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Obtaining the geometry and 5-axis machining paths for the manufacture of spiral bevel gear

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ABSTRACT

This study proposes a methodology for the design and manufacturing of spiral bevel gears using five-axis NC machines. Unlike traditional methods, which are rigid and specific, this technique offers a flexible approach based on customizable input parameters. The process includes generating gear geometry using a point cloud, 3D modelling in software such as SolidWorks, and creating optimized machining toolpaths with tools like the Darboux trihedron to ensure precision and efficiency. Additionally, the use of additive manufacturing enables prior physical simulations, improving collision detection and validating the toolpaths.

The experimental results, conducted on a five-axis machining center, demonstrate the feasibility and high accuracy of this methodology. The solution not only enables the manufacturing of customized gears in various sizes but also optimizes NC code generation, reducing code volume while maintaining precision. This represents a highly practical and robust alternative to conventional commercial tools, offering greater control and adaptability for industrial applications to effectively meet specific production needs.

Keywords: Spiral bevel gear, Darboux trihedron, Toolpaths generation, 5-axis machining, Parametric modelling

1. Introduction

Spiral bevel gears stand out for their smooth and quiet operation and their ability to uniformly transmit high loads. These characteristics make them ideal for various applications in sectors such as aviation, marine, and industrial, among others [1, 2]. In such demanding sectors, recent research has explored the use of five-axis milling machines to address the manufacturing of large spiral bevel gears, offering notable improvements in finishing quality and error control [3, 4].

The specialized design of these gears requires specific manufacturing methods to ensure the precision of the tooth profile and the smoothness of contact between them [5]. Depending on the required surface finish quality and production speed, various manufacturing methods are employed, including the Gleason milling method, the Klingelnberg method [6], the spiral generation method, broaching, as well as techniques such as forging and stamping.

However, these traditional methods manufacturing pose a significant limitation: limited flexibility [7]. The machinery and tooling used are specific to the production of a single type of gear, requiring substantial investments of time and resources to adjust or reconfigurations when manufacturing a spiral bevel gear with different characteristics, such as variations in size or the number of teeth. The emergence of universal five-axis CNC systems has enabled the integration of custom toolpaths and adaptable strategies, facilitating gear customization and reducing setup complexity [4, 8].

In parallel, the influence of toolpath strategies on surface integrity and machining forces has been addressed in various studies on complex free-form surfaces [9, 10], highlighting the importance of path optimization in achieving high-quality finishes while minimizing tool wear and cutting forces.

The present study proposes an alternative methodology to conventional methods for the geometric design of spiral bevel gears. This methodology is based on defining input parameters that allow for gear customization and subsequent manufacturing using five-axis NC milling techniques, providing a more flexible and efficient solution compared to traditional approaches.

2. Methodology

As a methodological basis, the AGMA 2005 standard [11] was used, covering design, manufacturing principles, and key parameters for bevel gears, including spiral bevel gears. This standard provides relevant technical information on fundamental dimensions and specific parameters of spiral bevel gears.

The developed methodology can be structured as follows:

1. Obtain the geometry of the spiral bevel gear.
 - 1.1. Obtain the envelope profile.
 - 1.2. Obtain the logarithmic spiral.
 - 1.3. Generate a point cloud of the discretized geometry.
2. Generate the machining toolpaths.
3. Generate the G-code.

3. Geometry generation

As described in the methodology section, the definition of gear geometry requires calculating the envelope profile. Subsequently, the logarithmic spiral is applied to this envelope to

discretize the geometry and generate a point cloud, from which the 3D CAD model is constructed.

3.1. Calculation of the tooth envelope profile

The determination of the points that define the tooth envelope profile is based on the methodologies described by Tsai and Chin [12]. This study establishes that generating the envelope profile on the cone surface requires unfolding the base cone surface.

The procedure involves keeping the unfolded surface in tangential contact with the base cone surface (Fig. 1). Based on the trajectory described during the unfolding process, it is possible to determine the tooth profile along the height of the cone. This approach ensures the geometric accuracy required to define the evolution of the tooth profile in the spiral bevel gear.

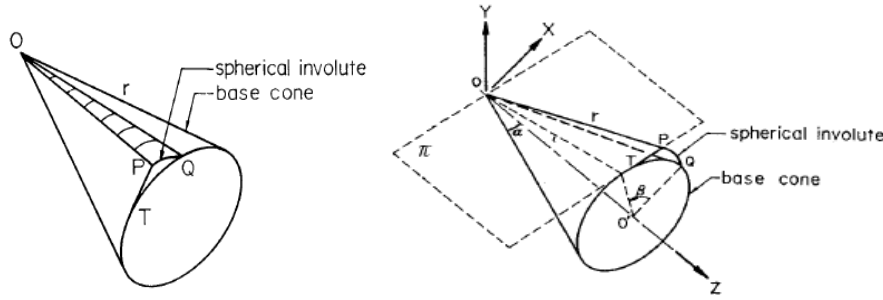


Fig. 1. Spherical envelope curve generated from the base cone surface and the coordinate system with the hypothetical tangent plane π [12].

The Cartesian coordinates of a point P contained in the tangent plane π are determined using Eq. 1 to 3:

$$X_p = r \sin \alpha \cos \beta \left(1 - \frac{\beta^2}{2} \sin^2 \alpha\right) + r \beta \sin \alpha \sin \beta \sqrt{1 - \frac{\beta^2}{4} \sin^2 \alpha} \quad (1)$$

$$Y_p = r \sin \alpha \sin \beta \left(1 - \frac{\beta^2}{2} \sin^2 \alpha\right) - r \beta \sin \alpha \cos \beta \sqrt{1 - \frac{\beta^2}{4} \sin^2 \alpha} \quad (2)$$

$$Z_p = r \cos \alpha \left(1 - \frac{\beta^2}{2} \sin^2 \alpha\right) \quad (3)$$

In these equations, all variables are known except for the angle β , which is required to generate the involute. However, it is important to note that the cone's involute is generated on a sphere, whose equation is given by Eq. 4:

$$x^2 + y^2 + z^2 = r^2 \quad (4)$$

where r represents the radius of the sphere corresponding to the development of the profile.

The height of the profile is directly related to the gear module. Specifically, the generated profile must have a height equivalent to 2.25 times the module (Fig. 2). This leads to the mathematical expression that describes the relationship given in Eq. 5:

$$h = 2.25m = r\delta \quad (5)$$

where h is the total height of the profile, which corresponds to the sum of the tooth head height and the tooth root height, and m is the gear module, defined as the ratio between the pitch diameter and the number of teeth.

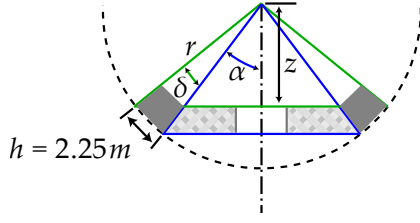


Fig. 2. Envelope Height.

Therefore, it can be stated that the Z coordinate, for the involute with a height $h=2.25m$ on the sphere, is given by Eq. 6:

$$z = r \cos(\alpha + \delta) = r \cos\left(\alpha + \frac{2.25m}{r}\right) \quad (6)$$

From this relationship, by equating z y Z_p , the equation that defines the angle β is obtained, as illustrated in Eq. 7:

$$\beta = \frac{1}{\sin \alpha} \sqrt{2 \left(1 - \frac{\cos\left(\alpha + \frac{2.25m}{r}\right)}{\cos \alpha}\right)} \quad (7)$$

3.2. Calculation of the spiral for the spiral bevel gear

The teeth of the spiral bevel gear exhibit a trace in the form of a logarithmic spiral. To calculate it, Eq. 7 is used, expressed in spherical coordinates, derived from the study conducted by Teixeira Alves et al [13].

$$\theta_{spiral} = \frac{\tan \lambda}{\sin \gamma} \log\left(\frac{r \sin \gamma}{a}\right) \quad (8)$$

where λ is the constant rotation angle of the spiral that defines the number of turns of the cone's spiral, θ_{spiral} is the angle that indicates the angular position of the spiral, γ is the cone's helix angle, r is the distance along the cone's generatrix between the apex and point P , and a is the constant calculated as $a = r_o \sin \gamma$, where r_o is the outer distance.

Eq. 8 can be developed, resulting in the relationship given by Eq. 9:

$$\theta_{spiral} = \frac{\tan \lambda}{\sin \gamma} \log\left(\frac{r \sin \gamma}{a}\right) = \frac{\tan \lambda}{\sin \gamma} \log\left(\frac{r \sin \gamma}{r_o \sin \gamma}\right) = \frac{\tan \lambda}{\sin \gamma} \log\left(\frac{r}{r_o}\right) \quad (9)$$

The constructive and sequential procedure for the spiral is shown in Fig. 3.

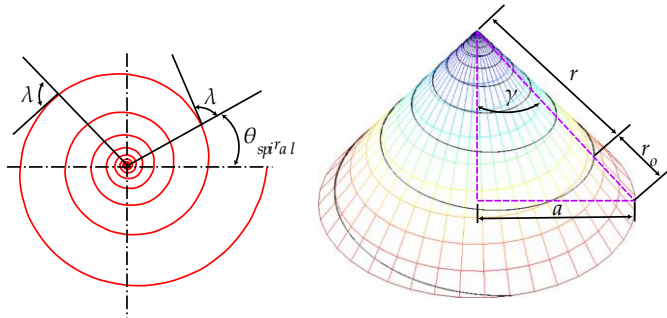


Fig. 3. Spiral construction procedure.

3.3. Calculation of the discretized geometry as a point cloud

This step involves calculating the points on the tooth envelopes, as well as the points corresponding to the tooth tips and the gaps between the teeth, in order to generate the spiral bevel gear.

3.3.1. Calculation of the tooth envelope

The calculation of the envelope points is performed using Eq. 1, 2, and 3. A rotation matrix is applied to these equations to impart a spiral trace to the teeth and position the profiles around the cone.

The envelope points are obtained by varying the parameters r and β . The values of r must be between the initial tooth height on the base cone (A_i) and the maximum tooth height at the upper cone tip (A_o). The values of β must range from 0 to the value obtained from Eq. 7.

For the calculation of the envelopes in the counterclockwise direction, the rotation matrix that defines the envelopes in that direction is applied, as shown in Eq. 10.

$$\begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} = \begin{bmatrix} \cos(i + \theta_{spiral}) & -\sin(i + \theta_{spiral}) & 0 \\ \sin(i + \theta_{spiral}) & \cos(i + \theta_{spiral}) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} X_p \\ Y_p \\ Z_p \end{Bmatrix} \quad (10)$$

where i is the angular position of the tooth, calculated according to Eq. 11, and θ_{spiral} is the angle that positions the tooth profiles to generate the spiral trace.

$$i = \frac{2\pi \cdot n^{\circ} \text{ of teeth}}{\text{Total number of teeth}} \quad (11)$$

For the calculation of the envelopes in the clockwise direction, the same procedure is used, but with the sign of the angle β modified and an additional rotation angle ζ (Fig. 4) defined according to Eq. 12:

$$\zeta = \frac{i}{2} + 2\varepsilon \quad (12)$$

where ε is the angle between the base point of the envelope and the point of intersection between the pitch circle and the envelope, and i is the angular position of the tooth.

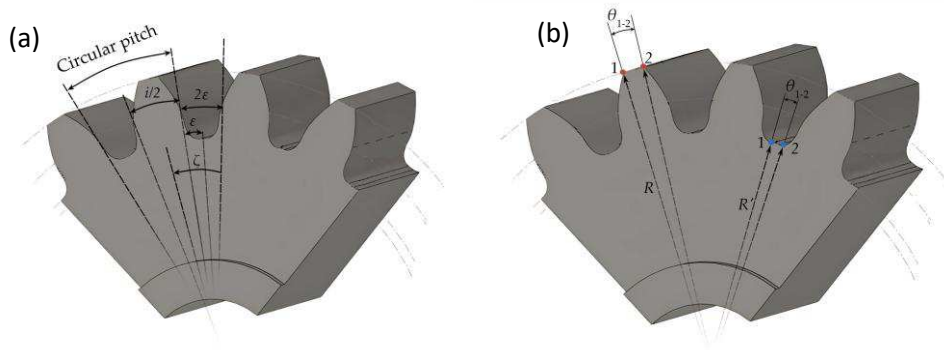


Fig. 4. (a) Rotation of the envelope, angles ζ and ε (b) Upper and lower profiles of the tooth.

The angular distance between the intersection points of the clockwise and counterclockwise envelopes with the pitch circle is given by 2ε . Additionally, $i/2$ represents the tooth thickness, which corresponds to the pitch circle.

To determine the angle ε it is necessary to calculate the X and Y coordinates of the intersection point between the envelope and the pitch circle. These coordinates are obtained using the previously mentioned equations without applying the rotation matrix.

Once the X and Y coordinates are calculated, the value of ε is determined by calculating the arctangent, as given by Eq. 13, and is as shown in Fig. 4 (a).

$$\varepsilon = \tan^{-1} \left(\frac{Y}{X} \right) \quad (13)$$

3.3.2. Calculation of the points defining the tooth tip and the gaps between teeth
The calculation of the profiles at the upper end of the teeth and the gaps is based on the points previously determined from the envelopes 1 and 2, as shown in Fig. 4 (b).

To calculate the points of the upper tooth profile, it is assumed that this profile is an arc with radius R , whose centre coincides with the centre of the pitch circle. When working in the XY plane, the Z coordinate remains constant for all points of the arc.

The radius R is determined from the module of one of the points previously calculated on the envelopes. The points of the arc are obtained using the system of equations in Eq. 14.

$$\begin{cases} X=R \cos \vartheta \\ Y=R \sin \vartheta \end{cases} \quad (14)$$

where ϑ is the angular variable used to calculate the points of the arc.

To incorporate the spiral trace of the tooth, the obtained points must undergo a transformation using the rotation matrix represented in Eq. 15.

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{bmatrix} \cos i & -\sin i & 0 \\ \sin i & \cos i & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} X_r \\ Y_r \\ Z \end{pmatrix} \quad (15)$$

The same procedure used to calculate the upper tooth profile is applied to determine the base of the gaps between teeth. In this case, the points corresponding to the base of the envelope, both clockwise and counterclockwise from the previous tooth, are selected, as shown in Fig. 5.

Using the procedure described earlier, a point cloud is obtained, which will serve as the basis for generating the geometry of the spiral bevel gear. To carry out this task, Microsoft Excel was used, programmed with VBA for Applications.

3.4. Generation of the 3D bevel gear

To carry out the 3D modelling of the spiral bevel gear geometry, the point cloud obtained in the previous section is imported into 3D software. In this case, the commercial software developed by Dassault Systèmes, SolidWorks, was used.

Once the points are imported, a surface is generated that connects all the points, and the geometry is filled in. This approach allows for an accurate and detailed visualization of the final shape of the spiral bevel gear, ensuring that all geometric parameters are respected and facilitating manufacturing through CAM techniques.

3.5. Generation of machining toolpaths

The machining toolpaths are generated for processing the gaps between the teeth of the gear, and this process is repeated for each of the gaps.

The previously calculated points are organized into a point matrix (as shown in Fig. 5), which represents the geometry of the gap. From this matrix, the entry and exit points are calculated, along with the auxiliary points needed to define the machining toolpaths. These points are essential for guiding the movement of the cutting tool along the machining paths, ensuring precision in the fabrication of the gap in the gear.

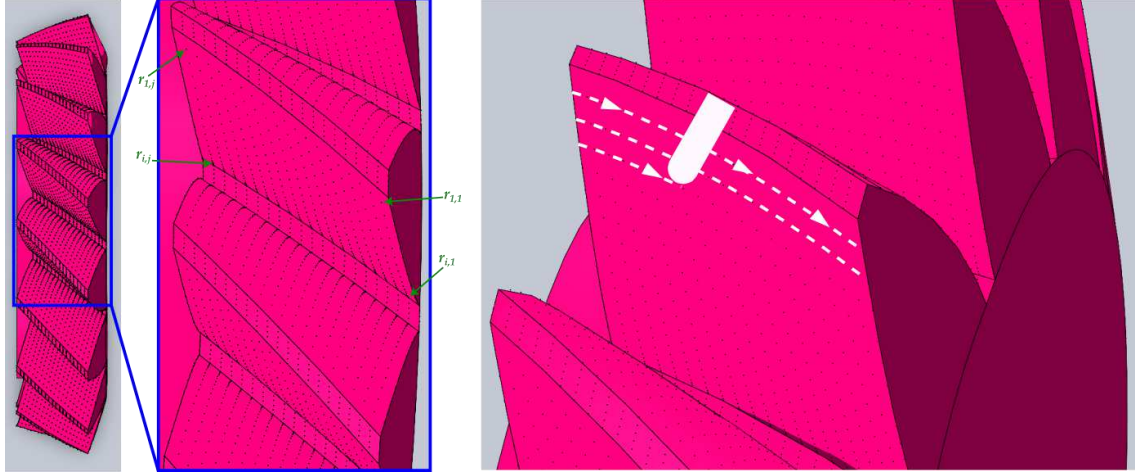


Fig. 5. Points on the Tooth Surface and Machining Toolpaths.

To define the orientation and position of the tool during machining, the Darboux trihedron is used. This consists of a set of orthonormal vectors located at a point P that belongs to an arc C drawn on a surface S [14].

This system of orthonormal vectors provides a convenient basis for representing the tool's orientation in three-dimensional space, allowing for precise calculation and control of the tool's movements along the machining toolpaths. The implementation of the Darboux trihedron is essential to ensure proper alignment of the tool with the gear geometry (Fig. 6 (a)), which is crucial for achieving precise finishes on the machined surfaces.

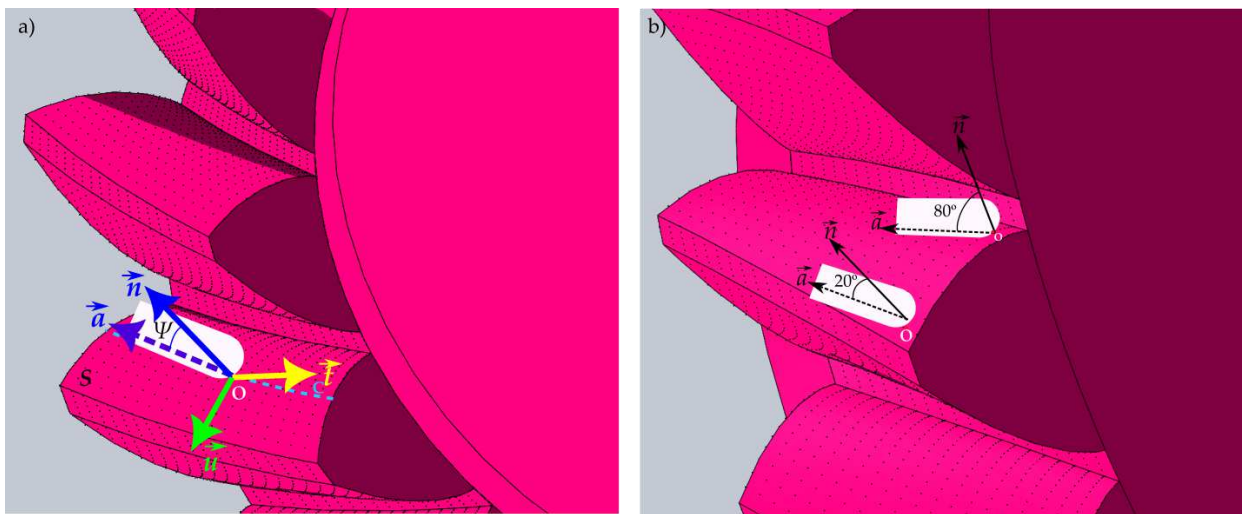


Fig. 6. (a) Darboux trihedron on the tooth surface, (b) Angle ψ at the base and tip of the tooth.

Three orthonormal vectors define the Darboux trihedron at a point P on the surface of the gear:

- The normal vector $\vec{n}$, which is perpendicular to the tooth surface.
- The tangent vector $\vec{t}$, which is tangent to the machining path.
- The perpendicular vector $\vec{u}$, which is perpendicular to the other two vectors.

All these vectors are unit vectors.

Instead of using the complex equations that describe the surface, the previously calculated points on the surface are utilized. This approach significantly simplifies the calculation process and improves the efficiency of the model. The equations for determining the vectors of the trihedron are presented in the system of Eq. 16.

$$\begin{aligned}\vec{t} &= \frac{r_{i+1,j} - r_{i,j}}{|r_{i+1,j} - r_{i,j}|} \\ \vec{n} &= \frac{\Delta(r_{i,j+1}; r_{i,j-1}) \wedge (r_{i+1,j} - r_{i,j})}{|\Delta(r_{i,j+1}; r_{i,j-1}) \wedge (r_{i+1,j} - r_{i,j})|} \\ \vec{u} &= \vec{n} \wedge \vec{t}\end{aligned}\quad (16)$$

To position and orient the spherical end mill, the spherical body of the tool must be in contact with point P on the surface, and the centre of the sphere must coincide with the normal vector $\vec{n}$. Due to this condition, the vector representing the tool's axis will form an angle ψ with respect to the normal vector $\vec{n}$, ensuring that the tool is correctly positioned along the machining path.

Angle ψ plays a key role in maintaining proper tool orientation during the machining process, allowing the tooth surface to be cut accurately.

After determining the center positions of the spherical tool, it is possible to calculate the points along the tool's path. Additionally, the angle ψ allows for adjusting the tool's orientation along the machining trajectory.

The tool positions are obtained using Eq. 17.

$$\vec{r}_{O_{i,j}} = \vec{r}_{i,j} + R\vec{n} \quad (17)$$

where $\vec{r}_{i,j}$ is a point belonging to the previously calculated point cloud, $\vec{n}$ is the normal vector at point $\vec{r}_{i,j}$ and R is the radius of the tool.

To calculate the angle ψ , a linear variation is assumed between the first depth of cut, where $\psi=20^\circ$, and the last depth of cut, where $\psi=80^\circ$ (Fig. 6 (b)). The intermediate values of angle ψ are calculated using linear interpolation, with Eq. 18:

$$\Psi_{int} = \Psi_{ini} + \left(\frac{\Psi_{final} - \Psi_{ini}}{\text{Depth of cut} - 1} \right) k \quad (18)$$

where $\Psi_{ini}=20^\circ$ is the angle value for the first depth of cut, $\Psi_{final} = 80^\circ$ is the angle value for the last depth of cut, and k is the index of the current pass, with $k = 0, 1, 2, \dots$ up to the total height of cut.

with the angle ψ and the vectors $\vec{n}$ y $\vec{u}$, the tool axis vector $\vec{a}$ is determined. This vector is calculated using Eq. 19 or alternatively Eq. 20:

$$\vec{a} = \vec{n} \cos \Psi + \vec{u} \sin \Psi \quad (19)$$

or

$$\vec{a} = \vec{n} \sin \Psi + \vec{u} \cos \Psi \quad (20)$$

The choice of the equation to be used depends on the range of motion allowed by the tool used for the machining process.

Additionally, it is important to consider that, depending on the type of NC used, the tool may have restrictions on certain movement axes. However, the calculated vector $\vec{a}$ has components along all three axes, so it is necessary to adapt these vectors to the milling machine's coordinate system.

To carry out this adaptation, an inverse kinematics strategy can be used, allowing the spindle holding the workpiece to rotate along the B and C axes. This enables the tool to perform machining according to the desired trajectory.

Once the trajectory points are adapted to the specific machine, the final step is to generate the G-code, which allows the machining process to be carried out according to the previously defined parameters. In our case, we programmed a VBA script that generates a string for each code block.

4. Results and discussion

To verify the validity of the methodology for obtaining the geometry and to check the machining trajectories, a 3D model of the spiral bevel gear was printed in PLA using FDM additive manufacturing technology. This printed model was clamped onto a Lagun L650 5-axis machining centre with Fagor 8065 control. To facilitate the clamping of the spiral bevel gear on the machining centre's table, grooves were made in the centre of the gear, allowing the placement of a chuck onto which it was secured in the chuck plate (Fig. 7).



Fig. 7. Spiral bevel gear clamped on a 5-axis machining centre.

To verify the machining trajectories, finishing paths were programmed with a step depth of 0.25 mm, a feed rate of 200 mm/min, and a spindle speed of 4,000 rpm.

Once the program was loaded and the necessary adjustments were made, the machining program was executed. It was observed that the tool made passes on each of the teeth of the spiral bevel gear, faithfully following the programmed trajectories and covering the entire geometry (Fig. 8). This demonstrated the accuracy and functionality of the methodology.

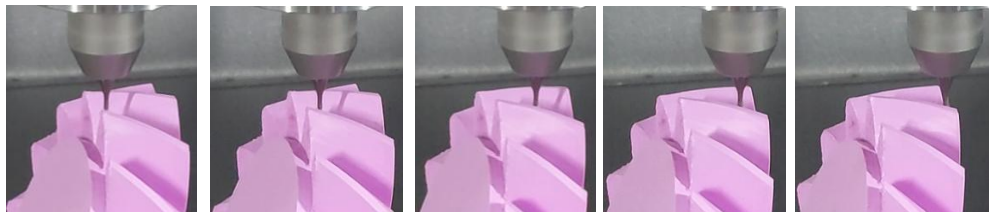


Fig. 8. Some finishing paths made by the tool on the spiral bevel gear [16].

The developed methodology provides a flexible, robust, and precise solution for the manufacturing of spiral bevel gears using 5-axis NC. It offers a practical and complementary alternative to existing commercial software, which in many cases are black boxes, lacking control or the ability to make modifications. Consequently, it allows the generation of gears with different sizes and characteristics, as well as their machining trajectories. Furthermore, it enables on-machine simulations of the machining paths on physically printed models via additive manufacturing, facilitating the detection of collisions between the workpiece, tool, and clamping system.

In addition to everything mentioned above, the proposed methodology opens the possibility of generating optimized NC codes, as it allows for a more efficient integration of the machine's inverse kinematics and the use of circular interpolation models instead of discretized linear codes. This aligns with the studies by Kawasaki et al [15], which highlight that the use of circular interpolation techniques can reduce the generated code by up to a factor of 1/7, while maintaining the same level of precision.

5. Conclusions

The developed methodology provides a flexible and precise solution for the design and manufacturing of spiral bevel gears using five-axis NC machines, overcoming the limitations of traditional methods and closed commercial software. By integrating key geometric parameters and using Darboux's trihedron for tool orientation, optimal control of the machining process is achieved, ensuring design customization and manufacturing accuracy. The experimental validation, through 3D printing and machining on a NC centre, demonstrated the effectiveness of the proposed approach, ensuring comprehensive geometry coverage and accurate execution of the programmed paths.

In addition, the methodology facilitates NC code optimization through circular interpolation, significantly reducing the code size without compromising precision. This, along with the ability to perform physical pre-simulations to identify collisions, opens new opportunities for the manufacturing of custom gears in sectors such as aerospace, maritime, and industrial. The proposal represents a significant contribution to the field of advanced manufacturing, providing a versatile and efficient tool for demanding applications.

Statements and Declaration

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Declaration of Competing Interest

The authors declare that they have no financial interest or personal relationships that could have appeared to influence the work reported in this paper.

Author Contributions

All authors contributed to the study conception and design. Material preparation, data collection and analysis were performed by Wilmer E. Cumbicus, Manuel Estrems and Álvaro Franco. The first draft of the manuscript was written by Álvaro Franco and all authors commented on previous versions of the manuscript. All authors read and approved the final manuscript.

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