequences are summarized in Table I3.

3.0.0.6 Three density laws. The picture above involves three distinct densities, each of which defines a halo radius (Section 6.0.0.1).

(i) The *fixed contrast*, a constant multiple of the background density, defines $r_h \propto m_h^{1/3} a$; it is tied to the expansion and is blind to dynamics. (ii) The *virial density*, shared by all haloes at a given epoch with $\rho_h^* m_h^* = \text{const}$ (Eq. (12)), defines the virial size $R_h \propto m_h^{1/3} a^\theta$; it records when haloes decouple from the flow. (iii) The *cascade*

law, $\rho \propto \varepsilon^{2/3} G^{-1} r^{-4/3}$, holds inside haloes and defines the scale radius r_s ; it is set by the energy flux rather than by collapse. Before a_* haloes virialize at fixed contrast, so $R_h = r_h$; after a_* decoupled haloes keep a fixed comoving size, so R_h falls behind r_h as $a^{-1/2}$. In real space the crossover is the moment when the virial size of cutoff haloes overtakes that of characteristic haloes, $R_h(M^*) = R_h^*$. The cascade law operates on both sides of a_* , because the radial cascade requires only a short internal dynamical time (Section 6). Reading the two dynamical fronts, m_h^* and M^* , through these three laws gives six halo radii, which together with the mass function show how haloes differ before and after a_* .

3.0.0.7 Injection and energy-containing scales. The two mass scales play the roles of the two scales of the dual cascade in two-dimensional turbulence. The characteristic mass m_h^* , where the linear bulk flow hands its energy to nonlinear virial motion ($\sigma_v = \sigma_h$), is the forcing scale. The cutoff M^* , where virial energy accumulates, is the growing energy-containing scale. Kinetic energy cascades inversely, as energy does in two dimensions, and potential energy directly, in place of enstrophy. With no large-scale sink the energy grows linearly, $\langle K_{pv} \rangle = |\varepsilon_u| t$, and the two scales separate as $M^*/m_h^* \propto a^{3/2}$ after crossover. The crossover a_* is then the epoch at which the energy-containing scale passes the forcing scale and the inverse inertial range opens. This explains why the rate $|\varepsilon_u|$ is time-invariant at m_h^* rather than at M^* . All virial energy originates as bulk kinetic energy converted at m_h^* , so by energy conservation the radial cascade in characteristic haloes carries the global rate ε_u .

4 UNIVERSAL SPECTRUM $P(k) \propto k^{-1}$ AFTER a_*

This section derives the universal range seen in Fig. 1 and the scalings of the five scales after the crossover a_* .

4.0.0.1 Three spectral ranges. After a_* the measured spectrum divides into three power-law ranges (Fig. 2),

$$\begin{aligned} \text{Linear: } \Delta_L^2(k, a) &\propto a^2 k^{3+n_i}, \quad k \leq k_{\text{NL1}}, \\ \text{Universal: } \Delta_X^2(k, a) &\propto a^b k^{3+n_X}, \quad k_{\text{NL1}} \leq k \leq k_{\text{NL2}}, \\ \text{Nonlinear: } \Delta_{\text{NL}}^2(k, a) &\propto a^{-n_a} k^{3+n_a}, \quad k \geq k_{\text{NL2}}, \end{aligned} \quad (16)$$

where n_i is the linear spectral index, b and n_X are the growth exponent and index of the universal range, and n_a is the nonlinear index, which stable clustering relates to the linear one [40],

$$n_a = -\frac{6}{5 + n_i}. \quad (17)$$

For scale-free spectra n_i is constant. For the CDM spectrum it is the local index $d \ln P_L / d \ln k$, which steepens with k , so the scale at which it is evaluated matters. Nonlinear modes descend from linear ones through the mapping $k_L = k / [1 + \Delta^2(k)]^{1/3}$ (Eq. (C1)). For $\Delta_X^2 = (k/k_{\text{NL1}})^2$ this gives $k_L \simeq (k k_{\text{NL1}}^2)^{1/3}$, so the entire universal range, which spans a factor of several in k , descends from a narrow band of the linear spectrum just above k_{NL1} . Both boundaries therefore originate from nearly the same linear scales, and a single effective index n_i , evaluated at k_{NL1} , enters both the continuity relations and Eq. (17).

Continuity at the two boundaries, $\Delta_L^2 = \Delta_X^2$ at k_{NL1} and $\Delta_X^2 = \Delta_{\text{NL}}^2$ at k_{NL2} , gives

$$k_{\text{NL1}} \propto a^{-\frac{2}{3+n_i}}, \quad b = \frac{2(3+n_X)}{3+n_i}, \quad k_{\text{NL1}}^{3+n_X} k_{\text{NL2}}^{n_a-n_X} \propto a^{n_a}. \quad (18)$$

For scale-free spectra b and n_X both depend on n_i (Appendix C).

For the CDM spectrum, the equality scale k_{eq} breaks self-similarity and selects $n_X = -1$ and $b = 3$, as we now show.

4.0.0.2 Shape: $n_X = -1$ from velocity matching. The linear bulk velocity variance smoothed on comoving scale R is [44]

$$\sigma_h^2(R, a) = \frac{a^2 H^2 f^2}{6\pi^2} \int_0^\infty P(k, a) W^2(kR) dk, \quad (19)$$

with $f = \frac{d \ln D}{d \ln a} \approx \Omega_m^{0.55}$ and the top-hat window $W(x) = 3(\sin x - x \cos x)/x^3$. For $kR \ll 1$, $W^2 \approx 1 - (kR)^2/5$, and the leading term is dominated by $k \sim k_{\text{eq}}$, $\int P dk \approx P(k_{\text{eq}}, 0) k_{\text{eq}} a^2$, with [45]

$$k_{\text{eq}} = \frac{H_0}{c} \sqrt{\frac{2\Omega_m^2}{\Omega_{\text{rad}}}}, \quad P(k_{\text{eq}}, 0) = \frac{2\pi^2 A_s}{k_{\text{eq}}^3 a_{\text{eq}}^2}, \quad (20)$$

where A_s is the primordial scalar amplitude and a_{eq} the scale factor at equality. This gives a mass-independent saturation value (Fig. 7),

$$\sigma_h^2(R \rightarrow 0, a) = \frac{a^4 H^2 f^2 A_s \Omega_{\text{rad}} c^2}{6 \Omega_m^2 H_0^2 a_{\text{eq}}^2} \propto a, \quad (21)$$

with a mass-dependent correction of order $R^2 \propto m_h^{2/3}$. On large scales, σ_h^2 instead tracks the density variance $\sigma_\delta^2(R, a) = (2\pi^2)^{-1} \int k^2 P W^2 dk$ through the exact relation

$$\sigma_h^2(R, a) = \frac{\beta_n}{3} a^2 H^2 f^2 \sigma_\delta^2(R, a) R^2, \quad (22)$$

$$\beta_n \equiv \frac{\int_0^\infty P W^2(kR) dk}{\int_0^\infty P(kR)^2 W^2(kR) dk}.$$

For a local power law $P \propto k^{n_{\text{eff}}}$ with $-1 < n_{\text{eff}} < 1$,

$$\beta_n = \frac{4(1 - n_{\text{eff}})}{(2 - n_{\text{eff}})(5 - n_{\text{eff}})(1 + n_{\text{eff}})}, \quad (23)$$

which depends on R only through $n_{\text{eff}}(R)$, since $W^2(kR)$ is sharply peaked near $k \sim 1/R$. For $n_{\text{eff}} > -1$ the large-scale branch scales as $\sigma_\delta^2 R^2 \propto R^{-(1+n_{\text{eff}})}$, whereas the saturated branch has zero slope (Fig. 7). Requiring the two branches to join near m_h^* in both value and logarithmic slope is possible only as $n_{\text{eff}} \rightarrow -1$:

$$\sigma_\delta^2(R, a) \propto R^{-2} \propto m_h^{-2/3} \implies P(k) \propto k^{-1}. \quad (24)$$

The matching argument is heuristic, but its prediction is tested directly by the plateau of Fig. 1. In this limit $\sigma_\delta^2 R^2$ becomes scale-independent, while the velocity integral of Eq. (19) stays finite because the CDM spectrum turns over below k_{eq} . The k^{-1} shape therefore follows from gravity and the existence of k_{eq} , not from the detailed initial spectrum, and the compensated spectrum $P(k)k$ is flat: the velocity variance per logarithmic interval is constant. An independent dimensional derivation is given in Appendix D.

4.0.0.3 Growth: $b = 3$ from pair conservation. Let $\xi(r, a)$ be the two-point correlation function at comoving separation r , $\bar{\xi}(r, a) = 3r^{-3} \int_0^r \xi(y, a) y^2 dy$ its volume average within r , and $\langle \Delta u_L \rangle$ the mean radial peculiar velocity of particle pairs at separation r . Pair conservation relates the two [40],

$$\frac{\langle \Delta u_L \rangle}{Har} = -\frac{1 + \bar{\xi}}{3(1 + \bar{\xi})} \frac{\partial \ln(1 + \bar{\xi})}{\partial \ln a}, \quad (25)$$

where Har is the Hubble velocity at physical separation ar . Three limits follow. In the linear regime ($\bar{\xi} \ll 1$, $\bar{\xi} \propto a^2$), $\langle \Delta u_L \rangle / Har \approx -2\bar{\xi}/3$. In stable clustering, pairs are frozen at fixed physical separation, $\langle \Delta u_L \rangle / Har = -1$, which the linear branch reaches at

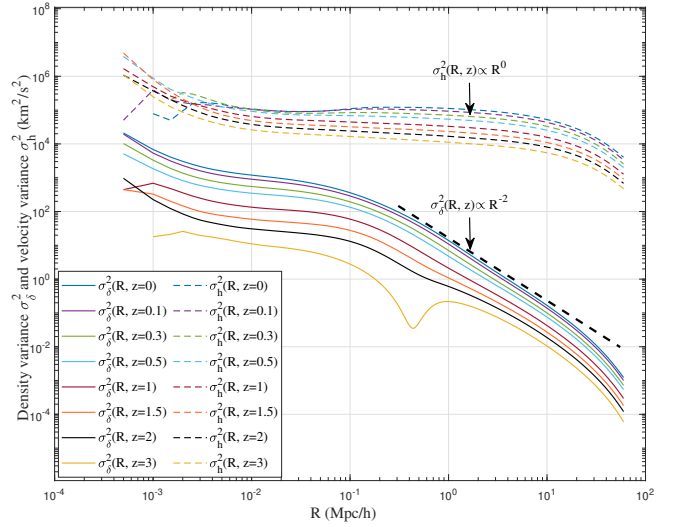


Figure 7. Density variance $\sigma_\delta^2(R, a)$ (solid) and bulk velocity variance $\sigma_h^2(R, a)$ (dashed) versus comoving scale R in Virgo SCDM [1]. On large scales $\sigma_h^2 \propto \sigma_\delta^2 H^2 R^2$ (Eq. (22)); on small scales σ_h^2 saturates (Eq. (21)). Matching the two in value and slope near m_h^* ($R \approx 1 h^{-1} \text{Mpc}$) requires $\sigma_\delta^2 \propto R^{-2}$, i.e. $P(k) \propto k^{-1}$.

$\bar{\xi} = 3/2$. In the universal range, $\xi \propto a^b r^{-2}$ gives $\bar{\xi} = 3\bar{\xi} \gg 1$ and $\langle \Delta u_L \rangle / Har = -b$. Matching the linear and universal branches at $r_{\text{NL1}} = 1/k_{\text{NL1}}$ gives $2\xi(r_{\text{NL1}}, a) = b$. Identifying this boundary with the same order-unity threshold, now applied to the local correlation, $\xi(r_{\text{NL1}}, a) = 3/2$, gives

$$b = 3, \quad P(k, a) \propto a^3 k^{-1}. \quad (26)$$

This matching is heuristic. The primary evidence for $b = 3$ is the direct measurement (Fig. 8), supported by physical argument below.

The a^3 growth has a simple origin. In stable clustering the number of pairs at fixed physical separation $r_{\text{phy}} = ar$ is constant while the mean density dilutes as a^{-3} , so $\xi(r_{\text{phy}}) \propto a^3$. For a local index n , converting to comoving wavenumber $k = ak_{\text{phy}}$ gives

$$\Delta_{\text{NL}}^2(k, a) \propto a^3 (k/a)^{3+n} = a^{-n} k^{3+n}. \quad (27)$$

The universal range adds linear amplification (a^2) to this dilution (a^3), so $\xi_X \propto a^5 r_{\text{phy}}^{-2} \propto a^3 r^{-2}$. For the same k^{-1} shape, the three ranges therefore grow as

$$\begin{aligned} \text{Linear: } \Delta_L^2 &\propto a^2 k^2, & \xi_L &\propto a^4 r_{\text{phy}}^{-2}, \\ \text{Universal: } \Delta_X^2 &\propto a^3 k^2, & \xi_X &\propto a^5 r_{\text{phy}}^{-2}, \\ \text{Nonlinear: } \Delta_{\text{NL}}^2 &\propto a k^2, & \xi_{\text{NL}} &\propto a^3 r_{\text{phy}}^{-2}, \end{aligned} \quad (28)$$

as confirmed by Δ^2 at fixed k (Fig. 8). Structures in the universal range are assembling but not yet virialized: they grow faster than linear perturbations by a factor a , and faster than stably clustered ones by a factor a^2 of quasi-linear amplification.

4.0.0.4 Boundary scales. With $n_X = -1$ and $b = 3$, Eq. (18) requires $3 + n_i = 4/3$, and Eq. (17), with the same effective index, then fixes the nonlinear index, which justifies the $k^{6/5}$ asymptote used to define k_{NL2} in Section 2:

$$n_i^{\text{eff}} = -\frac{5}{3} \text{ at } k_{\text{NL1}}, \quad n_a^{\text{eff}} = -\frac{9}{5} \text{ at } k_{\text{NL2}}, \quad (29)$$

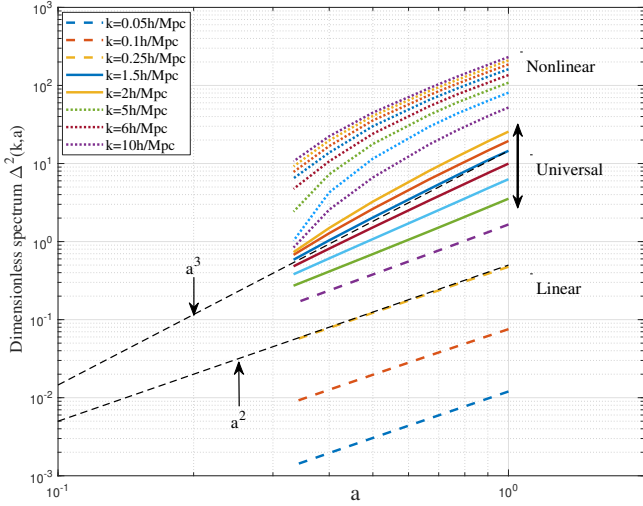


Figure 8. Evolution of $\Delta^2(k, a)$ at fixed comoving k in Virgo SCDM: $\propto a^2$ on linear scales, $\propto a^3$ in the universal range, and slower growth in the deeply nonlinear range, where virialization halts further amplification.

so that

$$k_{\text{NL1}}, k_{\text{NL2}} \propto a^{-3/2}, \quad M_{\text{NL1}}^*, M_{\text{NL2}}^* \propto a^{9/2}. \quad (30)$$

The universal range stays continuous with the linear spectrum only where the local linear index at k_{NL1} is close to $-5/3$. For the SCDM spectrum this holds near and after a_* , where $k_{\text{NL1}} \propto a^{-3/2}$ is observed (Fig. 4). The measured k_{NL2} evolves more slowly than Eq. (30), roughly as $a^{-0.9}$, because Virgo SCDM has a low mass resolution close to M_{NL2}^* .

4.0.0.5 Linear-extrapolation and kinematic scales coincide.

Two independent routes give the same scaling for the central scale. From the spectrum, threshold $\Delta_L^2(\infty) \propto a^2$ meets $\Delta_X^2 \propto a^3 k^2$ at

$$k_L^* \propto a^{-1/2}, \quad M_L^* \propto a^{3/2}. \quad (31)$$

From the kinematics, haloes of mass m_h^* have just decoupled from the flow, so their comoving size is fixed and their physical density scales as $\rho_h^* \propto m_h^* a^{-3}$. Combined with $\rho_h^* m_h^* = \text{const}$ (Eq. (12)), this gives $m_h^{*2} \propto a^3$, i.e. $m_h^* \propto a^{3/2}$, and a contrast $\Delta_h^* = \rho_h^*/\bar{\rho}(a) \propto a^{3/2}$. The physical size of a halo at this dynamical contrast, its virial size R_h (Section 6.0.0.1), and the dispersions are then

$$R_h = \left(\frac{3m_h}{4\pi\Delta_h^*\bar{\rho}(a)} \right)^{1/3} \propto m_h^{1/3} a^{1/2}, \quad \sigma_h^2(m_h^*, a) \propto a^{1/2}, \quad (32)$$

$$\sigma_v^2(m_h, a) \propto \frac{Gm_h}{R_h} \propto m_h^{2/3} a^{-1/2}, \quad \sigma_v^2(m_h^*) \propto m_h^{*1/3},$$

so $\theta = 1/2$, consistent with Eq. (12). Here $\sigma_h^2(m_h^*, a)$ is the bulk dispersion at the crossover, which lies below the small-mass plateau of Eq. (21) once the steady state is established (Fig. 3). The two scales agree after a_* (Fig. 4), which closes the argument: the spectral threshold and the velocity crossover pick out the same turnaround scale. In the universal range,

$$\Delta_X^2(k, a) \sim \delta_{\text{ta}}^2 a^2 (k/k_h^*)^2, \quad \Delta_X^2(k_h^*, a) \sim \delta_{\text{ta}}^2 a^2, \quad (33)$$

where $\delta_{\text{ta}} = (3\pi/4)^2 - 1 \approx 5.55$ is the turnaround overdensity. At $a = 1$ this recovers the spherical-collapse value, a consequence of normalizing σ_8 today. At earlier epochs the contrast $\Delta_h^* \sim \delta_{\text{ta}} a^{3/2}$

is smaller: haloes at $m_h^*(a)$ are lower-contrast proto-haloes that approach isolated spherical collapse only as $a \rightarrow 1$. In terms of density variance the crossover reads

$$\frac{\sigma_v^2(m_h, a)}{\sigma_h^2(m_h^*, a)} = \frac{\sigma_\delta^2(m_h^*, a)}{\sigma_\delta^2(m_h, a)}, \quad (34)$$

with $\sigma_\delta^2(m_h^*, a) \propto a^2$ and $\sigma_\delta^2(m_h, a) \propto m_h^{-2/3} a^3$.

4.0.0.6 Before a_* . Without a universal range, the rare haloes at $m_h^* > M^*$ have a fixed contrast, $\rho_h^* \propto a^{-3}$, so $m_h^* \propto a^3$ and $\theta = 1$ (Section 3), the early growth seen in Illustris-1-Dark (Fig. 5). The construction $\Delta^2(k_L^*) = \Delta_L^2(\infty)$ is then not tied to m_h^* : extrapolating the k^2 line from k_{NL1} gives $M_L^* \propto a^{6/(3+n_{\text{eff}})-3}$, set by the local linear index n_{eff} at k_{NL1} . In the SCDM run k_L^* happens to track K^* before a_* (Figs. 2 and 4), but this follows from the curvature of the CDM spectrum rather than from a general law.

5 HALO MASS FUNCTION AND MASS CASCADE

After a_* the halo population below the cutoff M^* is statistically steady, and mass and energy cascade through halo mass space at scale-independent rates, driven by gravitational merging and accretion rather than by viscosity and vortices. We measure the mass cascade, derive the mass function from a random walk of haloes on two clocks, one for m_h^* and one for M^* , and use it to obtain the cascade rates [43, 46]; the energy cascade in mass space is detailed in Appendix F0.0.1.

5.0.0.1 Mass flux in halo mass space. Let $M_h(t)$ be the total mass in haloes and $f_M(m_h, t)$ the halo mass function, normalized so that $M_h f_M dm_h$ is the mass in haloes with masses in $[m_h, m_h + dm_h]$. The cumulative mass function and the mass flux across m_h are

$$\Lambda_m(m_h, t) = \int_{m_h}^{\infty} M_h(t) f_M(m, t) dm, \quad (35)$$

$$\Pi_m(m_h, t) = -\frac{\partial \Lambda_m}{\partial t}.$$

Π_m is the rate at which halo mass is transferred from scales below m_h to scales above m_h ; the negative sign makes an upward (inverse) transfer negative. Only mergers whose remnant lies above m_h and at least one of whose progenitors lies below it contribute; mergers between two haloes already above m_h transfer no mass across m_h . Locality in mass space is not assumed, although it holds approximately (Appendix F0.0.2). Since $\Lambda_m(0, t) = M_h(t)$,

$$\Pi_m(m_h, t) = -\dot{M}_h + \frac{\partial}{\partial t} \int_0^{m_h} M_h f_M dm. \quad (36)$$

Unbound matter enters haloes only at the smallest mass, which we identify with the particle mass m_p ; accreting one particle onto a halo of mass m_h carries m_p across every scale between m_p and m_h . If the distribution below some mass is statistically steady, $\partial_t(M_h f_M) = 0$ there, and the flux becomes scale-independent,

$$\varepsilon_m(t) \equiv \Pi_m(m_h, t) = -\dot{M}_h(t), \quad (37)$$

which defines the mass cascade rate. Mass space then divides into a *propagation range*, where each scale passes on all the mass it receives, like a leak-free pipe at constant level, and a *deposition range*, where the flux decreases with mass and the cascaded mass is consumed to grow haloes.

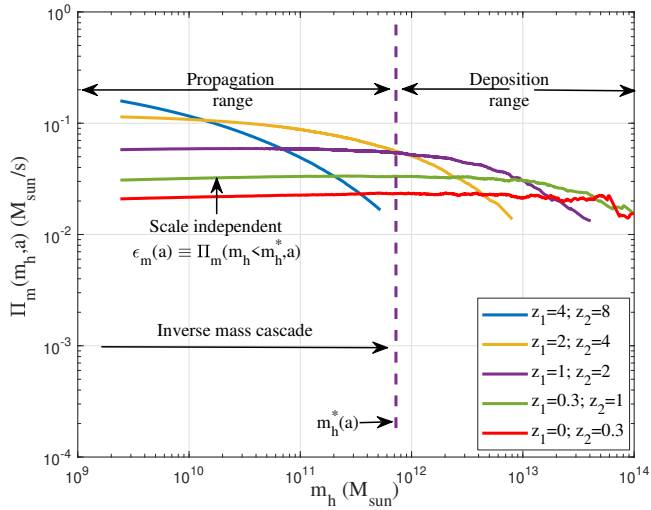


Figure 9. Mass flux $-\Pi_m(m_h, z)$ from Illustris-1-Dark at several redshifts, computed from Eq. (35). After the steady state is reached at $z \approx 1-2$, a propagation range forms with a scale-independent cascade rate $\epsilon_m(a) \propto a^{-1}$, reaching $\approx -0.02 M_\odot s^{-1}$ at $z = 0$. The negative flux indicates an inverse cascade. The vertical line marks m_h^* for illustration only; the edge of the plateau is set by M^* (Eq. (61)).

Figure 9 confirms this picture. In Illustris-1-Dark, Π_m computed from Eq. (35) develops a scale-independent plateau well below the cutoff once the steady state is reached at $z \approx 1-2$, with $\epsilon_m \approx -0.02 M_\odot s^{-1}$ at $z = 0$ and $\epsilon_m \propto a^{-1}$. The plateau extends to larger masses with time, faster than m_h^* ; Eq. (61) below shows that its edge is set by M^* . In Virgo the propagation range forms later, at $z \approx 0.3$ (not shown), consistent with its later crossover (Section 3). Before the steady state, the coherent flow keeps raising the abundance of haloes of every mass, $\partial_t(M_h f_M) \neq 0$, and the flux varies with halo mass.

5.0.0.2 Small-scale permanence. Let $m_g(m_h) \equiv M_h f_M m_p = n_h m_h$ be the halo group mass, the total mass of all n_h haloes of mass m_h in bins of one particle mass. Differentiating Eq. (35),

$$\frac{\partial \Pi_m}{\partial m_h} = \frac{1}{m_p} \frac{\partial m_g}{\partial t}, \quad (38)$$

so a scale-independent flux is equivalent to a time-invariant group mass: the mass flowing into each group balances the mass flowing out, and m_g depends on m_h only. This *small-scale permanence* holds once the steady state is established, independently of cosmology and initial conditions. It fixes the time invariance of m_g , not its shape. Figure 10 confirms it and shows, as a separate observation, that m_g at different redshifts collapses onto a single power law $m_g \propto m_h^{-\lambda}$ with local slope $\lambda \approx 0.9$, close to the effective slope $\lambda_1 = 5/6$ derived below (Eq. (58)).

5.0.0.3 Local slopes from the spectrum. The PS formalism predicts a small-mass slope $f_M \propto m_h^{-(3-n_i)/6}$, set entirely by the primordial index n_i . We attribute its known discrepancies with simulations, which the Sheth–Tormen model partially corrects through ellipsoidal collapse [22], to two effects: gravity changes the effective index in the deeply nonlinear regime, and a universal range with $n_{\text{eff}} = -1$ emerges near m_h^* . Replacing n_i by the local effective index, the walk exponent is

$$\lambda = 1 + \frac{d \ln \sigma_\delta}{d \ln m_h} = \frac{3 - n_{\text{eff}}}{6}. \quad (39)$$

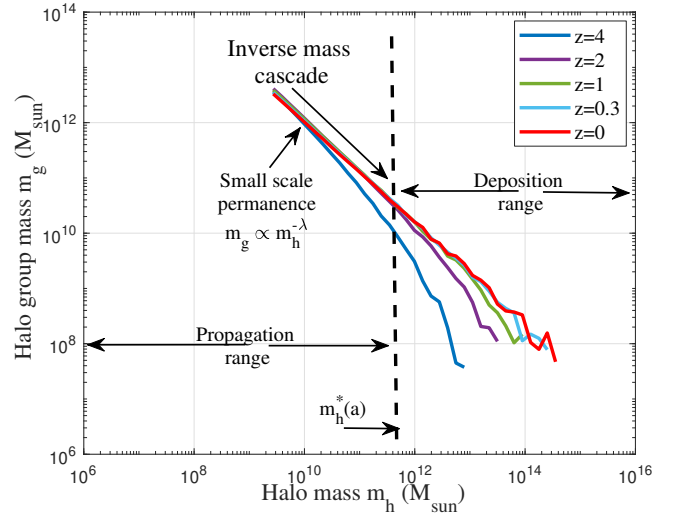


Figure 10. Halo group mass $m_g(m_h)$ at different redshifts from Illustris-1-Dark, demonstrating small-scale permanence (Eq. (38)). Once the steady state is established, m_g collapses onto a time-independent power law, $m_g \propto m_h^{-\lambda}$ with local slope $\lambda \approx 0.9$ at small masses. The time-invariant range extends to larger masses as $M^*(a)$ grows.

The three spectral ranges of Section 4, with $n_{\text{eff}} = -9/5$ (nonlinear), $-5/3$ (linear, at $k_{\text{NL}1}$) and -1 (universal), give $\lambda = 4/5, 7/9$ and $2/3$, respectively. These are local exponents of the walk. They do not appear as three separate slopes of f_M : below the cutoff M^* , the mass function has a single effective slope $\lambda_1 = 5/6$ fixed by permanence, and the gravitational value $\lambda_2 = 2/3$ appears only in the shape of the cutoff (see the double- λ mass function below).

5.0.0.4 Generalized barrier crossing. In the excursion-set picture, a halo of mass m_h forms at epoch a when the smoothed linear variance reaches the collapse barrier,

$$\sigma_\delta(m_h, a) = \sigma_\delta^L(m_h) D(a) = \delta_c, \quad (40)$$

where $D(a)$ is the growth factor. Because the spectrum spans three ranges, we generalize the condition to

$$\sigma_\Delta(m_h) D(a)^{\tau/2} = \delta_{\text{th}}, \quad \sigma_\Delta(m_h) \propto m_h^{\lambda-1}, \quad (41)$$

where δ_{th} is a generalized barrier and τ the growth exponent of the variance. In the three ranges, $\sigma_\Delta = \sigma_\delta^L(m_h)$ with $\tau = 2$, $\sigma_\Delta \propto m_h^{-1/3}$ with $\tau = 1$, and $\sigma_\Delta \propto m_h^{\lambda_a-1}$ with $\tau = -n_a$, and the corresponding barrier conditions are

$$\text{Linear: } \sigma_\delta^L(m_h) D(a) = \delta_c \approx 1.686,$$

$$\text{Universal: } \frac{\sigma_\delta^X(m_h, a)}{D(a)} \propto m_h^{-\frac{1}{3}} D(a)^{\frac{1}{2}} \sim \delta_{\text{ta}}, \quad (42)$$

$$\text{Nonlinear: } \sigma_\delta^{\text{NL}}(m_h, a) \propto m_h^{\lambda_a-1} D(a)^{-\frac{n_a}{2}} \sim \delta_{\text{NL}}.$$

The linear case is the PS condition. In the universal range, $\sigma_\delta^X \propto m_h^{-1/3} D^{3/2}$ follows from $\Delta_X^2 \propto D^3 k^2$ (Eq. (28)); because the amplitude at k_h^* grows as a^2 (Eq. (33)), the effective barrier for σ_δ^X grows as $\delta_{\text{ta}} D$, equivalent to $\delta_{\text{th}} = \delta_{\text{ta}}$. In the nonlinear range, $\lambda_a = (3 - n_a)/6$, $\tau = -n_a$ encodes stable clustering and δ_{NL} marks full virialization. The barrier values δ_{ta} and δ_{NL} are absorbed into the characteristic masses m_h^* and $M_{\text{NL}2}^*$, so only the exponents matter below. In every range a change in D maps onto a shift in m_h , so m_h can be treated as a stochastic variable performing a random walk in D -space.

5.0.0.5 Halo random walk in mass space. Define $u \equiv \sigma_\Delta(m_h)^{-1}$. By Eq. (41), $\langle u^2 \rangle \propto D^\tau$, so with the time-like variable $D_o \equiv D(a)^\tau$, the clock of the walk, u performs a random walk on the half-line $u > 0$ with constant diffusivity D_u ,

$$\frac{du}{dD_o} = \sqrt{2D_u} \zeta(D_o), \quad (43)$$

$$\langle \zeta(D_o) \rangle = 0, \quad \langle \zeta(D_o) \zeta(D'_o) \rangle = \delta(D_o - D'_o).$$

The boundary $u = 0$ corresponds to a halo dissolving into the background and is treated as reflecting. With $u \propto m_h^{1-\lambda}$, $\frac{du}{dm_h} = (1-\lambda)m_h^{-\lambda}$, and the chain rule gives the Langevin equation for m_h ,

$$\frac{dm_h}{dD_o} = \sqrt{2D_p(m_h)} \zeta(D_o), \quad (44)$$

$$D_p(m_h) = D_{p0} m_h^{2\lambda}, \quad D_{p0} = \frac{D_u}{(1-\lambda)^2}.$$

Physically, a halo of mass m_h gains a mass increment δm after a waiting time $\tau_D \propto m_h^{-\lambda}$, so λ encodes the mass dependence of the accretion rate. The variable D_o makes the equation autonomous; using physical time would introduce a factor $H(t)a(t)$ that conflates the intrinsic accretion rate with cosmic expansion.

5.0.0.6 Fokker–Planck solution. The physical noise has a short but finite correlation time and therefore converges, by the Wong–Zakai theorem, to the Stratonovich interpretation [47, 48]. Equation (44) is then equivalent to the Fokker–Planck equation

$$\frac{\partial f_M}{\partial D_o} = D_{p0} \frac{\partial}{\partial m_h} \left[m_h^\lambda \frac{\partial}{\partial m_h} (m_h^\lambda f_M) \right], \quad (45)$$

whose normalized solution is

$$f_M(m_h, D_o) = \frac{m_h^{-\lambda}}{\sqrt{\pi D_{p0} D_o}} \exp \left[-\frac{m_h^{2-2\lambda}}{4(1-\lambda)^2 D_{p0} D_o} \right], \quad (46)$$

a stretched Gaussian with a power-law tail $f_M \propto m_h^{-\lambda}$ at small masses and an exponential cutoff at large masses. Identifying the characteristic mass M_h^* with the root-mean-square displacement of the walk (Appendix E) gives the *front* of the walk,

$$M_h^* \propto D_o^{\frac{1}{2-2\lambda}} \propto D(a)^{\frac{\tau}{2-2\lambda}}. \quad (47)$$

For $\lambda < 1$ the mass dispersion $\langle m_h^2 \rangle \propto D_o^{1/(1-\lambda)}$ grows faster than linearly with D_o : the walk is super-diffusive. In terms of M_h^* ,

$$f_M(m_h, D_o) = \frac{1-\lambda}{\sqrt{\pi\eta_0} M_h^*} \left(\frac{m_h}{M_h^*} \right)^{-\lambda} \exp \left[-\frac{1}{4\eta_0} \left(\frac{m_h}{M_h^*} \right)^{2-2\lambda} \right], \quad (48)$$

where the constant η_0 is given in Appendix E. The evolving distribution continuously maximizes the Gibbs–Boltzmann entropy. This approach complements the extended PS framework beyond the linear regime: f_M is the solution of a stochastic transport equation in which haloes walk in mass space through mergers and accretion.

5.0.0.7 Three spectrum limits. The single- λ solution applies in each spectral range with the corresponding λ , τ and barrier (Table 2). In the linear range it reproduces the PS mass function with walk front $M_{PS}^* \propto M_{NL1}^*$; in the deeply nonlinear range its front is M_{NL2}^* , which marks the stable-clustering range of the spectrum and does not enter the mass function below (both limits in Appendix E). In the *universal limit*, near m_h^* , repeated mergers and smooth accretion

Table 2. Random walks in halo mass space (Eq. (44)), $\frac{dm_h}{dD_o} \propto m_h^\lambda \zeta$ on a clock $D_o = a^\tau$, the counterpart of Table 3. The front grows as $D_o^{1/(2-2\lambda)} = a^\chi$ with $\chi = \tau/(2-2\lambda)$ (Eqs. (47) and (80)), the cutoff is $\exp[-(\frac{m_h}{M_{\text{front}}})^{2-2\lambda}/4\eta_0]$, the effective index is $n = 3 - 6\lambda$, and the waiting time is $\tau_g \propto m_h^{-\lambda} a^{(3-\tau)/2}$ (Eq. (63) for $\tau = 1$). Only the cascade-clock walk follows the growth of individual haloes; the linear-clock walk describes the advance of the population cutoff.

Walk (front)	λ (n)	τ	χ	$2-2\lambda$	$\tau_g \propto$
<i>Spectral limits</i> (Appendix E)					
PS (M_{PS}^*)	$\frac{3-n_i}{6}$ (n_i)	2	$\frac{6}{3+n_i}$	$\frac{3+n_i}{3}$	–
Nonlin. (M_{NL2}^*)	4/5 (–9/5)	9/5	9/2	2/5	–
$a < a_*$: $\theta = 1$					
Cascade (m_h^*)	2/3 (–1)	2	3	2/3	$m_h^{-\frac{2}{3}} a^{\frac{1}{2}}$
Linear (M^*)	11/15 (–7/5)	2	15/4	8/15	–
$a > a_*$: $\theta = 1/2$					
Cascade (m_h^*)	2/3 (–1)	1	3/2	2/3	$m_h^{-\frac{2}{3}} a$
Linear (M^*)	2/3 (–1)	2	3	2/3	–
Total halo mass (M_h)	5/6 (–2)	1	1/2	–	–

erase memory of the initial conditions and λ reflects gravitational assembly alone. With the universal-range barrier of Eq. (42),

$$\nu \equiv \left(\frac{\delta_{\text{ta}} D(a)}{\sigma_\delta^X(m_h, a)} \right)^2 = \left(\frac{m_h}{m_h^*} \right)^{2-2\lambda_h} \simeq \frac{\sigma_v^2(m_h, a)}{\sigma_h^2(m_h^*, a)}, \quad (49)$$

$$\lambda_h = \frac{2}{3}, \quad m_h^* \propto D(a)^{3/2}.$$

The last equality follows from $\sigma_v^2 \propto m_h^{2/3} D^{-1/2}$ and $\sigma_h^2(m_h^*) \propto D^{1/2}$ (Eq. (32)). Thus $\lambda_h = 2/3$ is the natural dynamical value where virial and bulk motions become comparable, and $m_h^* \propto a^{3/2}$ is confirmed by simulations (Fig. 5).

5.0.0.8 Two clocks for m_h^* and M^* . The halo scales are fronts of walks on two clocks. The *cascade clock* $D_c = a^{\tau_c}$ carries the characteristic mass with the gravitational exponent $\lambda_h = 2/3$,

$$m_h^* \propto D_c^{\frac{1}{2-2\lambda_h}} = D_c^{3/2}, \quad \tau_c = 2 \ (a < a_*), \quad \tau_c = 1 \ (a > a_*), \quad (50)$$

which reproduces $m_h^* \propto a^{3\theta}$ (Eq. (12)): before a_* haloes at m_h^* grow coherently with the linear flow ($\tau_c = 2$), after a_* diffusively ($\tau_c = 1$, Eq. (49)). The *linear clock* $D_L = a^{\tau_L}$ with $\tau_L = 2$ carries the cutoff mass with the exponent of the local linear index at M^* ,

$$M^* \propto D_L^{\frac{1}{2-2\lambda_2}}, \quad \lambda_2 = \frac{3-n}{6} \Big|_{M^*} = \begin{cases} \frac{11}{15} & (a < a_*, n = -\frac{7}{5}), \\ \frac{2}{3} & (a > a_*, n = -1), \end{cases} \quad (51)$$

which reproduces $M^* \propto a^{6/(n+3)}$, i.e. $a^{15/4}$ and a^3 (Eq. (13)). After a_* the two exponents coincide, $\lambda_2 = \lambda_h = 2/3$: the universal index $n = -1$ is the one for which the linear walk exponent equals the gravitational one. The clocks differ, however, so the two fronts separate as $M^*/m_h^* \propto (D_L/D_c)^{3/2} \propto a^{3/2}$. The cascade clock governs the growth of individual haloes, whose accretion rate is set by their own potential (Eq. (64)); the linear clock governs the advance of the population cutoff, which is occupied by newly assembled haloes at each epoch (Section 6).

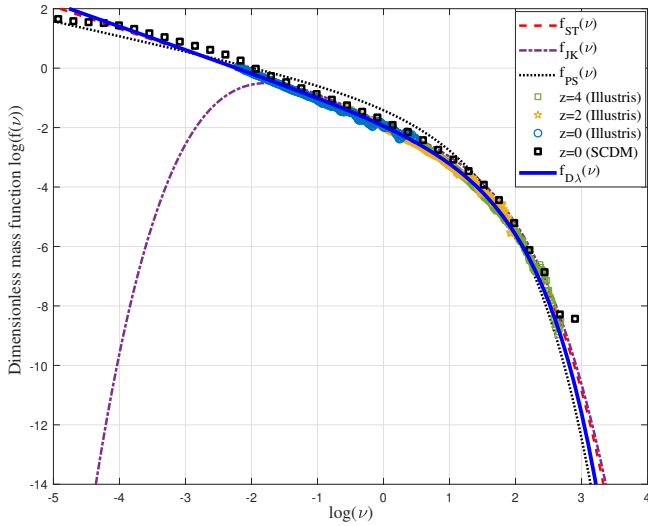


Figure 11. Halo mass functions from simulations at several redshifts (symbols) compared with the double- λ mass function, Eq. (52) (lines), and with the Press–Schechter, Sheth–Tormen, and JK forms (Appendix E).

5.0.0.9 Double- λ mass function. With the cutoff mass M^* as the only scale, the double- λ mass function is [43]

$$f_M(m_h, z) = \frac{2(1-\lambda_2)}{\Gamma(q/2)} (2\sqrt{\eta_0})^{-q} \frac{1}{M^*} \left(\frac{m_h}{M^*}\right)^{-\lambda_1} \times \exp\left[-\frac{1}{4\eta_0} \left(\frac{m_h}{M^*}\right)^{2-2\lambda_2}\right], \quad q = \frac{1-\lambda_1}{1-\lambda_2}. \quad (52)$$

The three ingredients have distinct origins. The cutoff mass M^* is fixed by energy: a time-independent energy cascade rate requires $\sigma_v^2(M^*, a) \propto t$ (Eq. (F6)). The cutoff exponent λ_2 is the walk exponent at the front, where haloes are still in exponential transit; it appears only in the shape of the cutoff. The slope λ_1 is the exponent of the stationary range below the cutoff; it is an *effective* slope, fixed by small-scale permanence below (Eq. (58)). Neither m_h^* nor M_{NL2}^* appears in Eq. (52); m_h^* enters only through the total mass M_h and the cascade rates. Because the local slope is $\frac{d \ln f_M}{d \ln m_h} = -\lambda_1 - (2-2\lambda_2)x$, with $x > 0$ as in Eq. (59), it is always steeper than $-\lambda_1$, and no mass range shows the slope $-2/3$.

In terms of the standard peak height the double- λ form is a generalized gamma distribution whose offset from M_{PS}^* evolves with redshift, so a strictly universal mass function is only an approximation (Appendix E). Define $\nu \equiv (m_h/M_{\text{PS}}^*)^{2-2\lambda_2}$, the dimensionless double- λ mass function is then a generalized gamma distribution with three parameters $\{\eta_0^*, p, q\}$,

$$f_{D\lambda}(\nu) = \frac{p (2\sqrt{\eta_0^*})^{-q}}{\Gamma(q/2)} \nu^{\frac{pq}{2}-1} \exp\left(-\frac{\nu p}{4\eta_0^*}\right), \quad (53)$$

$$\eta_0^*(a) = \eta_0 \left(\frac{M^*}{M_{\text{PS}}^*}\right)^{2-2\lambda_2}, \quad p = \frac{1-\lambda_2}{1-\lambda_1}, \quad q = \frac{1-\lambda_1}{1-\lambda_2}.$$

Here p sets the shape of the cutoff relative to the linear baseline, q the ratio of the two nonlinear exponents, and $\eta_0^*(a)$ the offset between M^* and M_{PS}^* . Figure 11 compares the double- λ mass function with simulated mass functions. Its enhancement over the linear (PS) form arises from the gravitational renormalization of the initial spectrum, through $p \neq 1$, and from the two nonlinear regimes, through $q \neq 1$.

5.0.0.10 Permanence fixes the slope λ_1 . For $m_h \ll M^*$ the exponential in Eq. (52) tends to unity, and

$$M_h f_M \rightarrow A_M m_h^{-\lambda_1}, \quad (54)$$

$$A_M = \frac{2(1-\lambda_2)}{\Gamma(q/2)} (2\sqrt{\eta_0})^{-q} M_h M^{\lambda_1-1}.$$

Small-scale permanence, $\partial_t(M_h f_M) = 0$ at fixed m_h , requires a constant amplitude A_M at fixed λ_1 , λ_2 and η_0 :

$$M_h(t) \propto M^*(t)^{1-\lambda_1}. \quad (55)$$

The total mass M_h follows the cascade clock. In the single- λ solution, Eq. (48), the power-law range is $f_M \propto M_h^{\lambda-1} m_h^{-\lambda}$, so permanence requires $M_h \propto M_h^{*1-\lambda}$, and with $M_h^* \propto D_o^{1/(2-2\lambda)}$ (Eq. (47))

$$M_h \propto D_o^{1/2} \quad \text{for any } \lambda. \quad (56)$$

The total halo mass grows diffusively with the clock. On the cascade clock after a_* ($D_c = a$),

$$M_h(t) \propto a^{1/2}, \quad \varepsilon_m = -\dot{M}_h \propto a^{-1}, \quad (57)$$

in agreement with Figs. 5, 9. Combining Eqs. (51), (55) and (56) with $M_h \propto D_c^{1/2}$,

$$1 - \lambda_1 = \frac{d \ln M_h}{d \ln M^*} = (1 - \lambda_2) \frac{\tau_c}{\tau_L} = \frac{1}{3} \cdot \frac{1}{2} \implies \lambda_1 = \frac{5}{6}. \quad (58)$$

The effective slope is set by the cutoff exponent and the ratio of the two clocks, with no primordial input. It lies between the local deeply nonlinear value $\lambda_a = 4/5$ and the measured local slope $\lambda \approx 0.9$ at small masses (Fig. 10). The two differ because Eq. (52) holds η_0 fixed; fixing the measured local slope instead would require η_0 to evolve. If the two clocks were equal, as before a_* , Eq. (58) would give $\lambda_1 = \lambda_2$, a single- λ (PS-like) form; before a_* the population is not steady (no small-scale permanence), so Eq. (55) does not apply.

5.0.0.11 Cascade rate and the propagation range. Substituting Eq. (52) into Eq. (35),

$$\Lambda_m(m_h, t) = \frac{M_h(t)}{\Gamma(q/2)} \Gamma\left(\frac{q}{2}, x\right), \quad x = \frac{1}{4\eta_0} \left(\frac{m_h}{M^*}\right)^{2-2\lambda_2}, \quad (59)$$

where $\Gamma(s, x)$ is the upper incomplete gamma function, and

$$\Pi_m(m_h, t) = -\dot{M}_h \left[\frac{\Gamma(q/2, x)}{\Gamma(q/2)} + \frac{\partial \ln M^*}{\partial \ln M_h} m_h f_M \right]. \quad (60)$$

Under permanence, $\partial \ln M^* / \partial \ln M_h = 1/(1 - \lambda_1)$, and with $m_h f_M = (2 - 2\lambda_2) x^{q/2} e^{-x} / \Gamma(q/2)$ the bracket combines exactly through $\Gamma(s + 1, x) = s\Gamma(s, x) + x^s e^{-x}$:

$$\Pi_m(m_h, t) = \varepsilon_m \frac{\Gamma(1 + q/2, x)}{\Gamma(1 + q/2)}, \quad \varepsilon_m = -\dot{M}_h. \quad (61)$$

The flux equals ε_m as $m_h \rightarrow 0$, and its first correction is of order $(m_h/M^*)^{5/3-\lambda_1}$. For $\lambda_1 = 5/6$ and $\eta_0 = 1/2$ the flux stays within 10% of ε_m up to $m_h \approx 0.2M^*$ and falls to half at $m_h \approx 2.5M^*$; both edges scale as $\eta_0^{3/2}$. The propagation range therefore extends up to about $0.2M^*$, and the deposition range spans roughly 0.2 to a few M^* . Its edge is set by the cutoff, not by m_h^* , which does not appear in Eq. (61); this is why the plateau of Fig. 9 extends well beyond m_h^* .

5.0.0.12 Group mass at m_h^* . In the propagation range $m_g = m_p A_M m_h^{-\lambda_1}$ is time-invariant, so at the moving characteristic mass the group mass and the number of haloes in the group are

$$m_g^* \propto m_h^{*-\lambda_1} \propto a^{-5/4}, \quad n_h^* = \frac{m_g^*}{m_h^*} \propto a^{-11/4}. \quad (62)$$

5.0.0.13 Energy cascade in mass space. The same mass function carries an energy cascade. Merging releases internal energy at the smallest scale (Eq. (B8)), and the mean specific virial energy of haloes above m_h grows through exchange across m_h , defining a virial-energy flux Π_{pv} (Eq. (F3)). For $m_h \ll M^*$ the flux is scale-independent and set by the specific virial energy at the cutoff, $K_{pv}^* \propto M^{2/3} a^{-\theta} \propto t$ (Eq. (13)), so the rate ε_u is time-independent both before and after a_* (Appendix F); its value follows from M^* (Eq. (72)). Illustris-1-Dark shows this plateau, $|\varepsilon_u| \approx (2-5) \times 10^{-7} \text{ m}^2 \text{ s}^{-3}$, below the cutoff (Fig. F1), consistent with the global rate of the cosmic energy equation (Appendix B). Part of this agreement is expected, since both rates are dominated by haloes near M^* ; the non-trivial content is that the energy flux is independent of both mass and time.

5.0.0.14 Waiting time near m_h^* . The random walk also fixes how fast individual haloes grow. In the universal range $D_o = a$ and $\lambda_h = 2/3$, so by Eq. (44) the rms mass increment over da scales as $m_h^{\lambda_h} (da)^{1/2}$: growth is diffusive with clock $a^{1/2}$, $\frac{dm_h}{d(a^{1/2})} \propto m_h^{\lambda_h}$. With $\frac{da}{dt} = aH \propto a^{-1/2}$ in the matter era, the mean time for a halo to gain one particle mass m_p is

$$\tau_g(m_h, a) \equiv \frac{m_p}{dm_h/dt} \propto m_h^{-\lambda_h} a. \quad (63)$$

The factor a makes the waiting time constant in conformal time: encounters are diluted by expansion. The mass dependence is gravitational. At a given time haloes share one dynamical density (Section 3), so their size scales as $m_h^{1/3}$, and an accretion rate proportional to the potential depth $\Phi \propto Gm_h/R_h$ gives

$$\tau_g \propto \frac{1}{\Phi} \propto \frac{R_h}{Gm_h} \propto m_h^{-2/3}, \quad (64)$$

which is Eq. (63) with $\lambda_h = 2/3$: the growth of an individual halo is governed by gravity rather than by the initial spectrum. This exponent describes the walk of a single halo. The abundance slope $\lambda_1 = 5/6$ is steeper because haloes are removed by mergers, so the population below M^* is a flux-driven steady state; and the total halo mass grows on the cascade clock while the cutoff advances on the linear clock. At the characteristic mass, $m_h^* \propto t$ makes the waiting time constant,

$$\tau_g^* \propto (m_h^*)^{-2/3} a \propto t^0, \quad r_h^* \propto m_h^{1/3} a \propto t, \quad (65)$$

where r_h^* is the fixed-contrast radius (Eq. (9)). Characteristic haloes therefore grow at a constant rate m_p/τ_g^* . These results are used in Section 6 and for the particle-mass bounds in Appendix H.

6 HALO DENSITY PROFILE AND ENERGY CASCADE

A fully virialized halo supports no net transfer of mass or energy across scales. Real haloes never reach this state: they keep accreting, and their interiors relax toward it. We use *cascade* here for the radial redistribution of mass and energy during this non-equilibrium relaxation, which shapes the density profile [43], underpins halo scaling relations [46], constrains dark matter particle properties [49], and extends to galaxy bulges [50]. In parallel with Section 5, we define three halo radii, measure the radial cascade and derive its scaling laws, obtain the profile from a particle random walk, and compare it with the halo mass function.

6.0.0.1 Three critical radii of a halo. A halo of mass m_h at epoch a is characterized by three radii, each defined by one condition and

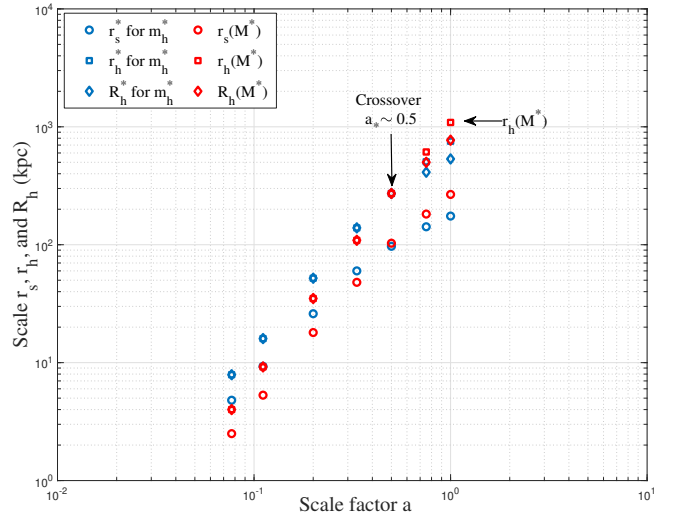


Figure 12. Characteristic radii versus scale factor measured in Illustris-1-Dark: scale radius r_s (circles), fixed-contrast radius r_h (squares) and virial size R_h (diamonds) for characteristic haloes m_h^* (blue) and cutoff haloes M^* (red). Predicted scalings are listed in Table 13. Before the crossover $a_* \approx 0.5$, $R_h^* = r_h^*$, and the radii of M^* grow faster than those of m_h^* ($a^{9/4}$ against a^2), catching up with them at a_* . After a_* the radii fan out, and each gap measures one physical effect: $r_h(M^*)/r_h^*$ the embedding of m_h^* in larger structures, r_h^*/R_h^* their decoupling from the expansion, and R_h^*/r_s^* their concentration.

tied to one of the three density laws of Section 3. (i) The *fixed-contrast radius* $r_h(m_h, a) \propto m_h^{1/3} a$ (Eq. (9)); it is a bookkeeping radius that links halo masses to the wavenumbers. (ii) The *virial size* $R_h(m_h, a)$ is the physical radius at dynamical contrast of haloes (Eq. (12)),

$$R_h(m_h, a) = \left(\frac{3m_h}{4\pi\Delta_h(a)\bar{\rho}(a)} \right)^{1/3} \propto m_h^{1/3} a^\theta, \quad (66)$$

$$\Delta_h = \Delta_{\text{vir}} \quad (a \leq a_*), \quad \Delta_h = \Delta_{\text{vir}} (a/a_*)^{3/2} \quad (a > a_*),$$

so that $\frac{Gm_h}{R_h} \propto \sigma_v^2(m_h) \propto m_h^{2/3} a^{-\theta}$. Before a_* ($\theta = 1$), $R_h = r_h$; after a_* ($\theta = 1/2$), $R_h = r_h(a_*/a)^{1/2}$, the radius of Eq. (32). (iii) The *scale radius* $r_s(m_h, a)$ is the radius where the logarithmic slope of the density profile equals -2 . The visible $-4/3$ range ends at a fixed fraction of it (Eq. (97)). For the characteristic halo we write $r_h^* \equiv r_h(m_h^*)$, $R_h^* \equiv R_h(m_h^*)$ and $r_s^* \equiv r_s(m_h^*)$, and for the cutoff halo $r_h(M^*)$, $R_h(M^*)$ and $r_s(M^*)$.

Each radius is a mass front of one of mass scales (m_h^* and M^*) read through one density law: the fixed contrast, the virial density, or the cascade law $\rho \propto \varepsilon^{2/3} G^{-1} r^{-4/3}$ (Eq. (77)),

$$r_h \propto m_h^{1/3} a, \quad R_h \propto m_h^{1/3} a^\theta, \quad r_s \propto \left(\frac{m_h}{|\varepsilon|^{2/3}} \right)^{3/5} \propto m_h^{1/3} m_h^{4/15}, \quad (67)$$

where the last form is derived below (Eq. (96)). Applied to the two fronts, m_h^* on the cascade clock and M^* on the linear clock (Eqs. (50) and (51)), the three laws give six radii, collected in Table 13 and shown for Illustris-1-Dark in Fig. 12. Three identities follow: $R_h = r_h$ for every halo before a_* ; $R_h(M^*) = R_h^*$ at a_* , the real-space form of the crossover; and $R_h(M^*) = r_h^*$ after a_* , when $R_h^* \propto a$ keeps a fixed comoving size. Figure 12 shows all six radii measured.

6.0.0.2 Universal $-4/3$ slope in simulations. Figure 13 shows compensated profiles $\rho_r(r) r^{4/3}$ for five independent high-resolution simulations of Milky-Way-scale haloes. All five show a plateau on

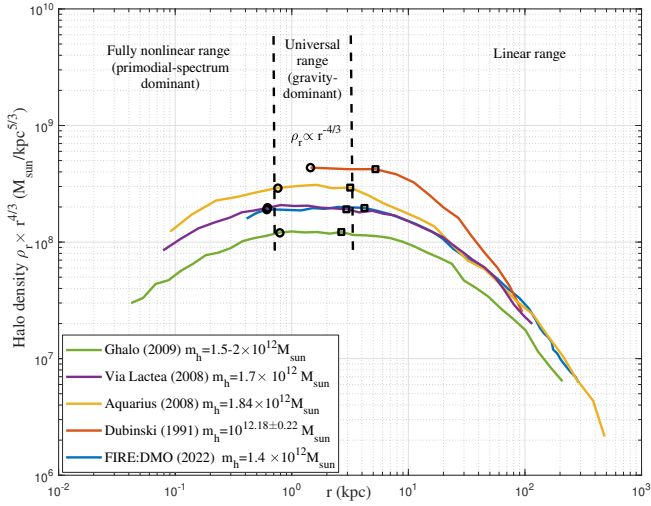


Figure 13. Compensated density profiles $\rho_r(r) r^{4/3}$ for five high-resolution N -body simulations of Milky-Way-scale haloes: (1) Ghalo [51]; (2) Via Lactea [52]; (3) Aquarius [53]; (4) Dubinski [54]; (5) FIRE:DMO [26]. A plateau ($\rho_r \propto r^{-4/3}$) appears on scales of several kpc (between circle and square symbols); at sub-kpc scales the profiles turn downward, revealing a shallower inner slope. The three regimes are labelled at the top.

scales of several kpc, i.e. $\rho_r \propto r^{-4/3}$, and turn downward at sub-kpc scales, indicating a shallower inner slope. The logarithmic slope $s = d \ln \rho / d \ln r$ (Fig. 14) shows three regimes, as the spectrum does (Fig. 1): a primordial-dominated regime at small r , where s differs between simulations; a gravity-dominated regime at $r \sim 1$ –6 kpc, where all profiles converge to $s \approx -4/3$; and an outer regime at large r , where s steepens through -2 at $r_s \approx 25$ kpc. The agreement across codes, resolutions and initial conditions indicates that the $-4/3$ slope at intermediate radii, inside the scale radius, is set by gravity, while the innermost slope retains the primordial spectrum below r_{in} . Because any profile whose slope steepens from about -1 to -3 must pass through $-4/3$, the evidence lies in the width of the plateau, measured by the ratio $r_{\text{out}}/r_{\text{in}}$ of the radii between which $|s + 4/3| \leq 0.1$. This ratio is only 2.1 for an NFW profile and 2.4 for an Einasto profile ($\alpha_E = 0.17$), but about 6 for the simulated profiles. The time invariance of the inner profile at m_h^* (Fig. 17) provides an independent test, and the inner edge r_{in} of the gravity-dominated regime is estimated in Eq. (79).

6.0.0.3 Radial energy cascade. By analogy with mass space (Section 5), we define cumulative profiles in radius for haloes of mass m_h . The mass and the virial kinetic energy outside r are

$$\begin{aligned} \Lambda_m^h(m_h, r, z) &= \int_r^\infty \rho_r(m_h, r', z) 4\pi r'^2 dr', \\ \Lambda_{pv}^h(m_h, r, z) &= \int_r^\infty K_{pv} \rho_r(m_h, r', z) 4\pi r'^2 dr', \end{aligned} \quad (68)$$

where ρ_r is the mean density profile and $K_{pv} = |\mathbf{v}_p|^2/2$ is the virial kinetic energy of Eq. (F1). The mean specific virial energy outside r , $K_{pv}^h = \Lambda_{pv}^h/\Lambda_m^h$, grows as kinetic energy is transferred outward, which defines the radial energy flux

$$\Pi_{pv}^h(m_h, r, z) = -\frac{\partial}{\partial t} \left(\frac{\Lambda_{pv}^h}{\Lambda_m^h} \right). \quad (69)$$

Inside the scale radius the dynamical time is much shorter than the time scale of large-scale evolution, so the inner halo communicates

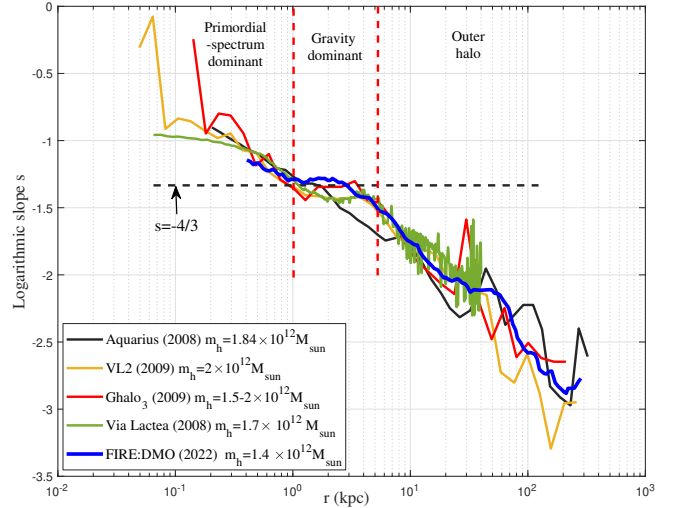


Figure 14. Logarithmic density slope $s = d \ln \rho / d \ln r$ for five simulated haloes of mass $\sim 10^{12} M_\odot$: Aquarius [53], VL2 and Ghalo₃ [51], Via Lactea [52], and FIRE:DMO [26]. Three regimes appear: primordial-dominated at small r , gravity-dominated at intermediate r (between the dashed red lines), and an outer regime at large r , where s passes through -2 at $r_s \approx 25$ kpc. In the gravity-dominated regime all profiles flatten to $s \approx -4/3$ (dashed black line) over $r \sim 1$ –6 kpc; its inner edge $r_{\text{in}} \approx 1$ kpc is estimated in Eq. (79).

with larger scales only through this flux. In a statistically steady state the flux is then independent of r , and it defines the cascade rate,

$$\varepsilon(m_h, z) \equiv \Pi_{pv}^h(m_h, r, z), \quad r < r_s. \quad (70)$$

Illustris confirms this plateau below r_s for two halo masses at all redshifts shown (Fig. 15), identifying an inverse cascade of virial kinetic energy. The plateau is already present at $z = 4$ –8, before a_* . The radial cascade is thus a steady state at the level of a single halo, which requires only a short internal dynamical time. The mass cascade of Section 5, by contrast, is a steady state of the population, which requires a steady halo population and forms only after a_* . For characteristic haloes the two rates coincide, $\varepsilon(m_h^*, z) = \varepsilon_u$ (Fig. F1), and at all epochs the rate follows a power law in mass (Fig. 16),

$$\varepsilon(m_h, z) = \varepsilon_u \left[\frac{m_h}{m_h^*(z)} \right]^{2/3}, \quad \varepsilon_u \approx -3 \times 10^{-7} \text{ m}^2 \text{ s}^{-3}. \quad (71)$$

6.0.0.4 Value and mass dependence of the cascade rate. Both the value of ε_u and the law (71) follow from three relations already established: the linear growth of the virial energy, $\langle K_{pv} \rangle \propto t$ (Fig. 6); the definition of the cutoff mass, $\langle K_{pv} \rangle = \frac{3}{2} \sigma_v^2(M^*)$; and the dispersion at fixed time, $\sigma_v^2(m_h) = \sigma_*^2(m_h/m_h^*)^{2/3}$ with $\sigma_*^2 \equiv \sigma_v^2(m_h^*) = \sigma_h^2(m_h^*)$. We identify the halo-level rate with the linear growth of the virial energy, $\langle K_{pv} \rangle = |\varepsilon_u|t$; the all-particle rate of Eq. (B5) is larger by a factor of about 1.5 because it also includes bulk motion and unbound particles (Fig. 6). Then

$$|\varepsilon_u| = \frac{3}{2} \frac{\sigma_v^2(M^*)}{t} = \frac{3}{2} \frac{\sigma_*^2}{t} \left(\frac{M^*}{m_h^*} \right)^{2/3} = \frac{3}{2} \frac{\sigma_*^2}{t} \frac{\langle K_{pv} \rangle}{\langle K_{ph} \rangle}, \quad (72)$$

where the last form uses Eq. (14). The rate is thus fixed at the energy-containing scale M^* . Two consequences follow. First, with $\sigma_*^2 \propto m_h^{*1/3}$ (Eq. (12)), a constant ε_u requires

$$m_h^* \propto M^{*2}/t^3 \propto a^{3\theta}, \quad (73)$$

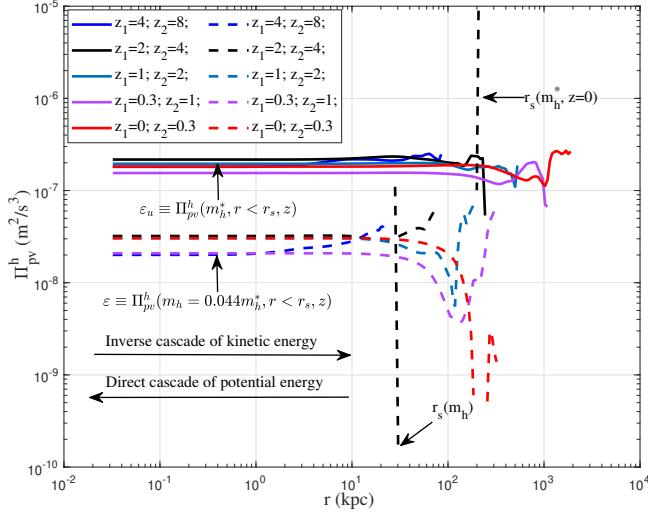


Figure 15. Radial energy flux $\Pi_{pv}^h(m_h, r, z)$ (Eq. (69)) at several redshifts for haloes of mass $m_h^*(z)$ and $0.044m_h^*(z)$ in Illustris. A scale-independent plateau $\varepsilon \equiv \Pi_{pv}^h(r < r_s)$, present at all redshifts including those before a_* , identifies an inverse cascade of virial kinetic energy from small to large r . For haloes of mass m_h^* , $\varepsilon_u \approx -(2-5) \times 10^{-7} \text{ m}^2 \text{ s}^{-3}$ (see Fig. F1).

which is Eq. (13). The linear clock, $M^* \propto t^{5/2}$ before and t^2 after a_* , then gives $m_h^* \propto t^2$ and t , i.e. $\theta = 1$ and $1/2$. Second, Eq. (72) defines a cascade time common to all haloes at a given epoch,

$$t_c \equiv \frac{\sigma_v^2}{|\varepsilon_u|} = \frac{2}{3} t \left(\frac{m_h^*}{M^*} \right)^{2/3} = \frac{2}{3} t_* \left(\frac{a}{a_*} \right)^\theta. \quad (74)$$

If every halo transfers its virial energy on this clock, i.e. if ε/σ_v^2 is independent of mass at fixed time, then

$$\varepsilon(m_h) = -\frac{\sigma_v^2(m_h)}{t_c} = \varepsilon_u \frac{\sigma_v^2(m_h)}{\sigma_v^2(m_h^*)} = \varepsilon_u \nu(m_h), \quad (75)$$

with ν the peak height of Eq. (49). This is Eq. (71), and it is normalized by $\varepsilon(m_h^*) = \varepsilon_u$ without further input. The common-clock condition is the single assumption; it is equivalent to the measured $m_h^{2/3}$ law for ε_u . The flux is drained on small scales by mergers, each of which releases $\frac{10}{21}$ of the host potential per unit accreted mass (Eq. (B8)). So ε is a continuum quantity and m_p sets only where the cascade ends (Appendix H).

6.0.0.5 Scaling laws from a constant cascade rate. In turbulence, a scale-independent cascade rate fixes universal scaling laws [35, 37]. The same holds for the radial cascade. With a characteristic velocity v_r and time t_r on scale r , a scale-independent rate

$$\varepsilon = \frac{v_r^2}{2t_r} = \frac{v_r^3}{3r} \quad (76)$$

gives the $2/3$ law $v_r^2 \propto (\varepsilon r)^{2/3}$, and the virial relation $v_r^2 = \gamma_r G m_r / r$, with γ_r a numerical factor, then gives the enclosed mass $m_r \propto r^{5/3}$ and density $\rho_r \propto r^{-4/3}$. Writing ε for its magnitude $|\varepsilon|$ in fractional powers, the complete set of scaling laws is

$$\begin{aligned} m_r &= \alpha_r \varepsilon^{2/3} G^{-1} r^{5/3}, & \rho_r &= \beta_r \varepsilon^{2/3} G^{-1} r^{-4/3}, \\ v_r^2 &= \beta_v^{-2/3} (\varepsilon r)^{2/3}, & t_r &= \frac{1}{2} \beta_v^{-2/3} \varepsilon^{-1/3} r^{2/3}, \\ v_r^2 &= \beta_v^{-2/3} \alpha_r^{-2/5} (\varepsilon G m_r)^{2/5}, \\ \dot{m}_r &= 4\pi \beta_v^{-1/3} \beta_r \varepsilon G^{-1} r, \end{aligned} \quad (77)$$

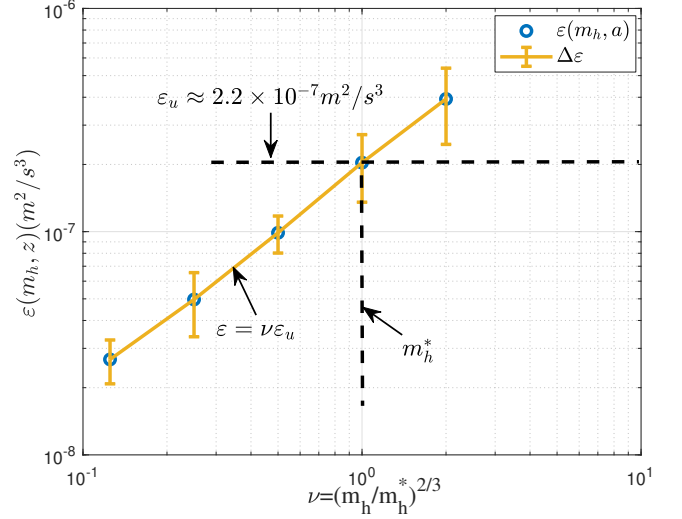


Figure 16. Radial cascade rate $\varepsilon(m_h, z)$ (Eq. (70)) versus $\nu = (m_h/m_h^*)^{2/3}$ at several redshifts. The rate scales as $\varepsilon = \varepsilon_u \nu$, with a time-independent $\varepsilon_u \approx -(2-5) \times 10^{-7} \text{ m}^2 \text{ s}^{-3}$ at $m_h = m_h^*$ (see also Fig. F1).

where $\dot{m}_r = 4\pi r^2 \rho_r v_r$ is the mass flux across r and the numerical factors are related by

$$\beta_v = \frac{1}{3}, \quad \gamma_r \alpha_r = \beta_v^{-2/3}, \quad \beta_r = \frac{5\alpha_r}{12\pi}. \quad (78)$$

Each quantity depends only on ε , G and r . Once a steady state is established, fast small-scale dynamics are decoupled from slow large-scale dynamics except through ε , so any quantity Q follows by dimensional analysis as $Q = \varepsilon^x G^y r^z$ [49, 55]; for the density this solution is unique. The laws hold where $r \ll r_s$. Near r_s the halo scale enters, $\rho_r = \beta_r \varepsilon^{2/3} G^{-1} r^{-4/3} F(r/r_s)$, and the slope steepens even where the flux is still nearly constant (Fig. 15). The microscopic origin of ε in terms of particle jumps is given in Appendix G.

6.0.0.6 Inner boundary of the cascade. Because kinetic energy cascades inversely, the $-4/3$ range begins at an injection scale r_{in} ; inside it the structure is inherited from the earliest progenitor, which collapsed from a primordial peak with characteristic density ρ_f . The cascade governs the profile only where its mean density, $\bar{\rho}(< r) = \frac{9}{5} \beta_r |\varepsilon|^{2/3} G^{-1} r^{-4/3}$, exceeds ρ_f :

$$r_{\text{in}} = \left(\frac{9\beta_r |\varepsilon|^{2/3}}{5G\rho_f} \right)^{3/4} = \left(\frac{9A_\rho}{5\rho_f} \right)^{3/4}, \quad (79)$$

with $A_\rho \equiv \rho_r r^{4/3}$ the plateau amplitude of Fig. 13. For the haloes of Fig. 14, $r_{\text{in}} \approx 1 \text{ kpc}$ encloses $\sim 10^9 M_\odot$ at a mean density $\approx 1.5 \times 10^{-20} \text{ kg m}^{-3}$. This equals the mean density inside the scale radius of a progenitor with $c \approx 5$ collapsing at $z_f \approx 10$, whose mass, a few $\times 10^9 M_\odot$, is within a factor of ~ 3 of the main progenitor of a Milky-Way accretion history at that epoch. If ρ_f is independent of host mass, $r_s/r_{\text{in}} \propto m_h^{3/5}$: the gravity-dominated range has a mass-independent width at fixed time and widens as $a^{9/10}$ after a_* .

6.0.0.7 Particle random walk. The profile follows from the stochastic accretion history. A halo consists of particles accreted at many earlier epochs, each deposited at the accretion boundary active at that time, so a particle's radius records when it was accreted and how massive the halo was then. We assume that: (i) smooth accretion dominates, with discrete mergers subdominant in setting the

mean profile; (ii) accretion is isotropic, so a single radial coordinate suffices; (iii) once accreted, a particle stays bound at a fixed position relative to the scale radius r_s , so the halo grows self-similarly (neglects orbital mixing: the profile is a time-averaged occupation); and (iv) the accretion radius is a power law in halo mass, $r = A_r m_h^{\beta_h}$.

Both mass scales are fronts of the walk of Eq. (44), each with its own exponent and clock (Eqs. (50) and (51)): m_h^* with (λ_h, D_c) and M^* with (λ_2, D_L) . Writing (λ, D_o) for either pair, with $D_o = a^\tau$, the front grows as

$$M_{\text{front}} \propto D_o^{\frac{1}{2-2\lambda}} \propto a^\chi, \quad \chi = \frac{\tau}{2-2\lambda}, \quad (80)$$

so that $\chi = 3\theta$ for m_h^* , and $\chi = 15/4$ before a_* and 3 after for M^* . The two walks differ in meaning. The cascade-clock walk is the equation of motion of individual haloes, whose accretion rate is set by their own potential (Eq. (64)); it is the walk that builds the profile. The linear-clock walk is an effective walk for the tail of the mass function, whose front M^* is set by the growth of the linear variance; no halo accretes on that clock. After a_* , a halo at the cutoff accreting at the gravitational rate slower than $\frac{d \ln M^*}{dt}$ and falls behind, so the cutoff is occupied by newly assembled haloes at each epoch. The radial construction below applies to both fronts and three density laws (Table 3), but only the cascade-clock walk yields a halo profile.

Mapping mass to radius. Each density law of Eq. (67) fixes a characteristic density shared by all haloes at a given epoch, $\rho \propto a^{-3\theta'}$, so its radius scales as $r \propto m_h^{1/3} a^{\theta'}$, with

$$\begin{aligned} \theta' &= 1 \quad (\text{fixed contrast, } r_h), \quad \theta' = \theta \quad (\text{virial, } R_h), \\ \theta' &= \frac{4}{5}\theta \quad (\text{cascade, } r_s). \end{aligned} \quad (81)$$

The cascade value follows from $\rho_s \propto m_h/r_s^3 \propto m_h^{*-4/5}$ (Eq. (67) and $m_h^* \propto a^{3\theta}$). Because θ' depends only on the law and the epoch, the same three densities apply to both fronts. Along a front $m_h \propto a^\chi$, the mapping exponent is

$$\beta_h = \frac{d \ln r}{d \ln m_h} = \frac{1}{3} + \frac{\theta'}{\chi} = \frac{1}{3} + (2-2\lambda) \frac{\theta'}{\tau}. \quad (82)$$

The fixed contrast is tied to the background rather than to m_h^* , so it is the only law with θ' unchanged at a_* . The cascade law holds only for $r \lesssim r_s$, so its mapping describes position relative to r_s , as in assumption (iii), rather than the deposition radius near the boundary.

Radial walk. Applying the chain rule to Eq. (44) gives a Langevin equation for the radius r of the most recently accreted shell,

$$\frac{dr}{dD_o} = \beta_h \sqrt{2D_{p0}} A_r^{\frac{1-\lambda}{\beta_h}} r^\gamma \zeta(D_o), \quad \gamma \equiv 1 - \frac{1-\lambda}{\beta_h}. \quad (83)$$

The diffusion amplitude grows as r^γ , so the residence time of the accretion layer near r scales as $\tau_{gr} \propto r^{-\gamma}$. The layer therefore lingers at small r , where the local kinetic energy is lower. The probability density of the accretion radius follows from Eq. (46) through the Jacobian of $r = A_r m_h^{\beta_h}$, $f_r \equiv f_M \left| \frac{\partial m_h}{\partial r} \right|$,

$$\begin{aligned} f_r(r, D_o) &= \frac{r^{-\gamma}}{\sqrt{\pi D_{r0} D_o}} \exp \left[-\frac{r^{2-2\gamma}}{4(1-\gamma)^2 D_{r0} D_o} \right], \\ D_{r0} &= \beta_h^2 D_{p0} A_r^{\frac{2(1-\lambda)}{\beta_h}}, \end{aligned} \quad (84)$$

so $f_r dr$ is the probability that, at epoch D_o , the accretion layer lies in $[r, r + dr]$. Since $2-2\gamma = 2(1-\lambda)/\beta_h$, its front, the radial counterpart of Eq. (47), is

$$r_{\text{front}} \propto (D_{r0} D_o)^{\frac{1}{2-2\gamma}} \propto D_o^{\frac{\beta_h}{2-2\lambda}} \propto a^{\chi \beta_h} = M_{\text{front}}^{1/3} a^{\theta'}. \quad (85)$$

Table 3. Unified mass-to-radius mappings (Eqs. (80)–(85)) for the two fronts: m_h^* with (λ_h, D_c) and M^* with (λ_2, D_L) . Here $\beta_h = 1/3 + \theta'/\chi$, $\gamma = 1 - (1-\lambda)/\beta_h$, and the radius grows as $a^{\chi \beta_h}$.

Law (radius)	$m_h^* (\lambda_h, D_c)$				$M^* (\lambda_2, D_L)$			
	θ'	β_h	γ	r	β_h	γ	r	
$a < a_*$: $\theta = 1$;	$\chi = 3, \lambda_h = 2/3$;				$\chi = 15/4, \lambda_2 = 11/15$			
Fixed contrast (r_h)	1	2/3	1/2	a^2	3/5	5/9	$a^{9/4}$	
Virial (R_h)	1	2/3	1/2	a^2	3/5	5/9	$a^{9/4}$	
Cascade (r_s)	4/5	3/5	4/9	$a^{9/5}$	41/75	21/41	$a^{41/20}$	
$a > a_*$: $\theta = 1/2$;	$\chi = 3/2, \lambda_h = 2/3$;				$\chi = 3, \lambda_2 = 2/3$			
Fixed contrast (r_h)	1	1	2/3	$a^{3/2}$	2/3	1/2	a^2	
Virial (R_h)	1/2	2/3	1/2	a	1/2	1/3	$a^{3/2}$	
Cascade (r_s)	2/5	3/5	4/9	$a^{9/10}$	7/15	2/7	$a^{7/5}$	

The radial front is the mass front read through the chosen density law. With constant A_r and D_{r0} , it reproduces all six radii of Table 13: three laws applied to two fronts (Table 3 and Fig. 12). The table also shows the identities of Section 6.0.0.1: the fixed-contrast and virial rows coincide before a_* , and $R_h(M^*)$ and r_h^* both grow as $a^{3/2}$ after. For the profile, which uses the cascade-clock walk, the fixed-contrast and virial mappings place the front at a halo boundary. Only the cascade mapping places it at a fixed multiple of r_s with constant D_{r0} ; the other two reach r_s only through a drifting amplitude via a halo concentration c , $D_{r0} \propto c^{-(2-2\gamma)}$ (Eq. (90) below).

6.0.0.8 Particle distribution. During each interval dD_o , one mass element is deposited at a radius sampled from f_r . Because accreted particles stay bound (assumption iii), the mass deposited near r depends on how often the layer visits r and how long it dwells there. Under isotropic accretion the geometric factor $4\pi r^2$ cancels between deposited mass and shell volume, so the number density is the residence-time-weighted visitation measure,

$$\begin{aligned} P_r &\propto \tau_{gr}(r) \bar{f}_r(r, D_o), \\ \bar{f}_r(r, D_o) &\equiv \frac{1}{D_o} \int_0^{D_o} f_r(r, D'_o) dD'_o, \end{aligned} \quad (86)$$

which integrates the entire accretion history. In the long-time limit, which describes the assembled interior, $\bar{f}_r \rightarrow 2f_r$ (Appendix G0.0.1), and

$$P_r \propto \tau_{gr}(r) f_r(r, D_o) \propto r^{-2\gamma} \exp \left(-\frac{r^{2-2\gamma}}{4(1-\gamma)^2 D_{r0} D_o} \right). \quad (87)$$

The extra factor $\tau_{gr} \propto r^{-\gamma}$ converts the probability of the accretion layer into a particle count: because the front advances slowly at small r , many events deposit particles in the same inner shell. Hence $\rho_r \propto r^{-2\gamma}$ and $m_r \propto r^{3-2\gamma}$ inside the front; the normalized distribution, and its equivalence to a single tracer performing a three-dimensional random walk, are given in Appendix G0.0.1.

6.0.0.9 Inner exponent from the cascade. Inside the halo the deposited particles join the steady cascade, and their residence time is set by the local potential, as for the halo waiting time of Eq. (64) and the particle jumps of Eq. (G9). With $m_r \propto r^{3-2\gamma}$, $\Phi \propto \frac{Gm_r}{r} \propto r^{2-2\gamma}$ and $\tau_{gr} \propto \Phi^{-1} \propto r^{2\gamma-2}$. Equating this with $\tau_{gr} \propto r^{-\gamma}$,

$$2\gamma - 2 = -\gamma \quad \Rightarrow \quad \gamma = \frac{2}{3}. \quad (88)$$

Thus $\rho_r \propto r^{-4/3}$ and $\tau_{gr} \propto r^{-2/3}$ in the interior, independent of the initial spectrum, in agreement with the cascade scaling laws of Eq. (77) and with Figs. 13 and 14; τ_{gr} acts as an inverse local temperature (Eq. (G9)). The interior exponent (2/3) differs from the

front exponent of the walk (Table 3): the inner power law and the cutoff are set by different processes.

6.0.0.10 Double- γ profile. By analogy with the double- λ mass function, where λ_1 sets the power law and λ_2 the cutoff, we let γ_1 set the inner power law and γ_2 the outer cutoff,

$$P_r \propto r^{-2\gamma_1} \exp \left[-\frac{r^{2-2\gamma_2}}{4(1-\gamma_2)^2 D_{r0} D_o} \right]. \quad (89)$$

On gravity-dominated scales $\gamma_1 = 2/3$ (Eq. (88)); inside r_{in} the measured slope is shallower due to primordial spectrum ($\gamma_1 < 2/3$; Eq. (79)). The cutoff exponent γ_2 is the front exponent γ of Eq. (83). The scale radius r_s , where the logarithmic slope equals -2 , satisfies

$$4(1-\gamma_2)^2 D_{r0} D_o = \frac{2-2\gamma_2}{2-2\gamma_1} r_s^{2-2\gamma_2}. \quad (90)$$

With $x = r/r_s(t)$ and the two shape parameters

$$\alpha = 2 - 2\gamma_2, \quad \beta = \frac{1-\gamma_2}{1-\gamma_1}, \quad (91)$$

the normalized distribution takes the compact form (Appendix G0.0.1)

$$P_r(x) = \frac{\alpha \beta^{-\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)}}{4\pi \Gamma\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)} x^{\frac{\alpha}{\beta}-2} \exp\left(-\frac{x^\alpha}{\beta}\right), \quad (92)$$

which reduces to the Einasto profile when $\alpha = 2\beta$ ($\gamma_1 = 0$). The double- γ density profile, with four parameters ($\rho_s, r_s, \alpha, \beta$), is

$$\rho_r(r, t) = \rho_s(m_h, t) \left(\frac{r}{r_s}\right)^{\frac{\alpha}{\beta}-2} \exp\left[\frac{1}{\beta} \left(1 - \left(\frac{r}{r_s}\right)^\alpha\right)\right], \quad (93)$$

where ρ_s is the density at r_s and the inner slope is $\alpha/\beta - 2 = -2\gamma_1$. The local slope at r_s equals -2 for any (α, β) , so the mapping affects only the shape of the cutoff, through γ_2 . For $\gamma_1 = 2/3$, the virial mapping gives $\gamma_2 = 1/2$, i.e. $\alpha = 1$ and $\beta = 3/2$ at all epochs; the fixed-contrast mapping gives the same values before a_* and $\gamma_2 = 2/3$ ($\alpha = 2/3, \beta = 1$) after; and the cascade mapping gives $\gamma_2 = 4/9$, i.e. $\alpha = 10/9$ and $\beta = 5/3$ at all epochs.

Equation (90) ties r_s rigidly to the walk front, $r_{\text{front}} = \beta^{1/\alpha} r_s$ with $r_{\text{front}}^\alpha \equiv 4(1-\gamma_2)^2 D_{r0} D_o$, so with constant D_{r0} , r_s grows as the front of Eq. (85). For the cascade mapping this is exactly the growth of r_s^* (Table 3). For the fixed-contrast and virial mappings the front grows as r_h^* or R_h^* , faster than the r_s fixed by the mass budget below, and Eq. (90) then requires a slowly decreasing $D_{r0} \propto c^{-\alpha}$, with $c \equiv R_h/r_s$ the concentration.

6.0.0.11 Gravity-dominated profile and the scale radius. Setting $\gamma_1 = 2/3$ ($\alpha/\beta = 2/3$) and replacing the free amplitude ρ_s by the cascade rate, as required by the scaling laws of Eq. (77), gives

$$\rho_r(r, m_h, z) = \beta_r \varepsilon^{\frac{2}{3}} G^{-1} r_s^{-\frac{4}{3}} \left(\frac{r}{r_s}\right)^{-\frac{4}{3}} \exp\left[-\frac{1}{\beta} \left(\frac{r}{r_s}\right)^{\frac{2\beta}{3}}\right], \quad (94)$$

$$\rho_s = \beta_r e^{-1/\beta} \varepsilon^{2/3} G^{-1} r_s^{-4/3} \propto \varepsilon_u^{6/5} G^{-9/5} m_h^{*-4/5},$$

where β_r is of order unity and $\varepsilon(m_h, z)$ is given by Eq. (71). With slope and amplitude both fixed by the cascade, the only remaining freedom is where the power law ends. The halo mass must fit inside,

$$m_h = \int_0^\infty \rho_r 4\pi r^2 dr = 4\pi \beta_r I(\beta) \varepsilon^{2/3} G^{-1} r_s^{5/3}, \quad (95)$$

$$I(\beta) = \frac{3}{2\beta} \beta^{\frac{5}{2\beta}} \Gamma\left(\frac{5}{2\beta}\right),$$

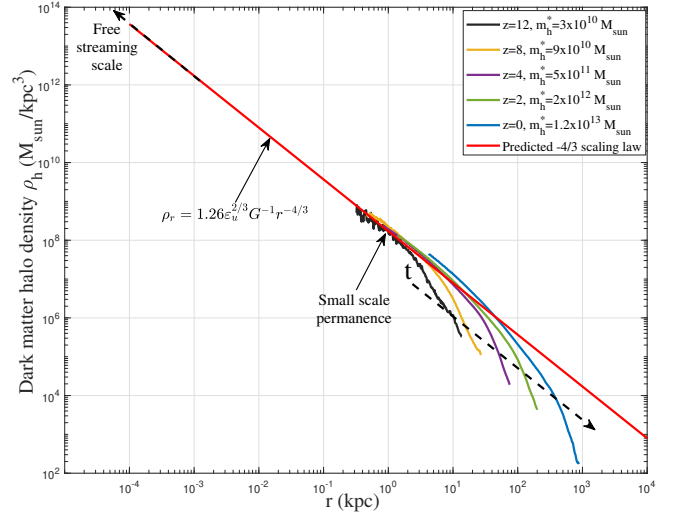


Figure 17. Density profiles $\rho_r(r)$ of haloes of characteristic mass $m_h^*(z)$ at several redshifts, before and after a_* , in Illustris. All collapse onto the predicted time-invariant $r^{-4/3}$ scaling (solid red line, Eq. (101)), demonstrating small-scale permanence. Profiles for other masses are shown in Fig. 18.

which fixes the scale radius,

$$r_s = \left[\frac{G m_h}{4\pi \beta_r I(\beta) |\varepsilon|^{2/3}} \right]^{3/5} \propto m_h^{1/3} m_h^{*4/15}, \quad (96)$$

where the last form uses Eq. (71) at fixed β . For characteristic haloes $r_s^* \propto m_h^{*3/5}$, i.e. $a^{9/5}$ before and $a^{9/10}$ after a_* ; for cutoff haloes $r_s(M^*) \propto M^{1/3} m_h^{*4/15}$ carries both clocks. At fixed time $r_s \propto m_h^{1/3}$: haloes of all masses share a common density at r_s .

The visible $-4/3$ range ends inside r_s . The local slope of Eq. (94) is $s = -4/3 - \frac{2}{3}(r/r_s)^\alpha$, which stays within 0.1 of $-4/3$ up to

$$r_{4/3} = r_s (0.15)^{3/(2\beta)}, \quad (97)$$

about $0.06 r_s$ for $\beta = 1$, $0.15 r_s$ for $\beta = 3/2$ and $0.18 r_s$ for $\beta = 5/3$. The slope steepens slowly because the exponent $\alpha = 2\beta/3$ is small. The simulated Milky-Way-mass haloes (Fig. 14) leave the $-4/3$ range at $r_{4/3} \approx 5-6$ kpc and reach $s = -2$ at $r_s \approx 25$ kpc, so $r_{4/3}/r_s \approx 0.2$, which requires $\beta \approx 2$, closest to the cascade mapping (5/3). The outer profiles instead favour $\beta \approx 1$ (Fig. 18): no single β fits both the transition near r_s and the slow steepening beyond it. Fitting β separately inside and outside r_s , and as a function of z , would distinguish the three mappings.

Dividing R_h by Eq. (96) gives, at fixed β ,

$$c \equiv \frac{R_h}{r_s} \propto \frac{a^\theta}{m_h^{*4/15}} \propto a^{\theta/5}, \quad (98)$$

independent of halo mass at fixed epoch and rising by only 7% between $a_* \approx 0.5$ and $a = 1$. Two definitions distinguish c from published values: $r_s = r_{-2}$ exactly, whereas most published values come from NFW fits whose fixed shape biases r_s ; and R_h is not a fixed-contrast radius, although $\Delta_h \bar{\rho} \approx (133-154) \rho_c$ for $0.5 \leq a \leq 1$ makes $R_h \approx R_{140c}$ at late times in Λ CDM. For $10^{12} M_\odot$, $R_h \approx 0.23$ Mpc and $r_s \approx 15-25$ kpc give $c \approx 9-15$, compared with $c_{200c} \approx 8$, or $c_{140c} \approx 9$ [56]. The weak growth along $m_h^*(z)$ agrees with the near-constancy of c at fixed peak height [57, 58, 59]. The mass independence matches the flattening of $c(M)$ at low masses [60] but not the decline $c \propto M^{-0.1}$ at $10^{11}-10^{15} M_\odot$ [61].

6.0.0.12 Effective mass–radius exponents. Along the characteristic track, writing $m_h^* \propto r^{3-2\gamma_{\text{eff}}}$ for any of the three radii gives, with Eq. (82),

$$3 - 2\gamma_{\text{eff}} = \frac{d \ln m_h^*}{d \ln r} = \frac{1}{\beta_h}, \quad (99)$$

the counterpart of $1 - \lambda_1 = \frac{d \ln M_h}{d \ln M^*}$. The scale radius reproduces the local slope, $-2\gamma_{\text{eff}} = -4/3$, whereas the virial size gives $-3/2$ and the fixed-contrast radius $-3/2$ before and -2 after a_* . Like λ_1 , these exponents compare haloes across epochs; they differ from $-4/3$ only because the concentration grows (Eq. (98)), and they are not slopes of the profiles in Fig. 14.

6.0.0.13 Small-scale permanence in density profiles. As $r \rightarrow 0$ the exponential of Eq. (94) tends to unity and

$$\begin{aligned} \rho_r(r, m_h, z) &= \beta_r \varepsilon^{2/3} G^{-1} r^{-4/3} \\ &= \beta_r |\varepsilon_u|^{2/3} (m_h/m_h^*)^{4/9} G^{-1} r^{-4/3}. \end{aligned} \quad (100)$$

At fixed mass the inner density falls as m_h^* grows. For haloes at the characteristic mass $\varepsilon = \varepsilon_u$ at all epochs, which gives the *small-scale permanence* of the profile,

$$\rho_r(r, m_h^*, z) = \beta_r \varepsilon_u^{2/3} G^{-1} r^{-4/3} \quad (r \rightarrow 0), \quad (101)$$

the density analogue of the time-invariant group mass m_g (Fig. 10). Like the radial cascade itself, it does not require the steady population that forms after a_* .

Figure 17 confirms Eq. (101): in Illustris the profiles of haloes of mass $m_h^*(z)$ from $z = 12$ to $z = 0$, before and after a_* , all converge onto the same time-invariant $r^{-4/3}$ curve on small scales. For characteristic haloes this scaling may extend down to the free-streaming cutoff, which depends on the particle properties [49]; the profiles shown start at the softening length, so this is an extrapolation (cf. r_{in} of Eq. (79) for Milky-Way-mass haloes). Figure 18 shows profiles at $z = 0$ over twenty orders of magnitude in mass, from 10^{-6} to $10^{14} M_\odot$, combining Illustris haloes averaged within $10^{\pm 0.1}$ of each target mass in $[10^{8.5}, 10^{14}] M_\odot$ [62] with earth-size haloes [63]. The profile of Eq. (94) with $\varepsilon = \varepsilon_u (m_h/m_h^*)^{2/3}$ and $\beta = 1$ fits the full radial and mass range, and the $-4/3$ range shifts to larger r for more massive haloes.

6.0.0.14 Mass function and density profile compared. Both distributions are normalized to unity and carry a separate total, $M_h \propto A_M M^*^{1-\lambda_1}$ (Eq. (54)) and $m_h \propto \beta_r \varepsilon^{2/3} G^{-1} r_s^{5/3}$ (Eq. (95)). The same two conditions, energy and permanence (the mass budget), close both but fix different quantities (Table 12): in mass space energy fixes the cutoff M^* and permanence the slope λ_1 ; in density profiles, energy fixes the slope $-4/3$ and the mass budget the scale radius r_s . The reason is whether the counted quantity gravitates in its own coordinate. In radius the density is the source of the potential, so ε , G and r determine ρ uniquely and only the extent remains free; in mass space the abundance of haloes does not change any halo's velocity, so one exponent remains free and the clocks fix it. Hence M^* is both the bend and the front of the mass function, with λ_1 absorbing the ratio of the clocks, whereas in a halo the bend r_s and the front R_h separate, with the concentration absorbing the difference between the two conditions. In both, the cutoff shape comes from the walk alone.

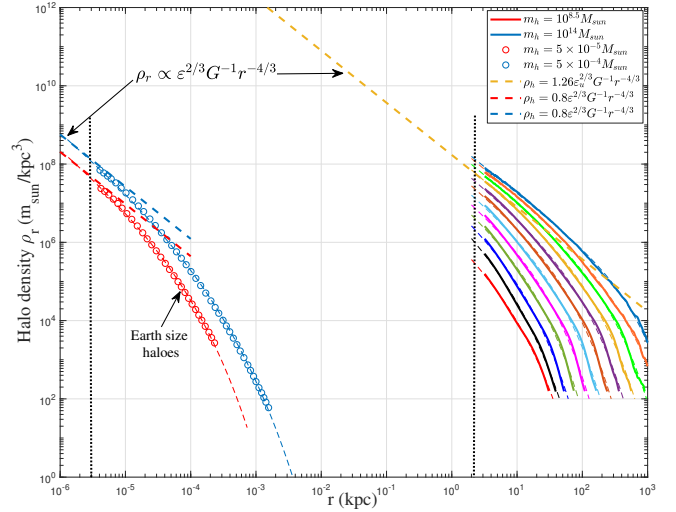


Figure 18. Density profiles $\rho_r(r)$ at $z = 0$ for haloes spanning 10^{-6} – $10^{14} M_\odot$. Solid lines: mean Illustris profiles for $m_h \in [10^{8.5}, 10^{14}] M_\odot$ [62]; symbols: earth-size haloes [63]; thin dashed lines: double- γ profile with $\gamma_1 = 2/3$ and $\beta = 1$ (Eq. (94)); dotted lines: gravitational softening length. Thick dashed lines show $\rho_r \propto r^{-4/3}$ with $\varepsilon = \varepsilon_u (m_h/m_h^*)^{2/3}$ (Eq. (71)).

7 CONCLUSIONS

We have developed a unified framework for nonlinear structure formation centred on the characteristic halo mass m_h^* . Five critical scales, each defined by one measurable condition, reorder at a crossover a_* , when the cutoff mass M^* overtakes m_h^* and multistream regions connect. Only afterwards does gravity establish a universal range with $P(k) \propto k^{-1}$, growing as a^3 , independent of the initial spectrum; it fixes the effective indices $-5/3$ at k_{NL1} and $-9/5$ at k_{NL2} and makes k_L^* coincide with $m_h^* \propto a^{3/2} \propto t$. The two halo masses are fronts of random walks on two clocks: m_h^* on a cascade clock that slows at a_* , and M^* on a linear clock that does not.

The universal spectrum sets the cascade at two levels. In mass space, haloes random-walk with waiting time $\tau_g \propto m_h^{-\lambda}$, giving the double- λ mass function without invoking spherical or ellipsoidal collapse: energy fixes its cutoff M^* , the gravitational exponent $\lambda_2 = 2/3$ its cutoff shape, and permanence with the ratio of the two clocks its effective slope $\lambda_1 = 5/6$. Within haloes, three density laws give three radii, and applied to the two fronts (Table 13). A radial cascade at rate $\varepsilon = \varepsilon_u (m_h/m_h^*)^{2/3}$, with ε_u fixed at M^* , gives $\rho \propto r^{-4/3}$ between an injection scale r_{in} and r_s ; the front of a particle random walk sets the cutoff, and the mass budget fixes $r_s \propto m_h^{1/3} m_h^{*4/15}$, so the concentration grows only as $a^{1/5}$ before and $a^{1/10}$ after a_* . The two levels are dual, because the counted density gravitates in radius but not in mass space (Table 12).

Both levels reach a non-equilibrium steady state with scale-independent rates. Mass flows upward at $\varepsilon_m \propto a^{-1}$ through a propagation range below about $0.2M^*$ and is deposited near and above the cutoff. Kinetic energy cascades inversely at ε_u and potential energy directly at $-1.4\varepsilon_u$, so the total energy, injected near m_h^* and contained in scales up to M^* , is "dissipated" by expansion at $-0.4\varepsilon_u$. The radial cascade is a halo-level steady state, present before a_* ; the mass cascade is a population-level steady state that forms only after a_* . Small-scale permanence follows at both levels: a time-invariant halo group mass and a time-invariant inner profile at m_h^* .

Scope and limitations. The five scales and the crossover are measured in a single Λ CDM simulation with a Bond & Efstathiou spectrum, and the cascade only in Illustris-1-Dark. The most important next test is therefore to confirm the crossover and the universal range in modern Λ CDM runs, where k_L^* needs an explicit threshold because the linear spectrum has no strict plateau. Three exponents rest on heuristic matching arguments and are supported mainly by measurement: the shape $n_X = -1$, the growth $b = 3$ and the pre-crossover index $-7/5$. Two predicted scalings are only approximately realized. The boundary k_{NL2} evolves as $a^{-0.9}$ rather than $a^{-3/2}$, probably because of limited resolution. Before a_* , M^* grows in Virgo as $a^{2.5} - a^{2.7}$, against the predicted and Illustris-measured $a^{15/4}$. In the halo profile, r_s , R_h and c follow from the clocks, and the amplitude of c agrees with standard fits once $R_h \simeq R_{140c}$ is taken into account. The predicted c is, however, independent of mass, whereas simulations give $c \propto M^{-0.1}$ above $10^{11} M_\odot$. The cutoff shape depends on the density law that maps mass to radius ($\beta = 5/3, 3/2$, or $3/2 \rightarrow 1$ across a_*) and has yet to be fitted directly. The inner edge r_{in} of the $-4/3$ range is estimated phenomenologically, and the particle random walk neglects orbital mixing. Finally, the bound $m_p \gtrsim 10^{12}$ GeV (Appendix H) rests on extrapolating the cascade to the single-particle scale and on a minimal-length postulate; it is not used elsewhere.

AUTHOR CONTRIBUTIONS

Z.J. Xu performed the conception, data analysis, and writing.

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DATA AVAILABILITY

Two datasets underlying this article, that is, halo-based and correlation-based statistics of dark matter flow, are available on Zenodo [64, 65], along with the presentation ‘A comparative study of dark matter flow & hydrodynamic turbulence and its applications’ [66]. All data are also available on GitHub [67].

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Table A1. Parameters of the two dark-matter-only simulations.

	Virgo SCDM	Illustris-1-Dark
Ω_m (collisionless)	1.0	0.2726
Ω_Λ	0	0.7274
Ω_b (initial spectrum only)	0	0.0456
h	0.5	0.704
σ_8	0.51	0.809
n_s	1.0	0.963
Γ	0.50	0.15
Box L [h^{-1} Mpc]	239.5	75
Particles	256 ³	1820 ³
Particle mass	$2.3 \times 10^{11} h^{-1} M_\odot$	$7.5 \times 10^6 M_\odot$
$k_{\min} = 2\pi/L$ [h Mpc ⁻¹]	0.026	0.084
A_s (implied)	3.4×10^{-10}	2.0×10^{-9}
a_{eq}	1.7×10^{-4}	3.1×10^{-4}

ety 341, 1311 (2003).

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APPENDIX A: N-BODY SIMULATIONS

We illustrate results with two dark-matter-only simulations: the Virgo SCDM run [68] and Illustris-1-Dark [62]. Their parameters are listed in Table A1. In Illustris-1-Dark all matter is collisionless; baryons enter only through the initial power spectrum. The two runs differ in cosmology, volume, resolution and spectrum.

APPENDIX B: COSMIC ENERGY AND ITS DISSIPATION

The global energy budget does not resolve transfer between scales, but its steady decline is the sink that sustains the mass and energy cascades of Sections 5 and 6.

B0.0.1 Cosmic energy equation. For N particles of mass m_p in comoving coordinates $\mathbf{x}$ [40],

$$\frac{d^2 \mathbf{x}_i}{dt^2} + 2H \frac{d\mathbf{x}_i}{dt} = -\frac{Gm_p}{a^3} \sum_{j \neq i}^N \frac{\mathbf{x}_i - \mathbf{x}_j}{|\mathbf{x}_i - \mathbf{x}_j|^3}. \quad (\text{B1})$$

The term $2H\dot{\mathbf{x}}_i$ acts as a damping force. Multiplying by the velocities and summing over particles gives the Layzer–Irvine equation [69, 70] for the specific kinetic energy K_p , potential energy P_y and total energy $E_y = K_p + P_y$ in physical coordinates,

$$\frac{\partial E_y}{\partial t} + H(2K_p + P_y) = 0, \quad (\text{B2})$$

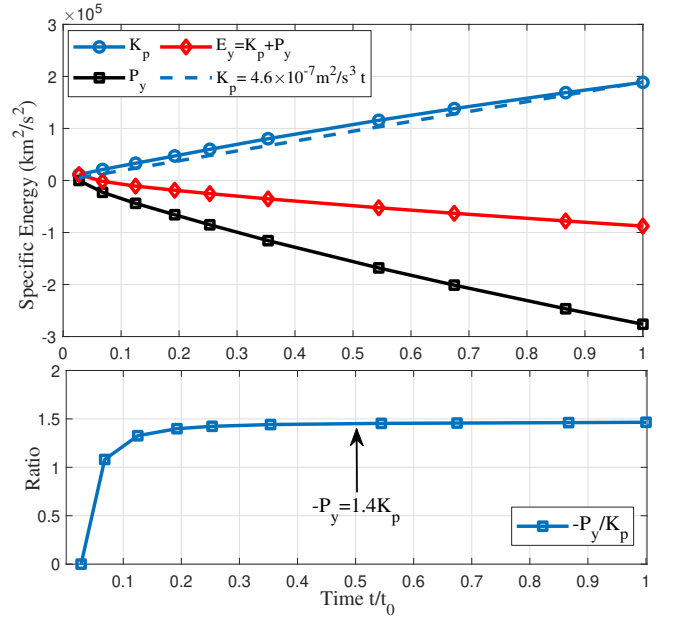


Figure B1. Specific kinetic energy K_p , potential energy P_y and total energy E_y ($\text{km}^2 \text{s}^{-2}$) versus t/t_0 in Virgo SCDM, with the solution of Eq. (B3) for $\varepsilon_u = -4.6 \times 10^{-7} \text{ m}^2 \text{s}^{-3}$. The fixed ratio $P_y = -1.4K_p$ gives the effective exponent $n_e = -10/7$ rather than the equilibrium value -1 .

where $H(2K_p + P_y)$ is the energy sink driven by expansion. The same equation follows in any rescaled time $ds/dt = a^\kappa$; for the matter era $\kappa = -3/2$ removes a from the equations of motion and leaves a constant damping coefficient $H_0/2$ [71]. In the matter era ($Ht = 2/3$), Eq. (B2) admits the exact linear solution

$$K_p = -\varepsilon_u t, \quad P_y = \frac{7}{5} \varepsilon_u t, \quad E_y = \frac{2}{5} \varepsilon_u t, \quad (\text{B3})$$

with $\varepsilon_u < 0$ the energy cascade rate. Figure B1 confirms $K_p \propto t$ and $P_y = -1.4K_p$ in Virgo SCDM.

B0.0.2 Non-equilibrium virial ratio and dissipation rate. In equilibrium, a potential $\propto r^{n_e}$ gives $2K_p = n_e P_y$ with $n_e = -1$ for gravity. Equation (B3) instead gives

$$n_e = \frac{2K_p}{P_y} = -\frac{10}{7}, \quad (\text{B4})$$

the signature of a system that never reaches virial equilibrium because energy cascades continuously across scales. Here n_e is an effective virial exponent fixed by $K_p \propto t$ and $Ht = 2/3$, not a spectral index. The total energy decreases linearly, $E_y = 2\varepsilon_u t/5$, the analogue of viscous dissipation with expansion as the sink, at the rate

$$\varepsilon_u = -\frac{K_p}{t} = -\frac{3}{2} \frac{v_0^2}{t_0} \approx -4.6 \times 10^{-7} \frac{\text{m}^2}{\text{s}^3}, \quad (\text{B5})$$

where $v_0 \approx 350 \text{ km s}^{-1}$ is the present one-dimensional peculiar-velocity dispersion of all particles (bound and unbound) and $t_0 \approx 13.7 \text{ Gyr}$; other simulations give slightly different v_0 . In the steady state, kinetic energy cascades inversely at ε_u , potential energy directly at $-1.4\varepsilon_u$, and total energy directly at $-0.4\varepsilon_u$, supplying the energy dissipated on small scales.

B0.0.3 Energy released by merging. The small-scale sink is merging. Two haloes M_1, M_2 of radii r_{h1}, r_{h2} merge into $M_3 = M_1 + M_2$ of radius r_{h3} . Haloes share a mass-independent bulk dispersion

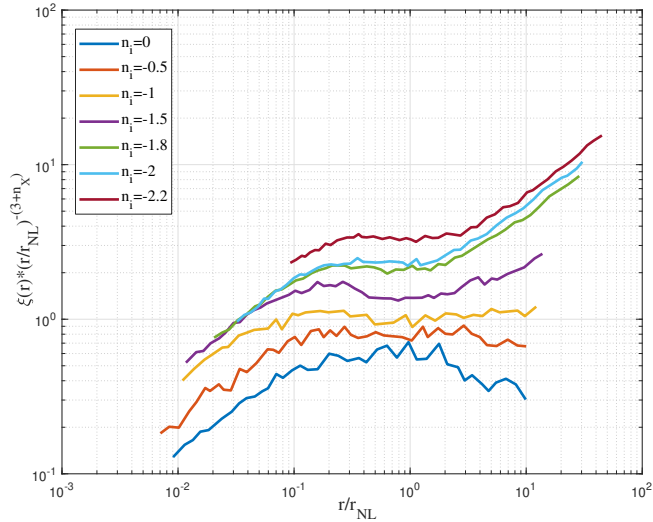


Figure B2. Compensated correlation function $\xi(r)(r/r_{\text{NL}})^{3+n_X}$ for scale-free simulations with $n_i \in \{0, -0.5, -1, -1.5, -1.8, -2, -2.2\}$ and $r_{\text{NL}} \propto a^{2/(n_i+3)}$. The plateau near $r \sim r_{\text{NL}}$ marks an n_i -dependent transition range, here with $n_X = -3/(4+n_i)$ ($\alpha_H = 3$ in Eq. (C2)).

(Eq. (21)), so bulk kinetic energy is conserved and the change is internal. With $\text{KE} = (n_e/2)\text{PE}$,

$$\Delta E_1 = \left(1 + \frac{n_e}{2}\right) G \left[\frac{M_1^2}{r_{h1}} + \frac{M_2^2}{r_{h2}} - \frac{M_3^2}{r_{h3}} \right]. \quad (\text{B6})$$

For a common virial density, $\frac{M_i}{r_{hi}} = \frac{4}{3}\pi\Delta_{\text{vir}}\bar{\rho}_{\text{DM}}$, and with $\mu = \frac{M_2}{M_1}$,

$$\Delta E_1 = \left(1 + \frac{n_e}{2}\right) \left(\frac{4}{3}\pi\Delta_{\text{vir}}\bar{\rho}_{\text{DM}}\right)^{\frac{1}{3}} G M_1^{\frac{5}{3}} \left(1 + \mu^{\frac{5}{3}} - (1 + \mu)^{\frac{5}{3}}\right) < 0, \quad (\text{B7})$$

since $1 + \mu^{5/3} < (1 + \mu)^{5/3}$ and $1 + n_e/2 > 0$. Merging always releases energy, so structure forms bottom-up; top-down formation would require energy input. For $\mu \ll 1$,

$$\Delta E_1 = -\frac{5}{3} \left(1 + \frac{n_e}{2}\right) \frac{G M_1}{r_{h1}} M_2 : \quad (\text{B8})$$

each accretion event releases the host potential times the accreted mass. In the smooth-accretion limit $M_2 = m_p$, every event removes energy on the smallest scale, and the cascade must replenish it.

APPENDIX C: SCALE-FREE INITIAL SPECTRA

Scale-free simulations show what happens without k_{eq} : the three ranges of Eq. (16) still exist, but b and n_X depend on n_i , so the transition range is not universal. In the ansatz of Hamilton et al. [72], as extended by Peacock & Dodds [73], the nonlinear spectrum is a universal function of the linear one,

$$\Delta^2(k_{\text{NL}}) = f_{\text{NL}} \left(\Delta_L^2(k_L) \right), \quad (\text{C1})$$

$$k_{\text{NL}} = \left[1 + \Delta^2(k_{\text{NL}}) \right]^{1/3} k_L,$$

with $f_{\text{NL}}(y) = y$ in the linear limit, $f_{\text{NL}} \propto y^{3/2}$ in stable clustering, and $f_{\text{NL}} \propto y^3$ in the quasi-linear regime from spherical collapse [74]. Inserting Eq. (16) with $f_{\text{NL}} \propto y^{\alpha_H}$ gives

$$b = \frac{6\alpha_H}{3 + (3+n_i)\alpha_H}, \quad n_X = -\frac{9}{3 + (3+n_i)\alpha_H}, \quad (\text{C2})$$

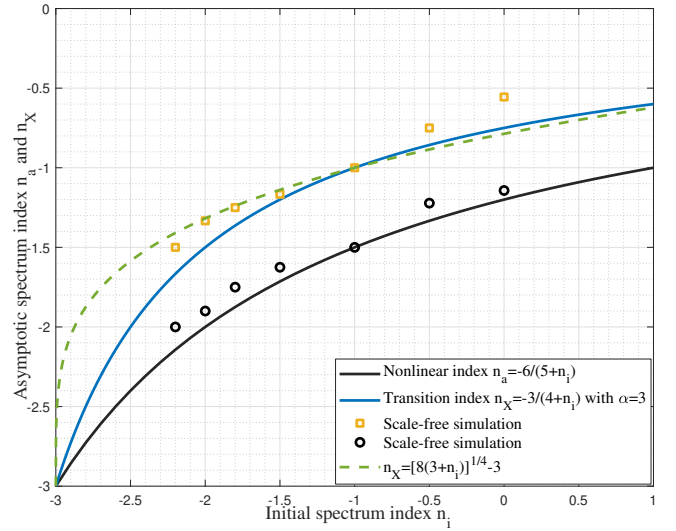


Figure C1. Nonlinear index n_a and transition index n_X versus n_i . Symbols: simulations (Fig. B2). Solid: $n_a = -6/(5+n_i)$ (black) and $n_X = -3/(4+n_i)$ (blue). Dashed green: empirical fit $n_X = [8(3+n_i)]^{1/4} - 3$, implying an n_i -dependent α_H .

and Eq. (17) for $\alpha_H = 3/2$. Scale-free simulations show an n_i -dependent transition range (Fig. B2). The Peebles n_a agrees well with them, and $n_X = -3/(4+n_i)$ for $\alpha_H = 3$ captures the trend but underestimates the steepening for $n_i \lesssim -2$ (Fig. C1). For CDM, $n_X = -1$ and $b = 3$ give $\alpha_H = 9/2$ and $n_i = -5/3$, the effective values of Eq. (29); $\alpha_H = 9/2$ is consistent with simulations [75].

APPENDIX D: DERIVATION OF SPECTRUM $P(k) \propto k^{-1}$

This appendix derives $n_X = -1$ independently of the velocity matching of Section 4. Write the universal spectrum about $k_h^* \sim 1/r_{hc}^*$, the comoving size of a characteristic halo at virial contrast, $m_h^* = \frac{4}{3}\pi r_{hc}^{*3} \Delta_{\text{vir}} \bar{\rho}_0$ with $H_0^2 = \frac{8}{3}\pi G \bar{\rho}_0$, as $\Delta_X^2(k, a) = \Delta_X^2(k_h^*, a) (k r_{hc}^*)^{c_X}$ with c_X to be determined. The amplitude at k_h^* is fixed by the collapse threshold, $\Delta_X^2(k_h^*, a) \propto (\delta_{\text{vir}}^{\text{lin}})^2 a^2$. In spherical collapse $\delta_{\text{vir}}^{\text{lin}} = \frac{3}{10} \Delta_{\text{vir}}^{1/3} \approx 1.686$, $\Delta_{\text{vir}} = 18\pi^2$ [10, 49], so

$$\Delta_X^2(k, a) \propto 2^{c_X/3} \Delta_{\text{vir}}^{\frac{2-c_X}{3}} a^2 \left(\frac{G m_h^*}{H_0^2} \right)^{c_X/3} k^{c_X}. \quad (\text{D1})$$

If gravity and expansion are the only physics of the universal range, the spectrum may depend only on G , m_h^* and H , so a^2 must combine with H_0^2 into $H^2 = H_0^2 a^{-3}$. This requires $c_X = 2$, for which Δ_{vir} also drops out:

$$\Delta_X^2(k, a) \propto \left(\frac{2G m_h^*}{H^2} \right)^{2/3} k^2, \quad (\text{D2})$$

i.e. $P(k) \propto k^{-1}$, in agreement with Eq. (24).

APPENDIX E: MASS FUNCTION: LIMITS, MOMENTS AND REFERENCE FORMS

E0.0.1 Moments. The characteristic mass of Eq. (47) is defined by the second moment of Eq. (46),

$$\langle m_h^2 \rangle = \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{3-\lambda}{2-2\lambda}\right) \left[4(1-\lambda)^2 D_{p0} D_o\right]^{\frac{1}{1-\lambda}} \equiv \gamma_0 M_h^{*2}, \quad (\text{E1})$$

with γ_0 a constant. Writing $4(1-\lambda)^2 D_{p0} D_o = 4\eta_0 M_h^{*2-2\lambda}$ gives Eq. (48) with

$$\eta_0 = \frac{1}{4} \left[\frac{\gamma_0 \sqrt{\pi}}{\Gamma((3-\lambda)/(2-2\lambda))} \right]^{1-\lambda}. \quad (\text{E2})$$

The cumulative moments of the double- λ mass function, Eq. (52), are

$$\int_{m_h}^{\infty} f_M m^k dm = \frac{M^{*k} (4\eta_0)^{\frac{k}{2-2\lambda_2}}}{\Gamma(q/2)} \Gamma\left(\frac{k+1-\lambda_1}{2-2\lambda_2}, x\right), \quad (\text{E3})$$

with x as in Eq. (59) and $\Gamma(s, x)$ the upper incomplete gamma function ($\Gamma(s, x) \rightarrow \Gamma(s) - x^s/s$ as $x \rightarrow 0$). The case $k=0$ gives Eq. (59), and $k=2/3$ gives the energy flux of Eq. (F4). The single- λ moments follow with $\lambda_1 = \lambda_2 = \lambda$, for which $q=1$ and $\Gamma(q/2) = \sqrt{\pi}$.

E0.0.2 Linear and deeply nonlinear limits. *Press–Schechter limit.* For a linear spectrum $P_i \propto k^{n_i}$, $\sigma_{\delta}^L(m_h) \propto m_h^{\lambda_i-1}$ with $\lambda_i = \frac{3-n_i}{6}$, and the peak height takes power-law form

$$\nu \equiv \left(\frac{\delta_c}{\sigma_{\delta}^L(m_h) D(a)} \right)^2 = \left(\frac{m_h}{M_{\text{PS}}^*} \right)^{2-2\lambda_i}, \quad (\text{E4})$$

$$M_{\text{PS}}^* \propto D(a)^{\frac{1}{1-\lambda_i}} = D(a)^{\frac{6}{3+n_i}},$$

where M_{PS}^* , defined by $\sigma_{\delta}^L(M_{\text{PS}}^*) D(a) = \delta_c$, is a Lagrangian mass. Substituting into Eq. (48),

$$f_{\lambda}(\nu) = \frac{1}{2\sqrt{\pi}\eta_0} \nu^{-1/2} \exp\left(-\frac{\nu}{4\eta_0}\right), \quad (\text{E5})$$

which reproduces the PS mass function for $\eta_0 = 1/2$. For a realistic CDM spectrum λ is the local logarithmic slope at m_h (Eq. (39)) and varies slowly, so the constant- λ remains a good local approximation.

Deeply nonlinear limit. On scales $m_h \ll m_h^*$ the relevant variance is the nonlinear one, $\sigma_{\delta}^{\text{NL}} \propto m_h^{-(3+n_a)/6} D^{-n_a/2}$, which encodes both the primordial spectrum and its gravitational renormalization. The peak height is

$$\nu \equiv \left(\frac{\delta_{\text{NL}}}{\sigma_{\delta}^{\text{NL}}(m_h, a)} \right)^2 = \left(\frac{m_h}{M_{\text{NL2}}^*} \right)^{2-2\lambda_a}, \quad \lambda_a = \frac{3-n_a}{6}, \quad (\text{E6})$$

$$M_{\text{NL2}}^* \propto D(a)^{\frac{6\lambda_a-3}{2(1-\lambda_a)}} = D(a)^{-\frac{3n_a}{3+n_a}}.$$

M_{NL2}^* is thus the front of the walk in this limit; it marks the stable-clustering range of the spectrum ($k > k_{\text{NL2}}$) and does not appear in the double- λ form, Eq. (52).

E0.0.3 Peak-height form. To connect the double- λ mass function, Eq. (52), with the standard peak height, define $\nu \equiv (m_h/M_{\text{PS}}^*)^{2-2\lambda_i}$ as in Eq. (E4). The dimensionless double- λ mass function is then a

generalized gamma distribution with three parameters $\{\eta_0^*, p, q\}$,

$$f_{D\lambda}(\nu) = \frac{p \left(2\sqrt{\eta_0^*}\right)^{-q}}{\Gamma(q/2)} \nu^{\frac{pq}{2}-1} \exp\left(-\frac{\nu^p}{4\eta_0^*}\right), \quad (\text{E7})$$

$$\eta_0^*(a) = \eta_0 \left(\frac{M^*}{M_{\text{PS}}^*} \right)^{2-2\lambda_2},$$

with

$$p = \frac{1-\lambda_2}{1-\lambda_i}, \quad q = \frac{1-\lambda_1}{1-\lambda_2}. \quad (\text{E8})$$

The parameter p sets the shape of the cutoff relative to the linear baseline, q the ratio of the two nonlinear exponents, and $\eta_0^*(a)$ the offset between M^* and M_{PS}^* . The redshift dependence of η_0^* implies that a strictly universal mass function is only an approximation, consistent with simulations [76]. For a general CDM spectrum the dimensional mass function follows as

$$f_M(m_h, z) = f_{D\lambda}(\nu) \frac{d\nu}{dm_h} = f_{D\lambda}(\nu) \frac{d \ln[\sigma_{\delta}^L(m_h)^{-1}]}{d \ln m_h} \frac{2\nu}{m_h}, \quad (\text{E9})$$

$$\nu = \left(\frac{\delta_c}{\sigma_{\delta}^L(m_h) D(a)} \right)^2,$$

with the ν - m_h mapping available in the literature [77].

E0.0.4 Reference mass functions. For comparison, the Press–Schechter, Sheth–Tormen and Jenkins et al. mass functions are

$$f_{\text{PS}}(\nu) = \frac{e^{-\nu/2}}{\sqrt{2\pi\nu}}, \quad (\text{E10})$$

$$f_{\text{ST}}(\nu) = \frac{\sqrt{2q_{\text{ST}}} [1 + (q_{\text{ST}}\nu)^{-p_{\text{ST}}}]}{\Gamma(\frac{1}{2}) + 2^{-p_{\text{ST}}}\Gamma(\frac{1}{2} - p_{\text{ST}})} \frac{e^{-q_{\text{ST}}\nu/2}}{2\sqrt{\nu}}, \quad (\text{E11})$$

$$f_{\text{JK}}(\nu) = \frac{0.315}{2\nu} \exp\left(-|\ln \sqrt{\nu}/\delta_c + 0.61|^{3.8}\right), \quad (\text{E12})$$

with $q_{\text{ST}} = 0.75$, $p_{\text{ST}} = 0.3$ [78] and $\delta_c = 1.6865$; the Jenkins et al. form is an empirical fit [79], is not strictly normalized, and should not be extrapolated beyond its fitting range.

APPENDIX F: ENERGY CASCADE AND LOCALITY

This appendix completes the mass-space cascade of Section 5: it formulates the energy flux, then shows from merger statistics that the transfer is local, which justifies a simple local form of both fluxes.

F0.0.1 Energy flux. Decompose the velocity of a halo particle as

$$\mathbf{v}_P = \mathbf{v}_h + \mathbf{v}'_P, \quad (\text{F1})$$

with $\mathbf{v}_h = \langle \mathbf{v}_P \rangle_h$ the halo mean velocity. The specific kinetic energy splits into a bulk part $K_{Ph} = \mathbf{v}_h^2/2$, set by inter-halo and large-scale gravity, and a virial part $K_{Pv} = \mathbf{v}'_P{}^2/2$, set by intra-halo dynamics. Because the bulk dispersion is nearly mass-independent (Eq. (21)), bulk energy is conserved in merging, and every merger releases internal energy (Eq. (B8)); a sustained virial-energy flux through mass space must replenish this loss. By analogy with Λ_m (Eq. (35)), the mean specific virial energy of haloes above m_h is

$$\overline{K_{Pv}}(m_h, t) = \frac{\Lambda_{Pv}}{\Lambda_m}, \quad K_{Pv} = \frac{3}{2} \sigma_v^2, \quad (\text{F2})$$

$$\Lambda_{Pv}(m_h, t) = \int_{m_h}^{\infty} M_h f_M K_{Pv} dm.$$

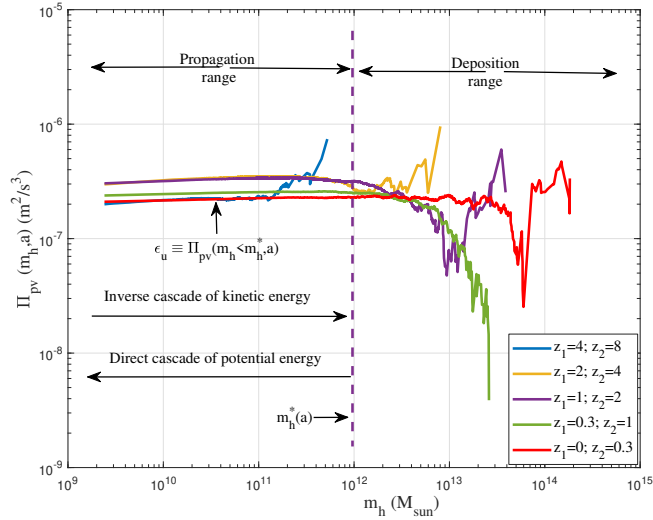


Figure F1. Virial energy flux $\Pi_{pv}(m_h, a)$ (Eq. (F3)) versus halo mass at several redshifts in Illustris-1-Dark. A scale-independent plateau ϵ_u below the cutoff, nearly constant in time, identifies an inverse cascade of virial kinetic energy, with $\epsilon_u \approx -2.5 \times 10^{-7} \text{ m}^2 \text{ s}^{-3}$ (ΛCDM); Virgo SCDM gives $-4.6 \times 10^{-7} \text{ m}^2 \text{ s}^{-3}$ (Appendix B).

If merging dominates the energy exchange and smooth accretion of the smallest unit m_p dominates merging, $\overline{K_{pv}}$ changes only through exchange across m_h , and the virial energy flux is

$$\Pi_{pv}(m_h, t) = -\frac{\partial \overline{K_{pv}}}{\partial t}. \quad (\text{F3})$$

With $K_{pv} = A_K m_h^{2/3} a^{-\theta}$ (Eq. (12)) and the moments of Eq. (E3),

$$\Pi_{pv}(m_h, t) = -(4\eta_0)^{\frac{2/3}{2-2\lambda_2}} \frac{\partial}{\partial t} \left[K_{pv}^* \frac{\Gamma\left(\frac{5/3-\lambda_1}{2-2\lambda_2}, x\right)}{\Gamma\left(\frac{1-\lambda_1}{2-2\lambda_2}, x\right)} \right], \quad (\text{F4})$$

where $K_{pv}^* \equiv K_{pv}(M^*, a)$ is the specific virial energy at the cutoff. For $m_h \ll M^*$ the gamma-function ratio tends to a constant, and the flux becomes the scale-independent rate

$$\epsilon_u = -(4\eta_0)^{\frac{2/3}{2-2\lambda_2}} \frac{\Gamma\left(\frac{5/3-\lambda_1}{2-2\lambda_2}\right)}{\Gamma(q/2)} \frac{\partial K_{pv}^*}{\partial t}. \quad (\text{F5})$$

After a_* ($\theta = 1/2$),

$$K_{pv}^* \propto M^{*2/3} a^{-1/2} \propto a^{3/2} \propto t, \quad (\text{F6})$$

so ϵ_u is time-independent; conversely, a constant ϵ_u requires $M^* \propto a^3$. Before a_* ($\theta = 1$) the same holds with $M^* \propto a^{15/4}$ (Eq. (13)). Figure F1 confirms a scale-independent, nearly time-independent plateau below the cutoff (Section 5).

F0.0.2 Locality of mass transfer. To test mass transfer locality in halo mass space, we group haloes by particle number n_p and follow mass between groups. The *forward* function $D_{FM}(z_1, n_{p1}, z_2, n_{p2})$ is the fraction of the mass of group n_{p1} at z_1 inherited from group n_{p2} at an earlier z_2 , and the *backward* function $D_{BM}(z_1, n_{p1}, z_3, n_{p3})$ the fraction passed to group n_{p3} at a later z_3 ; unbound matter is counted as $n_p = 1$. Mass conservation gives

$$\sum_{n_{p2}} D_{FM}(z_1, n_{p1}, z_2, n_{p2}) = 1, \quad (\text{F7})$$

$$\sum_{n_{p3}} D_{BM}(z_1, n_{p1}, z_3, n_{p3}) = 1.$$

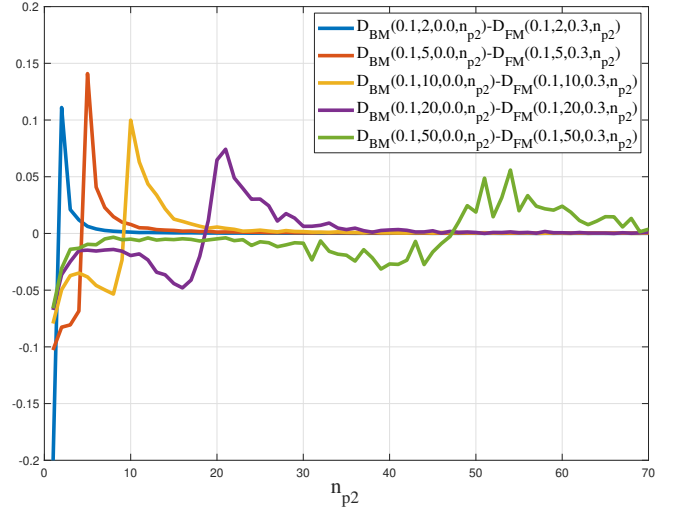


Figure F2. Net redistribution function D_{NM} (Eq. (F8)) versus n_{p2} for five group sizes n_{p1} . Negative values mark net donors to group n_{p1} ; the sign change at $n_{p2} = n_{p1}$ confirms an inverse mass cascade.

Both functions peak near $n_p \approx n_{p1}$ (similar-size groups) and near $n_p \approx 1$ (single-particle accretion), so the transfer is local and dominated by smooth and minor merging; larger groups exchange mass with a wider range of sizes. They are asymmetric: more mass is inherited from smaller groups than from larger ones, and more is passed upward than downward. The net transfer,

$$D_{NM}(z_1, n_{p1}, n_{p2}) \equiv D_{BM}(z_1, n_{p1}, z_3, n_{p2}) - D_{FM}(z_1, n_{p1}, z_2, n_{p2}), \quad (\text{F8})$$

taken at fixed $z_2 = 0.3$ and $z_3 = 0$ for $z_1 = 0.1$, sums to zero over n_{p2} and is negative for $n_{p2} < n_{p1}$ and positive for $n_{p2} > n_{p1}$ (Fig. F2): mass flows from small to large scales, an inverse cascade, opposite to the direct cascade of three-dimensional turbulence.

F0.0.3 Time scales and local fluxes. Let $\tau_h(m_h, a)$ be the mean time between single-merger events in the group of all n_h haloes of mass m_h . If each halo in the group is equally likely to be the one that merges, a given halo waits on average

$$\tau_g = \sum_{k=1}^{\infty} \frac{k\tau_h}{n_h} \left(1 - \frac{1}{n_h}\right)^{k-1} = n_h \tau_h, \quad (\text{F9})$$

which is both its mean lifespan and the time to pass the whole group mass $m_g = n_h m_h$ upward. Under locality the mass flux across m_h in the propagation range is then

$$\Pi_m = -\frac{m_h}{\tau_h} = -\frac{m_g}{\tau_g} = \epsilon_m, \quad (\text{F10})$$

independent of m_h in the steady state, consistent with Eq. (37): groups in the propagation range act as conduits with time-invariant m_g (Fig. 10). Cascading the entire halo mass takes $\tau_M = -M_h/\epsilon_m \sim t$. In each interval τ_h the cascade raises the specific virial energy of all haloes above m_h by $m_h \overline{K_{pv}}/\Lambda_m$, so the local energy flux is

$$\Pi_{pv} = \Pi_m \frac{\overline{K_{pv}}}{\Lambda_m} = \epsilon_u \quad (\text{propagation range}). \quad (\text{F11})$$

APPENDIX G: PARTICLE RANDOM WALK IN HALOES

This appendix supplies three details of Section 6, the exact sojourn average, the normalized profiles and the tracer equivalence, and then

a microscopic model of the particle jumps that underlie the radial cascade rate ε .

G0.0.1 Sojourn average, normalized profiles and tracer equivalence. Evaluating the time average of Eq. (86) with Eq. (84),

$$\bar{f}_r(r, D_o) = \frac{2r^{-\gamma}}{\sqrt{\pi D_{r0} D_o}} \exp\left(-\frac{r^{2-2\gamma}}{4(1-\gamma)^2 D_{r0} D_o}\right) - \frac{r^{1-2\gamma}}{(1-\gamma) D_{r0} D_o} \operatorname{Erfc}\left(\frac{r^{1-\gamma}}{2(1-\gamma)\sqrt{D_{r0} D_o}}\right). \quad (\text{G1})$$

At fixed r the first term decays as $D_o^{-1/2}$ and the second as D_o^{-1} , so in the long-time limit $\bar{f}_r \rightarrow 2f_r$, which gives Eq. (87). Normalized over volume, the single-exponent distribution is

$$P_r = \frac{(2-2\gamma)^{\frac{\gamma-2}{1-\gamma}} r^{-2\gamma}}{4\pi(D_{r0} D_o)^{\frac{3-2\gamma}{2-2\gamma}} \Gamma\left(\frac{3-2\gamma}{2-2\gamma}\right)} \exp\left(-\frac{r^{2-2\gamma}}{4(1-\gamma)^2 D_{r0} D_o}\right), \quad (\text{G2})$$

and its double- γ generalization (Section 6) is

$$P_r = \frac{(2-2\gamma_2)^{\frac{2\gamma_1-2-\gamma_2}{1-\gamma_2}} r^{-2\gamma_1}}{4\pi(D_{r0} D_o)^{\frac{3-2\gamma_1}{2-2\gamma_2}} \Gamma\left(\frac{3-2\gamma_1}{2-2\gamma_2}\right)} \exp\left(-\frac{r^{2-2\gamma_2}}{4(1-\gamma_2)^2 D_{r0} D_o}\right). \quad (\text{G3})$$

With $x = r/r_s$ and Eq. (90) this becomes

$$P_r(x) = \frac{(1-\gamma_2)x^{-2\gamma_1}}{2\pi \Gamma\left(\frac{3-2\gamma_1}{2-2\gamma_2}\right) \left(\frac{1-\gamma_2}{1-\gamma_1}\right)^{\frac{3-2\gamma_1}{2-2\gamma_2}}} \exp\left(-\frac{1-\gamma_1}{1-\gamma_2} x^{2-2\gamma_2}\right). \quad (\text{G4})$$

which is Eq. (92), with cumulative mass

$$\int_0^x P_r(y) 4\pi y^2 dy = 1 - \frac{\Gamma\left(\frac{1}{\alpha} + \frac{1}{\beta}, \frac{x^\alpha}{\beta}\right)}{\Gamma\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)}, \quad (\text{G5})$$

Because accreted particles are deposited one per step dD_o and remain bound, the ensemble of deposition radii samples the same process as a single tracer performing the isotropic walk

$$\frac{dX}{dD_o} = \sqrt{2D_{r0}} r^\gamma \xi(D_o), \quad r = |X|, \quad (\text{G6})$$

with ξ isotropic Gaussian white noise and r^γ inherited from m_h^λ through the mass-radius mapping. Unlike the mass-space walk of Section 5, whose Stratonovich form follows from the finite correlation time of the physical noise, this tracer walk is a statistical construct, so its convention is fixed by consistency: only the Itô form [80] reproduces the residence-time weighting of Eq. (87). Its Fokker-Planck equation,

$$\frac{\partial P_r}{\partial D_o} = D_{r0} \nabla^2 \left(r^{2\gamma} P_r \right), \quad (\text{G7})$$

has Eq. (G2) as its exact spherical solution, which the Stratonovich form does not. The equivalence is statistical only: interior particles do not random-walk; their positions relative to r_s are fixed once accreted.

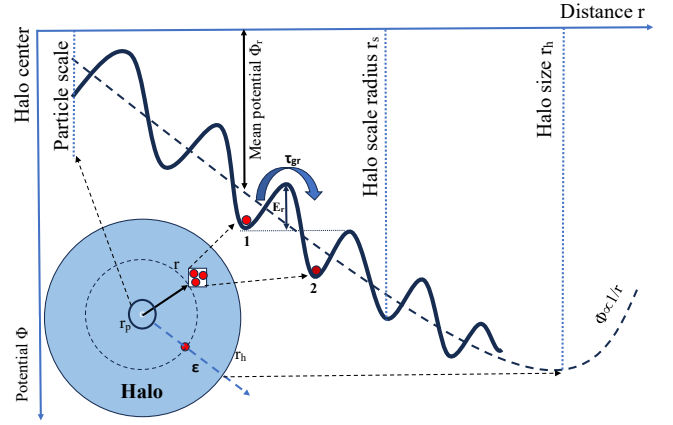


Figure G1. Particle random walk in a spherical halo. Particles diffuse in the granular potential $\Phi = \Phi_r + E_r$. An outward jump from “1” to “2” carries virial kinetic energy $v_r^2/2$ to a larger scale, after a mean interval $\tau_{gr} \propto |\Phi_r|^{-1}$ (Eq. (G9)). The result is an inverse cascade of kinetic energy at rate ε and a direct cascade of potential energy at -1.4ε .

G0.0.2 Microscopic jumps. Because haloes are made of discrete particles, the potential on scale r has a mean part and a granular fluctuation (Fig. G1),

$$\Phi_r(r) = -\frac{Gm_r}{r}, \quad E_r(r) = \frac{Gm_p}{d_r}, \quad \rho_r = \frac{m_p}{d_r^3}, \quad (\text{G8})$$

where m_r is the enclosed mass and d_r the inter-particle spacing. Treating the local kinetic energy as a temperature, $k_B T_r = m_p v_r^2 = m_p \gamma_r Gm_r/r$ with γ_r the virial factor of Eq. (76), a jump from “1” to “2” succeeds with probability $P_r = \exp(-m_p E_r/k_B T_r) \approx 1 - m_p E_r/k_B T_r$, since $m_p E_r \ll k_B T_r$. With attempt frequency Γ_r , the waiting time is

$$\tau_{gr}(r) = \frac{1 - P_r}{\Gamma_r} = \frac{1}{\gamma_r \Gamma_r d_r} \frac{Gm_p}{|\Phi_r|} = \frac{m_p}{u_h^*} \frac{r}{m_r}, \quad (\text{G9})$$

$$u_h^* \equiv \gamma_r \Gamma_r d_r,$$

inversely proportional to the potential depth, as assumed in Eq. (88). We take u_h^* independent of r . Then τ_{gr} is a collective interval, not the waiting time of one particle: at the boundary it reduces to Eq. (64), the interval between successive accretions onto the whole halo, far shorter than the attempt period $1/\Gamma_r$ (Table H1). For characteristic haloes, $dm_h^*/dt = m_p/\tau_{gr}(r_h^*)$ identifies u_h^* as the growth speed of the fixed-contrast radius,

$$u_h^* = \frac{dm_h^*}{dt} \frac{r_h^*}{m_h^*} = \frac{r_h^*}{t} = \frac{dr_h^*}{dt} \approx 75 \text{ km s}^{-1}, \quad (\text{G10})$$

using $m_h^* \propto t$ (Section 4) and $r_h^* \propto t$ (Eq. (65)), with $m_h^* \approx 3.2 \times 10^{13} M_\odot$, $r_h^* \approx 1.05 \text{ Mpc}$ and $t = 13.7 \text{ Gyr}$. Since u_h^* is constant while the density falls, a larger spacing d_r is offset by a lower Γ_r .

G0.0.3 Energy per jump and cascade rate. If the particle obeys the virial relation before and after a jump, its energy changes by

$$\Delta E_2 = \frac{1}{2} \left(1 + \frac{2}{n_e} \right) m_p dv_r^2 < 0 \quad (\text{G11})$$

for $n_e = -10/7$: each jump releases energy, as each merger does (Eq. (B6)). For a shell of thickness s_r containing $N_r = 4\pi r^2 \rho_r s_r/m_p$ particles, one particle crosses r every $\tau_{hr} = \tau_{gr}/N_r$ (cf. Eq. (F9)). Each crossing carries $dm_r = m_p$ and $dv_r^2 = \frac{2}{3} m_p v_r^2/m_r$ (from

$v_r^2 \propto m_r^{2/5}$, Eq. (77)). The cascade rate is therefore

$$\varepsilon = \frac{\Delta E_2}{m_p \tau_{hr}} = \frac{1}{5} \left(1 + \frac{2}{n_e} \right) \frac{m_p v_r^2}{m_r \tau_{hr}} = \frac{\gamma_r}{5} \left(1 + \frac{2}{n_e} \right) \frac{G m_p}{r \tau_{hr}}, \quad (\text{G12})$$

the microscopic counterpart of Eq. (76), with $\varepsilon < 0$ for outward transfer. Equating the migration flux m_p/τ_{hr} with $\dot{m}_r = 4\pi r^2 \rho_r v_r$ fixes the shell geometry,

$$\tau_{hr} = \frac{m_p}{4\pi r^2 \rho_r v_r}, \quad s_r = v_r \tau_{gr} = \frac{m_p}{u_h^*} \frac{r v_r}{m_r}. \quad (\text{G13})$$

This kinematic τ_{hr} agrees with the energetic one of Eq. (G12) up to the factor $|1 + 2/n_e| = 2/5$. Both channels, merging and migration, release energy at the rate set by ε , $\varepsilon \propto \Delta E_2/(m_p \tau_{hr}) \propto \Delta E_1/[m_h \tau_{hr}(r_h)]$.

G0.0.4 Scaling of microscopic quantities. With the scaling laws of Eq. (77), on scales $r \leq r_s$ and writing ε for $|\varepsilon|$,

$$\begin{aligned} d_r &= \beta_r^{-\frac{1}{3}} m_p^{\frac{1}{3}} \varepsilon^{-\frac{2}{9}} G^{\frac{1}{3}} r^{\frac{4}{9}}, & \Gamma_r &= \frac{u_h^*}{\gamma_r d_r}, \\ \tau_{gr} &= \frac{m_p}{\alpha_r u_h^*} \varepsilon^{-\frac{2}{3}} G r^{-\frac{2}{3}}, & \tau_{hr} &= \frac{\gamma_r}{5} m_p \varepsilon^{-1} G r^{-1}, \\ N_r &= \frac{5}{\alpha_r \gamma_r u_h^*} \varepsilon^{\frac{1}{3}} r^{\frac{1}{3}}, & s_r &= \frac{3^{1/3}}{\alpha_r u_h^*} m_p \varepsilon^{-\frac{1}{3}} G r^{-\frac{1}{3}}. \end{aligned} \quad (\text{G14})$$

Here τ_{hr} is the kinematic value of Eq. (G13). All depend explicitly on m_p , which is the basis of Appendix H. Consider characteristic haloes ($\varepsilon = \varepsilon_u$, u_h^* constant) evaluated at $r = r_s^* \propto a^{9/10}$ after a_* (Eq. (96)). Then $\rho_s^* \propto a^{-6/5}$, $d_s^* \propto a^{2/5}$, $\Gamma_s^* \propto a^{-2/5}$, $\tau_{gs}^* \propto a^{-3/5}$, $\tau_{hs}^* \propto a^{-9/10}$, $N_s^* \propto a^{3/10}$ and $s_s^* \propto a^{-3/10}$.

APPENDIX H: BOUNDS ON THE PARTICLE MASS

In characteristic haloes a single particle matters in two places: at the centre, where the radial cascade ends on one particle, and at the boundary, where accretion adds one particle at a time. At each, a physical limit on a single-particle quantity bounds m_p from below. None of the results of the main text depends on these bounds.

H0.0.1 Assumptions and postulates. (A1) *Steady state:* after a_* , $m_h^* \propto t$ and ε_u is constant (Section 4 and Appendix B).

(A2) *Centre:* the scaling laws of Eq. (77) with $\varepsilon = \varepsilon_u$ hold down to the radius enclosing one particle, with no free-streaming cutoff. This requires particles heavy, and therefore cold, enough that gravity controls all scales. It extrapolates the measured $-4/3$ range over about 35 decades in radius below r_s^* .

(A3) *Boundary:* particles are deposited one at a time at the fixed-contrast boundary $r_h^* \propto t$ (Eq. (65)). There the density is $\rho_b^* = \rho_{\text{crit}}/\alpha_c$, with α_c of order unity (Fig. H1); in matter domination $\rho_b^* \propto t^{-2}$.

(P1) *Quantum localization:* a particle carrying the cascade on scale r with speed v obeys $m_p v r \gtrsim \hbar/2$ (established physics).

(P2) *Minimal length:* no length defined by the metric can be resolved below the Planck length l_p [81, 82]. This is a speculative postulate of quantum gravity.

H0.0.2 Time-invariant boundary. A constant particle mass can be fixed only by time-invariant quantities. At the centre ε_u is constant. At the boundary, A1 and A3 give $r_h^* = u_h^* t$ with constant u_h^* and

$\rho_b^* \propto t^{-2}$. Hence the advance of the boundary per accreted particle and the waiting time,

$$s_h^* = \frac{m_p}{4\pi r_h^{*2} \rho_b^*}, \quad \tau_g^* = \frac{m_p}{\dot{m}_h^*} = \frac{m_p t}{m_h^*}, \quad (\text{H1})$$

are both constant. At the virial size R_h^* , which keeps a fixed comoving size after a_* and where the density is the virial density $\rho_h^* \propto t^{-1}$, the step instead decreases as $t^{-1/3}$. The bound is therefore imposed at r_h^* ; at $z = 0$ the two surfaces differ only by $R_h^*/r_h^* \approx 0.7$. With $m_h^* = \frac{4}{3} \pi r_h^{*3} \alpha_c \rho_b^* \Omega_{\text{DM}} \Delta_c$, where $\Delta_c \simeq \Delta_{\text{vir}}$ (Eq. (9)),

$$s_h^* = \frac{1}{3} \Omega_{\text{DM}} \Delta_c \alpha_c u_h^* \tau_g^*, \quad u_h^{*2} = \frac{2\alpha_c}{3H^2 t^2} \frac{G m_p}{s_h^*}. \quad (\text{H2})$$

The second relation is an identity, equivalent to $u_h^* = r_h^*/t$, in which m_p cancels. It acquires content only once s_h^* is bounded.

H0.0.3 Centre: quantum limit (A1, A2, P1). The cascade ends at the radius r_p enclosing one particle, $m_r(r_p) = m_p$, where $v_r(r_p) = (3|\varepsilon_u| r_p)^{1/3}$:

$$r_p = \left(\frac{G m_p}{\alpha_r |\varepsilon_u|^{2/3}} \right)^{3/5}. \quad (\text{H3})$$

The action $m_p v_r(r_p) r_p$ grows as $m_p^{9/5}$, so P1 gives a lower bound,

$$m_p \gtrsim m_X \equiv \left[\frac{(\hbar/2)^5 \alpha_r^4 |\varepsilon_u|}{3^{5/3} G^4} \right]^{1/9} \simeq M_{\text{Pl}} \left(\frac{|\varepsilon_u| t_p}{c^2} \right)^{1/9}, \quad (\text{H4})$$

which depends only on $\hbar$, G and ε_u [49]. Since $v_r^2 = \gamma_r G m_r/r$, the condition $m_p v_r r_p = \hbar$ is equivalent to $r_p = a_G/\gamma_r$, with

$$a_G = \frac{\hbar^2}{G m_p^3}, \quad v_G = \frac{G m_p^2}{\hbar} = c \left(\frac{m_p}{M_{\text{Pl}}} \right)^2 \quad (\text{H5})$$

the radius and speed of the gravitational ground state of a pair. The cascade thus ends on the most compact quantum configuration of two particles. Take $|\varepsilon_u| = (2-5) \times 10^{-7} \text{ m}^2 \text{ s}^{-3}$, $\alpha_r = 4-6$ and a threshold $\hbar/2-\hbar$. Then $m_X \approx (5-9) \times 10^{11} \text{ GeV}$, with $r_p \approx 2 \times 10^{-13} \text{ m}$ and $v_r(r_p) \approx 5 \times 10^{-7} \text{ m s}^{-1}$. The result is as reliable as A2.

H0.0.4 Boundary: minimal length (A1, A3, P2). The boundary is a gravitationally defined surface, so s_h^* is a change of geometry produced by one accreted particle. Under P2 a single accretion event is distinguishable only if $s_h^* \geq l_p$:

$$m_p \geq 4\pi r_h^{*2} \rho_b^* l_p = \frac{3(Ht)^2}{2\alpha_c} M_{\text{Pl}} \left(\frac{u_h^*}{c} \right)^2. \quad (\text{H6})$$

The particle mass in Planck units is set by $(u_h^*/c)^2 \approx 6 \times 10^{-8}$, about $1/25$ of the potential depth $|\Phi_h^*|/c^2 = \frac{1}{2} \Omega_{\text{DM}} \Delta_c (Ht)^2 (u_h^*/c)^2$. Both sides are time-invariant in matter domination. With $r_h^* \approx 1.05 \text{ Mpc}$ and $\rho_b^* \approx \rho_{\text{crit}}$, $m_p \gtrsim (1-2) \times 10^{-15} \text{ kg} \approx (0.6-1.1) \times 10^{12} \text{ GeV}$ for $\alpha_c = 1-2$.

At the bound, Eq. (H2) reads $u_h^{*2} \simeq G m_p/l_p$, with $\sqrt{G m_p/l_p} \approx 86 \text{ km s}^{-1}$ for $m_p = 10^{12} \text{ GeV}$. The growth speed is thus the orbital speed of a pair at the minimal separation, the geometric counterpart of the quantum state of Eq. (H5). Equivalently, $\tau_g^* = 3c t_p/(\Omega_{\text{DM}} \Delta_c \alpha_c u_h^*) \approx 2 \times 10^2 t_p$. This bound is speculative: it requires P2 in its fixed-minimal-length form, and that the relevant length be the geometric step s_h^* rather than a quantum length of the particle.

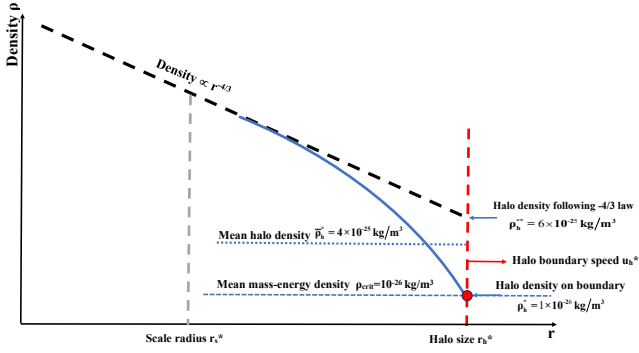


Figure H1. Densities near the boundary of characteristic haloes at $z = 0$ ($m_p^* = 3.2 \times 10^{13} M_\odot$, $r_h^* \approx 1.05 \text{ Mpc}$): the critical density $\rho_{\text{crit}} \approx 10^{-26} \text{ kg m}^{-3}$; the mean halo density $\bar{\rho}_h^* \approx 5 \times 10^{-25} \text{ kg m}^{-3}$; the boundary density $\rho_b^* \approx \rho_{\text{crit}}$; and the boundary density extrapolated from the $-4/3$ scaling, $\rho_h^{**} \approx (3-6) \times 10^{-25} \text{ kg m}^{-3}$ (Eq. (77)). See also Table H1.

H0.0.5 Consistency of the two limits. The centre and the boundary are the quantum and geometric two-body limits of gravity for the same particle, $(Gm_p/l_p)^2 = c^3 v_G$. They rest on different physics, yet both give $m_p \approx 10^{12} \text{ GeV}$. If both are saturated, eliminating m_p between Eqs. (H4) and (H6) gives

$$\frac{u_h^*}{c} \approx \left(\frac{2\alpha_c}{3(Ht)^2} \right)^{1/2} \left(\frac{|\varepsilon_u| t_p}{c^2} \right)^{1/18}, \quad (\text{H7})$$

i.e. $u_h^* \approx 50-70 \text{ km s}^{-1}$ ($H_0 t_0 \approx 0.99$) or $70-100 \text{ km s}^{-1}$ ($Ht = 2/3$), against the measured $\approx 75 \text{ km s}^{-1}$. Because of the $1/18$ power, this tests the two-limit structure rather than the value of ε_u ; it supports P2 but does not derive it.

H0.0.6 Caveats and summary. The centre bound extrapolates the $-4/3$ scaling to $r_p \sim 10^{-13} \text{ m}$ (A2). In simulated Milky-Way-mass haloes the $-4/3$ range ends at $r_{\text{in}} \approx 1 \text{ kpc}$ (Eq. (79)), so A2 assumes that the cascade, not an inherited core, controls the innermost structure of characteristic haloes. The boundary bound depends on P2, for which other quantum-gravity scenarios give different floors. It also depends on order-of-magnitude boundary values (r_h^* , ρ_b^* , α_c , Δ_c) and on the expansion history through Ht . The particle mass in N -body simulations is numerical, so the bounds concern the physical particle implied by the measured ε_u and u_h^* . Together the two arguments give $m_p \gtrsim (5-9) \times 10^{11} \text{ GeV}$ and $m_p \gtrsim (0.6-1.1) \times 10^{12} \text{ GeV}$, consistent with superheavy dark matter of mass $\sim 10^{12} \text{ GeV}$. Table H1 lists the corresponding quantities in characteristic haloes for $m_p = 10^{12} \text{ GeV}$.

APPENDIX I: TABLES OF SCALES AND PARAMETERS

Table I1 lists the main symbols. Table I2 shows what fixes each quantity of the mass function and of the density profile. Table I3 gives the scales and radii before and after the crossover a_* , and Table 3 the mass-to-radius mappings of the two fronts. Each scale is fixed by one of three inputs: the spectral index of its range (k_{NL1} , k_{NL2}); the velocity crossover on the cascade clock (m_h^* , the quantities attached to it, and the total halo mass M_h); or the constant energy cascade rate on the linear clock (M^*). Small-scale permanence then combines the two clocks into λ_1 and ε_m . The crossover switches θ from 1 to $1/2$, slows the cascade clock from $\tau_c = 2$ to 1, and opens the universal range.

[1]

Table H1. Order-of-magnitude values in characteristic haloes at $z = 0$ ($m_p = 10^{12} \text{ GeV}$, $|\varepsilon_u| = 3 \times 10^{-7} \text{ m}^2 \text{ s}^{-3}$, $\alpha_r = 5$). Centre values follow the cascade laws (Eqs. (77), (G14)). Boundary values follow from $\rho_b^* \approx \rho_{\text{crit}}$, $\dot{m}_r = dm_h^*/dt$ and $v_r^2 = \gamma_r Gm_h^*/r_h^*$. The scalings hold inside r_s , and the lower bound depends on m_p .

Quantity	Symbol	Scaling	r_p	r_h^*
Scale (m)	r		1.7×10^{-13}	3.2×10^{22}
Mass (kg)	m_r	$\propto r^{5/3}$	1.8×10^{-15}	6.4×10^{43}
Density (kg m^{-3})	ρ_r	$\propto r^{-4/3}$	4.7×10^{22}	9.3×10^{-27}
Time (s)	t_r	$\propto r^{2/3}$	5×10^{-7}	4.3×10^{17}
Velocity (m s^{-1})	v_r	$\propto r^{1/3}$	5.4×10^{-7}	2.3×10^5
Mass flow (kg s^{-1})	$\dot{m}_r$	$\propto r$	9×10^{-9}	1.5×10^{26}
Mean distance (m)	d_r	$\propto r^{4/9}$	3.4×10^{-13}	5.8×10^3
Frequency (s^{-1})	$\gamma_r \Gamma_r = u_h^*/d_r$	$\propto r^{-4/9}$	2.2×10^{17}	13
Waiting time (s)	τ_{gr}	$\propto r^{-2/3}$	2.3×10^{-18}	1.2×10^{-41}
Waiting time (s)	τ_{hr}	$\propto r^{-1}$	1.9×10^{-7}	1.2×10^{-41}

Table I1. Main symbols.

Symbol	Meaning
<i>Scales (wavenumber, virial mass)</i>	
$k_{\text{NL1}}, M_{\text{NL1}}^*$	linear-universal boundary, $\Delta_L^2 = 1$
$k_{\text{NL2}}, M_{\text{NL2}}^*$	universal-nonlinear boundary
k_h^*, m_h^*	kinematic scale, $\sigma_v^2 = \sigma_h^2$
K^*, M^*	cutoff of the mass function, $\sigma_v^2 = u^2$
k_L^*, M_L^*	linear-extrapolation scale, $\Delta^2 = \Delta_L^2(\infty)$
M_{PS}^*	Press-Schechter (Lagrangian) mass
M_h^*	front of the single- λ walk
M_h, m_g	total halo mass; halo group mass
a_*	crossover, $M^* = m_h^*$
<i>Halo radii and densities</i>	
r_h	fixed-contrast radius, $\Delta_{\text{vir}} = 18\pi^2$
R_h, Δ_h	virial size; dynamical contrast
ρ_h^*	virial density of m_h^* haloes, $\rho_h^* m_h^* = \text{const}$
$r_s, r_{4/3}$	scale radius (slope -2); outer edge of the $-4/3$ range
r_{in}	inner edge of the $-4/3$ range (injection scale)
θ'	$\rho \propto a^{-3\theta'}$ for a density law
c	concentration, R_h/r_s
<i>Kinematics and spectrum</i>	
σ_v^2, σ_h^2	virial and bulk dispersions of haloes
u^2	mean virial dispersion of all halo particles
θ	$\sigma_v^2 \propto m_h^{2/3} a^{-\theta}$, $\sigma_h^2(m_h^*) \propto a^\theta$
$\Psi_{\parallel}, R_L$	coherent displacement; Lagrangian radius
n_i, n_X, n_a	linear, universal, nonlinear indices
b	growth exponent of the universal range
<i>Random walks and cascades</i>	
λ	mass-walk exponent, $(3 - n_{\text{eff}})/6$
λ_i, λ_a	local linear and deeply nonlinear values
λ_h, λ_2	exponents of m_h^* and of the cutoff M^*
λ_1	effective small-mass slope
τ, D_o	barrier growth exponent; clock D^τ
$D_c, D_L; \tau_c, \tau_L$	cascade and linear clocks; their exponents
τ_g, τ_h	waiting time of a halo; of a halo group
$\gamma; \gamma_1, \gamma_2$	radial-walk exponent; inner, cutoff values
β_h	radius-mass exponent of the accretion layer
χ	growth exponent of a mass front, a^χ
t_c	common cascade time, $\sigma_v^2/ \varepsilon_u $
α, β	shape parameters of the double- γ profile
τ_{gr}, τ_{hr}	particle and shell intervals in haloes
$\varepsilon_m, \varepsilon_u$	mass and energy cascade rates in mass space
$\varepsilon(m_h, z)$	radial cascade rate in haloes
n_e	effective virial exponent, $2K_p/P_y = -10/7$
<i>Particle level (Appendices G and H)</i>	
$m_p; d_r, \Gamma_r$	particle mass; spacing, attempt frequency
u_h^*	growth speed, $r_h^*/t = \gamma_r \Gamma_r d_r$
N_r, s_r	particles in, and thickness of, the crossing shell
$r_p; m_X$	one-particle radius; quantum mass bound
$\rho_b^*, \alpha_c, \Delta_c$	boundary density $\rho_{\text{crit}}/\alpha_c$; contrast of r_h^*

Table I2. The quantities of the halo mass function and the halo density profile, what fixes each, and their time dependence after a_* . In mass space one scale, M^* , is both the bend and the front, and the slope λ_1 absorbs the ratio of the two clocks. In radius the slope is fixed by the cascade, so the bend r_s (mass budget) and the front R_h (walk front) separate, and the mass-independent concentration $c = R_h/r_s$ absorbs the difference.

Quantity	Value ($a > a_*$)	Mass function $M_h f_M$	Value ($a > a_*$)	Density profile $\rho_r = m_h P_r$
		Fixed by		Fixed by
Total	$M_h \propto a^{1/2}$	walk on the cascade clock, $M_h \propto D_c^{1/2}$	$m_h (m_h^* \propto a^{3/2})$	halo growth on the cascade clock
Power-law amplitude	A_M , constant	permanence (m_g frozen)	$\beta_r \varepsilon^{2/3} G^{-1}$	energy flux; constant for m_h^*
Slope	$\lambda_1 = 5/6$	permanence and clocks (Eq. (58))	$2\gamma_1 = 4/3$	energy flux and Poisson (unique)
Bend scale	$M^* \propto a^3$	energy, $K_{pV} \propto \varepsilon_u t$	$r_s \propto m_h^{1/3} m_h^{*4/15}$	mass budget (Eq. (95))
Front (cutoff)	M^* (same scale)	energy	$R_h \propto m_h^{1/3} a^{1/2}$	walk front (virial mapping)
Cutoff shape	$2 - 2\lambda_2 = 2/3$	walk front ($n = -1$ at M^*)	$\alpha = 2 - 2\gamma_2$	walk front
Absorbs the conflict	λ_1	—	$c = R_h/r_s \propto a^{1/10}$	—

Table I3. Scales and radii before and after the crossover a_* (matter domination, $D = a$, $t \propto a^{3/2}$; all masses from Eq. (8)). Here $\sigma_v^2 \propto m_h^{2/3} a^{-\theta}$ at fixed mass and $\sigma_v^2(m_h^*) \propto a^\theta$. The two fronts are $m_h^* \propto D_c^{3/2}$ on the cascade clock and $M^* \propto D_L^{1/(2-2\lambda_2)}$ on the linear clock (Eqs. (50) and (51)), with effective index $n \equiv 3 - 6\lambda$. Each radius is a front read through one density law (Table 3). Exponents marked $\dagger$ use $M^* \propto a^{15/4}$ before a_* , predicted and measured in Illustris-1-Dark (Fig. 5).

Quantity	$a < a_*$	$a > a_*$	Fixed by / remark
<i>The switch</i>			
Multistream regions; accreted matter	isolated; cold	connected; pre-processed	Section 3
Mass ordering; dominant energy	$M^* < m_h^*$; $\langle K_{ph} \rangle$	$m_h^* < M^*$; $\langle K_{pv} \rangle$	Eq. (14)
θ (n at M^*); cascade clock τ_c	1 (−7/5); 2	1/2 (−1); 1	Eqs. (12), (13), (50)
Universal range; halo population	absent; evolving	$k_{\text{NL1}} < k < k_{\text{NL2}}$; steady below M^*	Sections 4 and 5
Unchanged at a_*	$K_P = \varepsilon_u t$; $\rho_h^* m_h^* = \text{const}$; $\varepsilon(m_h^*) = \varepsilon_u$		radial cascade on both sides
<i>Spectrum (three ranges after a_*)</i>			
$k_{\text{NL1}} [M_{\text{NL1}}^*]$	faster than $a^{-3/2}$	$a^{-3/2} [a^{9/2}]$	$\Delta_L^2(k_{\text{NL1}}) = 1$; local index $n_i \rightarrow -5/3$, $\lambda_i = 7/9$
Universal range Δ_X^2	–	$a^3 k^2$	$n_X = -1$, $b = 3$, $\lambda_h = 2/3$; memory erased
$k_{\text{NL2}} [M_{\text{NL2}}^*]$	$> k_{\text{NL1}}$	$a^{-3/2} [a^{9/2}]$ predicted	$n_a = -9/5$, $\lambda_a = 4/5$; measured $\approx a^{-0.9}$
$k_L^* [M_L^*]$	tracks K^* (empirical)	$a^{-1/2} [a^{3/2}]$	threshold $\Delta_L^2(\infty) \propto a^2$ meets Δ_X^2
<i>Characteristic halo m_h^* (cascade clock, $\lambda_h = 2/3$)</i>			
m_h^*	$a^3 \propto t^2$	$a^{3/2} \propto t$	$m_h^* \propto a^{3\theta}$
$\sigma_v^2(m_h^*)$; contrast Δ_h^*	a ; const	$a^{1/2}$; $a^{3/2}$	$\rho_h^* m_h^* = \text{const}$
Waiting time τ_g^*	$a^{-3/2}$	const	$\tau_g^* = m_P / \dot{m}_h^* \propto t / m_h^*$ (Table 2)
r_h^* , R_h^* ; r_s^*	a^2 ; a^2 ; $a^{9/5}$	$a^{3/2}$; a ; $a^{9/10}$	three density laws (Eq. (67))
$c^* = R_h^*/r_s^*$; $-2\gamma_{\text{eff}}(R_h)$	$a^{1/5}$; $-3/2$	$a^{1/10}$; $-3/2$	Eqs. (98), (99); c independent of m_h
<i>Cutoff halo M^* (linear clock)</i>			
$\lambda_2(n)$; M^*	11/15 (−7/5); $a^{15/4\dagger}$	2/3 (−1); a^3	$\langle K_{pv} \rangle = \frac{3}{2} \sigma_v^2(M^*) \propto t$
M^*/m_h^* ; $\varepsilon(M^*)/\varepsilon_u$	$a^{3/4\dagger}$; $a^{1/2\dagger}$	$a^{3/2}$; a	ratio of the two clocks
$r_h(M^*)$; $R_h(M^*)$	$a^{9/4\dagger}$ (equal)	a^2 ; $a^{3/2}$ ($= r_h^*$)	
$r_s(M^*)$; $c(M^*)$	$a^{41/20\dagger}$; $a^{1/5}$	$a^{7/5}$; $a^{1/10}$	
<i>Population and cascades</i>			
M_h ; $\varepsilon_m = -\dot{M}_h$	not steady	$a^{1/2}$; a^{-1}	$M_h \propto D_c^{1/2}$ (Eq. (56))
$\lambda_1(n)$	–	5/6 (−2)	$1 - (1 - \lambda_2) \tau_c / \tau_L$ (Eq. (58))
Propagation; deposition range	absent	$m_h \lesssim 0.2M^*$; 0.2–few M^*	Eq. (61)
Inner density at m_h^*	time-invariant	time-invariant	$\varepsilon(m_h^*) = \varepsilon_u$ (Eq. (101))
<i>Profile of m_h^* haloes: $2\gamma_1 = 4/3$ in $r_{\text{in}} < r < r_s$; (γ_2, α, β) for the three mappings</i>			
Fixed contrast	1/2, 1, 3/2	2/3, 2/3, 1	
Virial	1/2, 1, 3/2	1/2, 1, 3/2	
Cascade law	4/9, 10/9, 5/3	4/9, 10/9, 5/3	
<i>Radial ordering, $a < a_*$: $r_s(M^*) < r_s^* < r_h(M^*) = R_h(M^*) < R_h^* = r_h^*$</i>			
<i>Radial ordering, $a > a_*$: $r_s^* < r_s(M^*) < R_h^* < R_h(M^*) = r_h^* < r_h(M^*)$</i>			