

Analytical solution for the nonlinear rotation of a spherical magnetic particle in a rotating magnetic field

Thomas Kampf · Martin A. Rückert · Anna Vilter · Volker J.F. Sturm ·
Patrick Vogel · Volker C. Behr

Received: date / Accepted: date

Abstract The application of a weak rotating magnetic field to magnetic particles, that show only Brownian dynamics and are suspended in a liquid leads to a nonlinear rotation if the magnetic field is not strong enough to overcome the rotational friction at the applied frequency. In this paper, the analytical solution to the equation of motion is derived for the case of negligible angular momentum and no random thermal interactions. The former is an accurate approximation in all typical applications such as magnetic particle imaging or magnetic particle based hyperthermia. Having

a closed solution for the theoretical edge case, where random thermal interactions go to zero, can be used as a verification tool for more generalized particle simulations and it can also be used as a basis for constructing approximations for more general cases.

Keywords Magnetic nanoparticles · Nonlinear rotation · Brownian relaxation · Rotational drift · Analytical solution

Thomas Kampf
Department of Diagnostic and Interventional Neuroradiology,
University of Würzburg, 97074, Würzburg, Germany
Department of Experimental Physics 5 (Biophysics),
University of Würzburg, 97074, Würzburg, Germany
E-mail: kampf_t@ukw.de

Martin A. Rückert (corresponding author)
Department of Experimental Physics 5 (Biophysics),
University of Würzburg, 97074, Würzburg, Germany
E-mail: martin.rueckert@physik.uni-wuerzburg.de

Anna Vilter
Department of Experimental Physics 5 (Biophysics),
University of Würzburg, 97074, Würzburg, Germany
E-mail: anna.vilter@physik.uni-wuerzburg.de

Volker F.J. Sturm
Department of Neuroradiology,
University of Heidelberg, 69120, Heidelberg, Germany
E-mail: volker.sturm@med.uni-heidelberg.de

Patrick Vogel
Department of Experimental Physics 5 (Biophysics),
University of Würzburg, 97074, Würzburg, Germany
phase Vision GmbH, 97222, Rimpf
E-mail: patrick.vogel@physik.uni-wuerzburg.de

Volker C. Behr
Department of Experimental Physics 5 (Biophysics),
University of Würzburg, 97074, Würzburg, Germany
E-mail: behr@physik.uni-wuerzburg.de

1 Introduction

Magnetic nanoparticles are used in a wide variety of biomedical applications, including magnetic particle imaging [1], magnetic drug targeting [2] and magnetic particle based hyperthermia (MHT) [3].

The dynamics of magnetic particles in rotating magnetic fields is of particular interest in, e.g., bio-assay applications [4] [5], for magnetization switching dynamics in single-domain particles [6], or for optimizing the heat dissipation in MHT [7] [8].

In [4], the nonlinear rotation of a magnetic microsphere in a weak rotating magnetic field is used as a highly sensitive detection method, e.g., for the detection of single bacteria. This nonlinear rotation occurs below a critical magnetic field strength, i.e., when the strength of the rotating magnetic field is too weak to rotate the microsphere synchronously. This nonlinear rotation has an average rotation rate below the rotation frequency of the external field and is strongly dependent on the viscous friction. By using particles with functionalized surfaces, corresponding analytes can bind to the surface and change the viscous friction of the particle. This binding can then be detected and quantified by the change of the average rotation rate of the particle.

Rotational drift spectroscopy [9] [10] aims to measure this nonlinear rotation inductively via the macroscopic magnetization of large ensembles of magnetic nanoparticles, that are suspended in a liquid and exhibiting only Brownian dynamics. Smaller particle sizes increase the relative change in viscous friction for a given analyte bound to a functionalized particle surface, but stochastic thermal interactions also become more important. In addition, the measurement of the nonlinear rotation via the macroscopic magnetization of large particle ensembles requires the alignment of all magnetic moments at the beginning of a measurement, because the orientation of the magnetic moment does not correlate with the rotating magnetic field if its amplitude is below the critical magnetic field strength. Otherwise, the magnetic moments will add up to zero macroscopic magnetization. In [9] this alignment is done by applying a magnetic field pulse before the measurement. This alignment is lost over time due to variations in rotational particle mobility and thermal random interactions, which limits the measurement time after the magnetic field pulse.

In [11] the dynamic of a magnetic particle in a rotating magnetic field showing only Brownian dynamics is simulated using the Langevin equation [13]. In [14] the rotation of the magnetic moment of a particle with uniaxial magnetic anisotropy is numerically studied by using the appropriate Fokker-Planck equation. This accounts for both Néel relaxation and Brownian relaxation. Extensive numerical frameworks are given in [15] for the Brownian dominated case and in [16] for accounting for the coupling of Néel relaxation and Brownian relaxation.

The presented work derives a closed solution to the deterministic part of the Langevin equation [11], i.e., the equation of motion for a spherical magnetic particle in a rotating magnetic field with the magnetization fixed inside the particle, with negligible angular momentum and without stochastic forces. This 3D vectorial differential equation can be expressed in the rotating frame in spherical coordinates as a set of two coupled nonlinear first-order differential equations. The analytical solution to this set of equations is derived in this work. It can be used as reference point for numerical simulations and as a basis for constructing numerical approximations for more general cases. When stochastic forces are included and the rotating axis is kept fixated, the dynamic of the magnetic pole of the particle becomes equivalent to the giant acceleration of free diffusion of a particle in a tilted periodic potential [12].

2 Methods

The basic equation to describe the dynamic of the magnetic particle in a viscous fluid neglecting the inertial term is given by [11]

$$\frac{d\mathbf{m}}{dt} = \frac{1}{\zeta}(\mathbf{m} \times \mathbf{B}) \times \mathbf{m} \quad (1)$$

where ζ describes the particle properties and is given by $\zeta = \eta\kappa V$ (η - dynamic viscosity, κ - shape factor, V - particle volume).

This equation of motion is the balances of the torque caused by the magnetic field and the torque caused by the viscous friction.

The presented work is restricted to $\mathbf{m}$ being fixated inside the particle with the norm of the magnetization vector $\mathbf{m}$ being preserved and $\mathbf{B}$ is a constant magnetic field vector rotating around a fixed axis with a constant rotating frequency ω_0 . Hence, the tip of the magnetization vector only moves on the surface of a sphere and Eq. (1) describes a rotation around the current axis of rotation $(\mathbf{m} \times \mathbf{B})/\zeta$. Thus, in the following the magnetization will be written in the form

$$\mathbf{m}[t] = m\mathbf{n}_m[t] \quad (2)$$

where m is the constant absolute value of the magnetization and $\mathbf{n}_m[t]$ is the time dependent unit vector in the direction of $\mathbf{m}$ and Eq. (1) can be given in the form

$$\frac{d\mathbf{n}_m}{dt} = \frac{m}{\zeta}(\mathbf{n}_m \times \mathbf{B}) \times \mathbf{n}_m. \quad (3)$$

2.0.1 Transition to the rotating frame of reference

For the problem under investigation it can be assumed without loss of generality that a magnetic field with constant amplitude B rotates around the z -axis with constant angular velocity ω_0 . For further treatment it is advantageous to change the frame of reference to the coordinate system rotating with the magnetic field vector where the external magnetic field becomes time independent. Hence the time derivative can be written in the form [17]

$$\left(\frac{d\mathbf{n}_m}{dt}\right)_r = \frac{d\mathbf{n}_m}{dt} - \omega_0\mathbf{e}_z \times \mathbf{n}_m \quad (4)$$

$$= \left(\frac{m}{\zeta}(\mathbf{n}_m \times \mathbf{B}) - \omega_0\mathbf{e}_z\right) \times \mathbf{n}_m. \quad (5)$$

As the external field vector has a constant amplitude B it can be written in the form $\mathbf{B} = B\mathbf{n}_B$. Introducing the dimensionless time $\tau = \omega_0 t$ with $d\tau = \omega_0 dt$ one may rewrite Eq. (5) in dimensionless variables

$$\left(\frac{d\mathbf{n}_m}{d\tau}\right)_r = \boldsymbol{\Omega} \times \mathbf{n}_m \quad (6)$$

with the vector $\boldsymbol{\Omega}$ given by

$$\boldsymbol{\Omega} = \alpha(\mathbf{n}_m \times \mathbf{n}_B) - \mathbf{e}_z \quad (7)$$

and the constant

$$\alpha = \frac{mB}{\omega_0 \zeta}. \quad (8)$$

It should be noted that the unit vector in z -direction is the same in the laboratory and the rotating frame of reference. As expected, Eq. (5) can be written as a differential equation of a rotation with rotation axis $\boldsymbol{\Omega}$. Thus, it is important to investigate the temporal behavior of $\boldsymbol{\Omega}$. Especially a constant direction of $\boldsymbol{\Omega}$ has direct consequences. Using

$$\boldsymbol{\Omega} = \Omega \mathbf{n}_\Omega \quad (9)$$

this would require

$$0 \stackrel{!}{=} \left(\frac{d\mathbf{n}_\Omega}{d\tau} \right)_r \quad (10)$$

$$\Rightarrow \left(\frac{d\boldsymbol{\Omega}}{d\tau} \right)_r = \mathbf{n}_\Omega \left(\frac{d\Omega}{d\tau} \right)_r, \quad (11)$$

where we used $\boldsymbol{\Omega} \cdot \boldsymbol{\Omega} = \Omega^2$. In this case $d\boldsymbol{\Omega}/d\tau \parallel \boldsymbol{\Omega}$ holds and Eq. (6) is the differential equation with an uni-axial rotation around $\mathbf{n}_\Omega$ but variable angular velocity $\Omega[t]$. As shown in Appendix A this holds if

$$0 = \omega_0 \cdot \mathbf{B} \quad (12)$$

holds. It must be emphasized that it is necessary to solve a differential equation for $\Omega[\tau]$ to obtain the absolute value of the current angular velocity. However, the vector $\mathbf{n}_\Omega$ can be easily obtained from the initial conditions:

$$\boldsymbol{\Omega}[\tau = 0] = \boldsymbol{\Omega}_0 = (\alpha(\mathbf{n}_{m0} \times \mathbf{n}_B)) - \mathbf{e}_z \quad (13)$$

$$\mathbf{n}_\Omega = \mathbf{n}_\Omega[t = 0] = \mathbf{n}_{\Omega_0} = \frac{\boldsymbol{\Omega}_0}{\sqrt{\boldsymbol{\Omega}_0 \cdot \boldsymbol{\Omega}_0}} = \frac{\boldsymbol{\Omega}}{\sqrt{\boldsymbol{\Omega} \cdot \boldsymbol{\Omega}}} \quad (14)$$

with the definitions $\mathbf{n}_{m0} = \mathbf{n}_m[\tau = 0]$. This is the simplest case for a rotating external magnetic field [4] and will be investigated closer.

For this purpose it is convenient to introduce an orthonormal basis in the rotating frame of reference. As Eq. (12) holds and ω_0 as well as $\mathbf{B}$ are constant in the rotating frame of reference their unit vectors $\mathbf{n}_B$ and ω_0 can be used to construct a frame of reference. Without loss of generality one can write:

$$\mathbf{e}_x = \mathbf{n}_B \quad (15)$$

$$\mathbf{e}_y = \mathbf{e}_z \times \mathbf{e}_x = \mathbf{n}_{\omega_0} \times \mathbf{n}_B \quad (16)$$

$$\mathbf{e}_z = \frac{\omega_0}{\omega_0} = \mathbf{n}_{\omega_0}. \quad (17)$$

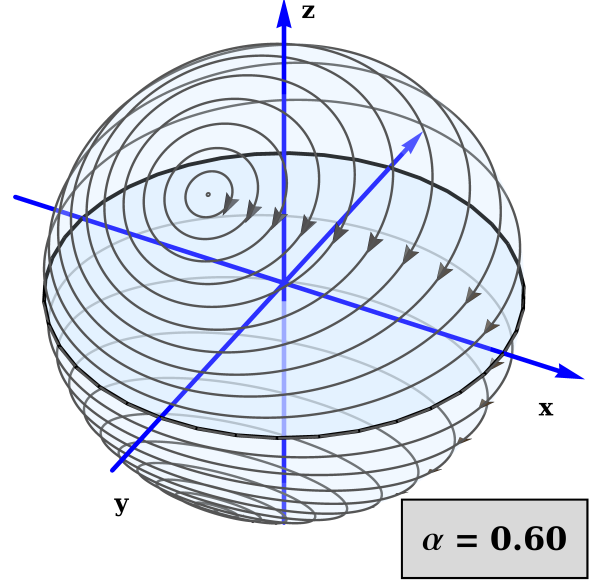


Fig. 1 In the rotating frame the motion of magnetic moments pointing outward the rotating frame move in circles. At a special angle the circle shrinks to a single point. Shown is an example for the normalized friction coefficient $\alpha = 0.6$.

These three vectors constitutes a rotating frame of reference suitable for the description of the problem as it does not depend on initial values of the magnetization and is hence easy to transform back into the laboratory frame. It should be noted that this rotating frame of reference is a special case of the frame of reference used to obtain Eq. (6).

For further investigation it is also helpful to transform Eq. (5) in the spherical coordinates derived from this frame of reference

$$\frac{d\theta}{d\tau} = \alpha \cos[\theta] \cos[\varphi] \quad (18)$$

$$\frac{d\varphi}{d\tau} = -\frac{\alpha}{\sin[\theta]} \sin[\varphi] - 1 \quad (19)$$

where $m[t] = m = \text{const.} \Rightarrow dm/d\tau = 0$ was used. These equations will be especially helpful in characterizing the types of motion in the rotating frame of reference and finding limiting cases of this motion.

3 Results

3.1 Stationary solutions and limiting cases in the rotating frame of reference

In this section the equations of motion in spherical coordinates shall be used to classify the type of motion and study the limiting cases of this motion more closely.

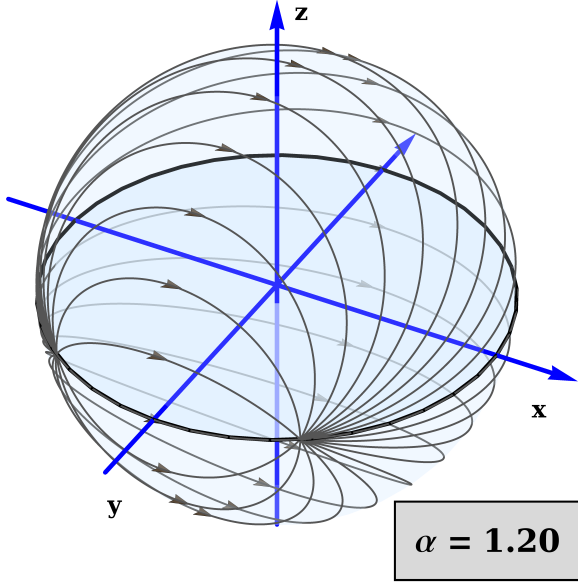


Fig. 2 For $\alpha > 1$ the magnetic moments move to a fix point in the rotating plane.

For stationary points both the angular velocities $d\theta/d\tau$ and $d\varphi/d\tau$ in Eqs. (18) and (19) must vanish simultaneously

$$0 \stackrel{!}{=} \alpha \cos[\theta] \cos[\varphi] \quad (20)$$

$$0 \stackrel{!}{=} \alpha \frac{\sin[\varphi]}{\sin[\theta]} + 1. \quad (21)$$

Hence, it is advantageous to start with the first equation. As the zeros can be easily found for this equation. It becomes zero for the following conditions:

$$\cos[\varphi] = 0 \Rightarrow \varphi_s \in \{\pi/2, 3\pi/2\} \quad (22)$$

$$\cos[\theta] = 0 \Rightarrow \theta_s = \pi/2. \quad (23)$$

Please note that the second solution for θ is outside the definition range of spherical coordinates. Thus, one can conclude that $d\theta/d\tau = 0$ if θ approaches $\pi/2$ for all values of φ . Thus the motion of the magnetization is confined to the hemisphere it was located at the beginning as it cannot cross the equatorial plane. Furthermore, magnetization which is initially located in the equatorial plane ($\theta[\tau = 0] = \pi/2$) remains there. It is important to note that these conditions can be fulfilled for any value of α . In the next step it is necessary to find the corresponding solutions of the other equation for stationary points so that both angular velocities vanish.

3.1.1 $\theta_s = \pi/2$ - condition

Substituting $\theta_s = \pi/2$ in Eq. (19) and setting the derivative to zero one finds

$$0 \stackrel{!}{=} \alpha \sin[\varphi] + 1 \Rightarrow \sin[\varphi] = -\frac{1}{\alpha} \quad (24)$$

$$\Rightarrow \varphi_{\theta,s} \in \left\{ -\arcsin\left[\frac{1}{\alpha}\right], \pi + \arcsin\left[\frac{1}{\alpha}\right] \right\}. \quad (25)$$

It must be emphasized that only real angles are allowed due to the conservation of m . Hence, the condition

$$1 \leq \alpha. \quad (26)$$

for the existence of a stationary point must be fulfilled. In this case both angle velocities become zero and the magnetization is locked in this point. To categorize the type of motion the behavior of the magnetization in the vicinity of this point is investigated.

$\alpha > 1$:

Introducing the distance ρ to the stationary point of interest $(\theta_s, \varphi_{\theta,s})$ in the φ - θ -plane with relative angles $\Delta\theta = \theta - \theta_s$ and $\Delta\varphi = \varphi - \varphi_{\theta,s}$ one yields

$$\rho = \sqrt{\Delta\varphi^2 + \Delta\theta^2} \quad (27)$$

$$\Rightarrow \frac{d\rho}{d\tau} = \frac{1}{2\rho} \left(2\Delta\varphi \frac{d\Delta\varphi}{d\tau} + 2\Delta\theta \frac{d\Delta\theta}{d\tau} \right) \quad (28)$$

Tailoring the ODEs (18) and (19) to the first order around these points for the corresponding pairs of θ_s and $\varphi_{\theta,s}$ one can write

$$\frac{d\Delta\theta}{d\tau} \approx \mp \alpha \sqrt{1 - \left(\frac{1}{\alpha}\right)^2} \Delta\theta. \quad (29)$$

The corresponding approximation to first order for $d\Delta\varphi/d\tau$ are

$$\frac{d\Delta\varphi}{d\tau} \approx \mp \alpha \sqrt{1 - \left(\frac{1}{\alpha}\right)^2} \Delta\varphi. \quad (30)$$

In both cases the upper sign corresponds to the stationary point $(\theta_s = \pi/2, \varphi_{\theta,s} = -\arcsin[1/\alpha])$ while the lower sign corresponds to $(\theta_s = \pi/2, \varphi_{\theta,s} = \pi + \arcsin[1/\alpha])$. Substituting Eqs. (29) and (30) in Eq. (28) results in

$$\lambda = \alpha \sqrt{1 - \left(\frac{1}{\alpha}\right)^2} \quad (31)$$

$$\frac{d\rho}{d\tau} = \mp \lambda \rho \quad (32)$$

with the solution

$$\rho[\tau] \propto e^{\mp \lambda \tau}. \quad (33)$$

The upper sign ("−") corresponds to a stable behavior as for increasing the distance ρ to this stationary point is reduced exponentially and magnetization accumulates. Hence, one can conclude that $(\theta_s = \pi/2, \varphi_{\theta,s} = -\arcsin[1/\alpha])$ is a stable fix point of the temporal evolution of the magnetization. The lower sign (+) marks an instable stationary point as any distance ρ in the vicinity to this stationary point increases exponentially in time.

It is important to note that these results are independent from the initial condition. As long as the magnetization vector is sufficiently close to the lock point in the φ - θ -plane, it will obey (33). Hence, all trajectories of the magnetization in the vicinity of the lock points will intersect at these very points and approach or descend them radially in the φ - θ -plane.

$\alpha = 1$:

This case has to be treated separately. From Eqs. (20) and (21) for $\alpha = 1$ one can see that both points of stability degenerate to a single stationary point at $\theta = \pi/2$ and $\varphi = 3\pi/2$. With the angles $\Delta\varphi = \varphi - 3\pi/2$ and $\Delta\theta = \theta - \pi/2$ a series expansion of Eqs. (18) and (18) in these variables can be performed. Contrary to the other cases, for $\alpha = 1$ the linear terms vanishes and one finds

$$\frac{d\Delta\theta}{d\tau} = -\Delta\theta \Delta\varphi \quad (34)$$

$$\frac{d\Delta\varphi}{d\tau} = \frac{1}{2} (\Delta\theta^2 - \Delta\varphi^2) \quad (35)$$

Introducing the complex variable $z = \Delta\varphi + i\Delta\theta$ the differential equation transforms into

$$\frac{dz}{d\tau} = -\frac{1}{2}z^2. \quad (36)$$

Thus the solution for Eqs. (34) and (35) is given by

$$\Delta\varphi = \frac{2(\tau + A)}{(\tau + A)^2 + B^2} \quad (37)$$

$$\Delta\theta = \frac{-2B}{(\tau + A)^2 + B^2}. \quad (38)$$

For $\tau \rightarrow \infty$ one finds that $\Delta\varphi = 2/\tau$ and $\Delta\theta = -2B/\tau^2$. Clearly, the magnetization vector approaches the lock point asymptotically on a parabolic curve with decreasing positive value of $\Delta\varphi$. The sign of $\Delta\theta$ depends solely on the sign of B . As in the case of $\alpha > 1$ it is interesting to find out how magnetization can potentially leave the lock point. As A can be interpreted as initial time τ_0 one can investigate this case by setting $A \rightarrow -\infty$. In this case one finds $\Delta\varphi = 2/(\tau + A)$ and $\Delta\theta = -2B/(\tau + A)^2$. Thus, the magnetization vector leaves the lock point with decreasing value τ . Furthermore, the sign of $\Delta\theta$ still only depends on the sign

of B . Thus one can conclude that the magnetization moves on parabolic curves from larger values of $\Delta\theta$ to smaller ones. However while the magnetization departs from the lock point for $\Delta\varphi < 0$ and approaches it for $\Delta\varphi > 0$. This can be viewed as the merging of both stationary points for $\alpha > 1$ to a single one, by keeping the motion direction of the magnetization vector.

3.1.2 $\varphi = 3\pi/2$ - condition

Using the φ -condition for vanishing angular velocity in the θ direction (see Eq. (22)) one finds for Eq. (21)

$$0 \stackrel{!}{=} \mp \frac{\alpha}{\sin[\theta]} - 1 \quad (39)$$

$$\Rightarrow \sin[\theta] = \mp \alpha \quad (40)$$

where the minus sign corresponds to $\varphi_s = \pi/2$ and the plus sign to $\varphi_s = 3\pi/2$. Keeping in mind that in spherical coordinates $\theta \in [0, \pi]$ the solution with the minus sign can be ignored and only the solution with $\varphi_s = 3\pi/2$ has to be considered. As only real angles are allowed ($m = \text{const.}$)

$$\alpha < 1 \quad (41)$$

follows. This is the opposite case to Eq. (26). In this case no stationary points exist in the equatorial plane. However, also here stationary points can occur. Taking the relevant solution from Eq. (40) one finds

$$\theta_{\varphi,s} \in \{\arcsin[\alpha], \pi - \arcsin[\alpha]\}. \quad (42)$$

which are both possible.

However, the interpretation of these points is important, e.g. if these are also points of intersection of the trajectories. A possible explanation is a vanishing net torque on the magnetization in ODE (6). One possibility is $\mathbf{\Omega} = 0$ which is the explanation for the stationary points Eq. (25) as can be seen from Eq. (7). The second is $\mathbf{n}_{m0} \parallel \mathbf{n}_{\Omega}$ which means their vector product must vanish. Using Eq. (14) one may write

$$0 \stackrel{!}{=} \mathbf{n}_{0m} \times \mathbf{n}_{\Omega} \quad (43)$$

$$= \mathbf{n}_{m0} \times (\alpha \mathbf{n}_{m0} \times \mathbf{n}_B - \mathbf{e}_z) \quad (44)$$

Using the Graßmann identity and the fact that $\mathbf{n}_B = \mathbf{e}_x$ this becomes

$$\begin{aligned} 0 = & \left(\alpha \left((\mathbf{n}_{m0} \cdot \mathbf{e}_x)^2 - 1 \right) - (\mathbf{n}_{m0} \cdot \mathbf{e}_y) \right) \mathbf{e}_x \\ & + (\mathbf{n}_{m0} \cdot \mathbf{e}_x) (\alpha (\mathbf{n}_{m0} \cdot \mathbf{e}_y) + 1) \mathbf{e}_y \\ & + \alpha (\mathbf{n}_{m0} \cdot \mathbf{e}_x) (\mathbf{n}_{m0} \cdot \mathbf{e}_z) \mathbf{e}_z. \end{aligned} \quad (45)$$

For the last component to vanish one of the scalar products in the z -component must vanish. The condition

which is the Bloch differential equation of an uniaxial rotation with time dependent angular velocity Ω . Obviously the magnetization parallel to $\mathbf{n}_\Omega$ given by $n'_{m\parallel} = n'_{mz}$ does not change. Hence it immediately follows that

$$n'_{mx}{}^2 + n'_{my}{}^2 = n_{m\perp}{}^2 = \text{const.} \quad (57)$$

and one can write

$$n'_{mx} = n_{m\perp} \cos[\varphi'] \quad (58)$$

$$n'_{my} = n_{m\perp} \sin[\varphi'] \quad (59)$$

where φ' is the angle between $\mathbf{n}_B$ and magnetization component transverse to $\mathbf{n}_\Omega$. Thus one find

$$\left(\frac{dn'_{mx}}{d\tau}\right)_l = -\Omega n_{m\perp} \sin[\varphi'] \quad (60)$$

$$\left(\frac{dn'_{my}}{d\tau}\right)_l = \Omega n_{m\perp} \cos[\varphi'] \quad (61)$$

which is fulfilled if the condition

$$\frac{d\varphi'}{d\tau} = \Omega = -(\alpha n_{m\perp} \sin[\varphi'] + (\mathbf{e}_z \cdot \mathbf{e}'_z)) \quad (62)$$

holds.

It is important to emphasize that this is indeed a nonlinear ODE as Ω is a nonlinear function of φ' . However, the right hand side of Eq. (62) does not contain the time explicitly and hence the ODE is separable and the formal solution is

$$\int_{\varphi'[0]}^{\varphi'[\tau]} \frac{d\phi'}{\Omega[\phi']} = \tau. \quad (63)$$

The solution of this differential equation is given in Appendix C and with the parameter

$$\alpha' = n_{m\perp} \frac{\alpha}{(\mathbf{e}_z \cdot \mathbf{e}'_z)} = n_{m\perp} \frac{\alpha}{(\mathbf{e}_z \cdot \mathbf{n}_\Omega)} = \quad (64)$$

$$\frac{\alpha \sqrt{1 - (\mathbf{n}_{m0} \cdot \mathbf{n}_\Omega)^2}}{(\mathbf{e}_z \cdot \mathbf{n}_\Omega)} \quad (65)$$

yields

$\alpha > 1$:

$$\varphi'[\tau] = -2 \arctan \left[\alpha' - \sqrt{\alpha'^2 - 1} \right. \\ \left. \times \tanh \left[\frac{1}{2} \sqrt{\alpha'^2 - 1} (\mathbf{e}_z \cdot \mathbf{e}'_z) \tau + C_0 \right] \right] \quad (66)$$

$$C_0 = \text{arctanh} \left[\frac{\alpha' + \tan[\varphi'_0/2]}{\sqrt{\alpha'^2 - 1}} \right] \quad (67)$$

$\alpha < 1$:

$$\varphi'[\tau] = -2 \arctan \left[\alpha' + \sqrt{1 - \alpha'^2} \right. \\ \left. \times \tan \left[\frac{1}{2} \sqrt{1 - \alpha'^2} (\mathbf{e}_z \cdot \mathbf{e}'_z) \tau - C_0 \right] \right] \quad (68)$$

$$C_0 = \arctan \left[\frac{\alpha' + \tan[\varphi'_0/2]}{\sqrt{1 - \alpha'^2}} \right] \quad (69)$$

$\alpha = 1$:

$$\varphi'[\tau] = \frac{3\pi}{2} + 2 \arctan [(\mathbf{e}_z \cdot \mathbf{e}'_z) \tau + C_0] \quad (70)$$

$$C_0 = \tan \left[\frac{\pi}{4} + \frac{\varphi'_0}{2} \right] \quad (71)$$

where φ'_0 is the angle at $\tau = 0$.

Hence, the solution of Eq. (6) can be written as

$$\mathbf{n}_m[\tau] = n_{m\parallel} \mathbf{e}'_z + n_{m\perp} (\cos[\varphi'[\tau]] \mathbf{e}'_x + \sin[\varphi'[\tau]] \mathbf{e}'_y) \quad (72)$$

$$= (\mathbf{n}_{m0} \cdot \mathbf{n}_{\Omega_0}) \mathbf{n}_{\Omega_0} + \sqrt{1 - (\mathbf{n}_{m0} \cdot \mathbf{n}_{\Omega_0})^2} \\ \times (\cos[\varphi'[\tau]] \mathbf{n}_B + \sin[\varphi'[\tau]] \mathbf{n}_{\Omega_0} \times \mathbf{n}_B) \quad (73)$$

where $\varphi'[\tau]$ obeys Eq. (62) with the initial condition $\varphi'_0 = \varphi'[\tau = 0]$ and the abbreviations

$$n_{m\parallel} = \mathbf{n}_m \cdot \mathbf{n}_{\Omega_0} = \mathbf{n}_{m0} \cdot \mathbf{e}'_z = \cos[\vartheta'_0] \quad (74)$$

$$n_{m\perp} = \sqrt{1 - (\mathbf{n}_{m0} \cdot \mathbf{n}_{\Omega_0})^2} = \sin[\vartheta'_0] \quad (75)$$

were used. Obviously the vector $\mathbf{n}_m$ moves on a cone with axis $\mathbf{n}_\Omega$, however, with variable angular velocity Ω .

It is interesting to note that the choice of the coordinate system Eqs. (53),(54) and,(55) lead to the fact that Ω must be positive as the direction of the local frame of reference is always along the direction of $\mathbf{n}_\Omega$. To understand this an important aspect of the solution in the local frame of reference must be considered. The coordinate system is defined in that way, that the $\mathbf{e}'_x = \mathbf{n}_B$ is the same as in the rotating frame of reference (given by Eqs. (15), (16), and, (17)) and $\mathbf{e}'_z = \mathbf{n}_\Omega$. The third vector is chosen by the condition to obtain an orthonormal system that obeys the right hand rule with $\mathbf{e}'_y = (\mathbf{n}_\Omega \times \mathbf{n}_B)$. In this system the value Ω is always positive, as any inversion in the direction is completely mapped in the unit vector. With this inversion in the direction of the vector $\mathbf{n}_\Omega$ with the change of the initial conditions, will also cause a inversion of the $\mathbf{e}'_y$ unit vector as the vector $\mathbf{n}_B$ is the same in the rotating frame and the local frame used to calculate the solution. This has to be kept in mind if the transformation

to a different frame of reference is performed which is independent from $\mathbf{n}_\Omega$.

3.2.1 Solution in the rotating frame of reference

Although the local frame of reference is advantageous to find the solution of the ODE (6) its orientation depends on the orientation of Ω and hence on the initial condition via Eqs. (13) and (14). Hence, it is advantageous to use a frame of reference independent of Ω . The rotating frame of reference (Eqs. (15), (16), and, (17)) can be used. To transform the solution Eq. (72) the unit vectors of the local frame of reference has to be transformed in the rotating frame of reference:

$$\begin{aligned}\mathbf{e}'_y &= (\mathbf{n}_\Omega \times \mathbf{e}_x) \\ &= (\mathbf{e}_z \cdot \mathbf{n}_\Omega) \mathbf{e}_y - (\mathbf{e}_y \cdot \mathbf{n}_\Omega) \mathbf{e}_z\end{aligned}\quad (76)$$

$$\begin{aligned}\mathbf{e}'_z &= \mathbf{n}_\Omega \\ &= (\mathbf{n}_\Omega \cdot \mathbf{e}_y) \mathbf{e}_y + (\mathbf{n}_\Omega \cdot \mathbf{e}_z) \mathbf{e}_z\end{aligned}\quad (77)$$

where again the cyclic property of the triple product was used. Furthermore, it must be kept in mind that the x -axis in both frame of references is the same (see Eq. (53)). This also implies that no x -component in the unit vectors $\mathbf{e}'_z$ and $\mathbf{e}'_y$ exist. Substitution of Eqs. (76) and (77) in Eq. (72) yields

$$\mathbf{e}_x + \mathbf{n}_{\Omega,r} = \begin{pmatrix} 1 \\ \mathbf{n}_\Omega \cdot \mathbf{e}_y \\ \mathbf{n}_\Omega \cdot \mathbf{e}_z \end{pmatrix}\quad (78)$$

$$\mathbf{M} = \begin{pmatrix} n_{m\perp} \cos[\varphi'[\tau]] & 0 & 0 \\ 0 & n_{m\parallel} & n_{m\perp} \sin[\varphi'[\tau]] \\ 0 & -n_{m\perp} \sin[\varphi'[\tau]] & n_{m\parallel} \end{pmatrix}\quad (79)$$

$$\mathbf{n}_{mr}[\tau] = \mathbf{M} \cdot (\mathbf{e}_x + \mathbf{n}_{\Omega,r})\quad (80)$$

where the subscript r denotes that the vector is given in the rotating frame of reference.

Alternatively one may write the result with the help of a rotation matrix. For this purpose one must recall that

$$(\mathbf{e}'_z \cdot \mathbf{e}_y) = (\mathbf{n}_\Omega \cdot \mathbf{e}_y) = \sin[-\vartheta_\Omega]\quad (81)$$

$$(\mathbf{e}'_z \cdot \mathbf{e}_z) = (\mathbf{n}_\Omega \cdot \mathbf{e}_z) = \cos[-\vartheta_\Omega]\quad (82)$$

where ϑ_Ω is the angle between $\mathbf{n}_\Omega$ and $\mathbf{e}_z$. The sign is necessary as the scalar product cannot properly define the direction of rotation and hence the orientation of the angles which, however, is fixed in the sense of rotation matrices and must be chosen correctly. Please note

that $\vartheta \in [0, \pi]$. Hence one can write Eq. (??)

$$\mathbf{n}_{ml}[\tau] = \begin{pmatrix} n_{m\perp} \cos[\varphi'[\tau]] \\ n_{m\perp} \sin[\varphi'[\tau]] \\ n_{m\parallel} \end{pmatrix}\quad (83)$$

$$\mathbf{R}_x[\vartheta_\Omega]\quad (84)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\vartheta_\Omega] & -\sin[\vartheta_\Omega] \\ 0 & \sin[\vartheta_\Omega] & \cos[\vartheta_\Omega] \end{pmatrix}\quad (85)$$

$$\mathbf{n}_{mr}[\tau] = \mathbf{R}_x[\vartheta_\Omega] \cdot \mathbf{n}_{ml}[\tau]\quad (86)$$

with the orientation of the magnetization in local coordinates given by $\mathbf{n}_{ml}[\tau]$ as well as the rotation matrix for a rotation around the x -axis with the angle ϑ_Ω given by $\mathbf{R}_x[\vartheta_\Omega]$.

The rotation matrix approach can be extended to describe the solution completely in the rotating frame. Introducing

$$\Delta\varphi'[\tau] = \varphi'[\tau] - \varphi'_0\quad (87)$$

one can write Eq. (72) as

$$\begin{aligned}\mathbf{n}_{ml}[\tau] &= n_{m\parallel} \mathbf{e}'_z \\ &+ n_{m\perp} (\cos[\Delta\varphi'[\tau]] \cos[\varphi'_0] - \sin[\Delta\varphi'[\tau]] \sin[\varphi'_0]) \mathbf{e}'_x \\ &+ n_{m\perp} (\sin[\Delta\varphi'[\tau]] \cos[\varphi'_0] + \cos[\Delta\varphi'[\tau]] \sin[\varphi'_0]) \mathbf{e}'_y\end{aligned}\quad (88)$$

$$\Rightarrow \mathbf{n}_{ml}[\tau] = \mathbf{R}_z[\Delta\varphi'[\tau]] \cdot \mathbf{n}_{ml}[0]\quad (89)$$

with

$$\mathbf{R}_z[\Delta\varphi'[\tau]] = \begin{pmatrix} \cos[\Delta\varphi'[\tau]] & -\sin[\Delta\varphi'[\tau]] & 0 \\ \sin[\Delta\varphi'[\tau]] & \cos[\Delta\varphi'[\tau]] & 0 \\ 0 & 0 & 1 \end{pmatrix}\quad (90)$$

was used. Knowing that the transformation between the local frame of reference and the rotating frame of reference is performed by a rotation around the x -axis as seen in Eq. (86), one can immediately write

$$\mathbf{n}_{mr}[\tau]\quad (91)$$

$$= \mathbf{R}_x[\vartheta_\Omega] \cdot \mathbf{R}_z[\Delta\varphi'[\tau]] \cdot \mathbf{R}_x[-\vartheta_\Omega] \cdot \mathbf{n}_{mr}[0]\quad (92)$$

$$(93)$$

expressing the result as three consecutive rotations applied on the initial magnetization. Here it is important to note that only the rotation in the local frame of reference is time dependent. The other rotations are completely given by the initial conditions.

3.2.2 Solution in the laboratory frame

To obtain the solution in the laboratory frame of reference one has to remember that the z -axis in the rotating frame of reference and the laboratory frame of reference is the same. The transformation is given by an uniaxial rotation around the z -axis with the angle $\varphi_L = \omega_0 t = \tau$. The corresponding coordinate transformation is given by

$$\mathbf{e}_x = \cos[\varphi_L] \mathbf{e}_{x,L} + \sin[\varphi_L] \mathbf{e}_{y,L} \quad (94)$$

$$\mathbf{e}_y = -\sin[\varphi_L] \mathbf{e}_{x,L} + \cos[\varphi_L] \mathbf{e}_{y,L}. \quad (95)$$

This can be written in Matrixform and one finds

$$\mathbf{n}_{mL}[\tau] \quad (96)$$

$$= \mathbf{R}_z[\tau] \cdot \mathbf{n}_{mr}[\tau] \quad (97)$$

$$\mathbf{n}_L[t] = \begin{pmatrix} n_{m\perp} \cos[\varphi'[\tau]] \cos[\tau] & -n_{m\parallel} \sin[\tau] \\ n_{m\perp} \cos[\varphi'[\tau]] \sin[\tau] & n_{m\parallel} \cos[\tau] \\ 0 & -n_{m\perp} \sin[\varphi'[\tau]] \\ & -n_{m\perp} \sin[\varphi'[\tau]] \sin[\tau] \\ & n_{m\perp} \sin[\varphi'[\tau]] \cos[\tau] \\ & n_{m\parallel} \end{pmatrix} \quad (98)$$

$$\mathbf{n}_{mL}[\tau] = \begin{pmatrix} \cos[\tau] & \cos[\vartheta_\Omega] \sin[\tau] \\ -\sin[\tau] & \cos[\vartheta_\Omega] \cos[\tau] \\ 0 & -\sin[\vartheta_\Omega] \\ & \sin[\vartheta_\Omega] \sin[\tau] \\ & \sin[\vartheta_\Omega] \cos[\tau] \\ & \cos[\vartheta_\Omega] \end{pmatrix} \cdot \mathbf{n}_{ml}[\tau]. \quad (99)$$

Substituting Eq. (93) and using the fact that transformation between the rotating frame and the laboratory frame is given by a rotation around the z -axis (see Eq. (97)) one finds

$$\mathbf{n}_{mL}[\tau] \quad (100)$$

$$(101)$$

$$= \mathbf{R}_z[\tau] \cdot \mathbf{R}_x[\vartheta_\Omega] \cdot \mathbf{R}_z[\Delta\varphi'[\tau]] \cdot \mathbf{R}_x[-\vartheta_\Omega] \cdot \mathbf{R}_z[-\tau] \cdot \mathbf{n}_{mL}[0]. \quad (102)$$

It should be noted that this solution requires the initial conditions

$$\mathbf{n}_{\omega_0} = \mathbf{e}_{z,L} \quad (103)$$

$$\mathbf{n}_B[t=0] = \mathbf{e}_{x,L} \quad (104)$$

as Eq. (12) must hold for this solution. However, in the laboratory frame neither the z -axis must be aligned to ω_0 nor has $\mathbf{B}$ to be aligned with the x -axis. However, this can be solved by time independent transformation of the laboratory frame to match the desired initial conditions. This does not bring further information on the dynamics in the laboratory frame and is not performed here.

3.3 stationary solutions and limiting cases in the local frame of reference

The characterization of the solution given in Eqs. (68), (66) and (70) is cumbersome and involves the analysis of inverse trigonometric functions. Thus, it is helpful to characterize the stationary solution the ODE (62). As shown in App. C, with the substitution $\tau' = (\mathbf{e}_z \cdot \mathbf{e}'_z) \tau$ these points are characterized by:

$$0 \stackrel{!}{=} \frac{d\varphi'}{d\tau'} = -(\alpha' \sin[\varphi'] + 1) \Rightarrow \sin[\varphi'] = -\frac{1}{\alpha'}. \quad (105)$$

This equation has the same form as Eq. (19) for $\theta = \pi/2$ and hence similar results for the stationary points are obtained. As only real angles are allowed and the norm of the magnetization is preserved stationary points only exist if

$$\left| \frac{n_{m\perp} \alpha}{(\mathbf{e}_z \cdot \mathbf{n}_\Omega)} \right| = |\alpha'| \geq 1. \quad (106)$$

The two stationary points given by Eq. (106) degenerate to a single point for $|\alpha'| = 1$, similar to the results in the rotating frame of reference. The angles of the stationary point are given by:

$$\sin[\varphi'] = -\frac{(\mathbf{e}_z \cdot \mathbf{n}_\Omega)}{n_{m\perp} \alpha} = -\frac{1}{\alpha'} \quad (107)$$

$$\Rightarrow \varphi'_s \in \left\{ -\arcsin\left[\frac{1}{\alpha'}\right], \pi + \arcsin\left[\frac{1}{\alpha'}\right] \right\} \quad (108)$$

It is important to note that the first angle in Eq. (108) is the angle closer to the x' -axis as the domain of the arcsin-function is $[-\pi/2, \pi/2]$.

As Eq. (62) has the same form as Eq. (19) for $\theta = \pi/2$ the same analysis as in the laboratory frame can be exploited. Thus performing the same case analysis is necessary.

Analogous to Eq. (30) one obtains by first order series expansion in $\Delta\varphi' = \varphi' - \varphi'_s$:

$$\lambda' = |\alpha'| \sqrt{1 - \frac{1}{\alpha'^2}} \quad (109)$$

$$\Rightarrow \Delta\varphi' \propto e^{\mp \lambda' \tau'} \quad (110)$$

with upper sign corresponding to $\varphi'_s = -\arcsin[\frac{1}{\alpha'}]$ while the lower sign corresponds to $\varphi'_s = \pi + \arcsin[\frac{1}{\alpha'}]$.

It is important to note that contrary to the rotating frame analysis in the local frame of reference $\alpha' < 0$ holds (see Eq. (196)). Following the same line of arguments as in the rotating frame the upper sign corresponds to a stable solution while the lower sign is an

instable point. Hence, one can write:

$$\varphi'[\tau \rightarrow \infty] = \begin{cases} \pi + \arcsin\left[\frac{1}{\alpha'}\right] & \text{if } \varphi'[0] = \pi + \arcsin\left[\frac{1}{\alpha'}\right] \\ -\arcsin\left[\frac{1}{\alpha'}\right] & \text{else.} \end{cases} \quad (111)$$

This result describes the fact that every magnetization that is not at the unstable stationary point in the beginning will eventually move to the stable stationary point and accumulate there. Thus the case where condition Eq. (106) is fulfilled will be called lock case.

Summarizing one can write (see also App. C)

$$\begin{aligned} \text{stable:} & \quad \varphi'_s = -\arcsin\left[\frac{1}{\alpha'}\right] \\ \text{unstable:} & \quad \varphi'_s = \pi + \arcsin\left[\frac{1}{\alpha'}\right] \end{aligned}$$

$\alpha = 1$:

For $\alpha' \rightarrow 1$ the stable and unstable point converges to a semi stable point at $\varphi'_s = \pi/2$. As in the corresponding case in the rotating frame of reference no first order approximation in $\Delta\varphi'$ is possible and one obtains analogous to Eq. (34) with $\Delta\theta = 0$

$$\Rightarrow \Delta\varphi' = \frac{2}{\tau'} \quad (112)$$

where for $\tau' \rightarrow \infty$ the magnetization vector approaches the lock point $\varphi'_s = \pi/2$ clock wise. Similar one can assume that it leaves this point clockwise for $\tau' \rightarrow -\infty$. This is in complete analogy to the rotating frame.

$\alpha < 1$:

On the other hand, if the condition (106) is not fulfilled, i.e.:

$$\left| \frac{n_{m\perp}\alpha}{(\mathbf{e}_z \cdot \mathbf{n}_\Omega)} \right| = |\alpha'| < 1 \quad (113)$$

no stationary points in φ' do exist for $n_{m\perp} \neq 0$.

However, the case $n_{m\perp} = 0$ represents itself a stationary point. In this case the magnetization is along $\mathbf{e}'_z = \mathbf{n}_\Omega$ and, thus, fixed at the Ω -axis. Hence, it does not precess around $\mathbf{n}_\Omega$. Hence this condition is fulfilled if $\mathbf{n}_{m0} \parallel \mathbf{e}'_z = \mathbf{n}_\Omega$. This type of stationary point corresponds to Eq. (42) in the laboratory frame.

Assuming $n_{m\perp} = 0$ the solution of the ODE (193) becomes

$$\Delta\varphi' = -\tau'. \quad (114)$$

It is important to note that this is the special case of the lock point with $(\mathbf{e}_z \cdot \mathbf{n}_\Omega) = -\cos[\theta_{\varphi,s}] = -\sqrt{1-\alpha^2}$ (see Eqs. (42) and (194)). Hence for with the help of Eq. (190) one may write $\varphi'[\tau] = \sqrt{1-\alpha^2}\tau$. Hence in this limit the motion is described by a uniaxial rotation with constant angular velocity $\sqrt{1-\alpha^2}$ which is analogous to the result in the rotating frame.

3.3.1 Comparison between local and rotating frame of reference - conditions for stationary solutions and critical conditions

In this section the equivalence of the stationary points in the local and rotating frame of reference is shown. Before continuing it is advantageous to determine how angles transform between the local and the rotating frame of reference. As only stationary points and critical condition are investigated only the transformation $\varphi \rightarrow \varphi'$ is studied. Hence it is convenient to express the sin functions as:

$$\sin[\varphi] = (\mathbf{n}_m \times \mathbf{e}_x) \cdot \mathbf{e}_z \quad (115)$$

$$\sin[\varphi'] = (\mathbf{n}_{m\perp} \times \mathbf{e}_x) \cdot \mathbf{e}'_z \quad (116)$$

$$= \frac{(\mathbf{n}_m \times \mathbf{e}_x) \cdot \mathbf{e}'_z}{|\mathbf{n}_m - (\mathbf{n}_m \cdot \mathbf{e}'_z) \mathbf{e}'_z|} \quad (117)$$

Before continuing it is helpful to rewrite the denominator of the last line. Using local coordinates one may write $(\mathbf{n}_m \cdot \mathbf{e}'_z) = n_{m\parallel}$ (see Eq (74)). From Eq. (56) one can see that the parallel component of magnetization along $\mathbf{e}'_z$ is independent of time. Hence $|\mathbf{n}_m - (\mathbf{n}_m \cdot \mathbf{e}'_z) \mathbf{e}'_z| = n_{m\perp}$ is the absolute value component of the unit vector in magnetization direction perpendicular to $\mathbf{e}'_z$. With the representation of $\mathbf{e}'_z$ in the rotating frame of reference (77) one can write

$$\sin[\varphi'] = \frac{\cos[\vartheta_\Omega]}{n_{m\perp}} \sin[\varphi] + \frac{\sin[\vartheta_\Omega]}{n_{m\perp}} \cos[\vartheta] \quad (118)$$

As only stationary points are of interest the second addend vanishes as in this case $\theta_s = \pi/2$. Substituting Eq. (24) one finds

$$\sin[\varphi'_s] = -\frac{(\mathbf{e}'_z \cdot \mathbf{e}_z)}{n_{m\perp}} \sin[\varphi_s] \quad (119)$$

$$\Rightarrow \frac{1}{\alpha'} = \frac{(\mathbf{e}'_z \cdot \mathbf{e}_z)}{n_{m\perp}} \frac{1}{\alpha} \quad (120)$$

which is exactly the stationary condition (107).

3.4 Mean frequencies and mean periods in the local frame of reference

In the previous sections it was shown that in the drift case $1 > |\alpha'| = n_{m\perp}\alpha/|\mathbf{e}_z \cdot \mathbf{n}_\Omega|$ the magnetization precess on closed orbits. Thus, the question of the mean frequency on these orbits arise. Using Eq. (200) one may write

$$T = -\frac{1}{(\mathbf{e}_z \cdot \mathbf{n}_\Omega)} \int_0^{2\pi} \frac{d\phi'}{\alpha' \sin[\phi'] + 1} \quad (121)$$

The solution is obtained using the same substitutions as in Eq. (68)

$$T = \frac{2\pi}{(\mathbf{e}_z \cdot \mathbf{n}_\Omega) \sqrt{1 - \alpha'^2}} \quad (122)$$

$$\Rightarrow \bar{\Omega} = \frac{2\pi}{T} = (\mathbf{e}_z \cdot \mathbf{n}_\Omega) \sqrt{1 - \alpha'^2}, \quad (123)$$

with the mean frequency $\bar{\Omega}$ in the local frame of reference.

Before continuing it is helpful to consider the explicit form of α and the sign of $(\mathbf{e}_z \cdot \mathbf{n}_\Omega)$. For the second it is advantageous to consider the explicit form of Ω_0 in the rotating frame of reference (see Eq. (7)):

$$\Omega_0 = \alpha \cos[\theta_0] \mathbf{e}_y - (\alpha \sin[\theta_0] \sin[\phi_0] + 1) \mathbf{e}_z \quad (124)$$

from which follows

$$\Omega_0 \cdot \mathbf{e}_y = \alpha (\mathbf{n}_{m0} \cdot \mathbf{e}_z) = \alpha \cos[\theta_0] \quad (125)$$

$$\Omega_0 \cdot \mathbf{e}_z = -(\alpha \sin[\theta_0] \sin[\phi_0] + 1) \quad (126)$$

Furthermore, one may write

$$\Omega_0^2 = (\mathbf{e}_z \cdot \Omega_0)^2 + (\mathbf{e}_y \cdot \Omega_0)^2 \quad (127)$$

as Ω_0 has no component in x -direction.

In the drift case one can see:

$$\alpha < 1 \Rightarrow \left(\sin[\theta_0] \sin[\phi_0] + \frac{1}{\alpha} \right) > 0 \quad (128)$$

$$\Rightarrow \text{sign}[\mathbf{e}_z \cdot \mathbf{n}_\Omega] = -1 \quad (129)$$

As Eq. (64) contains $\mathbf{n}_{m0} \cdot \mathbf{n}_\Omega$ this term must be investigated first. With Eqs. (13), (55) and (125) the scalar product becomes

$$\mathbf{n}_{m0} \cdot \mathbf{n}_\Omega = \frac{\alpha}{\Omega_0} \underbrace{\mathbf{n}_{m0} \cdot (\mathbf{n}_{m0} \times \mathbf{n}_B)}_{=0} - \frac{\mathbf{n}_{m0} \cdot \mathbf{e}_z}{\Omega_0} \quad (130)$$

$$= -\frac{\mathbf{n}_{m0} \cdot \mathbf{e}_z}{\Omega_0} \quad (131)$$

$$= -\frac{\Omega_0 \cdot \mathbf{e}_y}{\alpha \Omega_0} \quad (132)$$

where Eq. (125) was used and one may write

$$\alpha' = \frac{\alpha}{(\mathbf{e}_z \cdot \mathbf{n}_\Omega)} \sqrt{1 - \frac{(\Omega_0 \cdot \mathbf{e}_y)^2}{\alpha^2 \Omega_0^2}} \quad (133)$$

which can be substituted in Eq. (123)

$$\bar{\Omega} = -\sqrt{(\mathbf{e}_z \cdot \mathbf{n}_\Omega)^2 + \frac{(\Omega_0 \cdot \mathbf{e}_y)^2}{\Omega_0^2} - \alpha^2} \quad (134)$$

$$= -\sqrt{1 - \alpha^2}. \quad (135)$$

where Eq. (127) was used. This result immediately implies that the mean drift frequency is the same for all

trajectories given a constant value of α which does not depend on the initial value of $\mathbf{n}_{m0}$. The analogous analysis for $\alpha' > 1$ shows that with the help of Eqs. (109) and (190) one finds $\lambda' \tau' = \sqrt{\alpha'^2 - 1} \tau = \lambda \tau$. Thus, the limiting case (110) takes the form

$$\Delta\varphi' \propto e^{\mp \lambda \tau} \quad (136)$$

which is also independent from the initial conditions.

3.4.1 Comparison between local and rotating frame of reference - Mean precession frequencies in the rotating frame

It must be emphasized that the frequency given in Eq. (135) cannot be easily transformed into the laboratory frame. The reason is, that the axes of rotations are not parallel. Furthermore, it was shown that even in the drift case stationary points exist (see Eq. (42)) and the magnetization in its vicinity precesses around this point (see Eqs. (50) and (51)). Hence, after transforming into the rotating frame of reference the mean frequency becomes zero. This is equivalent to the fact that the orbit does not include the pole of the corresponding hemisphere. In this case the azimuthal angle in the rotating frame of reference will not span the full range of 2π during a single precession. Consequently, the magnetization will be locked and $\bar{\Omega}_r = 0$. Thus, despite this case is called drift case the magnetization is locked to the external field and the mean drift frequency in the laboratory frame will be

$$\bar{\omega}_L = \omega_0. \quad (137)$$

It is the mean rotation frequency of the magnetization in the x - y -plane of this frame of reference. In the case the orbit is around the pole the azimuthal angle gains the full phase of 2π during a single precession. Hence, a real drift of the magnetization is observable. In this case a mean frequency of

$$\bar{\omega}_L = \omega_0 \left(1 - \sqrt{1 - \alpha^2} \right) \quad (138)$$

is expected. Hence one must distinguish the following cases:

$$\begin{aligned} |n_{m\parallel}/m| < |\mathbf{e}_z \cdot \mathbf{e}'_z| &: \quad \bar{\omega}_L = \omega_0 \\ |n_{m\parallel}/m| > |\mathbf{e}_z \cdot \mathbf{e}'_z| &: \quad \bar{\omega}_L = \omega_0 (1 - \sqrt{1 - \alpha^2}) \end{aligned}$$

The case of $|n_{m\parallel}/m| = |\mathbf{e}_z \cdot \mathbf{e}'_z|$ is in that sense undetermined as one can argue both cases, depending on the question how much azimuthal angle is accumulated when hitting the pole.

It is important to emphasize that this dynamic lock is more a mathematical aspect than a real physical

property. Also in the dynamic lock case the magnetization performs a periodic motion on the rotating frame of reference. Hence, in the spectral analysis frequency components at the mean drift frequency $\omega_0\sqrt{1-\alpha^2}$ will be present in this case. The vanishing of the accumulated phase is caused by the fact that varying magnetization does not encircle the z -axis anymore which is more an attribute of the coordinate system rather than the physics of the system. Hence, the signal will vary smooth between both cases.

In the limiting cases one obtains ($|n_{m\parallel}/m| > |\mathbf{e}_z \cdot \mathbf{e}'_z|$):

$$\alpha \ll 1 : \quad \bar{\omega}_L \approx \omega_0 \alpha^2 \quad (139)$$

$$\alpha \approx 1 : \quad \bar{\omega}_L \approx \omega_0 \left(1 - \sqrt{2(1-\alpha)}\right). \quad (140)$$

Consequently, for small fields the drift frequency reaches zero, while for field strengths close to the critical $B_c = \zeta\omega_0/m$ the second addend in Eq. (140) vanishes and the drift frequency approaches the rotation frequency of the external field vector $\mathbf{B}$.

3.5 Special case $0 = \mathbf{n}_{m0} \cdot \mathbf{n}_\Omega$

In this part the special case when $\mathbf{m}_0$ is perpendicular to $\mathbf{n}_\Omega$ is studied. This corresponds to the case in [4], where the rotation axis is essentially fixed since the microparticle of the sensor is rotating on a substrate. In this case only the transverse component exists in Eq. (72). To find the initial condition of this special case one can write (see Eq. (131))

$$0 = \mathbf{n}_{m0} \cdot \mathbf{n}_\Omega = \mathbf{n}_{0m} \cdot \mathbf{e}'_z = \mathbf{n}_{0m} \cdot \mathbf{e}_z \quad (141)$$

Obviously this is fulfilled if the magnetization is initially in the x - y -plane. Keeping in mind that $\mathbf{B}$ is in the x - y -plane too, $\mathbf{n}_\Omega$ takes the form

$$\mathbf{n}_\Omega = \frac{d\varphi}{d\tau} \mathbf{n}_\Omega = -(\alpha \sin[\varphi] + 1) \mathbf{e}_z. \quad (142)$$

From this directly follows that

$$\mathbf{n}_\Omega = -\mathbf{e}_z \quad (143)$$

$$n_{m\perp} = 1 \quad (144)$$

and the differential equation for the angle

$$\frac{d\varphi}{d\tau} = -\alpha \sin[\varphi] - 1. \quad (145)$$

As φ is the angle compared to $\mathbf{B}$ the first addend has been taken with a minus sign.

Obviously, the motion of the magnetization is constrained to the x - y -plane, which is self consistent with the assumption made. In this case the local and the rotating frame of reference are essentially the same. Only

the unit vectors may differ in sign and, hence, direction along the coordinate axes. The ODE (145) is separabel and can be integrated directly (see also Eq. (200) and following equations)

$$\int_{\varphi[0]}^{\varphi[\tau]} \frac{d\varphi'}{-\alpha \sin[\varphi'] - 1} = \tau. \quad (146)$$

This is exactly the form of Eq. (200) $n_{m\perp} = 1$ and $\mathbf{e}_z \parallel \mathbf{e}'_z$. Hence, the solution of this equation can be found as:

$\alpha < 1$:

$$\varphi[\tau] = -2 \arctan \left[\alpha + \sqrt{1-\alpha^2} \tan \left[\frac{\sqrt{1-\alpha^2}\tau}{2} - C_0 \right] \right] \quad (147)$$

$$C_0 = \arctan \left[\frac{\alpha + \tan[\varphi_0/2]}{\sqrt{1-\alpha^2}} \right]. \quad (148)$$

$\alpha > 1$:

$$\varphi[\tau] = -2 \arctan \left[\alpha - \sqrt{\alpha^2-1} \tanh \left[\frac{\sqrt{\alpha^2-1}\tau}{2} + C_0 \right] \right] \quad (149)$$

$$C_0 = \operatorname{arctanh} \left[\frac{\alpha + \tan[\varphi_0/2]}{\sqrt{\alpha^2-1}} \right], \quad (150)$$

$\alpha = 1$:

In this case the integral considerably simplifies and no further substitution is needed

$$\varphi[\tau] = -2 \arctan \left[\frac{2}{\tau + C_0} - 1 \right] \quad (151)$$

$$C_0 = \frac{2}{1 + \tan[\varphi_0/2]}, \quad (152)$$

or

$$\varphi[\tau] = \frac{\pi}{2} - 2 \arctan \left[\tau + \tan \left[\frac{\pi}{4} - \frac{\varphi_0}{2} \right] \right] \quad (153)$$

Hence the 2D magnetization with Eqs. (147), (149) and (151) or (153)

$$\mathbf{n}_m[\tau] = (\cos[\varphi[\tau]] \mathbf{e}_x + \sin[\varphi[\tau]] \mathbf{e}_y) \quad (154)$$

which is essentially a rotation in the x - y -plane with variable angular velocity Ω . It must be emphasized that the constraint in Eq. (??) requires real values $\varphi[\tau]$ as complex values would change m over time.

The general treatment of the stationary points can be easily adapted in this case. As the motion is confined to the x - y -plane ($\theta = \pi/2$) the time derivative of the polar angle θ vanishes directly. Hence, only the ODE in φ has to be considered. As ODE (145) has the same form as Eq. (19) and the corresponding knowledge on the stationary points can be used. Obviously one of the two stationary points in the x - y -plane is stable while the

other is instable. Also a analogous treatment in the local reference system is possible. For the 2D problem the stationary points in the drift case do not exist. Hence to obtain these points only the lock case has to be considered. The result is the same as in Eq. (25).

As already mentioned only closed orbits exist in the drift case without any point of stability. As has been shown the mean frequencies are independent from the initial value of $\mathbf{n}_{m0}$ and the results for the 3D case can be used.

4 Conclusion and discussion

It is shown that the equation (1), which is a nonlinear differential equation describing the 3D movement of a magnetic moment in a rotating magnetic field, can be solved analytically. The parameter α , the normalized friction coefficient, is the key parameter for determining the characteristic of the motion of the magnetic moment in the rotating magnetic field. There are essentially three cases to be described: $\alpha > 1$ describes the locked case, showing a stable and an instable fixpoint in the rotating plane. All magnetic moments move into the stable fixpoint. $\alpha = 1$ describes the transition point between locked case and rotational drift case. In this case, the instable and stable fixpoint become one. There the rotational friction matches exactly the magnetic moment of the rotating magnetic field. This point is also instable. An infinitesimal small disturbance drives the magnetic moment out of the point into a movement of a full circle. Finally there is the rotational drift case $\alpha < 1$. The key properties of this case are illustrated in Fig. 4. In the rotating frame, the motion of the magnetic moment on the unit sphere results in tilted circles for $\alpha < 0$ outside the rotating plane. Any orientation of the magnetic moment results in a rotation around a fixed rotation axis. At $\alpha < 0$ there is no fixpoint, but there is the special case on the two hemispheres where the circle motion shrinks into a point without any movement. This is reflected in α' , which was defined in (64) as:

$$\alpha' = n_{m\perp} \frac{\alpha}{(\mathbf{e}_z \cdot \mathbf{e}'_z)} = n_{m\perp} \frac{\alpha}{(\mathbf{e}_z \cdot \mathbf{n}_\Omega)} \quad (155)$$

The most important interpretation of this equation is, that the circular motions depicted in ?? can be described in the same way as with the value α for the case, where the magnetic moment rotates in a plane. The case where the circular motion degenerates into a point is therefore expressed as $\alpha' = 1$, i.e., the rotational friction of the magnetic moment at that angle cancels out with the effective magnetic torque.

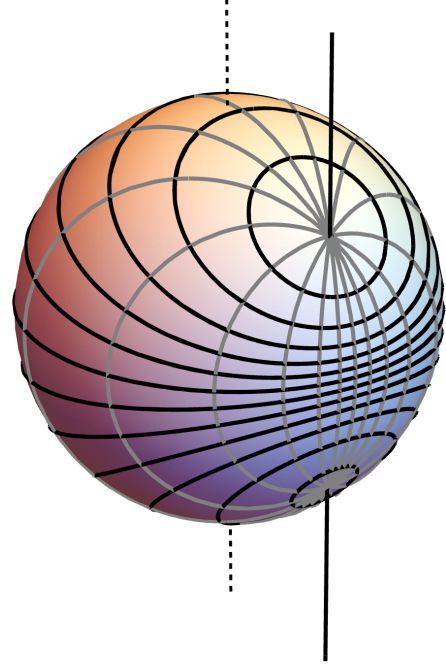


Fig. 4 The most interesting case is when $\alpha < 1$, i.e., when the magnetic moment is not in a locked state with the rotating magnetic field. Shown is an example for $\alpha = 0.8$. The black lines show the movements of the magnetic moment on the unit sphere for different orientations. The gray lines are isochrones. Their intersection coincides with the line going through the fix points. The angular velocity of all magnetic moment orientation is the same for the angle relative to this axis when projected onto the rotation plane.

The main result of this paper can be summarized in the three equations (66), (68) and (70), corresponding to the three cases $\alpha > 1$, $\alpha < 1$ and $\alpha = 1$:

$\alpha > 1$:

$$\varphi'[\tau] = -2 \arctan \left[\alpha' - \sqrt{\alpha'^2 - 1} \right] \times \tanh \left[\frac{1}{2} \sqrt{\alpha'^2 - 1} (\mathbf{e}_z \cdot \mathbf{e}'_z) \tau + C_0 \right] \quad (156)$$

$$C_0 = \operatorname{arctanh} \left[\frac{\alpha' + \tan[\varphi'_0/2]}{\sqrt{\alpha'^2 - 1}} \right] \quad (157)$$

$\alpha < 1$:

$$\varphi'[\tau] = -2 \arctan \left[\alpha' + \sqrt{1 - \alpha'^2} \right] \times \tan \left[\frac{1}{2} \sqrt{1 - \alpha'^2} (\mathbf{e}_z \cdot \mathbf{e}'_z) \tau - C_0 \right] \quad (158)$$

$$C_0 = \arctan \left[\frac{\alpha' + \tan[\varphi'_0/2]}{\sqrt{1 - \alpha'^2}} \right] \quad (159)$$

$\alpha = 1$:

$$\varphi'[\tau] = \frac{3\pi}{2} + 2 \arctan [(\mathbf{e}_z \cdot \mathbf{e}'_z) \tau + C_0] \quad (160)$$

$$C_0 = \tan \left[\frac{\pi}{4} + \frac{\varphi'_0}{2} \right] \quad (161)$$

The core finding for the most interesting case, i.e., the drifting case at $\alpha < 0$, is that the solution for the rotation in a plane with a fixed axis is generalized with the definition of the effective friction coefficient α' (Equ. (155)). The rotating vector of a magnetic moment pointing outside of the rotating plane is constant and defines a new rotating plane in the rotating frame, where the solution for the angular motion around this constant rotating vector has α' as effective rotational friction coefficient and is identical with the solution for the motion in the 2D plane with a fixed rotating axis.

Having a closed solution for the theoretical edge case, where random thermal interactions go to zero, can be used as a verification tool for more generalized particle simulations and it can also be used as a basis for constructing approximations for more general cases.

5 Author contributions

T.K., M.R. and A.V. wrote the main manuscript text and are responsible for the theoretical framework, A.V., V.S., P.V. and V.B. provided theoretical and simulation guidance. All authors reviewed the manuscript.

6 Funding

This work was supported by the German Research Council (DFG) (grant numbers: BE 5293/1-2, VO 2288/1-1).

A Invariance of $\mathbf{n}_\Omega$

In this appendix the preservation of $\mathbf{n}_\Omega$ in the rotating frame of reference is shown. Hence, the unit vectors $\mathbf{n}_B$ and $\mathbf{e}_z$ as well as the constant α do not depend on time and one find for the derivative of Ω :

$$\left(\frac{d\Omega}{d\tau} \right)_r = \frac{d}{d\tau} (\alpha(\mathbf{n}_m \times \mathbf{n}_B) - \mathbf{e}_z) \quad (162)$$

$$= \alpha \mathbf{n}_B \times (\mathbf{n}_m \times \Omega) \quad (163)$$

here Eq. (5) and Eq. (6) were used. Using the Graßmann identity the last line can be rewritten as

$$\left(\frac{d\Omega}{d\tau} \right)_r = \alpha (\mathbf{n}_m (\mathbf{n}_B \cdot \Omega) - \Omega (\mathbf{n}_m \cdot \mathbf{n}_B)). \quad (164)$$

The second term is parallel to Ω . The first term however has to be investigated more closely. Substituting the definition of Ω from Eq. (7) and Eq. (6) in the scalar product of the first addend of the last line and using the triple product identity one finds

$$\mathbf{n}_B \cdot \Omega = \mathbf{n}_B \cdot (\alpha(\mathbf{n}_m \times \mathbf{n}_B) - \mathbf{e}_z) \quad (165)$$

$$= -\mathbf{e}_z \cdot \mathbf{n}_B. \quad (166)$$

Obviously this leads to the condition

$$\mathbf{0} = \mathbf{e}_z \cdot \mathbf{n}_B \quad (167)$$

that has to be fulfilled for a uniaxial rotation around Ω . This is equivalent to the more general condition

$$\mathbf{0} = \omega_0 \cdot \mathbf{B} \quad (168)$$

with the vector of the constant angular velocity ω_0 where the orientation of the rotation is not limited to the x - y -plane as long as it is uniaxial. However, in this work for simplicity the rotation axis of the magnetic field $\mathbf{B}$ is along $\mathbf{e}_z$ without any loss of generality. This is equivalent to a $\mathbf{B}$ -field rotating with constant angular velocity in the x - y -plane. Under this condition Eq. (164) can be rewritten as

$$\left(\frac{d\Omega}{d\tau} \right)_r = -\alpha \Omega \mathbf{n}_\Omega (\mathbf{n}_m \cdot \mathbf{n}_B) \quad (169)$$

which corresponds to (see. Eq. (11))

$$\left(\frac{d\Omega}{d\tau} \right)_r = -\alpha \Omega (\mathbf{n}_m \cdot \mathbf{n}_B) \quad (170)$$

which has to be checked for completeness. The absolute value of the vector Ω is given by

$$\Omega = \sqrt{(\alpha(\mathbf{n}_m \times \mathbf{n}_B) - \mathbf{e}_z)^2} \quad (171)$$

Hence the time derivative can be written as

$$\left(\frac{d\Omega}{d\tau} \right)_r = \frac{\alpha}{\Omega} \Omega \cdot \left(\left(\frac{d\mathbf{n}_m}{d\tau} \right)_r \times \mathbf{n}_B \right) \quad (172)$$

With Eq. (6) one finds

$$\left(\frac{d\Omega}{d\tau} \right)_r = \frac{\alpha}{\Omega} \Omega \cdot ((\Omega \times \mathbf{n}_m) \times \mathbf{n}_B) \quad (173)$$

$$= -\alpha \Omega (\mathbf{n}_m \cdot \mathbf{n}_B) \quad (174)$$

where Eq. (167) and Eq. (166) have been used. Thus, the vector $\mathbf{n}_\Omega$ is constant and the motion is an uni-axial rotation around the Ω -axis.

B Transforming the equation of motion to spherical coordinates in the rotating frame of reference

In this appendix the transformation from Cartesian to spherical coordinates in the rotating frame of reference is shown. As implicated by Eq. (??), the magnetization moves on the surface of a sphere with radius m . Hence the description in spherical coordinates $\{r, \theta, \varphi\}$ with radius r , polar angle θ , and azimuthal angle φ . The ODE (6) in Cartesian coordinates of the rotating frame of reference is given by ($\mathbf{n}_B = \mathbf{e}_x$, $\mathbf{n}_{\omega_0} = \mathbf{e}_z$):

$$\left(\frac{dn_{mx}}{d\tau}\right)_r = n_{mz}^2 \alpha + n_{my}^2 \alpha + n_{mx} \quad (175)$$

$$\left(\frac{dn_{my}}{d\tau}\right)_r = -n_{mx} n_{my} \alpha - n_{mx} \quad (176)$$

$$\left(\frac{dn_{mz}}{d\tau}\right)_r = -n_{mz} m_{mx} \alpha. \quad (177)$$

With the definition of $\mathbf{n}_m$ in spherical coordinates

$$n_{mx} = \cos[\varphi] \sin[\theta] \quad (178)$$

$$n_{my} = \sin[\varphi] \sin[\theta] \quad (179)$$

$$n_{mz} = \cos[\theta] \quad (180)$$

one can rewrite Eq. (175), Eq. (176), and, Eq. (177) From the last equation immediately follows the differential equation for θ

$$\dot{\theta} = \alpha \cos[\theta] \cos[\varphi] \quad (181)$$

To obtain the differential equation for φ Eq. (??) is multiplied by $-\sin[\varphi]$ and Eq. (??) by $\cos[\varphi]$. By adding these new equations one obtains the differential equation for φ

$$\dot{\varphi} \sin[\theta] = -\alpha \sin[\varphi] - \sin[\theta] \quad (182)$$

$$\dot{\varphi} = -\frac{\alpha}{\sin[\theta]} \sin[\varphi] - 1 \quad (183)$$

Please note that in the second form the differential equation becomes singular at $\theta = \{0, \pi\}$. This has to be considered when dealing with these equations. It represents the fact that at the poles the angle φ has not meaning as the whole circle of latitude degenerates to a point.

C Solving the differential equation for φ'

To obtain the full solution it is necessary to solve the ODE (??). For this purpose it is advantageous to write the vector $\boldsymbol{\Omega}$ in the local frame of reference Eqs. (53), (54), and, (55) used to construct the solution. With Eq. (??) and (6) one may write (see Eq. (14))

$$\boldsymbol{\Omega} = \alpha (\mathbf{n}_m \times \mathbf{n}_B - \mathbf{e}_z) \quad (184)$$

$$\begin{aligned} &= n_{m\parallel} \alpha \mathbf{e}'_z \times \mathbf{n}_B - ((\mathbf{e}_z \cdot \mathbf{e}'_z) \mathbf{e}'_z + (\mathbf{e}_z \cdot \mathbf{e}'_y) \mathbf{e}'_y) \\ &\quad + n_{m\perp} \alpha \left(\underbrace{\cos[\varphi'] \mathbf{e}'_x \times \mathbf{n}_B}_{=0} + \sin[\varphi'] \mathbf{e}'_y \times \mathbf{n}_B \right) \end{aligned} \quad (185)$$

where Eq. (167) was used in second addend of the second line. The first term of the third line vanishes due to Eq. (53). With the help of the Eq. (53) one find

$$\mathbf{e}'_y \times \mathbf{n}_B = -\mathbf{e}'_z \quad (186)$$

$$\mathbf{e}'_z \times \mathbf{n}_B = \mathbf{e}'_y \quad (187)$$

and hence

$$\begin{aligned} \boldsymbol{\Omega} &= (n_{m\parallel} \alpha - (\mathbf{e}_z \cdot \mathbf{e}'_y)) \mathbf{e}'_y \\ &\quad - (n_{m\perp} \alpha \sin[\varphi'] + (\mathbf{e}_z \cdot \mathbf{e}'_z)) \mathbf{e}'_z. \end{aligned} \quad (188)$$

Obviously the first term must vanish as the vector $\boldsymbol{\Omega}$ has not perpendicular components to $\mathbf{n}_\Omega = \mathbf{e}'_z$. and one finally gets

$$\frac{d\varphi'}{d\tau} = \Omega = -(\alpha n_{m\perp} \sin[\varphi'] + (\mathbf{e}_z \cdot \mathbf{e}'_z)) \quad (189)$$

a nonlinear but separable ODE in φ' . By introducing the scaled time τ with

$$\tau' = (\mathbf{e}_z \cdot \mathbf{e}'_z) \tau \quad (190)$$

$$\frac{d\tau'}{d\tau} = (\mathbf{e}_z \cdot \mathbf{e}'_z) \quad (191)$$

$$\frac{d\varphi'}{d\tau} = (\mathbf{e}_z \cdot \mathbf{e}'_z) \frac{d\varphi'}{d\tau'} \quad (192)$$

the ODE (62) reads

$$\frac{d\varphi'}{d\tau'} = -(\alpha' \sin[\varphi'] + 1). \quad (193)$$

where α' is given in Eq. (64).

In Sec. 3.1 it was shown that a case analysis is necessary to distinguish between the different types of motion. Hence, before continuing it is necessary to show that this separation also holds for α' . From Eq. (64) it is clear that the behavior $\sqrt{1 - (\mathbf{n}_{m0} \cdot \mathbf{n}_\Omega)^2} / (\mathbf{e}_z \cdot \mathbf{n}_\Omega)$ has to be investigated. The second term under the square root is given by Eq. (131). For the denominator one finds

$$\mathbf{e}_z \cdot \mathbf{n}_\Omega = -\frac{\alpha \mathbf{n}_{m0} \cdot \mathbf{e}_y + 1}{|\boldsymbol{\Omega}_0|}. \quad (194)$$

Hence one finds

$$-\frac{\alpha'}{\alpha} = \frac{\sqrt{\Omega_0^2 - (\mathbf{n}_{m0} \cdot \mathbf{e}_z)^2}}{\alpha \mathbf{n}_{m0} \cdot \mathbf{e}_y + 1} \quad (195)$$

$$= \sqrt{1 - \frac{(1 - \alpha^2)(\mathbf{n}_{m0} \cdot \mathbf{e}_z)^2}{(\alpha \mathbf{n}_{m0} \cdot \mathbf{e}_y + 1)^2}}. \quad (196)$$

For $\alpha > 1$ one can see that that $(1 - \alpha^2) < 0$ and hence

$$\alpha > 1 : \quad |\alpha'| > |\alpha|. \quad (197)$$

For $\alpha = 1$ the second addend under the square root vanishes and one finds

$$\alpha = 1 : \quad |\alpha'| = |\alpha|. \quad (198)$$

For $\alpha < 1$ it is possible to show that

$$\alpha < 1 : \quad |\alpha'| < |\alpha|. \quad (199)$$

Concluding, the three cases found in the rotating frame of reference translates uniquely in the corresponding cases of the local frame of reference.

The formal solution of this equation is

$$-\int_{\varphi'[0]=\varphi'_0}^{\varphi'[\tau]} \frac{d\phi'}{\alpha' \sin[\phi'] + 1} = \tau' = (\mathbf{e}_z \cdot \mathbf{e}'_z) \tau. \quad (200)$$

At this point it is advantageous to do an case analysis.

$\alpha < 1$:

this case leads to the integral

$$\int \frac{d\phi'}{\alpha' \sin[\phi'] + 1} = \frac{2 \arctan[u]}{\sqrt{1 - \alpha'^2}}. \quad (201)$$

Hence, the solution of ODE (193) or equivalently Eq. (200) can be given analytically

$$\begin{aligned} \varphi'[\tau] = & -2 \arctan \left[\alpha' + \sqrt{1 - \alpha'^2} \right. \\ & \left. \times \tan \left[\frac{1}{2} \sqrt{1 - \alpha'^2} (\mathbf{e}_z \cdot \mathbf{e}'_z) \tau - C_0 \right] \right] \end{aligned} \quad (202)$$

$$C_0 = \arctan \left[\frac{\alpha' + \tan[\varphi'_0/2]}{\sqrt{1 - \alpha'^2}} \right], \quad (203)$$

where φ'_0 is the angle at $\tau = 0$.

$\alpha > 1$:

To keep all variables and constants real one finds

$$\int \frac{d\phi'}{\alpha' \sin[\phi'] + 1} = -\frac{2 \operatorname{arctanh}[u]}{\sqrt{\alpha'^2 - 1}}. \quad (204)$$

Thus, one can write the solution of the ODE (193) for the case $\alpha' > 1$ as

$$\begin{aligned} \varphi'[\tau] = & -2 \arctan \left[\alpha' - \sqrt{\alpha'^2 - 1} \right. \\ & \left. \times \tanh \left[\frac{1}{2} \sqrt{\alpha'^2 - 1} (\mathbf{e}_z \cdot \mathbf{e}'_z) \tau + C_0 \right] \right] \end{aligned} \quad (205)$$

$$C_0 = \operatorname{arctanh} \left[\frac{\alpha' + \tan[\varphi'_0/2]}{\sqrt{\alpha'^2 - 1}} \right]. \quad (206)$$

$\alpha = 1$:

In this case the integral considerably simplifies

$$\int \frac{d\phi'}{\sin[\phi'] + 1} = \frac{-2}{1 + \tan\left[\frac{\phi'}{2}\right]}. \quad (207)$$

Substituting this result in Eq. (200) one find for the solution of ODE (193)

$$\varphi'[\tau] = -2 \arctan \left[\frac{2}{(\mathbf{e}_z \cdot \mathbf{e}'_z) \tau + C_0} - 1 \right] \quad (208)$$

$$C_0 = \frac{2}{1 + \tan[\varphi'_0/2]}, \quad (209)$$

A different form of the solution can be found if one keeps in mind that without specified upper and lower bound the solution of Eq. (207) is ambiguous by a constant and an alternative solution is given by

$$\frac{-2}{1 + \tan[\phi'/2]} = -\left(\tan\left[\frac{\pi}{4} - \frac{\phi'}{2}\right] + 1 \right) \quad (210)$$

Substituting this result with the appropriate boundary values of ϕ' for the left hand side of Eq. (200) one finds for the solution of ODE (193)

$$\varphi'[\tau] = \frac{3\pi}{2} + 2 \arctan[(\mathbf{e}_z \cdot \mathbf{e}'_z) \tau + C_0] \quad (211)$$

$$C_0 = \tan\left[\frac{\pi}{4} + \frac{\varphi'_0}{2}\right], \quad (212)$$

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