A universal Fano-type scattering model and its experimental tests

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I. EQUIVALENT CIRCUIT MODEL OF A COPLANAR WAVEGUIDE HALF WAVELENGTH RESONATOR

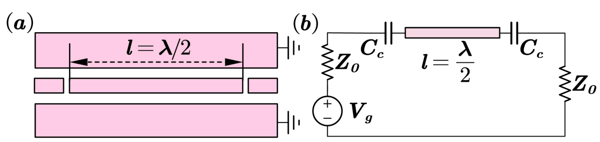

FIG. S1. (a) A typical $\lambda/2$ CPW distributed resonator, and (b) its input-output two-port equivalent circuit.

This section reviews how the $\lambda/2$ CPW resonator, as shown in Fig. S1(a), which is coupled to an external circuit of characteristic impedance Z_0 through a gap capacitor C_c (as shown in Fig. S1(b)), can be equivalent to a lumped-element resonator by a series circuit equivalences shown schematically in Fig. S2(a), and finally the S_{21} of a $\lambda/2$ resonator can be expressed effectively as:

$$S_{21,e} = \frac{V_2^-}{V_1^+}\bigg|_{V_2^+=0} = \frac{2V_2}{V_g}. \tag{S1}$$

Here, V_g is the voltage of the voltage source, V_1^+ and V_2^- are the input- and output voltages of the resonator,

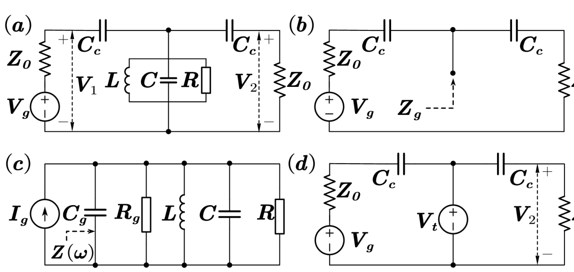

FIG. S2. A series of equivalent circuit models of a $\lambda/2$ CPW resonator for getting V_2; (a) The original parallel-RLC equivalent circuit of a $\lambda/2$ CPW resonator; (b) A equivalent circuit used to compute the Norton-equivalent source impedance Z_g; (c) The Norton-equivalent circuit to get the equivalent current source I_g; (d) The Norton-equivalent circuit to get the equivalent output voltage V_2, wherein the parallel RLC network is replaced by an ideal voltage source V_t.

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respectively. V_2 is the effective outport voltage of the resonator which is expected to be obtained.

According to Norton's theorem, any linear active two-port network can be equivalently represented by a current source model. Therefore, the circuit in Fig. S2(a) can be transformed into the current source model shown in Fig. S2(c). Using the equivalent circuit in Fig. S2(b), the Norton equivalent impedance can be calculated as:

$$Z_g = \frac{1 + Z_0 j\omega C_c}{2j\omega C_c}, \tag{S2}$$

where Z_0 is the characteristic impedance of the transmission line, and C_c is the coupling capacitance between the resonator and the transmission line. For the weak limit, the admittance of this equivalent impedance satisfies the condition: $Z_0\omega C_c \ll 1$, which yields:

$$Y_g = \frac{2j\omega C_c}{1 + Z_0 j\omega C_c} \approx 2j\omega C_c + \frac{1}{R_g(\omega)}, \tag{S3}$$

with

$$R_g(\omega) = \frac{1}{2Z_0(\omega C_c)^2}. \tag{S4}$$

From Fig. S2(c), we have $C_g = 2C_c$. Therefore, applying Norton's theorem once again, the source current of the equivalent current source in Fig. S2(c) can be computed as:

$$I_g = \frac{V_g}{Z_0 + 1/j\omega C_c} = \frac{V_g}{2Z_g}. \tag{S5}$$

Accordingly, the equivalent circuit in Fig. S2(a) can be further transformed into the Norton equivalent circuit shown in Fig. S2(c). On the basis of the above circuit equivalence, the quality factors of the can be represented as:

$$\begin{cases} Q_l = \omega_r R_{\|}(C + C_g) \approx \omega_r R_{\|} C, \\ Q_i = \omega_r R(C + C_g) \approx \omega_r RC, \\ Q_c = \omega_r R_g(\omega)(C + C_g) \approx \omega_r R_g(\omega) C, \end{cases} \tag{S6}$$

where $\omega_r^2 = 1/L(C + C_g)$ is the resonant angular frequency of the parallel RLC circuit, $R_{\|} = R\|R_g = 1/(1/R + 1/R_g)$ denotes the equivalent parallel resistance. Clearly, when the input signal frequency is around ω_r, i.e., $\omega = \omega_r + \delta\omega$ with $\delta\omega$ being a small detuning, we have:

$$\omega^2 - \omega_r^2 = 2\omega_r\delta\omega + \delta\omega^2 \approx 2\omega_r\delta\omega. \tag{S7}$$

Using Eq. (S6), the total impedance of the circuit shown in Fig. S2(c) can be obtained as:

$$Z(\omega) = \frac{R_{||}}{1 + 2jQ_l \frac{\omega-\omega_r}{\omega_r}}. \tag{S8}$$

Consequently, as shown in Fig. S2(d), the equivalent parallel RLC resonant circuit can be represented by an ideal voltage source V_t. Moreover, based on the circuit shown in Fig. S2(c), we obtain:

$$V_t = I_g Z(\omega) = \frac{V_g}{2Z_g} \frac{R_{||}}{1 + 2jQ_l \frac{\omega-\omega_r}{\omega_r}}. \tag{S9}$$

Consequently, from Fig. S2(d), we get:

$$V_2 = V_t \frac{Z_0}{Z_0 + 1/j\omega C_c}. \tag{S10}$$

On the other hand, with Eq. (S9) we get:

$$V_g = V_t \frac{Z_0 + 1/j\omega C_c}{Z(\omega)}. \tag{S11}$$

Accordingly, by invoking Eqs. (S1-S2), (S4), (S6), and (S8-S11), we obtain:

$$S_{21,e} = \frac{2V_2}{V_g} = \frac{K(\omega)}{1 + 2jQ_l \frac{\omega-\omega_r}{\omega_r}}, \tag{S12}$$

where

$$K(\omega) = \frac{2Z_0 R_{||}}{(Z_0 + 1/j\omega C_c)^2} = \frac{\frac{R}{R+R_g} \frac{1}{(\omega C_c)^2}}{Z_0^2 + \frac{2Z_0}{j\omega C_c} - \frac{1}{(\omega C_c)^2}}. \tag{S13}$$

In the weak-coupling regime, wherein $Z_0 \omega C_c \ll 1$ and $\omega = \omega_r + \delta\omega$, Eq. (S13) can be simplified as:

$$K(\omega) \approx -\frac{R}{R+R_g} = -\frac{Q_l}{Q_c}. \tag{S14}$$

Consequently,

$$S_{21,e} = -\frac{Q_l}{Q_c} \frac{1}{1 + 2jQ_l(\omega - \omega_r)/\omega_r}. \tag{S15}$$

This is just Eq. (1) in the main text. Basically, any lumped-element model works practically under the EDA and thus cannot faithfully capture the electromagnetic scattering of the distributed resonator. Therefore, when Eq. (1) is used to fit the experimentally observed scattering spectrum of a distributed resonator, an additional complex function term, i.e., S_{leak} in Eq. (2), is required to be artificially added.

II. TEMPORAL COUPLED MODE THEORY FOR THE SCATTERINGS OF THE LOADED DISTRIBUTED RESONATORS WITH INTRINSIC LOSSES

Ref. [1] presents a method i.e., the TCMT to deliver the electromagnetic scattering parameters of the distributed resonators without any intrinsic loss (i.e., in the limit $Q_i \to \infty$). While any practical resonator must exhibit the intrinsic loss. Thus, it is necessary to generalize the standard TCMT proposed in Ref. [1] for the case with the intrinsic loss. In this section, we provide such a generalization.

Let us reconsider the two-port inputoutput electromagnetic scattering model of a single-mode distributed resonator, illustrated typically in Fig. 1(b). The existing direct coupling loss between these two ports, due to the non-EDA interaction between the resonator and the scattered electromagnetic wave, can be characterized by the time constant $1/\tau_c$, while the intrinsic dissipation of the resonator is represented by $1/\tau_i$. Following Ref. [1], the dynamical equation for the amplitude of the resonant-mode in the resonator can be written as:

$$\frac{da}{dt} = \left(j\omega_0 - \frac{1}{\tau_c} - \frac{1}{\tau_i} \right) a + \left(\langle \kappa |^* \right) |s_+\rangle . \tag{S16}$$

Here, $|s_+\rangle = (s_{1+}, s_{2+})^T$ with $s_{1+,2+}$ being the amplitudes injected into the cavity from ports 1 and 2, respectively. The vector $\langle \kappa|^* = (\kappa_1, \kappa_2)$ collects the coupling coefficients between the resonator and ports 1 and 2. Accordingly, the outgoing field amplitude of the resonator can be expressed as:

$$|s_-\rangle = C\,|s_+\rangle + a|d\rangle, \tag{S17}$$

where s_{1-} and s_{2-} denote the outgoing field amplitudes at ports 1 and 2, respectively; ω_0 is the eigenmode of the resonator; and $|d\rangle = (d_1, d_2)^T$ collects the coefficients that map the cavity mode into the outgoing waves at ports 1 and 2, respectively. Eq. (S17) indicates that scattering by the resonator proceeds via two pathways: one is the resonant pathway via exciting the cavity mode and subsequently leaks out, and another one is the quasi direct inputoutput pathway described by the scattering matrix C, due to the non-EDA interaction between the scattered wave and the distributed resonator C must be unitary and symmetric. Finally, the total scattered field at the outport is superposed by the output fields through these two pathways.

Under continuous excitation by a monochromatic wave at angular frequency ω, the cavity mode in the resonator reaches a steady state, i.e., $a(t) = ae^{j\omega t}$. Setting $da/dt = 0$ in Eq. (S16) we get the steady-state solution:

$$a = \frac{(\langle \kappa\ |^*)\ |\ s_+\rangle}{j\,(\omega - \omega_0) + 1/\tau_c + 1/\tau_i}. \tag{S18}$$

for the above dynamical equation (S16). Substituting it into Eq. (S17), we obtain:

$$|s_-\rangle \equiv S\,|s_+\rangle = \left[C + \frac{|d\rangle\langle\kappa|^*}{j\,(\omega-\omega_0)+1/\tau_c+1/\tau_i}\right]|s_+\rangle\,, \tag{S19}$$

with

$$\langle\kappa|^*|s_+\rangle|d\rangle = (\kappa_1,\kappa_2)\begin{pmatrix} s_{1+}\\ s_{2+}\end{pmatrix}\begin{pmatrix} d_1\\ d_2\end{pmatrix} = |d\rangle\langle\kappa|^*|s_+\rangle. \tag{S20}$$

Since the scattering matrix S must be time-reversal symmetric, we have:

$$|d\rangle\langle\kappa|^* = |\kappa\rangle\langle d|^*. \tag{S21}$$

Accordingly, $|\kappa\rangle$ and $|d\rangle$ are not independent; they must satisfy the relation $d_1\kappa_2 = d_2\kappa_1$.

Clearly, in the absence of external excitation (i.e., $|s_+\rangle = 0$), energy conservation requires that the temporal change of the stored energy equals the power dissipated through the ports $|d\rangle$ and through the intrinsic loss channel $|\ell\rangle$; hence:

$$\begin{cases} \frac{da}{dt} = \left(j\omega_0 - \frac{1}{\tau_c} - \frac{1}{\tau_i}\right)a, \\ |s_-\rangle = \left|s_-^c\right\rangle + \left|s_-^i\right\rangle = a|d\rangle + a|\ell\rangle. \end{cases} \tag{S22}$$

In this case, the temporal evolution of the resonant-mode energy is given by:

$$\begin{aligned} \frac{d}{dt}|a|^2 &= \frac{d}{dt}(aa^*) = a\frac{da^*}{dt} + a^*\frac{da}{dt} \\ &= a\left(-j\omega_0 a^* - \frac{a^*}{\tau_c} - \frac{a^*}{\tau_i}\right) + a^*\left(j\omega_0 a - \frac{a}{\tau_c} - \frac{a}{\tau_i}\right) \\ &= -\frac{2}{\tau_c}|a|^2 - \frac{2}{\tau_i}|a|^2. \end{aligned} \tag{S23}$$

In the above expression, $-2|a|^2/\tau_c$ denotes the rate at which energy is coupled into the port channel $|s_-^c\rangle$ and $-2|a|^2/\tau_i$ denotes the rate of energy dissipated from the resonator into the intrinsic-loss channel $|s_-^i\rangle$, due to the radiation absorptions in the conductor and dielectric. This implies that,

$$-\frac{2}{\tau_c}|a|^2 = -\left\langle s_-^c \mid s_-^c\right\rangle = -|a|^2\langle d \mid d\rangle, \tag{S24}$$

and

$$-\frac{2}{\tau_i}|a|^2 = -\left\langle s_-^i \mid s_-^i\right\rangle = -|a|^2\langle \ell \mid \ell\rangle, \tag{S25}$$

thereby

$$\langle d \mid d\rangle = \frac{2}{\tau_c}, \langle\ell|\ell\rangle = \frac{2}{\tau_i}. \tag{S26}$$

Next, we apply a time-reversal transformation to the exponentially decaying dissipative process described above. Suppose an exponentially growing electromagnetic wave, with the complex frequency $\omega = \omega_0 - j\,(1/\tau_c + 1/\tau_i)$, is injected into the resonator whose amplitude at t=0 equals $|s_-^c\rangle^* + \left|s_-^i\right\rangle^*$, we obtain:

$$\left|s_-^c\right\rangle^* = a^*|d\rangle^*, \left|s_-^i\right\rangle^* = a^*|\ell\rangle^*. \tag{S27}$$

Then, invoking Eq. (S20), we get:

$$a^* = \frac{\left(\langle\kappa|s_-^c\rangle + \langle\kappa|s_-^{\mathrm{i}}\rangle\right)^*}{2\left(\frac{1}{\tau_c}+\frac{1}{\tau_i}\right)} = \frac{\left(\langle\kappa|d\rangle + \langle\kappa|\ell\rangle\right)a^*}{2\left(\frac{1}{\tau_c}+\frac{1}{\tau_i}\right)}, \tag{S28}$$

thereby:

$$\langle\kappa|d\rangle + \langle\kappa|\ell\rangle = 2\left(\frac{1}{\tau_c}+\frac{1}{\tau_i}\right). \tag{S29}$$

Using the relationship: $\langle\kappa \mid d\rangle = 2/\tau_c = (\langle\kappa \mid d\rangle)^*$,we have:

$$|\kappa\rangle = |d\rangle. \tag{S30}$$

Accordingly, the scattering matrix in Eq. (S19) can be rewritten as:

$$S = C + \frac{|d\rangle\langle d|^*}{j\,(\omega-\omega_0)+1/\tau_c+1/\tau_i}. \tag{S31}$$

Moreover, the time-reversed excitation $\left|s_-^c\right\rangle^*$ must satisfy the physical condition that no outgoing waves are produced under this excitation; therefore,

$$0 = C\left|s_-^c\right\rangle^* + a^*\,|d\rangle = a^*C\,|d\rangle^* + a^*\,|d\rangle\,. \tag{S32}$$

This implies that the coupling vector $|d\rangle$ must also satisfy the following condition:

$$C|d\rangle^* = -|d\rangle. \tag{S33}$$

This indicates that the coupling constants d_1 and d_2 cannot be chosen arbitrarily; rather, they are constrained by the form of the scattering matrix C associated with the direct coupling pathway. Specifically, for a resonator structure with mirror symmetry, one may place the reference planes symmetrically on either side of the structure with respect to the mirror plane, thereby making the two diagonal elements of the scattering matrix equal. In this case, the direct inputoutput coupling can be described by the following scattering matrix:

$$C = e^{j\varphi}\begin{bmatrix} r & jt\\ jt & r\end{bmatrix}, \tag{S34}$$

where t and r are the transmission and reflection coefficients of the direct coupling path, satisfying $|t|^2+|r|^2 = 1$; $e^{j\varphi}$ denotes the overall phase delay associated with the direct inputoutput pathway. Using Eq. (S33), we obtain:

$$\begin{cases} r^2|d_1|^2 + t^2|d_2|^2 = |d_1|^2, \\ t^2|d_1|^2 + r^2|d_2|^2 = |d_2|^2. \end{cases} \tag{S35}$$

which means $|d_2|^2 = |d_1|^2$ and thus $d_1 = \pm d_2$. Specifically, $d_1 = d_2$ indicates that no phase shift exists between the input and output fields in the resonant pathway; whereas $d_1 = -d_2$ signifies a π phase shift between the output and input fields which is consistent with most experimental phase measurements near resonance. Using Eq. (S26), we thus obtain $|d_2|^2 = |d_1|^2 = 1/\tau_c$. Hence, if $d_1 = d_2$, then $|d\rangle = \sqrt{1/\tau_c}[1,1]^T$; while, if $d_1 = d_2$, then $|d\rangle = \sqrt{1/\tau_c}[1,-1]^T$. Since $C|d\rangle^* = -|d\rangle$ and thus $d_1 = d_2$, we have:

$$e^{j\phi}(r + jt) = -1. \tag{S36}$$

On the contrary, when $d_1 = -d_2$, we get

$$e^{j\phi}(r - jt) = -1. \tag{S37}$$

Therefore, for $d_1 = d_2$ the scattering matrix S can be written as:

$$\begin{aligned} S &= C + \frac{|d\rangle\langle d|^*}{j\left(\omega-\omega_0\right)+1/\tau_c+1/\tau_i} \\ &= e^{j\varphi}\begin{bmatrix} r & jt \\ jt & r \end{bmatrix} + \frac{1/\tau_c}{j\left(\omega-\omega_0\right)+1/\tau_c+1/\tau_i}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\ &= e^{j\varphi}\left\{\begin{bmatrix} r & jt \\ jt & r \end{bmatrix}\right. \\ &\left. + \frac{1/\tau_c}{j\left(\omega-\omega_0\right)+1/\tau+1/\tau_i}\begin{bmatrix} -(r+jt) & -(r+jt) \\ -(r+jt) & -(r+jt) \end{bmatrix}\right\}. \end{aligned} \tag{S38}$$

While, for $d_1 = -d_2$, the scattering matrix S alternatively is:

$$\begin{aligned} S &= C + \frac{|d\rangle\langle d|^*}{j\left(\omega-\omega_0\right)+1/\tau_c+1/\tau_i} \\ &= e^{j\varphi}\begin{bmatrix} r & jt \\ jt & r \end{bmatrix} + \frac{1/\tau}{j\left(\omega-\omega_0\right)+1/\tau_c+1/\tau_i}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \\ &= e^{j\varphi}\left\{\begin{bmatrix} r & jt \\ jt & r \end{bmatrix}\right. \\ &\left. + \frac{1/\tau}{j\left(\omega-\omega_0\right)+1/\tau_c+1/\tau_i}\begin{bmatrix} -(r-jt) & r-jt \\ r-jt & -(r-jt) \end{bmatrix}\right\}. \end{aligned} \tag{S39}$$

Finally, Eqs. (S38) and (S39) can be unified as:

$$\begin{aligned} S =& e^{j\varphi}\left\{\begin{bmatrix} r & jt \\ jt & r \end{bmatrix}\right. \\ &\left. + \frac{1/\tau_c}{j\left(\omega-\omega_0\right)+1/\tau_c+1/\tau_i}\begin{bmatrix} -(r\pm jt) & \mp(r\pm jt) \\ \mp(r\pm jt) & -(r\pm jt) \end{bmatrix}\right\} \\ &= \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}. \end{aligned} \tag{S40}$$

with

$$S_{11} = S_{22} = e^{i\varphi}\left\{r - (r\pm jt)\frac{Q_l/Q_c}{2jQ_l(\omega-\omega_0)/\omega_0+1}\right\}, \tag{S41}$$

being the reflection coefficient, and

$$S_{12} = S_{21} = e^{i\varphi}\left\{jt \mp (r\pm jt)\frac{Q_l/Q_c}{2jQ_l(\omega-\omega_0)/\omega_0+1}\right\}. \tag{S42}$$

being the transmission coefficient. This is just Eq. (3) in the main text, where "-" ("+") in the first $\mp$ corresponds to $d_1 = d_2$ ($d_1 = -d_2$). Above, $Q_{i,c} = \omega_0\tau_{i,c}/2$ and $Q_l^{-1} = Q_i^{-1} + Q_c^{-1}$.

[1] S. Fan, W. Suh, and J. D. Joannopoulos, Temporal coupled-mode theory for the fano resonance in optical resonators, J. Opt. Soc. Am. A **20**, 569 (2003).