Supplemental Material for: Simultaneous multiply-accumulate operations in optical computing by Jacobi time-wave packets

Part 1: Recursive Equation Systems for the Jacobi Pulse Sequences (JPSs)

The recursive equation system for the Jacobi polynomials $\mathrm{JP}_n^{4,4}(t)$ is represented by

$$\mathrm{JP}_0^{4,4}(t) = 1 \tag{1}$$

for the first polynomial,

$$\mathrm{JP}_1^{4,4}(t) = \left(\frac{(2n+7)\cdot(n+4)\cdot t\cdot \mathrm{JP}_{n-1}^{4,4}(t)}{(n+8)\cdot n}\right)_{n=1} = 5t\cdot \mathrm{JP}_0^{4,4}(t) = 5t \tag{2}$$

for the second and

$$\mathrm{JP}_n^{4,4}(t) = \frac{(2n+7)\cdot(n+4)\cdot t\cdot \mathrm{JP}_{n-1}^{4,4}(t)}{(n+8)\cdot n} - \frac{\Gamma(n+5)\cdot \mathrm{JP}_{n-2}^{4,4}(t)}{(n+8)\cdot n\cdot \Gamma(n+3)} \tag{3}$$

For $n \geq 2$.

In analogy, the orders of the Jacobi pulses (JCP) can be calculated as:

$$\mathrm{JCP}_0(t) = \frac{3}{16}\sqrt{35}\cdot(t^2-1)^2\cdot\Pi(t)$$

for the first, (4)

$$\mathrm{JCP}_1(t) = \left(\frac{(2n+9)^{1/2}\cdot(2n+7)^{1/2}\cdot t\cdot \mathrm{JCP}_{n-1}(t)}{(n+8)^{1/2}\cdot n^{1/2}}\right)_{n=1} = \sqrt{11}\cdot t\cdot \mathrm{JCP}_0(t)$$

$$= \frac{3}{16}\sqrt{385}\cdot t\cdot(t^2-1)^2\cdot\Pi(t) \tag{5}$$

for the second and

$$\mathrm{JCP}_n(t) = \frac{(2n+9)^{1/2}\cdot(2n+7)^{1/2}\cdot t\cdot \mathrm{JCP}_{n-1}(t)}{(n+8)^{1/2}\cdot n^{1/2}} - \frac{(2n+9)^{1/2}\Gamma(n+9)^{1/2}\cdot\Gamma(n+1)^{1/2}\cdot \mathrm{JCP}_{n-2}(t)}{(n+8)\cdot n\cdot(2n+5)^{1/2}\Gamma(n+7)^{1/2}\cdot\Gamma(n-1)^{1/2}} \tag{6}$$

for $n \geq 2$. Here $\Pi(t)$ is a rectangular function with 1 between $t=-1$ and $t=1$, $\frac{1}{2}$ for $|t|=1$ and 0 elsewhere.

Let

$$\mathrm{JPS}_{n,\Delta f}(t) := 2^{1/2}\Delta f^{1/2}\sum_{m=-\infty}^{\infty}\mathrm{JCP}_n(2\Delta f\cdot t - 2m) \tag{7}$$

be the unmodulated JCP sequence associated with the repetition frequency Δf. Than the recursive equation system for the Jacobi pulse sequences (JPS) is:

$$\mathrm{JPS}_{0,\Delta f}(t) = 2^{1/2}\Delta f^{1/2}\sum_{m=-\infty}^{\infty}\mathrm{JCP}_0(2\Delta f\cdot t - 2m)\,, \tag{8}$$

for the first

$$\mathrm{JPS}_{1,\Delta f}(t) = \left(\frac{(2n+9)^{1/2}\cdot(2n+7)^{1/2}\mathrm{ST}_{\Delta f}(t)\cdot \mathrm{S}_{n-1,\Delta f}(t)}{(n+8)^{1/2}\cdot n^{1/2}}\right)_{n=1} = \sqrt{11}\cdot \mathrm{ST}_{\Delta f}(t)\cdot \mathrm{JPS}_{0,\Delta f}(t)$$

$$= 2^{1/2}\Delta f^{1/2}\sum_{m=-\infty}^{\infty}\mathrm{JCP}_1(2\Delta f\cdot t - 2m) \tag{9}$$

for the second and

$$\mathrm{JPS}_{n,\Delta f}(t) = \frac{(2n+9)^{1/2}\cdot(2n+7)^{1/2}\mathrm{ST}_{\Delta f}(t)\cdot \mathrm{JPS}_{n-1,\Delta f}(t)}{(n+8)^{1/2}\cdot n^{1/2}} - \frac{(2n+9)^{1/2}\Gamma(n+9)^{1/2}\cdot\Gamma(n+1)^{1/2}\cdot \mathrm{JPS}_{n-2,\Delta f}(t)}{(n+8)\cdot n\cdot(2n+5)^{1/2}\Gamma(n+7)^{1/2}\cdot\Gamma(n-1)^{1/2}} \tag{10}$$

for $n \geq 2$. With $\mathrm{ST}_{\Delta f}(t)$ as a triangular-shaped sawtooth wave defined via a modulo function as:

$$\mathrm{ST}_{\Delta f}(t) := 2\mathrm{Mod}\left(\Delta f\cdot t - \frac{1}{2}, 1\right) - 1\,. \tag{11}$$

The first order JPS can be generated from the zeroths order Eq. (9), and all higher order modes only require the two previous modes $\mathrm{JPS}_{n-1,\Delta f}(t)$ and $\mathrm{JPS}_{n-2,\Delta f}(t)$ and a sawtooth signal $\mathrm{ST}_{\Delta f}(t)$. All other factors in Eq. (10)

are time-independent and as such only numbers. Therefore, all JPS modes can be generated by simple analog electronics from the zeroth order and a sawtooth signal as input.

Part 2: Scalar Product Operation or MAC Operation Utilizing Jacobi Polynomial Pulses

The Jacobi pulse sequences are given by:

$$\mathrm{JPS}_{n,\Delta f}(t) \coloneqq 2^{1/2}\Delta f^{1/2} \sum_{m=-\infty}^{\infty} \mathrm{JCP}_n(2\Delta f \cdot t - 2m) \tag{7}$$

with n as the order of the sequence and Δf as the repetition rate. For the scalar product, the signal before the integration filter can be described by

$$s_{\Delta f}(t) = \left(\sum_{p=0}^{N} data1_p(t) \cdot \mathrm{JPS}_{p,\Delta f}(t)\right) \sum_{p=0}^{N} data2_p(t) \cdot \mathrm{JPS}_{p,\Delta f}(t)\ . \tag{12}$$

$data..(t)$ are the rectangular-shaped data signals to be added up. The filter transfer can be described by a sinc function, here defined as:

$$\mathrm{Sinc}(f) \coloneqq \begin{cases} \frac{\mathrm{Sin}(2\pi \cdot f)}{2\pi \cdot f}, & f \neq 0 \\ 1, & f = 0 \end{cases}\ . \tag{13}$$

resulting in the filter function:

$$\mathrm{Filter}_{\Delta f}(f) = \frac{\mathrm{Sinc}\left(\frac{f}{2\Delta f}\right)}{\Delta f}\ . \tag{14}$$

With the forward $\mathbf{F}_t$ and backward Fourier transform $\mathbf{F}_f^{-1}$, the filtered signal at the time instance t_v can be expressed as:

$$x_{\Delta f}(t_v) = \left[\mathbf{F}_f^{-1}\left(\left[\mathbf{F}_t\left(\mathrm{s}_{\Delta f}(t)\right)\right](f) \cdot \mathrm{Filter}_{\Delta f}(f)\right)\right](t_v)\ , \tag{15}$$

where

$$t_v = \frac{v}{\Delta f} \tag{16}$$

are the centred data points of the rectangular-shaped data signals for $v \in \mathbb{Z}$. After applying the convolution theorem, Eq. (15) becomes:

$$x_{\Delta f}(t_v) = \int_{-\infty}^{\infty} \mathrm{s}_{\Delta f}(t_v - \tau) \cdot \Pi(2\Delta f \cdot \tau)\mathrm{d}\tau$$

$$= \int_{-\infty}^{\infty} \left(\sum_{p=0}^{N} data1_p(t_v - \tau) \cdot \mathrm{JPS}_{p,\Delta f}(t_v - \tau)\right) \sum_{p=0}^{N} data2_p(t_v - \tau) \cdot \mathrm{JPS}_{p,\Delta f}(t_v - \tau) \cdot \Pi(2\Delta f \cdot \tau)\mathrm{d}\tau \tag{17}$$

Since the rectangular function is zero outside of its range, this can be simplified to

$$x_{\Delta f}(t_v) = \int_{-1/(2\Delta f)}^{1/(2\Delta f)} \left(\sum_{p=0}^{N} data1_p(t_v - \tau) \cdot \mathrm{JPS}_{p,\Delta f}(t_v - \tau)\right) \sum_{p=0}^{N} data2_p(t_v - \tau) \cdot \mathrm{JPS}_{p,\Delta f}(t_v - \tau)\, \mathrm{d}\tau\ . \tag{18}$$

Now, Δf for the integration range can be replaced by $1/T$ with the symbol duration T and the signum of τ can be changed without changing the integral for symmetry reasons leading to

$$x_{\Delta f}(t_v) = \int_{-T/2}^{T/2} \left(\sum_{p=0}^{N} data1_p(t_v + \tau) \cdot \mathrm{JPS}_{p,\Delta f}(t_v + \tau)\right) \sum_{p=0}^{N} data2_p(t_v + \tau) \cdot \mathrm{JPS}_{p,\Delta f}(t_v + \tau)\, \mathrm{d}\tau\ . \tag{19}$$

In the next step, we can omit the t_v terms in the pulse sequences because of their periodicity, which gives

$$x_{\Delta f}(t_v) = \int_{-T/2}^{T/2} \left(\sum_{p=0}^{N} data1_p(t_v + \tau) \cdot \mathrm{JPS}_{p,\Delta f}(\tau)\right) \sum_{p=0}^{N} data2_p(t_v + \tau) \cdot \mathrm{JPS}_{p,\Delta f}(\tau)\, \mathrm{d}\tau\ . \tag{20}$$

Additionally, the τ term for the data signals can be omitted due to the fact that t_v describes a centred data point of a rectangular-shaped data signal and does therefore not vary inside the integral range. So, we get

$$x_{\Delta f}(t_v) = \int_{-T/2}^{T/2} \left(\sum_{p=0}^{N} data1_p(t_v) \cdot \mathrm{JPS}_{p,\Delta f}(\tau)\right) \sum_{p=0}^{N} data2_p(t_v) \cdot \mathrm{JPS}_{p,\Delta f}(\tau)\, \mathrm{d}\tau$$

$$= \int_{-T/2}^{T/2} \left(\sum_{p=0}^{N} data1_p(t_v) \cdot 2^{1/2} \Delta f^{1/2} \sum_{m=-\infty}^{\infty} \mathrm{JCP}_p(2\Delta f \cdot \tau - 2m) \right) \sum_{p=0}^{N} data2_p(t_v) \cdot$$
$$2^{1/2} \Delta f^{1/2} \sum_{m=-\infty}^{\infty} \mathrm{JCP}_p(2\Delta f \cdot \tau - 2m) \, \mathrm{d}\tau \, , \tag{21}$$

whereby we also inserted the definition of the JCP sequences. The τ range of the integral is centred around zero and according to its length it follows, that from the sums with index m only $m = 0$ is inside the integration range. Hence, it results to:

$$x_{\Delta f}(t_v) = \int_{-T/2}^{T/2} \left(\sum_{p=0}^{N} data1_p(t_v) \cdot 2^{1/2} \Delta f^{1/2} \cdot \mathrm{JCP}_p(2\Delta f \cdot \tau) \right) \sum_{p=0}^{N} data2_p(t_v) \cdot$$
$$2^{1/2} \Delta f^{1/2} \cdot \mathrm{JCP}_p(2\Delta f \cdot \tau) \, \mathrm{d}\tau \, . \tag{22}$$

Utilizing the orthogonality of the JCPs this further reduces to

$$x_{\Delta f}(t_v) = \int_{-T/2}^{T/2} 2\Delta f \sum_{p=0}^{N} data1_p(t_v) \cdot data2_p(t_v) \cdot \mathrm{JCP}_p(2\Delta f \cdot \tau) \cdot \mathrm{JCP}_p(2\Delta f \cdot \tau) \, \mathrm{d}\tau$$
$$= \sum_{p=0}^{N} data1_p(t_v) \cdot data2_p(t_v) \, , \tag{23}$$

which exactly corresponds to the required functionality of the scalar product operation or MAC operation at any time instance t_v.