Supplementary Note: Detailed derivation of triangular localization of the 3γ annihilation point

Here, we derive the localization of the 3γ annihilation point in detail. The three-photon decay process conserves both energy and momentum. Accordingly, the total energy of the three emitted photons, E_1, E_2, and E_3, must satisfy:

$$E_1 + E_2 + E_3 = 1.022 \text{ MeV} \tag{1}$$

$$0 \text{ keV} \leq E_1, E_2, E_3 \leq 511 \text{ keV} \tag{2}$$

Due to momentum conservation, all three photons must be coplanar. The conservation of momentum in the E_1 and its vertical directions can be written as:

$$\begin{cases} E_1 + E_2 \cos\theta_{12} + E_3 \cos(\theta_{12} + \theta_{23}) = 0 \\ E_2 \sin\theta_{12} + E_3 \sin(\theta_{12} + \theta_{23}) = 0 \end{cases} \tag{3}$$

Eliminating θ_{23}, we get:

$$\begin{aligned} E_3^2 &= (E_1 + E_2 \cos\theta_{12})^2 + (E_2 \sin\theta_{12})^2 \\ &= E_1^2 + 2E_1E_2\cos\theta_{12} + E_2^2 \end{aligned} \tag{4}$$

Therefore,

$$\cos\theta_{12} = \frac{-E_1^2 - E_2^2 + E_3^2}{2E_1E_2} \tag{5}$$

This leads to generalized relationship between the emission angle and the photon energy:

$$\theta_{ij} = \arccos\left(\frac{-E_i^2 - E_j^2 + E_k^2}{2E_iE_j}\right) \tag{6}$$

$$(i, j, k) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\} \tag{7}$$

To solve this problem, we propose a method that reduces the problem to a quadratic equation, allowing for the straightforward elimination of extraneous solutions.

First, the 2D plane shown in figure 2(a) is mapped into a complex plane as shown in figure 2(b). Let P_1, P_2, and P_3 denote the detector positions, each detecting a photon with energies E_1, E_2, and E_3 respectively, in coincidence. The decay position Q can then be determined as the intersection point of two circles, defined in the complex plane, where P_1, P_2, and P_3 are arranged counterclockwise.

The angle θ_{ij} can be calculated from the measurements of energy E_1, E_2, and E_3. The positions of the photons detected with energies E_1, E_2, and E_3 are denoted as P_1, P_2, and P_3, and the reconstructed position is denoted as Q. After rearranging the situation, it can be seen that the coordinate system is comparatively simple. The coordinate system is a complex plane, and P_1, P_2, and P_3 are arranged around the counterclockwise direction. Furthermore, the point Q is located within the triangular shape formed by P_1, P_2, and P_3.

Two circles are considered: C_1, which subtends a central angle θ_{12} across the chord P_1P_2, and C_2, which subtends an angle θ_{23} across P_2P_3. The emission point Q corresponds to the intersection of these two circles, excluding the trivial intersection point at P_2. The centers of these two circles are complex numbers D_1 and D_2 with radii r_1 and r_2, respectively. The complex numbers are denoted by capital letters, while the radii are represented by lowercase letters.

Two cases can be considered: when D_1 is outside and when D_1 is inside the triangle $P_1P_2P_3$. The equations in both cases are as follows:

$$P_1 - D_1 = e^{-2i\theta_{12}}(P_2 - D_1) \tag{8}$$

After simplifying this, and doing so similarly for D_2, we get the following expression for D_1 and D_2:

$$D_1 = \frac{P_1 - P_2 e^{-2i\theta_{12}}}{1 - e^{-2i\theta_{12}}} \tag{9}$$

$$D_2 = \frac{P_2 - P_3 e^{-2\theta_{23} i}}{1 - e^{-2\theta_{23} i}} \tag{10}$$

Next, for r_1 and r_2, we can calculate the distances as follows:

$$r_1 = |P_1 - D_1| \tag{11}$$

$$r_2 = |P_2 - D_2| \tag{12}$$

Now, using these distances D_1, D_2, r_1, r_2, we can calculate Q as follows:

$$|Q - D_1| = r_1 \tag{13}$$

$$|Q - D_2| = r_2 \tag{14}$$

By expanding these two equations, we obtain:

$$|Q|^2 - Q\overline{D_1} - \overline{Q}D_1 + |D_1|^2 = r_1^2 \tag{15}$$

$$|Q|^2 - Q\overline{D_2} - \overline{Q}D_2 + |D_2|^2 = r_2^2 \tag{16}$$

Now, rearranging these expressions gives:

$$\overline{Q} = \frac{\overline{-D_{12}}Q + e}{D_{12}} \tag{17}$$

$$e \equiv |D_1|^2 - |D_2|^2 - (r_1^2 - r_2^2) \tag{18}$$

$$D_{12} \equiv D_1 - D_2 \tag{19}$$

Substituting into eq. (15) and simplifying with respect to Q, we get:

$$-\overline{D_{12}}^2 Q^2 + \left(e - \overline{D_1}D_{12} + D_1\overline{D_{12}}\right) Q - eD_1 + D_{12}(|D_1|^2 - r_1^2) = 0 \tag{20}$$

This equation can be solved for the real or imaginary parts of Q. By rewriting each complex variable as $Q = x + yi$ and $D_{12} = d_x + id_y$, the eq. (17) can be written as:

$$x + yi = \frac{-(d_x - d_y i)(x + yi) + e}{d_x + d_y i} \tag{21}$$

Solving for y gives:

$$y = \frac{-2d_x x + e}{2d_y} \tag{22}$$

For practical analysis, we assume that $d_y \neq 0$, which is generally the case in practice. Then, eq. (20) can be reduced to two real quadratic equations of x.

In order to avoid dealing with the messy coefficients, we consider a generalized case where

$$Z = x + yi \tag{23}$$

$$y = sx + t \tag{24}$$

$$A_1 Z^2 + A_2 Z + A_3 = 0 \tag{25}$$

Here, A_1, A_2, A_3 are arbitrary complex numbers:

$$A_1 = a_{x1} + a_{y1} i, \quad A_2 = a_{x2} + a_{y2} i, \quad A_3 = a_{x3} + a_{y3} i \tag{26}$$

Thus, we get:

$$(a_{x1} + a_{y1} i) Z^2 + (a_{x2} + a_{y2} i) Z + (a_{x3} + a_{y3} i) = 0 \tag{27}$$

Substituting $z = x + (sx + t)i$ and rearranging the real and imaginary parts, we get the following equations. For the real part:

$$(a_{x1} - a_{x1} s^2 - 2a_{y1} s) x^2 + (-2a_{x1} st - 2a_{y1} t + a_{x2} - a_{y2}) x - a_{x1} t^2 + a_{x3} = 0 \tag{28}$$

For the imaginary part:

$$(a_{y1} - a_{y1} s^2 + 2a_{x1} s) x^2 + (-2a_{x1} st + 2a_{x2} x - a_{x3}) x - a_{x2} t^2 + a_{y3} = 0 \tag{29}$$

Next, applying the generalized form in eqs. (20) and (22), we have:

$$A_1 = -\overline{D_{12}} \tag{30}$$

$$A_2 = e - \overline{D_1} D_{12} + D_1 \overline{D_{12}} \tag{31}$$

$$A_3 = -eD_1 + D_{12} \left(|D_1|^2 - r_1^2 \right) \tag{32}$$

$$s = -\frac{d_x}{d_y}, \quad t = \frac{e}{2d_y} \tag{33}$$

This allows us to calculate the coefficients in eqs. (28) and (29). Using a program to solve these two equations, we find that the solutions are almost identical, so the experiment was conducted using the assumption that the real part is equal to zero.

From the above process, the position of $Q(x, y)$ can be reconstructed from $E_1, E_2, E_3, P_1, P_2, P_3$. One of the solution corresponds to P_2, so the determination of Q is easy.

Finally, the conditions that are necessary in this method can be summarized as follows, and in the actual experiment, we perform the following adjustments in the program:

1. P_1, P_2, P_3 are arranged in counterclockwise order around the time circle.

2. d_y is not zero.