Supplementary Information: Fostering Sustainable Cooperation through Strategic Resource Allocation and Utilization on Social Networks

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This supplementary information provides a more detailed and comprehensive account of our analysis.

In Section 1, we present a novel framework of iterated multi-player games on hypergraphs to study repeated interactions in structured populations.

In Section 2, we extend the concept of the full cooperation feasible interval to hypergraphs. Specifically, we establish the necessary conditions for this concept to hold, and then, by examining three representative types of payoff functions, we derive the sufficient conditions under which full cooperation is feasible. Furthermore, we theoretically determine the minimum continuation probability threshold required to sustain full cooperation on any homogeneous hypergraph.

In Section 3, we shift our focus to the evolution of player strategies and examine the conditions that promote full cooperation among players.

1 Model Description

1.1 A framework of iterated multi-player games on hypergraphs

This paper introduces a general framework to investigate repeated multi-player games on hypergraph. Specifically, we consider a population of N players, denoted by $\mathcal{N} = \{1, 2, \ldots, N\}$, engaged in M repeated games, indexed by $\mathcal{M} = \{1, 2, \ldots, M\}$. The hypergraph is represented by an incidence matrix $A = \{a_{ij}\} \in R^{N,M}$, where $a_{ij} = 1$ if i participates in games j, and $a_{ij} = 0$ otherwise.

Each player i is initially assigned an endowment $e_i \geq 0$, representing their income or the time and effort they invest in the game. The vector of endowments is denoted as $\boldsymbol{e} = (e_1, e_2, \ldots, e_N)$, and without loss of generality, we assume that the endowment vector is normalized such that $\sum_{i=1}^{N} e_i = 1$. In the case of equal endowments among all players, the endowment vector simplifies to $\boldsymbol{e} = (\frac{1}{N}, \frac{1}{N}, \ldots, \frac{1}{N})$.

Each player may contribute a certain fraction of their endowment to the games in which they participate. We define the contribution matrix as $X = \{x_{ij}\} \in R^{N \times M}$, where x_{ij} represents the proportion of contributions from node i to game j. The constraints are $0 \leq x_{ij} \leq 1$ and $0 \leq \sum_{j=1}^{M} x_{ij} \leq 1$ for any i. We do not require $\sum_{j=1}^{M} x_{ij} = 1$ because individuals may retain a portion of their endowment without participating in the game. We introduce a shorthand notation, $X = \{\mathbf{1}\}$, to represent the full cooperation where each player contributes their entire endowment, meaning $\sum_{j=1}^{M} x_{ij} = 1$ for any i. In addition, the full cooperation is not unique.

The payoffs of public goods gaming are determined by endowments, contribution strategies, and individual player productivity. Specifically, given an endowment distribution $\boldsymbol{e}$ and a contribution matrix $X = \{x_{ij}\}$, players generate a payoff vector $u(\boldsymbol{e}, X) \in R^N$, where u represents the payoff function of the game. The element u_i in the i-th row corresponds to the benefit of player i. The total payoff of all game groups in the hypergraph is the sum of the benefits of all players, expressed as $U(\boldsymbol{e}, X) = \sum_{i=1}^{N} u_i(\boldsymbol{e}, X)$.

1.2 Public goods games create social dilemmas

In the classical public goods game, players face a dilemma when deciding their strategies. A brief overview of the public goods game problem is as follows: Each player must choose whether to invest a certain amount of money c into the public funds (cooperation strategy) or not (betray strategy). The public fund is then multiplied by a factor of r and distributed equally among all individuals. The payoff for each player can be expressed in the following form:

	1	$\ldots$	k	$\ldots$	$N-1$
cooperator	$\frac{rc}{N} - c$	$\ldots$	$\frac{rck}{N} - c$	$\ldots$	$\frac{rc(N-1)}{N} - c$
betrayer	$\frac{rc}{N}$	$\ldots$	$\frac{rck}{N}$	$\ldots$	$\frac{rc(N-1)}{N}$

Table 1: The payoffs of cooperators and betrayers in all situations in the public goods game. The first row shows the number of cooperators, while the second and third rows correspond to their cooperator payoffs and betrayer payoffs, respectively.

In the public goods game, as illustrated in Table 1, three properties need to be satisfied for the game to create a dilemma:

1. Positive Externality Property: A player desires their companions to cooperate, resulting in higher profits for themselves. When the number of cooperators is k, the betrayer's payoff is $\frac{rck}{N}$. If one betrayer becomes a cooperator, the betrayer's payoff becomes $\frac{rc(k+1)}{N}$. The positive externality property is satisfied by ensuring $r > 0$.

2. Incentive of Free-rides Property: Players choose the betrayal strategy to obtain higher benefits. When $k+1$ individuals cooperate, the cooperator's payoff is $\frac{rc(k+1)}{N} - c$. If one of the cooperators chooses to betray, their payoff becomes $\frac{rck}{N}$. This property is satisfied if $\frac{rck}{N} > \frac{rc(k+1)}{N} - c$, which implies $r < N$.

3. Optimality of Cooperation Property: From a collective perspective, players need to fully cooperate to achieve higher benefits for all players. When the number of cooperators is k, the collective payoff is $k(\frac{rck}{N} - c) + (N-k)\frac{rck}{N} = kc(r-1)$. Full cooperation results in the highest payoff when $r > 1$.

Therefore, the game creates a dilemma whenever $1 < r < N$. In such situations, players face a conflict between their incentives to betray and the collective benefits of cooperation.

In this paper, we study a public goods game that generates a dilemma, similar to the one discussed above. We consider a public goods game with the following four properties:

- **(C) Continuity:** The payoff function $u(\boldsymbol{e}, X)$ is continuous with respect to the parameters $\boldsymbol{e}$ and X.
- **(PE) Positive Externality:** Given two contribution matrices X and X', where $x_{ik} = x'_{ik}$ and $x_{jk} \geq x'_{jk}$ for all $j \neq i$ and $k = \{1, 2, \ldots, M\}$, we can conclude that $u_i(\boldsymbol{e}, X) \geq u_i(\boldsymbol{e}, X')$ for all endowment distributions $\boldsymbol{e}$. The strict inequality $u_i(\boldsymbol{e}, X) > u_i(\boldsymbol{e}, X')$ holds if and only if there exists at least one node j in the hyperedge k associated with node i satisfying $a_{ik}a_{jk} > 0$, $e_j > 0$ and $x_{jk} > x'_{jk}$.
- **(IF) Incentive of Free-rides:** Given two contribution matrices X and X', where $x_{ik} < x'_{ik}$ and $x_{jk} = x'_{jk}$ for all $j \neq i$ and $k = \{1, 2, \ldots, M\}$, we can conclude that $u_i(\boldsymbol{e}, X) \geq u_i(\boldsymbol{e}, X')$ for all endowment distributions $\boldsymbol{e}$. The strict inequality $u_i(\boldsymbol{e}, X) > u_i(\boldsymbol{e}, X')$ holds if and only if player i's endowment is positive, $e_i > 0$.
- **(OC) Optimality of Cooperation:** Given two contribution matrices X and X', where $x_{ik} \geq x'_{ik}$ and $x_{jk} \geq x'_{jk}$ for all $j \neq i$ and $k = \{1, 2, \ldots, M\}$, we can conclude that $U(\boldsymbol{e}, X) \geq U(\boldsymbol{e}, X')$ for all endowment distributions $\boldsymbol{e}$. The strict inequality $U(\boldsymbol{e}, X) > U(\boldsymbol{e}, X')$ holds if and only if there is a player i with $e_i > 0$ and $x_{ik} > x'_{ik}$.

1.3 Repeated games

This paper examines a scenario where different individuals interact for the same number of rounds across various games. Interactions are repeated over time with a continuation probability δ, which represents the likelihood of proceeding to the next round after each iteration. The expected number of rounds follows a geometric distribution, with the expected number of rounds being $\frac{1}{1-\delta}$. All players receive the same endowment in each round, with the initial endowment distribution denoted as $\boldsymbol{e}$. However, the contribution matrix X changes based on the outcome of each round of the game. Player i determines the contribution $x_{ik}(t+1)$ for the next round based on their previous contribution $x_{ik}(t)$ and the payoff obtained.

The player's strategy in the repeated game is determined by the percentage of contribution at each moment. If we define the contribution matrix at the moment t as $X(t) \in R^{N \times M}$, the payoff of the repeated game can be defined as a weighted average:

$$\pi_i = (1-\delta) \sum_{t=0}^{\infty} \delta^t u_i(\boldsymbol{e}, X(t)). \tag{1}$$

When $\delta \to 1$, the player payoff is obtained from the limit of the average payoff in each round

[1]:

$$\pi_i = \lim_{T \to +\infty} \frac{1}{T} \sum_{t=0}^{T-1} u_i(\boldsymbol{e}, X(t)). \tag{2}$$

2 Static equilibrium analysis

In this section, we explore the concept of subgame perfect equilibrium in repeated games. In a one-shot public goods game, the Nash equilibrium results in zero contribution; however, this outcome can be altered in a repeated setting [1, 2]. We focus on subgame perfect equilibrium, where players contribute their entire endowment in each round of the game. Given the game's history, no player has the incentive to deviate from the strategy of contributing their entire endowment, making this strategy profile a subgame perfect equilibrium [3]. Subgame perfect equilibrium is a more refined concept than Nash equilibrium. While every subgame perfect equilibrium is a Nash equilibrium, the reverse is not necessarily true [4].

2.1 General condition for full cooperation

The feasible interval of full cooperation represents the range of intervals in which all participants will contribute their entire endowment in each round, given the payoff function u and the continuation probability δ. In other words, given u and δ, it determines what endowment distribution $\boldsymbol{e}$ would allow participants to maintain full cooperation at all times (subgame perfect equilibrium). Let $E_u(\delta) = \{\boldsymbol{e} = (e_1, e_2, \dots, e_N)\}$ denote the set of all endowment distributions that allow for full cooperation. We consider the payoff function u for a generalized linear public goods game:

$$u_i(\boldsymbol{e}, X) = \sum_{k=1}^{M} \frac{a_{ik}}{|l_k|} \sum_{j=1}^{N} r_{jk} x_{jk} e_j + (1 - \sum_{k=1}^{M} x_{ik}) e_i, \tag{3}$$

where r_{jk} is the productivity factor of node j on the k-th hyperedge and $|l_k|$ is the number of nodes on the k-th hyperedge. We denote by $X = \{\mathbf{1}_{-i}\}$ the situation in which all players except i contribute their full endowments, while player i contributes nothing; that is, $\sum_{k=1}^{M} x_{ik} = 0$ and $\sum_{k=1}^{M} x_{jk} = 1, \forall j \neq i$. We propose two assumptions.

Assumption 2.1 $\forall r_{j1} = r_{j2} = \cdots = r_{jM} = r_j, \forall j \in N$.*(Asymmetric productivity)*

Assumption 2.2 $\forall |l_1| = |l_2| = \cdots = |l_M| = \sigma$.*(Uniform hypergraph)*

Lemma 2.1 *Consider a public goods game with a payoff function u of the form given in Eq.(3) and a continuation probability δ, under a given endowment distribution $\boldsymbol{e}$. Assuming that Assumptions2.1 and 2.2 hold, the following three conditions are equivalent:*

1. $\boldsymbol{e} \in E_u(\delta)$,

2. *The following condition holds for all players i with $e_i > 0$,*

$$\delta(u_i(\boldsymbol{e}, \{\mathbf{1}_{-i}\}) - u_i(\boldsymbol{e}, O)) \geq u_i(\boldsymbol{e}, \{\mathbf{1}_{-i}\}) - u_i(\boldsymbol{e}, \{\mathbf{1}\}), \tag{4}$$

3. *The strategy profile where all players apply the strategy Grim is a subgame perfect equilibrium for the given endowment distribution $\boldsymbol{e}$.*

Proof *1 ⇒ 2* Satisfying $\boldsymbol{e} \in E_u(\delta)$ indicates that under a given endowment allocation scheme, each player opts for the full cooperation, i.e., $X = \{\mathbf{1}\}$. First, we need to prove that a player i altering their full cooperation does not impact their payoff, thus $u_i(\boldsymbol{e}, \{\mathbf{1}\}) = u_i(\boldsymbol{e}, \{\mathbf{1}^{'}\})$. Here, $\{\mathbf{1}^{'}\}$ implies that while all x_{jk} remain constant, only $x_{ik}^{'}$ varies with $\sum_{k=1}^{M} x_{ik}^{'} = 1$. If this condition is not met, any change in player i's strategy will either increase or decrease its payoff, failing to achieve subgame perfect equilibrium. Considering two different full cooperation strategies $X = \{\mathbf{1}\}$ and $X' = \{\mathbf{1}^{'}\}$, player i's payoff is calculated as:

$$\begin{aligned}
u_i(\boldsymbol{e}, \{\mathbf{1}\}) &= \sum_{k=1}^{M} \frac{a_{ik}}{|l_k|} \sum_{j=1}^{N} r_{jk} x_{jk} e_j + (1 - \sum_{k=1}^{M} x_{ik}) e_i \\
&= \sum_{k=1}^{M} \frac{a_{ik}}{|l_k|} \sum_{j \neq i}^{N} r_{jk} x_{jk} e_j + \sum_{k=1}^{M} \frac{a_{ik}}{|l_k|} r_{ik} x_{ik} e_i + (1 - \sum_{k=1}^{M} x_{ik}) e_i \\
&= \sum_{k=1}^{M} \frac{a_{ik}}{|l_k|} \sum_{j \neq i}^{N} r_{jk} x_{jk} e_j + e_i \sum_{k=1}^{M} \frac{r_{ik}}{|l_k|} x_{ik}.
\end{aligned} \tag{5}$$

When assumptions 2.1 and 2.2 are satisfied, the payoff $u_i(\boldsymbol{e}, \{\mathbf{1}\})$ simplifies to:

$$\begin{aligned}
u_i(\boldsymbol{e}, \{\mathbf{1}\}) &= \sum_{k=1}^{M} \frac{a_{ik}}{\sigma} \sum_{j \neq i}^{N} r_j x_{jk} e_j + e_i \sum_{k=1}^{M} \frac{r_i}{\sigma} x_{ik} \\
&= \sum_{k=1}^{M} \frac{a_{ik}}{\sigma} \sum_{j \neq i}^{N} r_j x_{jk} e_j + \frac{e_i r_i}{\sigma} \sum_{k=1}^{M} x_{ik}.
\end{aligned} \tag{6}$$

This leads us to conclude that $u_i(\boldsymbol{e}, \{\mathbf{1}^{'}\}) = \sum_{k=1}^{M} \frac{a_{ik}}{\sigma} \sum_{j \neq i}^{N} r_j x_{jk} e_j + \frac{e_i r_i}{\sigma} \sum_{k=1}^{M} x_{ik}^{'}$. Given that $\sum_{k=1}^{M} x_{ik} = \sum_{k=1}^{M} x_{ik}^{'} = 1$ and x_{jk} remains constant, it follows that $u_i(\boldsymbol{e}, \{\mathbf{1}\}) = u_i(\boldsymbol{e}, \{\mathbf{1}^{'}\})$. Next, we introduce a "mutant" who deviates from the full cooperation, its payoff in the subsequent round is $u_i(\boldsymbol{e}, \{\mathbf{1}_{-i}\})$. By the (PE) property, the "mutant's" minimum payoff would be $u_i(\boldsymbol{e}, O)$, leading to the conclusion:

$$\pi_M \geq (1 - \delta) u_i(\boldsymbol{e}, \{\mathbf{1}_{-i}\}) + \delta u_i(\boldsymbol{e}, O). \tag{7}$$

As the full cooperation is a subgame perfect equilibrium, it follows that:

$$u_i(\boldsymbol{e}, \{\mathbf{1}\}) \geq (1-\delta)u_i(\boldsymbol{e}, \{\mathbf{1}_{-i}\}) + \delta u_i(\boldsymbol{e}, O), \tag{8}$$

or equivalently,

$$\delta(u_i(\boldsymbol{e}, \{\mathbf{1}_{-i}\}) - u_i(\boldsymbol{e}, O)) \geq u_i(\boldsymbol{e}, \{\mathbf{1}_{-i}\}) - u_i(\boldsymbol{e}, \{\mathbf{1}\}). \tag{9}$$

2 $\Rightarrow$ *3* It is essential to note that the Grim strategy, in the context of repeated multi-player games on hypergraphs, implies that initially, all players adopt the full cooperation. Should any player choose betrayal, every player within the related hyperedge will also switch to betray. This effect cascades to the remaining players and so forth.

Our objective is to demonstrate that the Grim strategy constitutes a subgame perfect equilibrium. Employing the one-time deviation principle [4], we observe that an individual's betrayal initially impacts only their immediate neighbors. As the game progresses, this effect subtly spreads, ultimately resulting in either universal betrayal or game termination, under the presumption that all adhere to the Grim strategy. Importantly, the initial advantage gained by player i from betrayal is confined to its adjacent connections, influencing their responses till the game's conclusion. Hence, the betrayal payoff for node i aligns with $u_i(\boldsymbol{e}, O)$. The immediate payoff following a "mutant" intervention is $u_i(\boldsymbol{e}, \{\mathbf{1}_{-i}\})$, with all subsequent rounds approximated as $u_i(\boldsymbol{e}, O)$. The continuous payoffs, therefore, are:

$$\pi_M = (1-\delta)u_i(\boldsymbol{e}, \{\mathbf{1}_{-i}\}) + \delta u_i(\boldsymbol{e}, O). \tag{10}$$

Adherence to the inequality condition in *2* ensures that players gain no additional benefit from a one-time deviation under the Grim strategy, solidifying its status as a subgame perfect equilibrium.

3 $\Rightarrow$ *1* With all players adhering to the Grim strategy, they each consistently choose the full cooperation in every round. This collective commitment to the Grim strategy inherently establishes a subgame perfect equilibrium. Consequently, full cooperation is feasible among all participants. $\square$

Under the condition that Lemma 2.1 is satisfied, the following Lemmas 2.2 and 2.3 can be obtained according to Ref. [5].

Lemma 2.2 *We consider that a public good game with payoff function* u *has the form of Eq. (3) and Assumption 2.1 and 2.2 are satisfied.*

1. *Suppose* δ *and* δ' *are two continuation probabilities with* $\delta < \delta'$. *Then* $E_u(\delta) \subset E_u(\delta')$.

2. *There is a $\delta' < 1$ such that $E_u(\delta) \neq 0$ for all $\delta \geq \delta'$.*

Lemma 2.3 *Consider a public good game with continuation probability $\delta < 1$ and endowment distribution $\mathbf{e}$. For any player i, there is an $\epsilon_i > 0$ such that if player $i's$ endowment exceeds $e_i = 1 - \epsilon_i$ then full cooperation is infeasible.*

Lemma 2.3 shows that hypergraphs cannot sustain full cooperation if too much of the initial endowment is concentrated on a single individual. A similar argument suggests that equal endowment distribution is most conducive to cooperation. In Ref. [5], it was shown that the payoff function u satisfies a linear symmetric payoff function. However, this conclusion does not hold when u is nonlinear or asymmetric. In this paper, we find that even a linear symmetric payoff function does not guarantee that equal endowments are the most favorable for cooperation in the hypergraph.

2.2 Linear symmetric payoff

In the following, we consider the public goods game payoff function u that satisfies the conditions of Assumptions 2.1 and 2.2 as a linear symmetric payoff function.

$$u_i(\boldsymbol{e}, X) = \sum_{k=1}^{M} \frac{a_{ik}}{\sigma} \sum_{j=1}^{N} r x_{jk} e_j + (1 - \sum_{k=1}^{M} x_{ik}) e_i, \tag{11}$$

where r is the productivity coefficient common to all nodes. We first determine the range of parameters based on the three properties **(PE), (IF), (OC)**.

1.**(PE)** property satisfies the condition:

$$\begin{aligned} u_i(\boldsymbol{e}, X) &= \sum_{k=1}^{M} \frac{a_{ik}}{\sigma} \sum_{j=1}^{N} r x_{jk} e_j + (1 - \sum_{k=1}^{M} x_{ik}) e_i \\ &= \frac{r}{\sigma} \sum_{k=1}^{M} a_{ik} \sum_{j \neq i}^{N} x_{jk} e_j + \frac{r}{\sigma} \sum_{k=1}^{M} x_{ik} e_i + (1 - \sum_{k=1}^{M} x_{ik}) e_i. \end{aligned} \tag{12}$$

The expression shows that we need to make $r > 0$ to satisfy the condition.

2.**(IF)** property satisfies the condition:

$$\begin{aligned}
u_i(\boldsymbol{e}, X) &= \sum_{k=1}^{M} \frac{a_{ik}}{\sigma} \sum_{j=1}^{N} r x_{jk} e_j + (1 - \sum_{k=1}^{M} x_{ik}) e_i \\
&= \frac{r}{\sigma} \sum_{k=1}^{M} a_{ik} \sum_{j \neq i}^{N} x_{jk} e_j + \frac{r}{\sigma} \sum_{k=1}^{M} x_{ik} e_i + (1 - \sum_{k=1}^{M} x_{ik}) e_i \\
&= \frac{r}{\sigma} \sum_{k=1}^{M} a_{ik} \sum_{j \neq i}^{N} x_{jk} e_j + \sum_{k=1}^{M} (\frac{r}{\sigma} - 1) x_{ik} e_i + e_i.
\end{aligned} \tag{13}$$

The expression shows that we need to make $r < \sigma$ to satisfy the condition.

3.**(OC)** property satisfies the condition:

$$\begin{aligned}
U(\boldsymbol{e}, X) = \sum_{i=1}^{N} u_i(\boldsymbol{e}, X) &= \sum_{i=1}^{N} [\sum_{k=1}^{M} \frac{a_{ik}}{\sigma} \sum_{j=1}^{N} r x_{jk} e_j + (1 - \sum_{k=1}^{M} x_{ik}) e_i] \\
&= \sum_{i=1}^{N} [\frac{r}{\sigma} \sum_{k=1}^{M} a_{ik} \sum_{j \neq i}^{N} x_{jk} e_j + \frac{r}{\sigma} \sum_{k=1}^{M} x_{ik} e_i + (1 - \sum_{k=1}^{M} x_{ik}) e_i] \\
&= \sum_{i=1}^{N} [\frac{r}{\sigma} \sum_{k=1}^{M} a_{ik} \sum_{j \neq i}^{N} x_{jk} e_j + \sum_{k=1}^{M} (\frac{r}{\sigma} - 1) x_{ik} e_i + e_i] \\
&= \frac{r}{\sigma} \sum_{k=1}^{M} a_{1k} \sum_{j \neq 1}^{N} x_{jk} e_j + \frac{r}{\sigma} \sum_{k=1}^{M} x_{1k} e_1 - \sum_{k=1}^{M} x_{1k} e_1 + e_1 \\
&+ \frac{r}{\sigma} \sum_{k=1}^{M} a_{2k} \sum_{j \neq 1, j \neq 2}^{N} x_{jk} e_j + \frac{r}{\sigma} \sum_{k=1}^{M} a_{2k} x_{1k} e_1 + \frac{r}{\sigma} \sum_{k=1}^{M} x_{2k} e_2 - \sum_{k=1}^{M} x_{2k} e_2 + e_2 \\
&+ \dots \\
&+ \frac{r}{\sigma} \sum_{k=1}^{M} a_{Nk} \sum_{j \neq 1, j \neq N}^{N} x_{jk} e_j + \frac{r}{\sigma} \sum_{k=1}^{M} a_{Nk} x_{1k} e_1 + \frac{r}{\sigma} \sum_{k=1}^{M} x_{Nk} e_N - \sum_{k=1}^{M} x_{Nk} e_N + e_N \\
&= e_1 [\frac{r}{\sigma} \sum_{k=1}^{M} (a_{2k} x_{1k} + a_{3k} x_{1k} + \cdots + a_{Nk} x_{1k} + x_{1k} - \frac{\sigma}{r} x_{1k}) + 1] \\
&+ e_2 [\frac{r}{\sigma} \sum_{k=1}^{M} (a_{1k} x_{2k} + a_{3k} x_{2k} + \cdots + a_{Nk} x_{2k} + x_{2k} - \frac{\sigma}{r} x_{2k}) + 1] \\
&+ \dots \\
&+ e_N [\frac{r}{\sigma} \sum_{k=1}^{M} (a_{1k} x_{Nk} + a_{2k} x_{Nk} + \cdots + a_{(N-1)k} x_{Nk} + x_{Nk} - \frac{\sigma}{r} x_{Nk}) + 1].
\end{aligned} \tag{14}$$

The expression shows that we need to make $\sum_{j=1}^{N} a_{jk} > \frac{\sigma}{r}$ with $\forall k$ to satisfy the condition. Where $\sum_{j=1}^{N} a_{jk} = \sigma$, the property **(OC)** is satisfied as long as there is $r > 1$.

In summary, the three properties are satisfied, and the parameter r needs to satisfy $1 < r < \sigma$. Next, we start to discuss the feasible interval of full cooperation in the public goods game under the linear symmetric payoff function.

Theorem 2.1 *The payoff function of a public goods game is given by Eq. (11). For a contribution matrix X and a productivity coefficient $1 < r < \sigma$, $e \in E_u(\delta)$ holds if the continuation probability $\delta \geq \frac{\max\{e_i\}(\sigma - r)}{r \min\{e_i\} \sum_{k=1}^{M} \sum_{j \neq i}^{N} a_{ik} x_{jk}}$ with $\forall i$, and the equal endowments $e = (\frac{1}{N}, \frac{1}{N}, \ldots, \frac{1}{N})$ also belong to $e \in E_u(\delta)$.*

Proof According to Lemma 2.1, we first calculate the three payoff functions.

$$\begin{aligned} u_i(\boldsymbol{e}, \{\mathbf{1}_{-i}\}) &= \frac{r}{\sigma} \sum_{k=1}^{M} a_{ik} \sum_{j \neq i}^{N} x_{jk} e_j + e_i, \\ u_i(\boldsymbol{e}, O) &= e_i, \\ u_i(\boldsymbol{e}, \{\mathbf{1}\}) &= \frac{r}{\sigma} \sum_{k=1}^{M} a_{ik} \sum_{j \neq i}^{N} x_{jk} e_j + \frac{r}{\sigma} e_i. \end{aligned} \tag{15}$$

Applying Lemma 2.1, we obtain sufficient conditions under which the endowment vector $\boldsymbol{e}$ belongs to the full cooperation feasible set $E_u(\delta)$, as follows:

$$\begin{aligned} &\delta(u_i(\boldsymbol{e}, \{\mathbf{1}_{-i}\}) - u_i(\boldsymbol{e}, O)) \geq u_i(\boldsymbol{e}, \{\mathbf{1}_{-i}\}) - u_i(\boldsymbol{e}, \{\mathbf{1}\}) \\ \Rightarrow\ & \delta(\frac{r}{\sigma} \sum_{k=1}^{M} a_{ik} \sum_{j \neq i}^{N} x_{jk} e_j + e_i - e_i) \geq \frac{r}{\sigma} \sum_{k=1}^{M} a_{ik} \sum_{j \neq i}^{N} x_{jk} e_j + e_i - \frac{r}{\sigma} \sum_{k=1}^{M} a_{ik} \sum_{j \neq i}^{N} x_{jk} e_j - \frac{r}{\sigma} e_i \\ \Rightarrow\ & \delta \frac{r}{\sigma} \sum_{k=1}^{M} a_{ik} \sum_{j \neq i}^{N} x_{jk} e_j \geq (1 - \frac{r}{\sigma}) e_i \\ \Rightarrow\ & \delta \geq \frac{(\sigma - r) e_i}{r \sum_{k=1}^{M} a_{ik} \sum_{j \neq i}^{N} x_{jk} e_j}, \end{aligned} \tag{16}$$

where it is required that inequality (16) holds for all players i with $e_i > 0$. Enlarging the right-hand side of inequality (16) can satisfy all i holds:

$$\delta \geq \frac{\max\{e_i\}(\sigma - r)}{r \min\{e_i\} \sum_{k=1}^{M} \sum_{j \neq i}^{N} a_{ik} x_{jk}} \geq \frac{(\sigma - r) e_i}{r \sum_{k=1}^{M} a_{ik} \sum_{j \neq i}^{N} x_{jk} e_j} = \frac{(\sigma - r)}{r \sum_{k=1}^{M} a_{ik} \sum_{j \neq i}^{N} x_{jk} \frac{e_j}{e_i}}. \tag{17}$$

Thus $\boldsymbol{e}$ is feasible for full cooperation as long as inequality (17) is satisfied. Equal endowments

$\boldsymbol{e} = (\frac{1}{N}, \frac{1}{N}, \ldots, \frac{1}{N})$ must also be feasible for full cooperation.

$$\delta \geq \frac{\max\{e_i\}(\sigma - r)}{r \min\{e_i\} \sum_{k=1}^{M} \sum_{j \neq i}^{N} a_{ik} x_{jk}} \geq \frac{(\sigma - r)\frac{1}{N}}{r\frac{1}{N} \sum_{k=1}^{M} a_{ik} \sum_{j \neq i}^{N} x_{jk}}, \tag{18}$$

where the inequality holds for $max\{e_i\} \geq \frac{1}{N}$ and $min\{e_i\} \leq \frac{1}{N}$. □

Inequality (18) in Theorem 2.1 indicates that node i is most likely to deviate from full cooperation when the other nodes in all the hyperedges that node i participates in contribute the lowest sum to these hyperedges. In other words, under the equal endowment condition, if the hyperdegree of node i is 1 and the number of participating contributing nodes is $\sigma - 1$, it will have a higher probability of deviating from full cooperation easily. In a hypergraph, nodes with lower hyperdegree (players involved in fewer public goods games) will have a slightly higher probability of deviating from full cooperation. Turning to inequality (17), for an arbitrary endowment distribution, node i is more likely to deviate from full cooperation when its endowment is low and hyperdegree is small.

Theorem 2.1 is only a sufficient condition, it does not mean that other δ that do not satisfy the condition are not feasible for full cooperation. Moreover, Theorem 2.1 shows that equal endowments are not feasible for full cooperation as long as the condition is satisfied. In the following, we present an example in which equal endowments are not feasible for full cooperation, yet a non-empty feasible interval for full cooperation still exists.

Consider a set of linear symmetric payoff functions:

$$\begin{aligned}
u_1(\boldsymbol{e}, X) &= \frac{r}{3}(x_{11}e_1 + x_{21}e_2 + x_{31}e_3) + (1 - x_{11})e_1, \\
u_2(\boldsymbol{e}, X) &= \frac{r}{3}(x_{11}e_1 + x_{21}e_2 + x_{31}e_3 + x_{22}e_2 + x_{32}e_3 + x_{42}e_4) + (1 - x_{21} - x_{22})e_2, \\
u_3(\boldsymbol{e}, X) &= \frac{r}{3}(x_{11}e_1 + x_{21}e_2 + x_{31}e_3 + x_{22}e_2 + x_{32}e_3 + x_{42}e_4) + (1 - x_{31} - x_{32})e_3, \\
u_4(\boldsymbol{e}, X) &= \frac{r}{3}(x_{22}e_2 + x_{32}e_3 + x_{42}e_4) + (1 - x_{42})e_4.
\end{aligned} \tag{19}$$

Let $x_{11} = 1, x_{21} = \frac{1}{3}, x_{22} = \frac{2}{3}, x_{31} = \frac{2}{3}, x_{32} = \frac{1}{3}, x_{42} = 1$. Then for equal endowments $\boldsymbol{e} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$, δ need to satisfy the following condition:

$$\begin{aligned}
\delta_1 &\geq \frac{3 - r}{r}, \\
\delta_2 &\geq \frac{3 - r}{3r}, \\
\delta_3 &\geq \frac{3 - r}{3r}, \\
\delta_4 &\geq \frac{3 - r}{r}.
\end{aligned} \tag{20}$$

Thus it is only necessary to make $\delta \geq \frac{3-r}{r}$ so that equal distribution is feasible for full cooperation. In the following, we apply a perturbation to the equal endowments and the perturbed endowments $\boldsymbol{e}' = (\frac{1}{4}-\epsilon, \frac{1}{4}+2\epsilon, \frac{1}{4}, \frac{1}{4}-\epsilon)$. Below we calculate the δ conditions it has to satisfy:

$$\begin{aligned} \delta_1 &\geq \frac{3-r}{r\frac{1+\frac{8\epsilon}{3}}{1-4\epsilon}}, \\ \delta_2 &\geq \frac{3-r}{r\frac{3-8\epsilon}{1+8\epsilon}}, \\ \delta_3 &\geq \frac{3-r}{3r}, \\ \delta_4 &\geq \frac{3-r}{r\frac{1+4\epsilon}{1-4\epsilon}} \end{aligned} \tag{21}$$

The perturbed endowments only need to satisfy $\delta \geq \frac{3-r}{r\frac{1+\frac{8\epsilon}{3}}{1-4\epsilon}}$ to make it feasible for full cooperation. Comparing the two values, we find that whenever we take $\delta \in [\frac{3-r}{r\frac{1+\frac{8\epsilon}{3}}{1-4\epsilon}}, \frac{3-r}{r}]$, we make the equal endowments not feasible for full cooperation and will still have feasible intervals of full cooperation. We obtain the following corollary:

Corollary 1 *There exists a linear symmetric public goods game function* u *such that* $E_u(\delta) \neq \emptyset$*, but* $\boldsymbol{e} = (\frac{1}{N}, \frac{1}{N}, \ldots, \frac{1}{N}) \neq E_u(\delta)$.

In addition, we find that the total payoff is the same for any feasible interval of full cooperation under the linear symmetric payoff function.

Corollary 2 *The payoff function* u *of the public goods game under any linear symmetric payoff function satisfies* $\forall e_1, e_2 \in E_u(\delta)$ *and* $X(0) = \{\mathbf{1}\}$*, we can get* $\sum_{i=1}^{N} \pi_i(\boldsymbol{e}_1, \{\mathbf{1}\}) = \sum_{i=1}^{N} \pi_i(\boldsymbol{e}_2, \{\mathbf{1}\}) = r$.

Proof Since the initial moment $X(0) = \{\mathbf{1}\}$ and $e \in E_u(\delta)$, the initial contribution matrix $X(t) = X(0)$ is maintained for all subsequent moments. It is straightforward to calculate the total payoff:

$$\sum_{i=1}^{N}\pi_i(\boldsymbol{e},\{\mathbf{1}\})=\sum_{i=1}^{N}(1-\delta)\sum_{t=0}^{\infty}\delta^t u_i(\boldsymbol{e},\{\mathbf{1}\})=\sum_{i=1}^{N}u_i(\boldsymbol{e},\{\mathbf{1}\})=U(\boldsymbol{e},\{\mathbf{1}\}).$$

$$\begin{aligned}U(\boldsymbol{e},\{\mathbf{1}\})&=e_1[\frac{r}{\sigma}\sum_{k=1}^{M}(a_{2k}x_{1k}+a_{3k}x_{1k}+\cdots+a_{Nk}x_{1k}+x_{1k}-\frac{\sigma}{r}x_{1k})+1]\\&+e_2[\frac{r}{\sigma}\sum_{k=1}^{M}(a_{1k}x_{2k}+a_{3k}x_{2k}+\cdots+a_{Nk}x_{2k}+x_{2k}-\frac{\sigma}{r}x_{2k})+1]\\&+\dots\\&+e_N[\frac{r}{\sigma}\sum_{k=1}^{M}(a_{1k}x_{Nk}+a_{2k}x_{Nk}+\cdots+a_{(N-1)k}x_{Nk}+x_{Nk}-\frac{\sigma}{r}x_{Nk})+1]\\&=e_1[\frac{r}{\sigma}\sum_{k=1}^{M}(\sum_{j=1}^{N}a_{jk}-\frac{\sigma}{r})x_{1k}+1]+\cdots+e_N[\frac{r}{\sigma}\sum_{k=1}^{M}(\sum_{j=1}^{N}a_{jk}-\frac{\sigma}{r})x_{Nk}+1]\\&=e_1[\frac{r}{\sigma}(\sigma-\frac{\sigma}{r})\sum_{k=1}^{M}x_{1k}+1]+\cdots+e_N[\frac{r}{\sigma}(\sigma-\frac{\sigma}{r})\sum_{k=1}^{M}x_{Nk}+1]\\&=r(\sum_{i=1}^{N}e_i)=r\end{aligned}\tag{22}$$

□

In Ref. [5] it was shown that the equal endowments are the most favorable for cooperation in a public goods game with a linear symmetric payoff function. However, Corollary 1 reveals that, in multi-player public goods games on hypergraphs, even equal endowments under linear symmetric payoff functions are not the most favorable for promoting cooperation. In addition, Corollary 2 finds that as long as the distribution of endowments falls within the feasible interval of full cooperation and the initial state is fully cooperative, the total payoff of the game is the same as the highest.

2.3 Linear asymmetric payoff

We extend the symmetric payoff function so that the productivity coefficients need only satisfy Assumption 2.1. Consider the public goods game payoff function u:

$$u_i(\boldsymbol{e},X)=\sum_{k=1}^{M}\frac{a_{ik}}{\sigma}\sum_{j=1}^{N}r_j x_{jk}e_j+(1-\sum_{k=1}^{M}x_{ik})e_i.\tag{23}$$

We still first discuss the range of values of the parameter r_j.

1.**(PE)** property satisfies the condition:

$$\begin{aligned} u_i(\boldsymbol{e}, X) &= \sum_{k=1}^{M} \frac{a_{ik}}{\sigma} \sum_{j=1}^{N} r_j x_{jk} e_j + (1 - \sum_{k=1}^{M} x_{ik}) e_i \\ &= \frac{1}{\sigma} \sum_{k=1}^{M} a_{ik} \sum_{j \neq i}^{N} r_j x_{jk} e_j + \frac{1}{\sigma} \sum_{k=1}^{M} r_i x_{ik} e_i + e_i - \sum_{k=1}^{M} x_{ik} e_i. \end{aligned} \tag{24}$$

The expression shows that we need to make $r_j > 0$ with $\forall j$ to satisfy the condition.

2.**(IF)** property satisfies the condition:

$$\begin{aligned} u_i(\boldsymbol{e}, X) &= \sum_{k=1}^{M} \frac{a_{ik}}{\sigma} \sum_{j=1}^{N} r_j x_{jk} e_j + (1 - \sum_{k=1}^{M} x_{ik}) e_i \\ &= \frac{1}{\sigma} \sum_{k=1}^{M} a_{ik} \sum_{j \neq i}^{N} r_j x_{jk} e_j + \frac{1}{\sigma} \sum_{k=1}^{M} r_i x_{ik} e_i + e_i - \sum_{k=1}^{M} x_{ik} e_i \\ &= \frac{1}{\sigma} \sum_{k=1}^{M} a_{ik} \sum_{j \neq i}^{N} r_j x_{jk} e_j + \sum_{k=1}^{M} (\frac{r_i}{\sigma} - 1) x_{ik} e_i + e_i. \end{aligned} \tag{25}$$

The expression shows that we need to make $r_j < \sigma$ with $\forall j$ to satisfy the condition.

3.**(OC)** property satisfies the condition:

$$\begin{aligned} U(\boldsymbol{e}, X) = \sum_{i=1}^{N} u_i(\boldsymbol{e}, X) &= \sum_{i=1}^{N} [\sum_{k=1}^{M} \frac{a_{ik}}{\sigma} \sum_{j=1}^{N} r_j x_{jk} e_j + (1 - \sum_{k=1}^{M} x_{ik}) e_i] \\ &= e_1 [\frac{r_1}{\sigma} \sum_{k=1}^{M} (a_{2k} x_{1k} + a_{3k} x_{1k} + \cdots + a_{Nk} x_{1k} + x_{1k} - \frac{\sigma}{r_1} x_{1k}) + 1] \\ &+ e_2 [\frac{r_2}{\sigma} \sum_{k=1}^{M} (a_{1k} x_{2k} + a_{3k} x_{2k} + \cdots + a_{Nk} x_{2k} + x_{2k} - \frac{\sigma}{r_2} x_{2k}) + 1] \\ &+ \dots \\ &+ e_N [\frac{r_N}{\sigma} \sum_{k=1}^{M} (a_{1k} x_{Nk} + a_{2k} x_{Nk} + \cdots + a_{(N-1)k} x_{Nk} + x_{Nk} - \frac{\sigma}{r_N} x_{Nk}) + 1]. \end{aligned} \tag{26}$$

The expression shows that we need to make $\sum_{j=1}^{N} a_{jk} > \frac{\sigma}{r_i}$ with $\forall k, i$ to satisfy the condition. Where $\sum_{j=1}^{N} a_{jk} = \sigma$, the property **(OC)** is satisfied as long as there is $r_i > 1$ with $\forall i$.

In summary, the three properties hold provided that the parameter r_i satisfies $1 < r_i < \sigma$ with $\forall i$. In the following, we discuss the feasible interval of full cooperation under the linear asymmetric payoff function.

Theorem 2.2 *The payoff function of a public goods game is given by Eq. (23). For a contribution matrix X and productivity coefficient $1 < r_i < \sigma$ with $\forall i$, $e \in E_u(\delta)$ holds if the continuation probability $\delta \geq \frac{\max\{e_i(\sigma - r_i)\}}{\min\{e_i r_i\} \sum_{k=1}^{M} \sum_{j \neq i}^{N} a_{ik} x_{jk}}$ with $\forall i$.*

Proof Similar to the proof of Theorem 2.1. According to Lemma 2.1, we calculate the 3 payoff functions.

$$
\begin{aligned}
u_i(\boldsymbol{e}, \{\mathbf{1}_{-i}\}) &= \frac{1}{\sigma} \sum_{k=1}^{M} a_{ik} \sum_{j \neq i}^{N} r_j x_{jk} e_j + e_i, \\
u_i(\boldsymbol{e}, O) &= e_i, \\
u_i(\boldsymbol{e}, \{\mathbf{1}\}) &= \frac{1}{\sigma} \sum_{k=1}^{M} a_{ik} \sum_{j \neq i}^{N} r_j x_{jk} e_j + \frac{r_i}{\sigma} e_i.
\end{aligned} \tag{27}
$$

Applying Lemma 2.1, we derive sufficient conditions under which the endowment vector $\boldsymbol{e}$ lies within the full cooperation feasible set $E_u(\delta)$, as follows:

$$
\begin{aligned}
& \delta(u_i(\boldsymbol{e}, \{\mathbf{1}_{-i}\}) - u_i(\boldsymbol{e}, O)) \geq u_i(\boldsymbol{e}, \{\mathbf{1}_{-i}\}) - u_i(\boldsymbol{e}, \{\mathbf{1}\}) \\
\Rightarrow\ & \delta\Big(\frac{1}{\sigma} \sum_{k=1}^{M} a_{ik} \sum_{j \neq i}^{N} r_j x_{jk} e_j + e_i - e_i\Big) \geq \frac{1}{\sigma} \sum_{k=1}^{M} a_{ik} \sum_{j \neq i}^{N} r_j x_{jk} e_j + e_i - \frac{1}{\sigma} \sum_{k=1}^{M} a_{ik} \sum_{j \neq i}^{N} r_j x_{jk} e_j - \frac{r_i}{\sigma} e_i \\
\Rightarrow\ & \delta \frac{1}{\sigma} \sum_{k=1}^{M} a_{ik} \sum_{j \neq i}^{N} r_j x_{jk} e_j \geq \Big(1 - \frac{r_i}{\sigma}\Big) e_i \\
\Rightarrow\ & \delta \geq \frac{(\sigma - r_i) e_i}{\sum_{k=1}^{M} a_{ik} \sum_{j \neq i}^{N} r_j x_{jk} e_j},
\end{aligned} \tag{28}
$$

where it is required that inequality (28) holds for all players i with $e_i > 0$. Enlarging the right-hand side of inequality (28) can satisfy all i holds:

$$
\delta \geq \frac{\max\{(\sigma - r_i) e_i\}}{\min\{e_i r_i\} \sum_{k=1}^{M} \sum_{j \neq i}^{N} a_{ik} x_{jk}} \geq \frac{(\sigma - r_i) e_i}{\sum_{k=1}^{M} a_{ik} \sum_{j \neq i}^{N} r_j x_{jk} e_j}. \tag{29}
$$

Thus $\boldsymbol{e}$ is fully cooperative as long as inequality (29) is satisfied. □

Remark 1 *The sufficient conditions in Theorem 2.2 do not guarantee that equal endowments are also feasible for full cooperation.*

Theorem 2.2 gives a sufficient condition for the continuation probability δ of full cooperation under the asymmetric payoff function. The key factor for node i to deviate from full cooperation that can be obtained from inequality 29 is the combined $\sum_{j \neq i} r_j x_{jk} e_j$ value of the associated hyperedge of node i. When node i is less associated and has a lower production coefficient r_i, it will be more likely to deviate from cooperation.

Similar to Corollary 1 and Corollary 2, we can obtain similar conclusions under the asymmetric payoff function.

Corollary 3 *There exists a linear asymmetric public goods game payoff function u such that $E_u(\delta) \neq \emptyset$, but $\boldsymbol{e} = (\frac{1}{N}, \frac{1}{N}, \dots, \frac{1}{N}) \neq E_u(\delta)$.*

We still consider the example in Corollary 1 by simply varying the productivity coefficient r. Consider a set of asymmetric payoff functions:

$$\begin{aligned}
u_1(\boldsymbol{e}, X) &= \frac{1}{3}(r_1 x_{11} e_1 + r_2 x_{21} e_2 + r_3 x_{31} e_3) + (1 - x_{11}) e_1, \\
u_2(\boldsymbol{e}, X) &= \frac{1}{3}(r_1 x_{11} e_1 + r_2 x_{21} e_2 + r_3 x_{31} e_3 + r_2 x_{22} e_2 + r_3 x_{32} e_3 + r_4 x_{42} e_4) + (1 - x_{21} - x_{22}) e_2, \\
u_3(\boldsymbol{e}, X) &= \frac{1}{3}(r_1 x_{11} e_1 + r_2 x_{21} e_2 + r_3 x_{31} e_3 + r_2 x_{22} e_2 + r_3 x_{32} e_3 + r_4 x_{42} e_4) + (1 - x_{31} - x_{32}) e_3, \\
u_4(\boldsymbol{e}, X) &= \frac{1}{3}(r_2 x_{22} e_2 + r_3 x_{32} e_3 + r_4 x_{42} e_4) + (1 - x_{42}) e_4.
\end{aligned} \tag{30}$$

Let $x_{11} = 1, x_{21} = \frac{1}{3}, x_{22} = \frac{2}{3}, x_{31} = \frac{2}{3}, x_{32} = \frac{1}{3}, x_{42} = 1, r_1 = 2+\epsilon, r_2 = 2, r_3 = 2, r_4 = 2+\epsilon$. Then for equal endowments $\boldsymbol{e} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$, δ need to satisfy the following condition:

$$\begin{aligned}
\delta_1 &\geq \frac{3 - 2 - \epsilon}{2\frac{1}{3} + 2\frac{2}{3}} = \frac{1 - \epsilon}{2}, \\
\delta_2 &\geq \frac{3 - 2}{2 + \epsilon + 2 + 2 + \epsilon} = \frac{1}{6 + 2\epsilon}, \\
\delta_3 &\geq \frac{3 - 2}{2 + \epsilon + 2 + 2 + \epsilon} = \frac{1}{6 + 2\epsilon}, \\
\delta_4 &\geq \frac{3 - 2 - \epsilon}{2\frac{2}{3} + 2\frac{1}{3}} = \frac{1 - \epsilon}{2}.
\end{aligned} \tag{31}$$

Thus it is only necessary to make $\delta \geq \frac{1-\epsilon}{2}$ so that equal distribution is feasible for full cooperation. In the following, we apply a perturbation to the equal endowments and the perturbed endowments $\boldsymbol{e}' = (\frac{1}{4}-\epsilon, \frac{1}{4}+2\epsilon, \frac{1}{4}, \frac{1}{4}-\epsilon)$. Below we calculate the δ conditions it has to satisfy:

$$\begin{aligned}
\delta_1 &\geq \frac{(3-2-\epsilon)(\frac{1}{4}-\epsilon)}{2(\frac{1}{4}+2\epsilon)\frac{1}{3}+2\frac{1}{4}\frac{2}{3}} = \frac{(1-\epsilon)(1-4\epsilon)}{2+\frac{16}{3}\epsilon}, \\
\delta_2 &\geq \frac{(3-2)(\frac{1}{4}+2\epsilon)}{(2+\epsilon)(\frac{1}{4}-\epsilon)+2\frac{1}{4}+(2+\epsilon)(\frac{1}{4}-\epsilon)} = \frac{1+8\epsilon}{6+2\epsilon-(2+\epsilon)(4\epsilon)}, \\
\delta_3 &\geq \frac{(3-2)\frac{1}{4}}{(2+\epsilon)(\frac{1}{4}-\epsilon)+2(\frac{1}{4}+2\epsilon)+(2+\epsilon)(\frac{1}{4}-\epsilon)} = \frac{1}{6+2\epsilon+16\epsilon-(2+\epsilon)(4\epsilon)}, \\
\delta_4 &\geq \frac{(3-2-\epsilon)(\frac{1}{4}-\epsilon)}{2(\frac{1}{4}+2\epsilon)\frac{2}{3}+2\frac{1}{4}\frac{1}{3}} = \frac{(1-\epsilon)(1-4\epsilon)}{2+\frac{32}{3}\epsilon}
\end{aligned} \tag{32}$$

The perturbed endowments only need to satisfy $\delta \geq \frac{(1-\epsilon)(1-4\epsilon)}{2+\frac{16}{3}\epsilon}$ to make it feasible for full cooperation. Comparing the two values, we find that whenever we take $\delta \in [\frac{(1-\epsilon)(1-4\epsilon)}{2+\frac{16}{3}\epsilon}, \frac{1-\epsilon}{2}]$, we make the equal endowments not feasible for full cooperation and will still have feasible intervals of full cooperation.

Corollary 4 *The payoff function u of the public goods game under any linear asymmetric payoff function satisfies $\forall e \in E_u(\delta)$ and $X(0) = \{\mathbf{1}\}$, we can get $\sum_{i=1}^{N} \pi_i(\boldsymbol{e}_1, \{\mathbf{1}\}) = \sum_{i=1}^{N} r_i e_i$.*

Proof Since the initial moment $X(0) = \{\mathbf{1}\}$ and $e \in E_u(\delta)$, the initial contribution matrix $X(t) = X(0)$ is maintained for all subsequent moments. It is straightforward to calculate the total payoff:

$$\begin{aligned}
U(\boldsymbol{e}, \{\mathbf{1}\}) = {} & e_1[\frac{r_1}{\sigma}\sum_{k=1}^{M}(a_{2k}x_{1k}+a_{3k}x_{1k}+\cdots+a_{Nk}x_{1k}+x_{1k}-\frac{\sigma}{r_1}x_{1k})+1] \\
& + e_2[\frac{r_2}{\sigma}\sum_{k=1}^{M}(a_{1k}x_{2k}+a_{3k}x_{2k}+\cdots+a_{Nk}x_{2k}+x_{2k}-\frac{\sigma}{r_2}x_{2k})+1] \\
& + \dots \\
& + e_N[\frac{r_N}{\sigma}\sum_{k=1}^{M}(a_{1k}x_{Nk}+a_{2k}x_{Nk}+\cdots+a_{(N-1)k}x_{Nk}+x_{Nk}-\frac{\sigma}{r_N}x_{Nk})+1] \\
& = e_1[\frac{r_1}{\sigma}\sum_{k=1}^{M}(\sum_{j=1}^{N}a_{jk}-\frac{\sigma}{r_1})x_{1k}+1]+\cdots+e_N[\frac{r_N}{\sigma}\sum_{k=1}^{M}(\sum_{j=1}^{N}a_{jk}-\frac{\sigma}{r_N})x_{Nk}+1] \\
& = e_1[\frac{r_1}{\sigma}(\sigma-\frac{\sigma}{r_1})\sum_{k=1}^{M}x_{1k}+1]+\cdots+e_N[\frac{r_N}{\sigma}(\sigma-\frac{\sigma}{r_N})\sum_{k=1}^{M}x_{Nk}+1] \\
& = \sum_{i=1}^{N} r_i e_i
\end{aligned} \tag{33}$$

$\square$

Corollary 4 shows that different initial endowments within the full cooperation interval $E_u(\delta)$ also have different total payoffs.

2.4 An example of a nonlinear symmetric payoff

We consider the example of a nonlinear symmetric public goods game with a payoff function u_i of the following form:

$$u_i(\boldsymbol{e}, X) = c \max_{p,q \in N(i)} \{\sum_{k=1}^{M}(x_{pk}e_p + x_{qk}e_q)\} + \frac{r}{\sigma}\sum_{k=1}^{M} a_{ik}\sum_{j=1}^{N} x_{jk}e_j - (1 - \sum_{k=1}^{M} x_{ik})e_i, \tag{34}$$

where $N(i)$ denotes the set of neighboring nodes of node i associated with hyperedges and $c > 0$. The nonlinear part means that the two highest actual values of endowments around node i will have an additional effect on it. Lemma 2.1 is satisfied by the condition that the payoff function u_i form is linear. We are considering a nonlinear payoff function and need to verify that $u_i(\boldsymbol{e}, \mathbf{1}) = u_i(\boldsymbol{e}, \mathbf{1}')$.

$$\begin{aligned} u_i(\boldsymbol{e}, \{\mathbf{1}\}) &= c \max_{p,q \in N(i)} \{\sum_{k=1}^{M}(x_{pk}e_p + x_{qk}e_q)\} + \frac{r}{\sigma}\sum_{k=1}^{M} a_{ik}\sum_{j=1}^{N} x_{jk}e_j - (1 - \sum_{k=1}^{M} x_{ik})e_i \\ &= c \max_{p,q \in N(i)} \{e_p + e_q\} + \frac{r}{\sigma}\sum_{k=1}^{M} a_{ik}\sum_{j\neq i}^{N} x_{jk}e_j + \frac{e_i r_i}{\sigma}\sum_{k=1}^{M} x_{ik} = u_i(\boldsymbol{e}, \{\mathbf{1}'\}). \end{aligned} \tag{35}$$

Here it can also be extended to consider a nonlinear asymmetric payoff function, whose nonlinear part is denoted as $\max\limits_{p,q \in N(i)} \{\sum_{k=1}^{M}(r_p x_{pk}e_p + r_q x_{qk}e_q)\}$. It still satisfies the condition that $u_i(\boldsymbol{e}, \{\mathbf{1}\}) = u_i(\boldsymbol{e}, \{\mathbf{1}'\})$. We verify the parameter conditions the three properties satisfy for a nonlinear payoff function of the form Eq.(34).

1.**(PE)** property satisfies the condition:

$$\begin{aligned} u_i(\boldsymbol{e}, X) &= c \max_{p,q \in N(i)} \{\sum_{k=1}^{M}(x_{pk}e_p + x_{qk}e_q)\} + \sum_{k=1}^{M}\frac{a_{ik}}{\sigma}\sum_{j=1}^{N} r x_{jk}e_j + (1 - \sum_{k=1}^{M} x_{ik})e_i \\ &= c \max_{p,q \in N(i)} \{\sum_{k=1}^{M}(x_{pk}e_p + x_{qk}e_q)\} + \frac{r}{\sigma}\sum_{k=1}^{M} a_{ik}\sum_{j\neq i}^{N} x_{jk}e_j + \frac{r}{\sigma}\sum_{k=1}^{M} x_{ik}e_i + (1 - \sum_{k=1}^{M} x_{ik})e_i. \end{aligned} \tag{36}$$

The expression shows that we need to make $r > 0$ to satisfy the condition.

2.**(IF)** property satisfies the condition:

$$\begin{aligned}
u_i(\boldsymbol{e}, X) &= c \max_{p,q \in N(i)} \{\sum_{k=1}^{M}(x_{pk}e_p + x_{qk}e_q)\} + \sum_{k=1}^{M} \frac{a_{ik}}{\sigma} \sum_{j=1}^{N} r x_{jk} e_j + (1 - \sum_{k=1}^{M} x_{ik}) e_i \\
&= c \max_{p,q \in N(i)} \{\sum_{k=1}^{M}(x_{pk}e_p + x_{qk}e_q)\} + \frac{r}{\sigma} \sum_{k=1}^{M} a_{ik} \sum_{j \neq i}^{N} x_{jk} e_j + \frac{r}{\sigma} \sum_{k=1}^{M} x_{ik} e_i + (1 - \sum_{k=1}^{M} x_{ik}) e_i \\
&= c \max_{p,q \in N(i)} \{\sum_{k=1}^{M}(x_{pk}e_p + x_{qk}e_q)\} + \frac{r}{\sigma} \sum_{k=1}^{M} a_{ik} \sum_{j \neq i}^{N} x_{jk} e_j + \sum_{k=1}^{M} (\frac{r}{\sigma} - 1) x_{ik} e_i + e_i.
\end{aligned} \tag{37}$$

The expression shows that we need to make $r < \sigma$ to satisfy the condition.

3.**(OC)** property satisfies the condition:

$$\begin{aligned}
U(\boldsymbol{e}, X) = \sum_{i=1}^{N} u_i(\boldsymbol{e}, X) &= \sum_{i=1}^{N} [c \max_{p,q \in N(i)} \{\sum_{k=1}^{M}(x_{pk}e_p + x_{qk}e_q)\} + \sum_{k=1}^{M} \frac{a_{ik}}{\sigma} \sum_{j=1}^{N} r x_{jk} e_j + (1 - \sum_{k=1}^{M} x_{ik}) e_i] \\
&= e_1 [\frac{r}{\sigma} \sum_{k=1}^{M} (a_{2k}x_{1k} + a_{3k}x_{1k} + \cdots + a_{Nk}x_{1k} + x_{1k} - \frac{\sigma}{r} x_{1k}) + 1] \\
&+ e_2 [\frac{r}{\sigma} \sum_{k=1}^{M} (a_{1k}x_{2k} + a_{3k}x_{2k} + \cdots + a_{Nk}x_{2k} + x_{2k} - \frac{\sigma}{r} x_{2k}) + 1] \\
&+ \dots \\
&+ e_N [\frac{r}{\sigma} \sum_{k=1}^{M} (a_{1k}x_{Nk} + a_{2k}x_{Nk} + \cdots + a_{(N-1)k}x_{Nk} + x_{Nk} - \frac{\sigma}{r} x_{Nk}) + 1] \\
&+ c \sum_{i=1}^{N} \max_{p,q \in N(i)} \{\sum_{k=1}^{M}(x_{pk}e_p + x_{qk}e_q)\}.
\end{aligned} \tag{38}$$

The expression shows that we need to make $\sum_{j=1}^{N} a_{jk} > \frac{\sigma}{r}$ with $\forall k$ to satisfy the condition. Where $\sum_{j=1}^{N} a_{jk} = \sigma$, the property **(OC)** is satisfied as long as there is $r > 1$.

In summary, the nonlinear part is multiplied by a productivity factor $c > 0$, which leads to the satisfaction of the three property principle parameters r and linear symmetric payoff function consistent with $1 < r < \sigma$. Next, we discuss the conditions satisfied by the fully cooperative feasible interval of the nonlinear symmetric payoff function.

Theorem 2.3 *The payoff function of a public goods game is given by Eq. (34). For a contribution matrix X and productivity coefficient $1 < r < \sigma$, $e \in E_u(\delta)$ holds if the continuation probability $\delta \geq \frac{(\sigma - r)e_i}{\sigma c \max\limits_{p,q \in N(i)} \{e_p + e_q\} + r \sum_{k=1}^{M} a_{ik} \sum_{j \neq i}^{N} x_{jk} e_j}$ with $\forall i$.*

Proof According to Lemma 2.1, we calculate the 3 payoff functions.

$$\begin{aligned} u_i(\boldsymbol{e},\{\mathbf{1}_{-i}\}) &= c\max_{p,q\in N(i)}\{e_p+e_q\}+\frac{r}{\sigma}\sum_{k=1}^{M}a_{ik}\sum_{j\neq i}^{N}x_{jk}e_j+e_i,\\ u_i(\boldsymbol{e},O) &= e_i,\\ u_i(\boldsymbol{e},\{\mathbf{1}\}) &= c\max_{p,q\in N(i)}\{e_p+e_q\}+\frac{r}{\sigma}\sum_{k=1}^{M}a_{ik}\sum_{j\neq i}^{N}x_{jk}e_j+\frac{r}{\sigma}e_i. \end{aligned} \tag{39}$$

Applying Lemma 2.1, we derive sufficient conditions under which the endowment vector $\boldsymbol{e}$ lies within the full cooperation feasible set $E_u(\delta)$, as follows:

$$\begin{aligned} &\delta(u_i(\boldsymbol{e},\{\mathbf{1}_{-i}\})-u_i(\boldsymbol{e},O))\geq u_i(\boldsymbol{e},\{\mathbf{1}_{-i}\})-u_i(\boldsymbol{e},\{\mathbf{1}\})\\ \Rightarrow\ &\delta(c\max_{p,q\in N(i)}\{e_p+e_q\}+\frac{r}{\sigma}\sum_{k=1}^{M}a_{ik}\sum_{j\neq i}^{N}x_{jk}e_j)\geq(1-\frac{r}{\sigma})e_i\\ \Rightarrow\ &\delta\geq\frac{(\sigma-r)e_i}{\sigma c\max\limits_{p,q\in N(i)}\{e_p+e_q\}+r\sum_{k=1}^{M}a_{ik}\sum_{j\neq i}^{N}x_{jk}e_j}, \end{aligned} \tag{40}$$

where it is required that inequality (40) holds for all players i with $e_i>0$. □

Theorem 2.3 shows that the nonlinear part of the payoff function affects the feasible interval of full cooperation mainly in the distribution of endowment inequality. When two of the remaining nodes in the hyperedge associated with a node have relatively high endowments, it gives the node a higher probability of full cooperation. This is because the value of the nonlinear part $\max\limits_{p,q\in N(i)}\{e_p+e_q\}$ will be relatively larger, making the conditions for the continuation probability δ to be satisfied relatively weaker.

Corollary 5 *There exists a nonlinear symmetric public goods game function u such that $E_u(\delta)\neq\emptyset$, but $\boldsymbol{e}=(\frac{1}{N},\frac{1}{N},\ldots,\frac{1}{N})\neq E_u(\delta)$.*

We consider the example in Corollary 1. Consider a set of nonlinear symmetric payoff functions:

$$\begin{aligned} u_1(\boldsymbol{e},X) &= \frac{r}{3}(x_{11}e_1+x_{21}e_2+x_{31}e_3)+(1-x_{11})e_1+c((x_{21}+x_{22})e_2+(x_{31}+x_{32})e_3),\\ u_2(\boldsymbol{e},X) &= \frac{r}{3}(x_{11}e_1+x_{21}e_2+x_{31}e_3+x_{22}e_2+x_{32}e_3+x_{42}e_4)+(1-x_{21}-x_{22})e_2+c\max_{p,q\in N(2)}\{e_p+e_q\},\\ u_3(\boldsymbol{e},X) &= \frac{r}{3}(x_{11}e_1+x_{21}e_2+x_{31}e_3+x_{22}e_2+x_{32}e_3+x_{42}e_4)+(1-x_{31}-x_{32})e_3+c\max_{p,q\in N(3)}\{e_p+e_q\},\\ u_4(\boldsymbol{e},X) &= \frac{r}{3}(x_{22}e_2+x_{32}e_3+x_{42}e_4)+(1-x_{42})e_4+c((x_{21}+x_{22})e_2+(x_{31}+x_{32})e_3). \end{aligned} \tag{41}$$

Let $x_{11}=1, x_{21}=\frac{1}{3}, x_{22}=\frac{2}{3}, x_{31}=\frac{2}{3}, x_{32}=\frac{1}{3}, x_{42}=1$. Then for equal endowments

$\boldsymbol{e} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$, δ needs to satisfy the following condition:

$$\begin{aligned}
\delta_1 &\geq \frac{(3-r)\frac{1}{4}}{3c\frac{1}{2} + r(\frac{1}{3}\frac{1}{4} + \frac{2}{3}\frac{1}{4})} = \frac{3-r}{6c+r}, \\
\delta_2 &\geq \frac{(3-r)\frac{1}{4}}{3c\frac{1}{2} + r(1\frac{1}{4} + 1\frac{1}{4} + 1\frac{1}{4})} = \frac{3-r}{6c+3r} \\
\delta_3 &\geq \frac{(3-r)\frac{1}{4}}{3c\frac{1}{2} + r(1\frac{1}{4} + 1\frac{1}{4} + 1\frac{1}{4})} = \frac{3-r}{6c+3r}, \\
\delta_4 &\geq \frac{(3-r)\frac{1}{4}}{3c\frac{1}{2} + r(\frac{2}{3}\frac{1}{4} + \frac{1}{3}\frac{1}{4})} = \frac{3-r}{6c+r}.
\end{aligned} \tag{42}$$

Thus it is only necessary to make $\delta \geq \frac{3-r}{6c+r}$ so that equal distribution is feasible for full cooperation. In the following, we apply a perturbation to the equal endowments and the perturbed endowments $\boldsymbol{e}' = (\frac{1}{4} - \epsilon, \frac{1}{4} + 2\epsilon, \frac{1}{4} - \epsilon, \frac{1}{4})$. Below we calculate the δ conditions it has to satisfy:

$$\begin{aligned}
\delta_1 &\geq \frac{(3-r)(\frac{1}{4}-\epsilon)}{3c(\frac{1}{2}+\epsilon) + r(\frac{1}{3}(\frac{1}{4}+2\epsilon) + \frac{2}{3}(\frac{1}{4}-\epsilon))} = \frac{(3-r)(1-4\epsilon)}{6c+8c\epsilon+r}, \\
\delta_2 &\geq \frac{(3-r)(\frac{1}{4}+2\epsilon)}{3c(\frac{1}{2}-\epsilon) + r(\frac{1}{4}-\epsilon+\frac{1}{4}-\epsilon+\frac{1}{4})}, \\
\delta_3 &\geq \frac{(3-r)(\frac{1}{4}-\epsilon)}{2c(\frac{1}{2}+2\epsilon) + r(\frac{1}{4}-\epsilon+\frac{1}{4}+2\epsilon+\frac{1}{4})}, \\
\delta_4 &\geq \frac{(3-r)\frac{1}{4}}{3c(\frac{1}{2}+\epsilon) + r(\frac{2}{3}(\frac{1}{4}+2\epsilon) + \frac{1}{3}(\frac{1}{4}-\epsilon))} = \frac{(3-r)}{6c+8c\epsilon+r+4r\epsilon}.
\end{aligned} \tag{43}$$

The perturbed endowments only need to satisfy $\delta \geq \max\{\frac{(3-r)(1-4\epsilon)}{6c+8c\epsilon+r}, \frac{(3-r)}{6c+8c\epsilon+r+4r\epsilon}\}$ to make it feasible for full cooperation. Comparing the two values, we find that whenever we take $\delta \in [\max\{\frac{(3-r)(1-4\epsilon)}{6c+8c\epsilon+r}, \frac{(3-r)}{6c+8c\epsilon+r+4r\epsilon}\}, \frac{3-r}{6c+r}]$, we make the equal endowments not feasible for full cooperation and will still have feasible intervals of full cooperation.

Corollary 6 *The payoff function u of the public goods game under any nonlinear symmetric payoff function satisfies $\forall e \in E_u(\delta)$ and $X(0) = \{\mathbf{1}\}$, we can get $\sum_{i=1}^{N} \pi_i(\boldsymbol{e}_1, \{\mathbf{1}\}) = r + c\sum_{i=1}^{N} \max\limits_{p,q \in N(i)} \{e_p + e_q\}$.*

Proof Since the initial moment $X(0) = \{\mathbf{1}\}$ and $e \in E_u(\delta)$, the initial contribution matrix $X(t) = X(0)$ is maintained for all subsequent moments. It is straightforward to calculate the total payoff:

$$\begin{aligned} U(\boldsymbol{e},\{\mathbf{1}\}) = e_1[\frac{r_1}{\sigma}\sum_{k=1}^{M}(a_{2k}x_{1k}+a_{3k}x_{1k}+\cdots+a_{Nk}x_{1k}+x_{1k}-\frac{\sigma}{r_1}x_{1k})+1] \\ + e_2[\frac{r_2}{\sigma}\sum_{k=1}^{M}(a_{1k}x_{2k}+a_{3k}x_{2k}+\cdots+a_{Nk}x_{2k}+x_{2k}-\frac{\sigma}{r_2}x_{2k})+1] \\ + \dots \\ + e_N[\frac{r_N}{\sigma}\sum_{k=1}^{M}(a_{1k}x_{Nk}+a_{2k}x_{Nk}+\cdots+a_{(N-1)k}x_{Nk}+x_{Nk}-\frac{\sigma}{r_N}x_{Nk})+1] \\ + c\sum_{i=1}^{N}\max_{p,q\in N(i)}\{\sum_{k=1}^{M}(x_{pk}e_p+x_{qk}e_q)\} \\ = r + c\sum_{i=1}^{N}\max_{p,q\in N(i)}\{e_p+e_q\}. \end{aligned} \quad (44)$$

□

Corollary 6 suggests that different endowment distributions in the fully cooperative interval $E_u(\delta)$ will have different total payoffs.

2.5 Maximally cooperative endowment distribution

This subsection will present a distribution of endowments for the most favorable cooperation. First, the smallest continuation probability of full cooperation is defined. When the continuation probability is large enough, there is always a feasible interval for full cooperation. By Lemma 2.2, the feasible interval of full cooperation will gradually become smaller as δ becomes smaller. There will be a critical smallest continuation probability, which we define as

$$\delta_u^* = inf\{\delta \in [0,1] | E_u(\delta) \neq \emptyset\}. \quad (45)$$

The maximal cooperative endowment distribution concept can be defined with the above definition of minimal continuation probability.

Definition 1 *Given a public goods game with payoff function u, a endowment distribution $\boldsymbol{e}^* = \{e_1^*, \dots, e_N^*\}$ is the maximal cooperative endowment distribution satisfying $\boldsymbol{e}^* \in E_u(\delta_u^*)$.*

The above definition combined with Lemma 2.2 shows that the maximal cooperative endowment distribution $\boldsymbol{e}^*$ must be in the fully cooperative feasible interval whenever the continuation probability $\delta \geq \delta^*$, due to $E_u(\delta^*) \subset E_u(\delta)$. But we want to find this smallest continuation probability as an optimization problem, and we consider a simple problem in our research model, the two-player multiple public goods game problem.

Theorem 2.4 *Consider a linear asymmetric payoff function* u *for a two-player dual public goods game as*

$$\begin{aligned} u_1 &= \frac{1}{2}(r_1x_{11}e_1 + r_1x_{12}e_1 + r_2x_{21}e_2 + r_2x_{22}e_2) + (1 - x_{11} - x_{12})e_1, \\ u_2 &= \frac{1}{2}(r_1x_{11}e_1 + r_1x_{12}e_1 + r_2x_{21}e_2 + r_2x_{22}e_2) + (1 - x_{21} - x_{22})e_2. \end{aligned} \tag{46}$$

The smallest continuation probability δ_u^* *and the maximal cooperative endowment distuibution* $\boldsymbol{e}^* = (e_1^*, e_2^*)$ *are given by*

$$\delta_u^* = \sqrt{\frac{(2-r_1)(2-r_2)}{r_1r_2}} \qquad and \qquad \frac{e_1^*}{e_2^*} = \sqrt{\frac{(2-r_2)r_2}{(2-r_1)r_1}}. \tag{47}$$

Proof For any continuation probability δ and endowment distribution $\boldsymbol{e} = (e_1, e_2)$, $\boldsymbol{e} \in E_u(\delta)$ if and only if

$$\begin{aligned} \delta &\geq \frac{(2-r_1)e_1}{r_2e_2}, \\ \delta &\geq \frac{(2-r_2)e_2}{r_1e_1}. \end{aligned} \tag{48}$$

The interval range of $\frac{e_1}{e_2}$ is obtained by transforming the inequality

$$\frac{2-r_2}{\delta r_1} \leq \frac{e_1}{e_2} \leq \frac{\delta r_2}{2-r_1}. \tag{49}$$

The sufficient condition for $\boldsymbol{e} \in E_u(\delta)$ to exist is $\frac{2-r_2}{\delta r_1} \leq \frac{\delta r_2}{2-r_1}$. When this condition is satisfied, $\delta \geq \delta_u^*$. A critical condition for the lower bound is simply to let both sides of the inequality be taken equally. We can get $\delta_u^* = \sqrt{\frac{(2-r_1)(2-r_2)}{r_1r_2}}$, and also $\frac{e_1^*}{e_2^*} = \sqrt{\frac{(2-r_2)r_2}{(2-r_1)r_1}}$. □

The result in Theorem 2.4 eventually degenerates into a linear asymmetric model result for the two-person public goods game [5].

2.6 Smallest continuation probability for homogeneous hypergraphs

In the following discussion, we address the issue of smallest continuation probability within the framework of homogeneous hypergraph. The term "homogeneous" refers to a broad generalization encompassing three distinct aspects: the structure of the hypergraph, the productivity of the individuals, and the configuration of the contribution matrix. Our demonstration of the smallest continuation probability establishes that all three sub-properties must uniformly satisfy this homogeneity criterion.

We postulate that the average hyperdegree of the network is k (i.e., each individual is connected to k hyperedges). The productivity expectations for all individuals are symmetrical. In the contribution matrix X, any non-zero element is valued at $\frac{1}{k}$. Provided these three homogeneity conditions are met, the following theorem can be derived.

Theorem 2.5 *Consider a linear symmetric payoff function u in a public goods game involving σ players, where each player participates in k public goods games. All players contribute a proportion $\frac{1}{k}$ to each of the k public goods games in which they are involved. The smallest continuation probability δ_u^* and the continuation probability for equal endowments δ_{equal}^* are identical, given by:*

$$\delta_u^* = \delta_{equal}^* = \frac{\sigma - r}{r(\sigma - 1)}. \tag{50}$$

Proof First, we compute the continuation probability δ_{equal}^* under equal endowment, based on the linear symmetric case. The calculation is as follows:

$$\delta_{\text{equal}}^* = \frac{(\sigma - r)e_i}{r\sum_{k=1}^{M} a_{ik} \sum_{j\neq i}^{N} x_{jk}e_j} = \frac{(\sigma - r)\frac{1}{N}}{rk\frac{1}{k}(\sigma - 1)\frac{1}{N}} = \frac{\sigma - r}{r(\sigma - 1)}. \tag{51}$$

In the following, we aim to demonstrate that for any perturbation from the equal endowments scenario, there exists at least one node i where δ_i^* exceeds δ_{equal}^*. Consequently, no perturbation can reduce δ_u^* below δ_{equal}^*, establishing it as the smallest continuation probability. We consider a perturbed endowment vector **e** defined as $\left\{ \underbrace{\frac{1}{N} + \epsilon_{i_1}, \frac{1}{N} + \epsilon_{i_2}, \ldots, \frac{1}{N} + \epsilon_{i_m}}_{m}, \underbrace{\frac{1}{N} - \epsilon_{i_{m+1}}, \ldots, \frac{1}{N} - \epsilon_{i_N}}_{N-m} \right\}$, consisting of m positive and $N-m$ negative perturbations. These perturbations satisfy $\sum_{k=1}^{m} \epsilon_{i_k} = \sum_{k=m+1}^{N} \epsilon_{i_k}$, where $\epsilon_{i_k} \in [0, \frac{1}{N}]$ for all $k \in [m+1, N]$ and $\epsilon_{i_k} \in [0, \frac{N-m}{N}]$ for all $k \in [1, m]$. The sequence i_1 to i_N represents any permutation of 1 through N. Next, we calculate $\delta_{i_m}^*$ for the initial m positive perturbations:

$$
\begin{aligned}
\delta_{i_1}^* &= \frac{k(\sigma-r)\left[\frac{1}{N}+\epsilon_{i_1}\right]}{r\left[\sum_{j\neq\{i_1,\ldots,i_m\}}^{N} P_j^{(i_1)}\left(\frac{1}{N}-\epsilon_j\right)+\sum_{j=\{i_2,\ldots,i_m\}} P_j^{(i_1)}\left(\frac{1}{N}+\epsilon_j\right)\right]} \\
&= \frac{k(\sigma-r)\left[\frac{1}{N}+\epsilon_{i_1}\right]}{r\left[k(\sigma-1)\frac{1}{N}-\sum_{j\neq\{i_1,\ldots,i_m\}}^{N} P_j^{(i_1)}\epsilon_j+\sum_{j=\{i_2,\ldots,i_m\}} P_j^{(i_1)}\epsilon_j\right]} \\
&= \frac{(\sigma-r)}{r(\sigma-1)}+\frac{\frac{\sigma-r}{r(\sigma-1)}Nr\left[\sum_{j\neq\{i_1,\ldots,i_m\}}^{N} P_j^{(i_1)}\epsilon_j-\sum_{j=\{i_2,\ldots,i_m\}} P_j^{(i_1)}\epsilon_j\right]+k(\sigma-r)N\epsilon_{i_1}}{kr(\sigma-1)-Nr\left[\sum_{j\neq\{i_1,\ldots,i_m\}}^{N} P_j^{(i_1)}\epsilon_j-\sum_{j=\{i_2,\ldots,i_m\}} P_j^{(i_1)}\epsilon_j\right]}.
\end{aligned}
\tag{52}
$$

Here, $P_j^{(i_1)}$ denotes the number of nodes j associated with node i_1 through hyperedges, which is an unspecified quantity. Nevertheless, it is known that the sum of these associations for nodes not included in the set $\{i_1,\ldots,i_m\}$ and those within the set $\{i_2,\ldots,i_m\}$ totals $k(\sigma-1)$. We can then express this relationship as follows:

$$
\sum_{j\neq\{i_1,\ldots,i_m\}}^{N} P_j^{(i_1)}\epsilon_j \leq \frac{1}{N}\sum_{j\neq\{i_1,\ldots,i_m\}}^{N} P_j^{(i_1)} \leq \frac{1}{N}k(\sigma-1). \tag{53}
$$

Thus, it must hold that

$$
\begin{aligned}
&kr(\sigma-1)-Nr\left[\sum_{j\neq\{i_1,\ldots,i_m\}}^{N} P_j^{(i_1)}\epsilon_j-\sum_{j=\{i_2,\ldots,i_m\}} P_j^{(i_1)}\epsilon_j\right] \\
&\geq kr(\sigma-1)-Nr\sum_{j\neq\{i_1,\ldots,i_m\}}^{N} P_j^{(i_1)}\epsilon_j \\
&\geq kr(\sigma-1)-Nr\frac{1}{N}k(\sigma-1) \\
&\geq 0.
\end{aligned}
\tag{54}
$$

We have proven that the denominator in Eq. (52) is non-negative. Now, it remains to demonstrate that the numerator is also positive. We enumerate the continuation probabilities for the remaining $m-1$ perturbations, $\delta_{i_m}^*$, as follows:

$$
\begin{cases}
\delta_{i_1}^* = \frac{(\sigma-r)}{r(\sigma-1)}+\frac{\frac{\sigma-r}{r(\sigma-1)}Nr\left[\sum_{j\neq\{i_1,\ldots,i_m\}}^{N} P_j^{(i_1)}\epsilon_j-\sum_{j=\{i_2,\ldots,i_m\}} P_j^{(i_1)}\epsilon_j\right]+k(\sigma-r)N\epsilon_{i_1}}{kr(\sigma-1)-Nr\left[\sum_{j\neq\{i_1,\ldots,i_m\}}^{N} P_j^{(i_1)}\epsilon_j-\sum_{j=\{i_2,\ldots,i_m\}} P_j^{(i_1)}\epsilon_j\right]}, \\
\vdots \\
\delta_{i_m}^* = \frac{(\sigma-r)}{r(\sigma-1)}+\frac{\frac{\sigma-r}{r(\sigma-1)}Nr\left[\sum_{j\neq\{i_1,\ldots,i_m\}}^{N} P_j^{(i_m)}\epsilon_j-\sum_{j=\{i_1,\ldots,i_{m-1}\}} P_j^{(i_m)}\epsilon_j\right]+k(\sigma-r)N\epsilon_{i_m}}{kr(\sigma-1)-Nr\left[\sum_{j\neq\{i_1,\ldots,i_m\}}^{N} P_j^{(i_m)}\epsilon_j-\sum_{j=\{i_1,\ldots,i_{m-1}\}} P_j^{(i_m)}\epsilon_j\right]}.
\end{cases}
\tag{55}
$$

We employ a counterfactual hypothesis, assuming that if all the numerators in these m equations are negative, then their sum must also be negative. Proving that the total is positive confirms the existence of a positive term. First, we sum all the numerators, hypothesizing that they are negative:

$$\sum_{q=1}^{m}\left\{\frac{\sigma-r}{r(\sigma-1)}Nr\left[\sum_{j\neq\{i_1,\ldots,i_m\}}^{N}P_j^{(i_q)}\epsilon_j-\sum_{j=\{i_1,\ldots,i_{q-1},i_{q+1},\ldots,i_m\}}P_j^{(i_q)}\epsilon_j\right]+k(\sigma-r)N\epsilon_{i_q}\right\}<0. \tag{56}$$

Next, we extract the coefficients for one of the terms, ϵ_{i_1}:

$$\begin{aligned}&\frac{\sigma-r}{r(\sigma-1)}Nr\left[-P_{i_1}^{(i_2)}\epsilon_{i_1}-P_{i_1}^{(i_3)}\epsilon_{i_1}-\cdots-P_{i_1}^{(i_m)}\epsilon_{i_1}\right]+k(\sigma-r)N\epsilon_{i_1}\\&=\left\{-\frac{N(\sigma-r)}{\sigma-1}\sum_{k=1}^{m}P_{i_1}^{(i_k)}+k(\sigma-r)N\right\}\epsilon_{i_1}.\end{aligned} \tag{57}$$

Given the hypergraph structure and the assumption that the hyperdegree is k, player i_1 can be associated with at most $k(\sigma-1)$ nodes. Therefore:

$$k(\sigma-r)N-\frac{N(\sigma-r)}{\sigma-1}\sum_{k=1}^{m}P_{i_1}^{(i_k)}\geq k(\sigma-r)N-\frac{N(\sigma-r)}{\sigma-1}k(\sigma-1)\geq 0 \tag{58}$$

This inequality establishes that the sum of the m numerators is greater than zero. Thus, at least one $\delta^*_{i_m}$ must be greater than δ^*_{equal}. Any perturbation will increase δ^*_u, confirming that equal endowments represent the smallest continuation probability. □

Theorem 2.5 offers a broad generalization, illustrating that within any homogeneous hypergraph, the smallest continuation probability is determined solely by the productivity and the dimensions of the hyperedges, rather than the average number of hyperedges. From Theorem 2.5, we derive the following corollary.

Corollary 7 *In any homogeneous hypergraph, the smallest continuation probability monotonically decreases with respect to the productivity factor r and monotonically increases with respect to the hyperedge dimension σ.*

Proof Define the functions $f(r)=\frac{\sigma-r}{r(\sigma-1)}$ and $g(\sigma)=\frac{\sigma-r}{r(\sigma-1)}$. We calculate the derivatives of $f(r)$ and $g(\sigma)$ with respect to r and σ respectively:

$$\begin{aligned}f'(r)&=\frac{-\sigma^2+\sigma}{(r\sigma-r)^2},\\g'(\sigma)&=\frac{r(r-1)}{(r\sigma-r)^2}.\end{aligned} \tag{59}$$

Given the conditions $1 < r < \sigma$ and $\sigma \geq 2$, it follows that $f'(r) < 0$ and $g'(\sigma) > 0$, indicating that $f(r)$ is monotonically decreasing and $g(\sigma)$ is monotonically increasing. Evaluating the limits of these functions, we find $\lim_{r \to 1^+} f(r) = 1$, $\lim_{r \to \sigma^-} f(r) = 0$, and $\lim_{\sigma \to \infty} g(\sigma) = \frac{1}{r}$.

$\square$

3 Evolutionary Process Analysis

In the previous section, we concentrated on the interval range of $\boldsymbol{e}$ that remains feasible for full cooperation in the initial state $X(0) = \{\mathbf{1}\}$. In this subsection, we investigate the outcomes of the system's evolution before the player reaches a steady state.

3.1 Strategy update process

In this subsection, the strategy update refers to the update of the contribution matrix $X(t)$. We employ an introspection dynamic model [5, 6] for strategy updating. Specifically, node i is selected with equal random probability and the last round of payoff π_i^{old} is computed. Subsequently, a strategy is chosen from the remaining strategy space with equal random probability, and the new payoff π_i^{new} is computed. Note that the dimension of the strategy space of node i is s_i. The transition probabilities of the new and old strategies are:

$$P(\pi_i^{old} \to \pi_i^{new}) = \frac{1}{N}\frac{1}{s_i - 1}\frac{1}{1 + e^{-\beta(\pi_i^{new} - \pi_i^{old})}}. \tag{60}$$

The parameter β reflects the strength of selection. In the case of $\beta = 0$, the probability $P = \frac{1}{2}$ indicates that the strategy shift is randomized. In the case of strong selection $\beta \to +\infty$, the player will adopt the new strategy if the new strategy yields a higher payoff than the old one. We choose this update strategy over the traditional pairwise comparison or DB update for two reasons. First, the number of hyperedges each node is involved in is not necessarily the same. After comparing the payoffs, it is not possible to exactly replicate the elements in the contribution matrix $X(t)$ when performing the strategy update. Second, if it enters an absorbing state at some point, it is impossible to evolve out of it. However, these two drawbacks can be overcome with the introspection dynamic model. Additionally, we consider the update process as a Moran [7] process, meaning that only one node performs the strategy update at each step.

3.2 Memory-one strategy

In the analysis of the memory-one strategy, we make two assumptions. First, we assume that the strategy space is limited in each round. Specifically, each contribution matrix element $X_{ij}(t)$ can only choose from a limited number of strategies. The simplest of these is to consider that each row has at most one $X_{ij}(t) = 1$, and the rest are all 0. This implies that nodes can only choose to contribute all their endowments to a particular hyperedge or retain all of their endowments. Second, each player decides the next round strategy based solely on the previous round's result, independent of all previous results.

We consider the simplest hypergraph model we discussed earlier, where nodes 1, 2, and 3 together form a hyperedge j_1 and nodes 2, 3, and 4 together form a hyperedge j_2. The contribution matrix is such that at most one element is 1 and the rest are 0. At this point, the state space totals $2 * 3 * 3 * 2 = 36$. However, not every state has a transition probability, and it needs to be ensured that there is only one strategy change from one state to another. With the above definitions and strategy update rules, we can obtain the transition probability matrix $P = \{p_{ij}\} \in R^{36*36}$. Given an initial strategy distribution $\mathbf{v}^0$, we can calculate:

$$\mathbf{v} = (1-\delta)\mathbf{v}^0(I-\delta P)^{-1}, \tag{61}$$

where I is the identity matrix. When $\delta \to 1$, the vector $\mathbf{v}$ is close to the left eigenvector corresponding to the 1 eigenvalue of the transition probability matrix P. Through the payoff vector $\mathbf{u}$, the obtained state vector $\mathbf{v}$, and the given endowment distribution $\boldsymbol{e}$, the total payoff Π (Extended Data Fig.5) can be obtained as

$$\Pi = \sum_{i=1}^{36} v_i * u_i(\boldsymbol{e}, X). \tag{62}$$

Furthermore, we assume that players' decisions are noisy. For any round, there is a small probability $\epsilon > 0$ that each player will have a memory error. For example, a player chose cooperation in the previous round but remembered choosing defection. We note that the dimension of the state space is b. For the elements of the transition probability matrix, we would have the following re-representation:

$$p_{ij}^{error} = (1-\epsilon)p_{ij} + \sum_{k\neq i}^{b} \frac{\epsilon}{b-1} p_{kj}. \tag{63}$$

We stipulate that memory errors are chosen with equal probability in the strategy space. This setup ensures that the row sum of the transition probability matrix remains equal to 1.

$$\begin{aligned}
\sum_{j=1}^{b} p_{ij}^{error} &= (1-\epsilon)\sum_{j=1}^{b} p_{ij} + \frac{\epsilon}{b-1}\sum_{j=1}^{b}\sum_{k\neq i}^{b} p_{kj} \\
&= (1-\epsilon) + \frac{\epsilon}{b-1}\sum_{k\neq i}^{b}\sum_{j=1}^{b} p_{kj} \\
&= (1-\epsilon) + \frac{\epsilon}{b-1}\sum_{k\neq i}^{b} 1 \\
&= (1-\epsilon) + \epsilon \\
&= 1.
\end{aligned} \tag{64}$$

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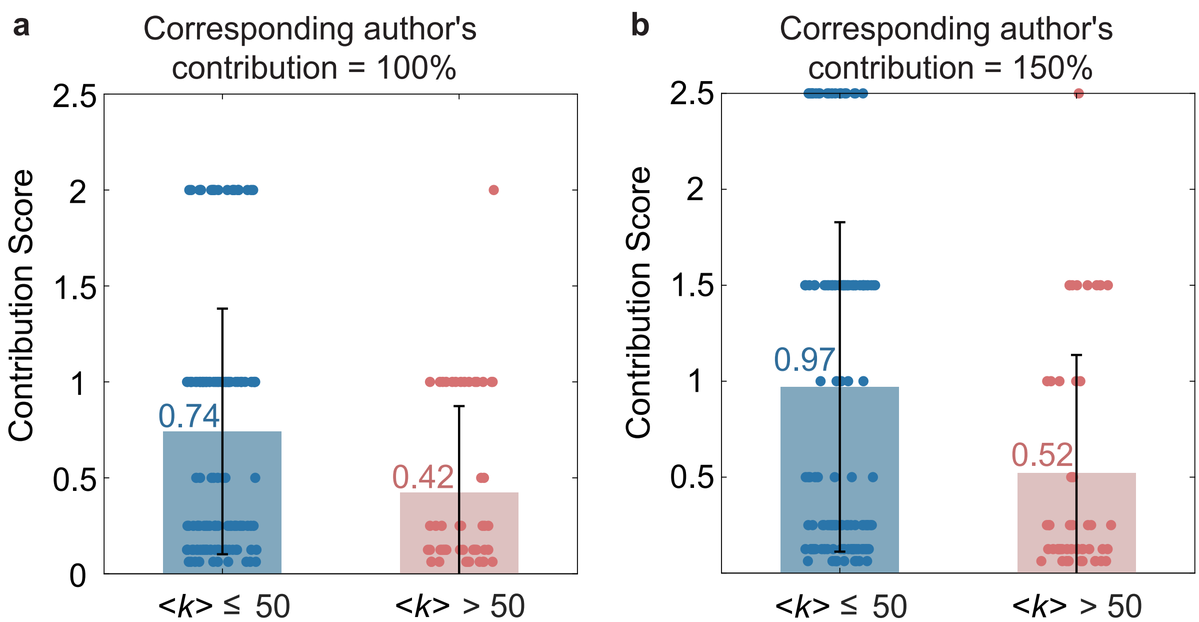
ID in Fig. 7	Context	Reference
DBLP	co-authorship on DBLP papers	[8]
TMS	tags on math.stackexchange.com	[8]
NDC	substances making up drugs	[8]
DAWN	drugs used by ER patients	[8]
CB	congresspersons cosponsoring bills	[8–10]
EEu	email addresses on emails (full)	[8, 11, 12]
EEn	email addresses on emails (subset)	[8]
CHS	high school contact groups	[8, 13]
CPS	primary school contact groups	[8, 14]
TAU	tags on askubuntu.com	[8]

Supplementary Table 1: Summary of the real-world hypergraphs used in Fig. 7.



Supplementary Fig. 1: Effect of corresponding author contribution. In the main text (Fig. 7**d**), we examined the case where the corresponding author's contribution was set to 50%. In panels **a** and **b**, we varied the corresponding author's contribution to 100% and 150%, respectively. The results show that the level of corresponding author contribution does not affect the general pattern that higher hyperdegree researchers tend to contribute more to lower hyperdegree collaborators.