Supplement file:Theoretical Derivation from Structural Tensor Conservation and Data–Driven Derivation of Tensor Conservation from MEST Fits

September 4, 2025

1 Theoretical Derivation from Structural Tensor Conservation

1.1 Concise covariant derivation and its weak-field control

We augment the main conservation statement $\nabla^\mu T_{\mu\nu} = 0$ (Eq. (9)) with a canonical structural sector

$$\mathcal{L}_{\mathrm{MEST}} = \frac{K_0}{2}\,\nabla_\mu\psi\,\nabla^\mu\psi \;-\; U(\psi),$$

which yields the field equation

$$K_0\,\Box\psi \;-\; U'(\psi) \;=\; 0,$$

and the stress tensor

$$T^{(\psi)}_{\mu\nu} = K_0\,\nabla_\mu\psi\,\nabla_\nu\psi - g_{\mu\nu}\left(\frac{K_0}{2}\,\nabla_\alpha\psi\,\nabla^\alpha\psi - U(\psi)\right).$$

In static spherical symmetry, one has

$$\Box\psi = e^{-2\Lambda}\left[\psi'' + \left(\Phi' - \Lambda' + \tfrac{2}{r}\right)\psi'\right],$$

which leads to the exact radial identity

$$\frac{d}{dr}\left(\tfrac{1}{2}K_0\,e^{-2\Lambda}\,\psi'^2 \;-\; U\right) \;+\; K_0\,e^{-2\Lambda}\,\psi'^2\left(\Phi' + \tfrac{2}{r}\right) - \tfrac{1}{2}K_0\,e^{-2\Lambda}\,\psi'^2\,\Lambda' \;=\; 0. \tag{1}$$

Invoking the circular–orbit relation $a_r = V^2/r = \Phi'(r)$ (Eq. (10)) and the weak-field limits $e^{-2\Lambda}\simeq 1$, $\Lambda'\simeq 0$, one recovers the working conservation law (Eq. (14)) and the structural equation

$$U'(\psi) = K_0\left[\psi'' + \left(\Phi' + \tfrac{2}{r}\right)\psi'\right], \tag{2}$$

which is used in the fits. Linearizing $U(\psi) = \frac{1}{2}m_{\mathrm{eff}}^2\psi^2 + \cdots$ near the center gives the characteristic scale relation

$$\frac{\alpha}{r_0} = \sqrt{\frac{m_{\mathrm{eff}}^2}{2K_0}}, \tag{3}$$

which, under single–scale self-similarity, implies the $b = 1$ law and constant k_α.

To control the weak-field reduction, we use the exact-minus-approximate residual Δ (Eq. (5)) and bound the fractional error ε (Eq. (6)). Post-Newtonian estimates yield the conservative bound (Eq. (8)), $\varepsilon \lesssim |\Phi| + \frac{3}{4}(V^2/c^2)$, i.e. $\lesssim 10^{-6}$ for galaxies and $\lesssim 2\times 10^{-5}$ for clusters. Consequently, the induced biases satisfy $\delta k_\alpha/k_\alpha = O(\varepsilon)$ and $\delta b = O(\varepsilon)$, well below measurement errors in all regimes considered.

1.2 Weak-field error bound for the radial conservation law

Starting from the exact static, spherically symmetric identity in Eq. (1), the weak-field, quasi-Newtonian reduction used in the main text (Eq. (14)) neglects $e^{-2\Lambda}\simeq 1$ and $\Lambda'\simeq 0$. To quantify the approximation error, define the difference

$$\Delta \;\equiv\; \Big[\text{exact LHS in Eq. (1)}\Big] \;-\; \Big[\text{approx. LHS in Eq. (14)}\Big]. \tag{4}$$

A direct rearrangement yields

$$\Delta = K_0\,\psi'^2\Big[(e^{-2\Lambda}-1)\Big(\Phi'+\tfrac{2}{r}\Big)-\tfrac{1}{2}\,e^{-2\Lambda}\Lambda'\Big] + \tfrac{1}{2}\,K_0\,\psi'^2\,\frac{d}{dr}\Big(e^{-2\Lambda}-1\Big)\,. \tag{5}$$

Dividing by the *dominant* geometric term $K_0\psi'^2\,(2/r)$ gives the fractional error

$$\varepsilon \;\equiv\; \frac{|\Delta|}{K_0\psi'^2\,(2/r)} \;\le\; \tfrac{1}{2}\,|e^{-2\Lambda}-1|\left(1+\frac{|\Phi'|}{2/r}\right)+\tfrac{1}{4}\,e^{-2\Lambda}\,\frac{|\Lambda'|}{1/r}+\tfrac{1}{4}\,\frac{\left|\frac{d}{dr}(e^{-2\Lambda}-1)\right|}{1/r}. \tag{6}$$

Post-Newtonian estimates. In the weak field ($|\Phi|\ll 1$) one has $e^{2\Lambda}\simeq 1-2\Phi$, so $|e^{-2\Lambda}-1|\lesssim 2|\Phi|$ and $|\Lambda'|\simeq|\Phi'|$. Using the circular-orbit identification $|\Phi'|=V^2/r$ (Eq. (10)) and the small parameter $(V/c)^2\ll 1$, the three ratios in Eq. (6) scale as

$$\frac{|\Phi'|}{2/r}=\frac{V^2}{2c^2},\qquad \frac{|\Lambda'|}{1/r}=\frac{V^2}{c^2},\qquad \frac{\left|\frac{d}{dr}(e^{-2\Lambda}-1)\right|}{1/r}\lesssim 2\,\frac{V^2}{c^2}. \tag{7}$$

Hence the conservative bound

$$\varepsilon \;\lesssim\; |\,\Phi\,|+\frac{3}{4}\,\frac{V^2}{c^2}, \tag{8}$$

which yields $\varepsilon_{\rm gal}\lesssim 1.3\times10^{-6}$ for typical spirals ($V\sim 200$ km s^{-1}) and $\varepsilon_{\rm cl}\lesssim 1.8\times10^{-5}$ for rich clusters ($V\sim 1000$ km s^{-1}).

Impact on k_α and b. Treating ε as a multiplicative perturbation of the conservation law implies a fractional bias $\delta k_\alpha/k_\alpha=\mathcal{O}(\varepsilon)$ and a slope bias $\delta b=\mathcal{O}(\varepsilon)$ in the log–log α–r_0 regression. Thus, at galaxy scales $\delta b\lesssim 10^{-6}$, and even for clusters $\delta b\lesssim 2\times10^{-5}$, far below the observational uncertainties quoted in this work. Accordingly, enforcing the exact $e^{-2\Lambda}$ and Λ' terms would not change the recovered $b=1$ and k_α within our error bars.

2 Data–Driven Derivation of Tensor Conservation from MEST Fits

In this section we show that the *existence* of a divergence–free structural tensor is not merely posited but can be *inferred from the data–calibrated MEST profiles*. Starting from fitted profiles $\psi(r)$ and the reconstructed structural potential $\Phi(r)$, we demonstrate that the observed fields satisfy the radial conservation identity implied by the covariant law

$$\nabla^\mu T_{\mu\nu}=0, \tag{9}$$

and hence justify the use of Eq. (9) for deriving the constants $b=1$ and k_α.

2.1 Observables to field variables

We adopt a static, spherically averaged patch and use the usual weak–field line element $ds^2=-(1+2\Phi)dt^2+(1-2\Phi)(dr^2+r^2d\Omega^2)$ with $|\Phi|\ll 1$. The observable that anchors $\Phi'(r)$ depends on the system:

- **Rotation curves:** the centripetal relation

$$a_r(r)=\frac{V^2(r)}{r}=\Phi'(r) \tag{10}$$

  fixes $\Phi'(r)$ directly.

- **Strong lensing:** the azimuthally averaged deflection obeys $\theta(r)\propto\partial_r\Phi_{\rm lens}$; up to a known geometric factor, we absorb the proportionality in K_0 below so that Φ' is fixed modulo a global scale (irrelevant for the conservation identity).

- **CMB cold/hot spots and polarization:** the structural potential Φ is inferred from the best–fit MEST profile that reproduces the radial temperature (or polarization–amplitude) contrast; again, an overall scale is absorbed in K_0.

Independently, the *structural profile* $\psi(r)$ is obtained from the same MEST fit (e.g. MEST$_2$, MEST$_{2n}$, or MEST$_{n2}$), and is normalized consistently across systems.

2.2 Stress tensor from a minimal structural Lagrangian

Consider the MEST structural Lagrangian

$$\mathcal{L} = \frac{1}{2}K(\psi)\, g^{\mu\nu}\partial_\mu\psi\,\partial_\nu\psi - U(\psi), \tag{11}$$

which yields the structural stress tensor

$$T_{\mu\nu} = K(\psi)\,\partial_\mu\psi\,\partial_\nu\psi - \frac{1}{2}g_{\mu\nu}\left[K(\psi)\,(\nabla\psi)^2 - 2U(\psi)\right]. \tag{12}$$

The field equation obtained from $\nabla^\mu T_{\mu\nu} = 0$ is equivalent to

$$\nabla_\mu\bigl(K\,\partial^\mu\psi\bigr) - \frac{1}{2}K'(\psi)(\nabla\psi)^2 + U'(\psi) = 0. \tag{13}$$

In the static, spherically symmetric weak–field limit and for slowly varying $K(\psi)$, Eq. (13) reduces to the *radial conservation identity*

$$\frac{d}{dr}\left(\frac{1}{2}K\,\psi'^2 - U\right) + K\,\psi'^2\,\Phi'(r) + \frac{2}{r}\,K\,\psi'^2 = 0, \tag{14}$$

and its equivalent differential form for $U'(\psi)$,

$$U'(\psi) = K(\psi)\left[\psi'' + \left(\Phi'(r) + \frac{2}{r}\right)\psi'\right] - \frac{1}{2}K'(\psi)\,\psi'^2. \tag{15}$$

When $K(\psi) \approx K_0$ is (locally) constant, Eq. (15) simplifies to the working relation used in our fits,

$$U'(\psi) = K_0\left[\psi'' + \left(\Phi'(r) + \frac{2}{r}\right)\psi'\right], \tag{16}$$

which is Eq. (2) in the main text.

2.3 Data–driven reconstruction and the conservation test

Given *fitted* $\psi(r)$ and reconstructed $\Phi'(r)$ from the observables, we compute:

1. The inferred $U'_{\rm inf}(\psi)$ from Eq. (16) (or Eq. (15) if a mild $K'(\psi)$ is retained).
2. The residual of the conservation law,

$$\mathcal{R}(r) \equiv \frac{d}{dr}\left(\frac{1}{2}K\,\psi'^2 - U\right) + K\,\psi'^2\,\Phi'(r) + \frac{2}{r}\,K\,\psi'^2. \tag{17}$$

We then (i) check the near–center linearity of $U'_{\rm inf}(\psi)$ to extract $m^2_{\rm eff} \equiv dU'/d\psi|_{\psi\to 0}$, and (ii) quantify $\mathcal{R}$ by its RMS and max norms over the fit domain. Across analytic control tests and observed profiles (galaxies, lenses, CMB spots), the conservation residual remains consistent with zero within numerical precision (see Table **??**), thereby *empirically* validating Eq. (14) and, hence, the covariant conservation law (9).

2.4 Consequences: the constants $b = 1$ and k_α

With $m^2_{\rm eff}$ measured from $U'_{\rm inf}$ and K_0 fixed by normalization, the structural scale–slope ratio follows

$$\frac{\alpha}{r_0} = \sqrt{\frac{m^2_{\rm eff}}{2K_0}}, \tag{18}$$

which immediately yields the *power–law constant* $b = 1$ (i.e. $\alpha \propto r_0^{-1}$) and the *linear compactness* $k_\alpha \equiv \alpha r_0$ = const across systems. These two constants are therefore not free assumptions but *data–driven consequences* of the empirically verified conservation identity (14).

Remark (weak–field control). Let $\varepsilon \equiv \max\{|\Phi|, |r\Phi'|\}$ over the fit range. The neglected post–Newtonian corrections dress Eq. (14) by a factor $(1 + O(\varepsilon))$, and the corresponding fractional error in Eq. (18) is bounded by $O(\varepsilon)$ (see the weak–field error bound derived in Sec. 1.2). For galaxies and lenses, $\varepsilon \lesssim 10^{-6}$–$10^{-5}$, well below observational uncertainties, ensuring that the derived b and k_α are robust to metric corrections.