# Supplement file: Supplementary Note S2: Why two constants suffice across scales

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## 1 Supplementary Note S2: Why two constants suffice across scales

### 1. Physical invariances

Two fundamental invariances reduce the structural description to only two constants.

**(A) Mass/flux conservation.** In the transition region ($x \approx r_0$), define the dimensionless coordinate $y = x/r_0$. The effective transition width is $\Delta y \sim 1/\alpha$, hence the physical width $\Delta x \sim r_0/\alpha$. The "contrast mass" or structural flux in this region,

$$\Delta M \ \sim \ \int_{r_0-\Delta x/2}^{r_0+\Delta x/2} 4\pi r^2 \rho(r)\, dr \ \approx \ 4\pi r_0^2 \rho_{\rm tr}\, \Delta x,$$

remains in a stable domain under continuity constraints. Since $\Delta x \sim r_0/\alpha$, one finds

$$\Delta M \propto \frac{\rho_{\rm tr}}{\alpha}\, r_0^3.$$

Meanwhile, the physical slope $S_{\rm phys} \equiv \left|\frac{dU}{dx}\right|_{x \approx r_0} \propto \alpha/r_0$ is observed to remain stable across systems. Thus we conclude

$$\boxed{\alpha \ \propto \ r_0}$$

as the first universal relation.

**(B) Action-phase invariance.** The characteristic time is $\tau \sim r_0/c$, with energy scale $E_0$. Requiring the action-phase jump $\Delta\phi$ across the transition to remain in the same equivalence class $O(1)$ gives

$$\Delta\phi \sim \frac{E_0 \tau}{\hbar} \ \sim \ \frac{E_0 r_0}{\hbar c} \ \approx \ O(1),$$

hence

$$\boxed{E_0\, r_0 \ \approx \ \hbar c} \qquad (b = 1).$$

### 2. Mathematical structure

**Proposition 1 (Shape-family reduction).** The canonical MEST solutions (MEST-2, MEST-2n, MEST-n2), under smoothness and finite-curvature boundary conditions, are homeomorphic to a universal class of smooth transition kernels (tanh/logistic/arctan). Thus any observable profile can be written as

$$U(x) \ = \ U_\infty\, \mathcal{F}\!\left(\frac{x - r_0}{r_0/\alpha}\right),$$

with a single dimensionless shape parameter $\alpha$ and one scale parameter $r_0$.

**Proposition 2 (Dual mapping).** Relativistic correspondence maps $r_0 \mapsto \tau = r_0/c$ and $E_0$, constrained by phase invariance. Hence

$$(\alpha, r_0) \ \longleftrightarrow \ (\alpha, E_0), \qquad E_0 r_0 \approx \hbar c.$$

### 3. Statistical support

- **Model comparison:** Two-parameter fits already minimize AIC/BIC; adding parameters yields negligible or negative improvement.
- **Hierarchical Bayes:** Posteriors of $\alpha$ overlap strongly across systems, supporting universality.
- **Cross-validation:** Hold-out and LOO errors remain stable, showing robustness of the dual-constant law.

## 4. Falsifiability and predictivity

- Any systematic deviation from slope $b = 1$ in $\log E_0$ vs. $\log r_0$, or breakdown of $\alpha/r_0$ stability, would falsify the law.
- Given either $r_0$ or $E_0$, the other is directly predicted with no extra freedom.

## 5. Conclusion

Through conservation and action-phase invariances, MEST equations reduce structural fits across galaxies, lensing, CMB, and scattering data to two universal constants: the scaling law $\alpha \propto r_0$ and the power-law duality $E_0 r_0 \approx \hbar c$. These invariants persist across more than twenty orders of magnitude, demonstrating that two constants suffice to unify micro- and macro-scale phenomena.