Supplementary Material: Spline Interpolation on Compact Riemannian Manifolds

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Abstract

This document presents the computation of the mass and stiffness matrices for **2**-manifolds in $\mathbb{R}^3$, using natural local coordinate charts such as cylindrical or spherical coordinates.

1 Introduction and notations

Here, we aim to detail the computation of

$$M_{ii} = \langle \psi_i, 1 \rangle_{L^2(\mathcal{M})}, \qquad F_{ij} = \langle \nabla\psi_i, \nabla\psi_j \rangle_{L^2(\mathcal{M})},$$

which are the entries of the mass matrix (with mass lumping) and the stiffness matrix, respectively. Here, we focus on the case of 2-dimensional manifolds embedded in $\mathbb{R}^3$, which is the situation detailed in the article. Anisotropies are taken into account in the local coordinates of the manifold $(\mathcal{M}, g)$, for instance using cylindrical coordinates (θ, z) for a cylinder, or spherical coordinates (θ, ϕ) for a sphere.

The manifold is triangulated, and for each triangle we introduce the following associated shapes, illustrated in Figure 1:

- the curved triangle $\mathcal{T}$ on the manifold,
- the polyhedral approximation T, which is a flat triangle embedded in $\mathbb{R}^3$,
- the reference triangle $T_0 \subset \mathbb{R}^2$, whose corner points are $(1, 0)$, $(0, 1)$, and $(0, 0)$.

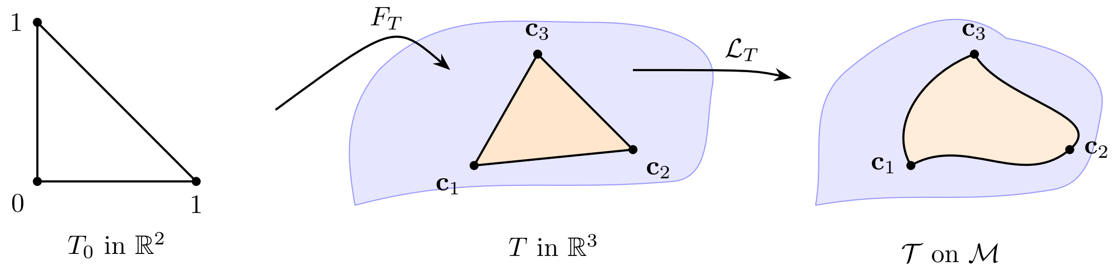

Fig. 1 The three representations of a triangle: the reference triangle T_0, the flat triangle $T \subset \mathbb{R}^3$, and the curved triangle $\mathcal{T} \subset \mathcal{M}$. The mappings F_T and $\mathcal{L}_T$ connect them.

These shapes are connected by the following functions

- The function $F_T : T_0 \to T$, which maps a point with barycentric coordinates in T_0 to the corresponding point in the flat triangle T.
- The function $L_T : T \to \mathcal{T}$, which maps a point in the flat triangle T to the corresponding point in the curved triangle $\mathcal{T} \subset \mathcal{M}$.

Let $\mathscr{T}$ denote the set of all triangles $\mathcal{T}$. For a node $\mathbf{c}_j$, we define

$$\mathscr{T}^{(j)} = \{\mathcal{T} \in \mathscr{T} : \mathbf{c}_j \text{ is one of the vertices of } \mathcal{T}\}.$$

Consider a triangle $\mathcal{T}$ with vertices $\mathbf{c}_{j_1}, \mathbf{c}_{j_2}, \mathbf{c}_{j_3}$, where $(j_1, j_2, j_3) \in \{1, \dots, m\}^3$. We denote by $1 \leq k_j \leq 3$ the index such that $j = j_{k_j}$. For instance, if $j_1 = 12$, then $k_{12} = 1$.

With these notations, the affine mapping from the reference triangle T_0 to $\mathcal{T}$ is

$$F_T(y_1, y_2) = \mathbf{c}_{j_3} + M_T \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \qquad M_T = (\mathbf{c}_{j_1} - \mathbf{c}_{j_3},\ \mathbf{c}_{j_2} - \mathbf{c}_{j_3}) \in \mathbb{R}^{3x2} \tag{1}$$

Moreover, for $j \in \{j_1, j_2, j_3\}$, the restriction of the basis function ψ_j to $\mathcal{T}$ is

$$\psi_{j|\mathcal{T}} = p_0^{(k_j)} \circ F_T^{-1} \circ L_T^{-1}, \tag{2}$$

where $p_0^{(k)}$ is the standard basis polynomial on T_0 which takes value 1 at vertex k:

$$p_0^{(k)}(y_1, y_2) = \begin{cases} y_k, & \text{if } k \in \{1, 2\}, \\ 1 - y_1 - y_2, & \text{if } k = 3. \end{cases}$$

2 Computation of the integrals

For a given curved triangle $\mathcal{T}$, we define $x_{\mathcal{T}} = F_T^{-1} \circ L_T^{-1}$ that associates a point in $\mathcal{T}$ to the barycentric coordinates in T_0. Then, $(\mathcal{T}, x_{\mathcal{T}})$ defines a coordinates chart.

The integral of a function f over $\mathcal{M}$ can be split into a sum of integrals over each triangle $\mathcal{T}$:

$$\int_{\mathcal{M}} f d\mu_g = \sum_{\mathcal{T}\in\mathscr{T}} \int_{\mathcal{T}} f d\mu_g = \sum_{\mathcal{T}\in\mathscr{T}} \int_{T_0} f \circ L_T \circ F_T(\mathbf{y}) |\mathbf{G}^{x\mathcal{T}} \left(L_T \circ F_T(\mathbf{y})\right)|^{\frac{1}{2}} d\mathbf{y},$$

where $\mathbf{G}^{x\mathcal{T}}(\cdot)$ is the matrix tensor at a given point of $\mathcal{M}$ expressed in the coordinate chart $(\mathcal{T}, x)$.

In practice, the matrix tensor $\mathbf{G}^{x\mathcal{T}}$ is assumed to be constant across each triangle:

$$\forall \mathcal{T} \in \mathscr{T}, \forall \mathbf{s} \in \mathcal{T}, \mathbf{G}^{x\mathcal{T}}(\mathbf{s}) = \mathbf{G}_{\mathcal{T}}.$$

2.1 Computation of M_{ii}

$$\begin{aligned}
M_{ii} &= \sum_{\mathcal{T}\in\mathscr{T}^{(i)}} \int_{T_0} \psi_i \circ L_T \circ F_T(\mathbf{y}) |\mathbf{G}^{x\mathcal{T}} \left(L_T \circ F_T(\mathbf{y})\right)|^{\frac{1}{2}} d\mathbf{y} \\
&= \sum_{\mathcal{T}\in\mathscr{T}^{(i)}} \int_{T_0} p_0^{(k_i)} |\mathbf{G}^{x\mathcal{T}} \left(L_T \circ F_T(\mathbf{y})\right)|^{\frac{1}{2}} d\mathbf{y} \\
&= \sum_{\mathcal{T}\in\mathscr{T}^{(i)}} |\mathbf{G}_{\mathcal{T}}|^{\frac{1}{2}} \int_{T_0} p_0^{(k_j)} d\mathbf{y} \\
&= \sum_{\mathcal{T}\in\mathscr{T}^{(i)}} \frac{|\mathbf{G}_{\mathcal{T}}|^{\frac{1}{2}}}{6}
\end{aligned}$$

2.2 Computation of F_{ij}

$$F_{ij} = \sum_{\mathcal{T}\in\mathscr{T}^{(i)}} \int_{\mathcal{T}} \Big(\nabla_{x\mathcal{T}} \psi_i(\mathbf{s})\Big)^{\top} \left[\mathbf{G}^{x\mathcal{T}}(\mathbf{s})\right]^{-1} \nabla_{x\mathcal{T}} \psi_j(\mathbf{s})\, d\mu_g,$$

where the gradient with respect to the coordinates $(\mathcal{T}, x_{\mathcal{T}})$ is given by

$$\nabla_{x\mathcal{T}} \psi_i(\mathbf{s}) = \boldsymbol{\omega}_{k_i},$$

with $\boldsymbol{\omega}_k$ the k-th canonical vector of $\mathbb{R}^2$ for $k \in \{1, 2\}$, and $\boldsymbol{\omega}_3 = -\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2$. Then,

$$\begin{aligned}
F_{ij} &= \sum_{\mathcal{T}\in\mathscr{T}^{(i)}} \int_{\mathcal{T}} \boldsymbol{\omega}_{k_i}^{\top} \left[\mathbf{G}^{x\mathcal{T}}(\mathbf{s})\right]^{-1} \boldsymbol{\omega}_{k_j} d\mu_g \\
&= \sum_{\mathcal{T}\in\mathscr{T}^{(i)}} \boldsymbol{\omega}_{k_i}^{\top} \left[\mathbf{G}_{\mathcal{T}}\right]^{-1} \boldsymbol{\omega}_{k_j} \int_{T_0} |\mathbf{G}^{x\mathcal{T}} \left(L_T \circ F_T(\mathbf{y})\right)|^{\frac{1}{2}} d\mathbf{y} \\
&= \sum_{\mathcal{T}\in\mathscr{T}^{(i)}} \frac{1}{2} \boldsymbol{\omega}_{k_i}^{\top} \left[\mathbf{G}_{\mathcal{T}}\right]^{-1} \boldsymbol{\omega}_{k_j} |\mathbf{G}^{\mathcal{T}}|^{\frac{1}{2}}
\end{aligned}$$

3 Matrix tensor defined in the natural local coordinates charts

The objective here is to define $\mathbf{G}^{x_{\mathcal{T}}}(\mathbf{s})$ from a representative matrix $\mathbf{G}^y$ expressed in another coordinate chart $(\mathcal{T}, y)$.

Here, y denotes spherical coordinates in the case of the sphere, or cylindrical coordinates when studying a cylinder, thus providing interpretable representative matrices $\mathbf{G}^y$.

More generally, we consider y of the form

$$y = \Psi \circ \Phi,$$

where Φ is the natural injection into $\mathbb{R}^3$, and $\Psi : \mathbb{R}^3 \to \mathbb{R}^2$ represents the change of coordinates (for instance, from Cartesian coordinates in $\mathbb{R}^3$ to spherical coordinates).

Following [1], we have

$$\mathbf{G}^{x_{\mathcal{T}}}(\mathbf{s}) = J_{y \circ x_{\mathcal{T}}^{-1}}\big(x(\mathbf{s})\big)^\top \, \mathbf{G}^y(\mathbf{s}) \, J_{y \circ x_{\mathcal{T}}^{-1}}\big(x(\mathbf{s})\big),$$

where $J_{y \circ x_{\mathcal{T}}^{-1}}\big(x(\mathbf{s})\big)$ denotes the Jacobian matrix of $y \circ x_{\mathcal{T}}^{-1}$ evaluated at $x(\mathbf{s})$.

We have

$$\begin{aligned} J_{y \circ x_{\mathcal{T}}^{-1}}\big(x(\mathbf{s})\big) &= J_{\Psi \circ \Phi \circ L_T \circ F_T}\left(x(\mathbf{s})\right) \\ &= J_\Psi\left(\Phi \circ L_T \circ F_T(x(\mathbf{s}))\right) J_{\Phi \circ L_T}\left(F_T(x(\mathbf{s}))\right) J_{F_T}(x(\mathbf{s})) \\ &= J_\Psi\left(\Phi(\mathbf{s})\right) J_{\Phi \circ L_T}\left(F_T(x(\mathbf{s}))\right) J_{F_T}\left(x(\mathbf{s})\right). \end{aligned}$$

For sufficiently fine triangulations, we can approximate $\Phi \circ L_T$ by the identity map, i.e. $\Phi \circ L_T \approx \mathrm{Id}$. Then in the following we consider

$$J_{y \circ x_{\mathcal{T}}^{-1}}\big(x(\mathbf{s})\big) = J_\Psi\left(\Phi(\mathbf{s})\right) J_{F_T}\left(x(\mathbf{s})\right).$$

Then, we obtain

$$\mathbf{G}_{\mathcal{T}} = \mathbf{M}_T^\top J_\Psi\left(T\right)^\top \mathbf{G}_{\mathcal{T}}^y J_\Psi\left(T\right) \mathbf{M}_T \tag{3}$$

with $\mathbf{G}_{\mathcal{T}}^y$ the constant approximation of $\mathbf{G}^y(\mathbf{s})$ on T, $J_\Psi(T)$ the Jacobian matrix of Ψ evaluated at the barycenter of T, and $\mathbf{M}_T$ is the Jacobian of F_T (see Equation 1).

Typically, the matrix $\mathbf{G}_{\mathcal{T}}^y$ can be represented as a local deformation of the manifold in the chosen coordinate chart (spherical or cylindrical) by combining a rotation and a diagonal scaling. More precisely,

$$\mathbf{G}_{\mathcal{T}}^y = \mathbf{R}(\theta_{\mathcal{T}}) \begin{pmatrix} \rho_1^{\mathcal{T}} & 0 \\ 0 & \rho_2^{\mathcal{T}} \end{pmatrix}^2 \mathbf{R}(\theta_{\mathcal{T}})^\top,$$

where

$$\mathbf{R}(\theta_{\mathcal{T}}) = \begin{pmatrix} \cos\theta_{\mathcal{T}} & -\sin\theta_{\mathcal{T}} \\ \sin\theta_{\mathcal{T}} & \cos\theta_{\mathcal{T}} \end{pmatrix}$$

is the rotation matrix by an angle $\theta_{\mathcal{T}}$, and $\rho_1^{\mathcal{T}}, \rho_2^{\mathcal{T}} > 0$ are local scaling factors along the principal directions.

3.1 Specific case of the sphere

On the sphere, the change of coordinates Ψ_1 is defined as

$$\Psi_1(x,y,z) = (\theta, \phi), \quad \text{with} \quad \theta = \arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right) + k_\theta(z), \quad \phi = \arctan\left(\frac{y}{x}\right) + k_\phi(x,y),$$

where $\theta \in [0, \pi]$ is the polar angle (from the z-axis), $\phi \in [0, 2\pi)$ is the azimuthal angle in the xy-plane, k_θ and k_ϕ are piecewise constant functions to adjust for the correct quadrant.

Then the Jacobian matrix is

$$J_{\Psi_1} = \begin{pmatrix} \dfrac{xz}{(x^2+y^2+z^2)\sqrt{x^2+y^2}} & \dfrac{yz}{(x^2+y^2+z^2)\sqrt{x^2+y^2}} & -\dfrac{\sqrt{x^2+y^2}}{x^2+y^2+z^2} \\ -\dfrac{y}{x^2+y^2} & \dfrac{x}{x^2+y^2} & 0 \end{pmatrix}.$$

3.2 Specific case of the cylinder

The change of coordinates Ψ_2 from 3D Cartesian coordinates to cylindrical coordinates is defined by

$$\Psi_2(x,y,z) = (\theta, z), \quad \theta = \arctan\left(\frac{y}{x}\right) + k_\theta(x,y), \quad z = z,$$

where $\theta \in [0, 2\pi)$ is the azimuthal angle around the cylinder, k_θ is a piecewise constant function to adjust for the correct quadrant, and z is the height along the cylinder's axis.

The Jacobian matrix of Ψ_2 is

$$J_{\Psi_2}(x,y,z) = \begin{pmatrix} -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

References

[1] Pereira, M.: Generalized random fields on Riemannian manifolds : theory and practice. Theses, Université Paris sciences et lettres (November 2019). https://pastel.hal.science/tel-02499376