Supplementary Information for *Transversal fault tolerant distributed quantum computing operations*

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This document contains supplementary information to the article: *Transversal fault tolerant distributed quantum computing operations.*

CONTENTS

1. QUANTUM ERROR CORRECTION (QEC)

Let $\mathcal{P} = \langle X, Y, Z \rangle$ denote the single-qubit Pauli group. The $n-$qubit Pauli group is then defined as $\mathcal{P}_n = \{P_1 \otimes P_2 \otimes \cdots \otimes P_n | P_i \in \mathcal{P}\}$. An $[\![n, k, d]\!]$ stabilizer code, defined by the stabilizer group $\mathcal{S} \subset \mathcal{P}_n$, encodes k qubits of logical information into a n-qubit physical qubit block and can correct up to $\lfloor (d-1)/2 \rfloor$ errors, with an encoding rate of k/n. The code space $\mathcal{C}$ is the common $+1$ eigenspace of $\mathcal{S}$ given by

$$\mathcal{C} = \{|\psi\rangle \mid s\,|\psi\rangle = +\,|\psi\rangle\}. \tag{1}$$

To construct a valid stabilizer code, the stabilizer group $\mathcal{S}$ must be an Abelian subgroup of the n-qubit Pauli group $\mathcal{P}_n$ such that $-I \notin \mathcal{S}$. For qubit stabilizer codes, each generator $g \in \mathcal{S}$ can be written as a tensor product $g = P_1 \otimes \cdots \otimes P_n$, where $P_i \in \{I, X, Y, Z\}$. Each generator can be mapped to a binary vector $\mathbf{g} = [\mathbf{g}^X \mid \mathbf{g}^Z]$ of length $2n$ using the symplectic representation. The mapping from each single-qubit Pauli operator P_i to a binary pair (g_i^X, g_i^Z) is defined as

$$(g_i^X, g_i^Z) = \begin{cases} (0,0) & \text{if } P_i = I \\ (1,0) & \text{if } P_i = X \\ (1,1) & \text{if } P_i = Y \\ (0,1) & \text{if } P_i = Z. \end{cases} \tag{2}$$

Collecting these binary vectors as rows yields the stabilizer check matrix

$$H = \left[\, H_X \,\middle|\, H_Z \,\right] \in \mathbb{F}_2^{(n-k)\times 2n}, \tag{3}$$

where $H_X, H_Z \in \mathbb{F}_2^{(n-k)\times n}$ correspond to the X and Z components of the stabilizer generators in binary form.

Calderbank-Steane-Shor (CSS) codes form a special subclass of stabilizer codes in which the stabilizer group $\mathcal{S}$ can be decomposed into two disjoint subsets containing only X-type and Z-type stabilizers. Consequently, the parity-check matrix of a CSS code takes the block-diagonal form

$$H = \left[\begin{array}{c|c} H_X & \mathbf{0} \\ \mathbf{0} & H_Z \end{array}\right]. \tag{4}$$

Since $\mathcal{S}$ is an Abelian group, the commutativity condition requires that $H_X H_Z^T = \mathbf{0}$ must be satisfied to ensure the validity of the CSS code.

An error that exceeds a code's correction capability can cause a logical error. That is, after error correction, the residue error acts as a logical operator and changes the logical state of the code. Such errors are undetectable as they commute with all stabilizers.

Decoding: A QEC cycle involves a syndrome extraction round, measurement of stabilizer qubits and then correction of inferred errors (either physically, or in a Pauli frame). Each stabilizer qubit corresponds to a row of either H_X or H_Z. The result of its measurement indicates the parity of the data qubits of that row.

Using these measurements, we can therefore produce a syndrome vector $\mathbf{s} \in \mathbb{F}_2^{n-k}$. Assuming that CSS codes are used for error correction and that errors are of the form $E = \bigotimes_{i=1}^{n} P_i$, represented as a binary vector $\mathbf{e} = (\mathbf{e}_X|\mathbf{e}_Z) \in \mathbb{F}_2^{2n}$, we can infer that the errors that have occurred on the data qubits related to the syndrome $\mathbf{s}$ by the following equations:

$$\begin{aligned} \mathbf{s}_X = H_X \odot \mathbf{e}_Z = H_X \mathbf{e}_Z \mod 2 \\ \mathbf{s}_Z = H_Z \odot \mathbf{e}_X = H_Z \mathbf{e}_X \mod 2. \end{aligned} \tag{5}$$

Given this relation, a decoder for an $[\![n,k,d]\!]$ stabilizer code is a classical algorithm that infers a recovery operation $\hat{E} \in \mathcal{P}_n$ from measured syndromes $\mathbf{s}_X, \mathbf{s}_Z$ that removes the syndrome. The decoder solves the inverse problem, $H_{\{X,Z\}} \odot \hat{\mathbf{e}}_{\{Z,X\}} = \mathbf{s}_{\{Z,X\}}$, to estimate $\hat{\mathbf{e}}$. The same syndrome can correspond to multiple errors $\mathbf{e}$ as H has fewer rows than columns. In these cases, a decoder typically chooses the error with the lowest Hamming weight, $\hat{\mathbf{e}}$, i.e., the most likely error (highest probability).

Threshold: The threshold of a QEC code family is the physical error rate past which increasing the code distance decreases the logical error rate [1].

2. SURFACE CODES

The rotated surface code (SC) is a topological QEC code defined on a square lattice of $d \times d$ data qubits (aka. plaquette) with parameters $[\![d^2, 1, d]\!]$. The stabilizer group $\mathcal{S}$ consists of X- and Z-type operators acting on each plaquette, with bulk stabilizers of weight four and edge stabilizers of weight two. For a coordinate (i,j) in the rotated lattice, the X-stabilizers $S_X^{(i,j)}$ and Z-stabilizers $S_Z^{(i,j)}$ are defined as

$$S_X^{(i,j)} = \bigotimes_{(k,l)\in\partial(i,j)} X_{k,l}, \quad S_Z^{(i,j)} = \bigotimes_{(k,l)\in\partial(i,j)} Z_{k,l}, \tag{6}$$

where $\partial(i,j)$ denotes the qubits adjacent to plaquette (i,j). This code is also a CSS code. Logical operators are pairs of $\overline{X}$ or $\overline{Z}$ strings spanning the lattice's diagonal with minimum weight d. Under independent circuit-level Pauli noise, the code achieves a threshold $p_{\text{th}} \approx 1\%$ [2]. The rotated surface code's constant weight stabilizers and nearest-neighbor connectivity mean that it is well suited to many hardware implementations and has been featured in many physical experiments [3, 4].

3. QLDPC BIVARIATE BICYCLE (BB) CODES

Bivariate bicycle (BB) codes form a class of CSS codes defined by two integers, l, m, and two bivariate polynomials: $a(x,y), b(x,y)$, over the quotient ring $\mathbb{F}_2[x,y]/(x^l - 1, y^m - 1)$ [5]. Let I_i denote the identity matrix of size i, and let S_j be the cyclic permutation matrix of size j, defined as $S_j = I_j >> 1$, where $>>$ is a cyclic shift. We can identify the variates x and y as matrices

$$x = S_l \otimes I_m, \quad y = I_l \otimes S_m. \tag{7}$$

This representation allows the polynomials $a(x,y)$ and $b(x,y)$ to be naturally expressed as $lm \times lm$ matrices A and B, respectively. Since $xy = yx$ holds due to the mixed-product property of the Kronecker product, matrix multiplication among these representations remains commutative. The parity-check matrices for BB codes are then given by

$$H_X = [A|B], \quad H_Z = [B^T|A^T]. \tag{8}$$

This construction guarantees a valid CSS code because the commutativity of A and B ensures that $H_X H_Z^T = AB + BA = 2AB = 0$ in $\mathbb{F}_2$, satisfying the CSS code commutativity condition.

BB codes' main advantage over the surface code is their high encoding as a function of code distance. Encoding rate $r = \frac{k}{n+c}$ is the number of logical qubits encoded in a code over the number of physical n and stabilizer qubits c used [5]. The surface code has encoding rate $r \approx \frac{1}{2d^2} = \frac{1}{2n}$. But BB codes have far better encoding rates. The codes considered in this paper have encoding rates up to 10 times higher than similar distance surface codes. Additionally, they have low-weight stabilizers that can be fixed to degree 6 [5] (unlike some other qLDPC codes) and can be efficiently implemented on neutral atom computers [6]. The BB codes we use in this paper are from [7] and [5]. Their specific parameters are in Tab. I in Appendix 5. Instructions on how to generate the code and syndrome extraction circuit, given these parameters, are in [5].

4. TRANSVERSAL GATES

A transversal gate in QEC is a logical operation that can be implemented by applying the corresponding physical gate to each data qubit in a code block. Logical operations are those that preserve distance and the code space. For a code with n-qubit logical states, applying a physical operation U to each data qubit acts as $U^{\otimes n}$ (single-block), or for paired interactions such as a CNOT, $U_{\text{CNOT}}^{\otimes n}$ between blocks. Transversal operations do not propagate errors within a code block and are fault tolerant, as they preserve the code space.

By the Eastin-Knill theorem [8], no QEC code can perform *universal* fault tolerant quantum computation using only transversal gates. CSS codes, however, have sufficient transversal gates and measurement to carry out non-local CNOTs and teleportation. All CSS codes have transversal CNOT, Hadamard, Pauli gates and logical measurement in Z or X basis. Implementation of a transversal Hadamard requires some care as it requires swapping X and Z operators. T-gates, however, are known to be non-transversal for CSS codes, yet are required for universal quantum computing and realized either by distillation factories or code switching. We restrict our work to transversal gates.

5. BIVARIATE BICYCLE CODE PARAMETERS

TABLE I. Parameters for generating the BB codes used in this paper.

Code	l	m	$a(x,y)$	$b(x,y)$
$[\![18,4,4]\!]$	3	3	$1+x+y$	$1+x^2+y^2$
$[\![54,4,8]\!]$	3	9	$x+y+y^3$	$1+x^2+y^2$
$[\![144,12,12]\!]$	12	6	x^3+y+y^2	$x+x^2+y^3$

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