Supplementary Information

1 Supplementary Figures

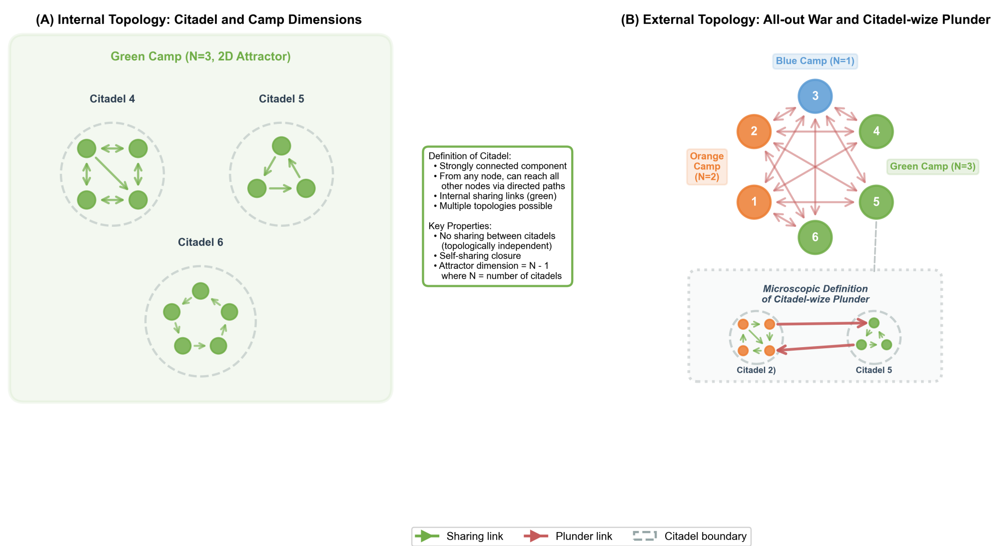

Figure 1: **Topological Design Principles of the "Camp-Citadel" Model. (A) Internal Topology: Citadel and Camp Dimensions.** This panel illustrates how attractor dimension is determined by a camp's internal topology. A "Camp" (e.g., "Green Camp", N=3) is composed of multiple (N) independent "Citadels" (e.g., Citadels 4, 5, 6). A "Citadel" is a "sharing-cohesive unit," with its internal micro-nodes strongly connected by **green "Sharing links"**. Critically, there are no sharing links between citadels, making them "topologically independent." The attractor dimension is determined by $N - 1$ (here, 2D). **(B) External Topology: All-out War and Citadel-wise Plunder.** This panel illustrates how multistability is achieved. The "all-out war" topology is established via **red "Plunder links"**, connecting all citadels that belong to different camps (e.g., Blue N=1, Orange N=2, Green N=3). The magnified "Microscopic Definition" box details the "citadel-wise plunder" rule: the plunder relationship is defined at the micro-node level, where a link from at least one node in Citadel 2 to at least one node in Citadel 5 (and vice-versa) constitutes the macro-level plunder link.

2 Supplementary Note: Mathematical Basis for Abstracting a Citadel as a Macro-Node

2.1 Argument

In our model, a "Citadel" is defined as a **strongly connected graph** of "sharing" links (see Figure S1A). We mathematically demonstrate here that for such a subsystem, when its internal dynamics (sharing) are much faster than its external dynamics (plundering), the resources X_i of all its constituent "micro-nodes" will necessarily converge to a **fixed ratio**.

This property is the mathematical foundation that allows us to abstract a Citadel (composed of many micro-nodes) into a single "macro-node" whose state is described solely by its total resource C_{citadel}, as described in Methods 4.4.

2.2 Definition of the Dynamical Matrix (A)

Consider an isolated Citadel of N micro-nodes with only "sharing" dynamics. From the main text's dynamical equations, the rate of change for node k is:

$$\frac{dX_k}{dt} = \sum_{i \neq k} (S_{ki} X_i - S_{ik} X_k)$$

where $S_{ij} \geq 0$ is the sharing rate from node j to node i. We note that this sharing matrix S **does not need to be symmetric** (i.e., $S_{ki} \neq S_{ik}$ is allowed).

This linear system can be written in matrix form as $\frac{d\mathbf{X}}{dt} = A\mathbf{X}$, where $\mathbf{X} = [X_1, \ldots, X_N]^T$ is the resource vector. The elements of the dynamical matrix A are defined as:

- **Off-diagonal ($k \neq i$):** $A_{ki} = S_{ki}$
- **Diagonal ($k = k$):** $A_{kk} = -\sum_{i \neq k} S_{ik}$ (Note: the sum is over the index i, representing the total *outgoing* sharing rate from node k).

2.3 Proof: Eigenvalue Stability (Re$(\lambda) \leq 0$)

We first prove that all eigenvalues λ of A have non-positive real parts. We use the **Gershgorin Circle Theorem** (applied to columns).

1. **Theorem:** All eigenvalues of A are located in the union of N disks D_k in the complex plane, where k is the column index.

2. **Disk Definition:** The k-th disk D_k is defined by:

   - **Center C_k:** The diagonal element A_{kk}.

   $$C_k = A_{kk} = -\sum_{i \neq k} S_{ik}$$

   - **Radius R_k:** The sum of the absolute values of the off-diagonal elements in column k.

   $$R_k = \sum_{i \neq k} |A_{ik}|$$

3. **Radius Calculation:** By our definition of A, $A_{ik} = S_{ik}$ (for $i \neq k$). Since all sharing rates $S_{ik} \geq 0$, we have:

   $$R_k = \sum_{i \neq k} S_{ik}$$

4. **Compare Center and Radius:** We find that for any disk D_k, the center C_k is exactly the negative of the radius R_k:

   $$C_k = -\sum_{i \neq k} S_{ik} = -R_k$$

5. **Conclusion:** The k-th Gershgorin disk D_k is centered at $-R_k$ (a non-positive real number) and has radius R_k. Any point z on this disk satisfies $|z - (-R_k)| \leq R_k$. The rightmost point of this disk on the real axis is $z = C_k + R_k = (-R_k) + R_k = 0$. This means *every* disk D_k is fully contained in the left half of the complex plane (including the imaginary axis). Since all

eigenvalues λ must lie in the union of these disks, all eigenvalues must satisfy $\text{Re}(\lambda) \leq 0$. This proves the sharing subsystem is stable and does not spontaneously diverge.

2.4 Proof: Simplicity of the Zero Eigenvalue

1. From the calculation in step 3.4, we showed that for every column k, $A_{kk} = -\sum_{i \neq k} S_{ik}$ and $R_k = \sum_{i \neq k} S_{ik}$. This means the sum of all elements in **column** k is zero:

$$\sum_{i=1}^{N} A_{ik} = A_{kk} + \sum_{i \neq k} A_{ik} = \left(-\sum_{i \neq k} S_{ik} \right) + \left(\sum_{i \neq k} S_{ik} \right) = 0$$

2. A matrix where every column sums to zero is equivalent to $\mathbf{1}^T A = \mathbf{0}^T$, where $\mathbf{1}^T = [1, 1, \ldots, 1]$.

3. This proves that $\mathbf{1}^T$ is a **left eigenvector** of A with a corresponding eigenvalue of $\lambda = 0$. Thus, $\lambda_1 = 0$ must be an eigenvalue.

4. By the Perron-Frobenius theorem (as applied to graph Laplacians), for a **strongly connected** directed graph (our definition of a Citadel), the zero eigenvalue of its Laplacian (which A represents) is **simple** (i.e., non-degenerate, with a multiplicity of 1).

5. Combining this with the stability proof, we conclude that the system has exactly one eigenvalue $\lambda_1 = 0$, and all other $N - 1$ eigenvalues λ_i ($i \geq 2$) have strictly negative real parts ($\text{Re}(\lambda_i) < 0$).

2.5 Proof: Convergence to a Fixed Ratio

1. The general solution to the linear system $\frac{d\mathbf{X}}{dt} = A\mathbf{X}$ is a linear combination of its right eigenvectors $\mathbf{v}_i$:

$$\mathbf{X}(t) = c_1 e^{\lambda_1 t} \mathbf{v}_1 + \sum_{i=2}^{N} c_i e^{\lambda_i t} \mathbf{v}_i$$

2. Substituting $\lambda_1 = 0$ and letting $\mathbf{v}_1 = \mathbf{v}_{ss}$ (the steady-state eigenvector, which is strictly positive by Perron-Frobenius):

$$\mathbf{X}(t) = c_1\mathbf{v}_{ss} + \sum_{i=2}^{N} c_i e^{\lambda_i t}\mathbf{v}_i$$

3. As $t \to \infty$, all terms in the summation decay to zero, since $\text{Re}(\lambda_i) < 0$ for all $i \geq 2$.

4. Therefore, the system's final steady state $\mathbf{X}_{ss}$ must converge to:

$$\mathbf{X}_{ss} = \lim_{t\to\infty} \mathbf{X}(t) = c_1\mathbf{v}_{ss}$$

   This proves the final state is proportional to the single, unique steady-state eigenvector $\mathbf{v}_{ss}$.

2.6 Solving for the Proportionality Coefficient (c_1)

1. We use the system's conservation law to find c_1. The total resource $C_{\text{total}} = \sum_k X_k = \mathbf{1}^T\mathbf{X}$ is conserved, because (as shown in 4.2) $\frac{d}{dt}(\mathbf{1}^T\mathbf{X}) = (\mathbf{1}^T A)\mathbf{X} = \mathbf{0}^T\mathbf{X} = 0$.

2. We start with the initial condition: $\mathbf{X}(0) = c_1\mathbf{v}_{ss} + \sum_{i=2}^{N} c_i\mathbf{v}_i$.

3. We left-multiply by the left eigenvector for $\lambda_1 = 0$, which is $\mathbf{u}_1^T = \mathbf{1}^T$:

$$\mathbf{1}^T\mathbf{X}(0) = \mathbf{1}^T\left(c_1\mathbf{v}_{ss} + \sum_{i=2}^{N} c_i\mathbf{v}_i\right) = c_1(\mathbf{1}^T\mathbf{v}_{ss}) + \sum_{i=2}^{N} c_i(\mathbf{1}^T\mathbf{v}_i)$$

4. By the biorthogonality of left and right eigenvectors, $\mathbf{u}_1^T\mathbf{v}_i = \mathbf{1}^T\mathbf{v}_i = 0$ for all $i \geq 2$. The summation term vanishes.

5. We are left with: $C_{\text{total}} = c_1(\mathbf{1}^T\mathbf{v}_{ss})$.

6. Solving for c_1:

$$c_1 = \frac{C_{\text{total}}}{\mathbf{1}^T\mathbf{v}_{ss}} = \frac{C_{\text{total}}}{\sum_j v_j}$$

where $\sum_j v_j$ is the sum of the elements of the steady-state eigenvector (a fixed positive scalar).

2.7 Conclusion

Substituting c_1 back into the steady-state equation $\mathbf{X}_{ss} = c_1 \mathbf{v}_{ss}$, we get:

$$\mathbf{X}_{ss} = \left(\frac{C_{\text{total}}}{\sum_j v_j} \right) \mathbf{v}_{ss}$$

This mathematically proves that regardless of the initial resource distribution $\mathbf{X}(0)$, as long as the total resource is C_{total}, the system will always converge to the exact same steady state $\mathbf{X}_{ss}$. In this state, the resource of each micro-node k, X_k, is proportional to v_k, and the ratio between any two nodes $\frac{X_i}{X_j} = \frac{v_i}{v_j}$ is a fixed constant.

This demonstrates that all micro-nodes are dynamically "locked" together as a "sharing-cohesive unit," fully justifying our abstraction of the entire Citadel as a single macro-node.