Supplementary Materials for
Oscillatory Hierarchical Reservoirs for Human-like Rhythm Perception and Anticipation

Supplementary Note 1: Approach for Modelling Beta-Band Activity

System Configuration for 2D-FDTD Using the Wave Equation

The original system is built based on :

$$\frac{\partial p}{\partial t} + k_p p + c^2 \nabla \cdot \mathbf{o} = 0, \tag{S1}$$

$$\frac{\partial \mathbf{o}}{\partial t} + \mathbf{k} \cdot \mathbf{o} + \nabla p = 0, \tag{S2}$$

where $\mathbf{o} = [o_x, o_y]^\top$ is a 2D vector. In 2D, the divergence and gradient operators are:

$$\nabla \cdot \mathbf{o} = \frac{\partial o_x}{\partial x} + \frac{\partial o_y}{\partial y}, \quad \nabla p = \begin{bmatrix} \frac{\partial p}{\partial x} \\ \frac{\partial p}{\partial y} \end{bmatrix}. \tag{S3}$$

Introducing the state vector:

$$\mathbf{u} = \begin{bmatrix} p \\ o_x \\ o_y \end{bmatrix}. \tag{S4}$$

Fourier Domain Representation

The Fourier transform expresses a function in terms of its frequency components. For a scalar field $f(\mathbf{r}, t)$, where $\mathbf{r}$ represents spatial coordinates, the Fourier transform is defined as:

$$\tilde{f}(\mathbf{k}, \omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\mathbf{r}, t) e^{-i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \, d\mathbf{r} \, dt, \tag{S5}$$

where $\mathbf{k} = [k_x, k_y]$ is the wave vector in the spatial frequency domain, and ω is the angular frequency in the time domain.

The inverse Fourier transform is given by:

$$f(\mathbf{r},t) = \frac{1}{(2\pi)^{n+1}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{f}(\mathbf{k},\omega) e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\, d\mathbf{k}\, d\omega. \tag{S6}$$

For a time-dependent field $f(t)$, the time derivative $\frac{\partial f}{\partial t}$ in the physical domain transforms as:

$$\begin{aligned} \mathcal{F}\left(\frac{\partial f}{\partial t}\right) &= \int_{-\infty}^{\infty} \frac{\partial f}{\partial t} e^{-i\omega t}\, dt \\ \texttt{Using integration by parts: } &= \left[f(t)e^{-i\omega t}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(t)\frac{\partial}{\partial t}\left(e^{-i\omega t}\right)\, dt. \end{aligned} \tag{S7}$$

The boundary term $\left[f(t)e^{-i\omega t}\right]_{-\infty}^{\infty}$ vanishes if $f(t)$ is sufficiently well-behaved (e.g., $f(t) \to 0$ as $t \to \pm\infty$). The remaining term simplifies as:

$$\begin{aligned} \mathcal{F}\left(\frac{\partial f}{\partial t}\right) &= -\int_{-\infty}^{\infty} f(t)\left(-i\omega e^{-i\omega t}\right)\, dt \\ &= -i\omega \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\, dt \\ &= -i\omega \tilde{f}(\omega). \end{aligned} \tag{S8}$$

Then the time derivative $\frac{\partial f}{\partial t}$ can be represented as follows:

$$\frac{\partial f}{\partial t} \xrightarrow{\text{Fourier Transform}} -i\omega\tilde{f}. \tag{S9}$$

Similarly, for spatial derivatives of a field $f(\mathbf{r})$, the gradient and divergence operators in the physical domain transform for the gradient will be:

$$\nabla f(\mathbf{r}) \xrightarrow{\text{Fourier Transform}} i\mathbf{k}\tilde{f}. \tag{S10}$$

Here, $\mathbf{k}$ is the wave vector, which has components $[k_x, k_y]$. This transformation comes from:

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} \xrightarrow{\mathcal{F}} \begin{bmatrix} ik_x\tilde{f} \\ ik_y\tilde{f} \end{bmatrix}. \tag{S11}$$

For $\mathbf{o}$, the divergence will be like:

$$\nabla \cdot \mathbf{o}(\mathbf{r}) \xrightarrow{\text{Fourier Transform}} i\mathbf{k}\cdot\tilde{\mathbf{o}}, \tag{S12}$$

where $\tilde{\mathbf{o}}$ is the Fourier transform of the vector field $\mathbf{o}$.

Using the Fourier transform, the spatial derivatives become:

$$\frac{\partial}{\partial t} \rightarrow -i\omega, \quad \nabla \rightarrow i\mathbf{k}, \quad \mathbf{k} = \begin{bmatrix} k_x \\ k_y \end{bmatrix}. \tag{S13}$$

The equations in Fourier space become:

$$\begin{aligned} -i\omega p + k_p p + c^2(ik_x o_x + ik_y o_y) &= 0, \\ -i\omega o_x + k_x o_x + ik_x p &= 0, \\ -i\omega o_y + k_y o_y + ik_y p &= 0. \end{aligned} \tag{S14}$$

These can be written as a matrix equation:

$$\begin{bmatrix} -i\omega + k_p & c^2 ik_x & c^2 ik_y \\ ik_x & -i\omega + k_x & 0 \\ ik_y & 0 & -i\omega + k_y \end{bmatrix} \begin{bmatrix} p \\ o_x \\ o_y \end{bmatrix} = 0. \tag{S15}$$

Efficient Eigenvalue Estimation

Since directly calculating the eigenvalues is complex, an approximation is employed. For small values of c or k, the coupling matrix can be expressed as:

$$\mathbf{A} = \begin{bmatrix} k_p & c^2 ik_x & c^2 ik_y \\ ik_x & k_x & 0 \\ ik_y & 0 & k_y \end{bmatrix}. \tag{S16}$$

The eigenvalues λ are determined by solving:

$$\det(\mathbf{A} - i\omega\mathbf{I}) = 0, \tag{S17}$$

The determinant gives the characteristic polynomial for the eigenvalues:

$$\det \begin{bmatrix} -i\omega + k_p & c^2 ik_x & c^2 ik_y \\ ik_x & -i\omega + k_x & 0 \\ ik_y & 0 & -i\omega + k_y \end{bmatrix} = 0. \tag{S18}$$

Expanding the determinant leads to a cubic equation in ω, with terms depending on k_p, c^2, and the wave vector components k_x and k_y.

$$c^2k_x^2k_y - ic^2k_x^2\omega + c^2k_xk_y^2 - ic^2k_y^2\omega + k_pk_xk_y - ik_pk_x\omega - ik_pk_y\omega - k_p\omega^2 - ik_xk_y\omega - k_x\omega^2 - k_y\omega^2 + i\omega^3 = 0. \tag{S19}$$

For small damping (k_p, k_x and k_y are small):

$$i\omega^3 - \left(k_p + k_x + k_y\right)\omega^2 - i\left(c^2k_x^2 + c^2k_y^2 + k_pk_x + k_pk_y + k_xk_y\right)\omega + c^2k_x^2k_y + c^2k_xk_y^2 + k_pk_xk_y = 0. \tag{S20}$$

Here, ω is complex ($\omega = \omega_r + i\omega_i$, where ω_r is the real part and ω_i is the imaginary part). Two key conclusions can be derived from Eq. (S20):

- **The real part of ω (ω_r) is dominated by the damping terms:**

  The damping terms are k_p, k_x, and k_y, which appear in the quadratic term:

  $$-\left(k_p + k_x + k_y\right)\omega^2. \tag{S21}$$

  These terms directly contribute to the real part of ω. Their presence ensures that as k_p, k_x, or k_y increase, the damping grows, leading to a larger ω_r. The dominant cubic term ($i\omega^3$) influences the frequency behaviour but does not directly contribute to damping. Thus, the damping is primarily governed by the term $-\left(k_p + k_x + k_y\right)$.

- **The imaginary part of ω (ω_i) is related to wave propagation speed:**

  The propagation speed c and the wave vector $|\mathbf{k}|$ dominate the imaginary part of ω:

  $$i\left(c^2k_x^2 + c^2k_y^2\right)\omega. \tag{S22}$$

  This term arises from the coupling of the wave propagation terms c^2 with k_x and k_y. Its leading-order contribution to ω is proportional to $\pm c|\mathbf{k}|$, where:

  $$|\mathbf{k}| = \sqrt{k_x^2 + k_y^2}. \tag{S23}$$

  In the absence of damping (i.e., $k_x = k_y = 0$), the equation simplifies to:

  $$i\omega^3 - ic^2|\mathbf{k}|^2\omega = 0, \tag{S24}$$

  leading to:

  $$\omega \approx \pm c|\mathbf{k}|, \tag{S25}$$

  as the dominant imaginary part.

Thus, the eigenvalues are approximately:

$$\omega \approx -\mathrm{Re}(k_p) \pm ic|\mathbf{k}|. \tag{S26}$$

In the parameter settings for all layers of the model, the initial values are defined as $k_x = k_y = 0$, ensuring that the damping contributions are governed solely by k_p. Initially, k_p is set as a small positive constant to maintain stable, damped oscillations. To activate the inhibition function, k_p is adjusted to a negative value, which introduces exponential growth and alters the system dynamics.