Supplementary Materials to "*Model-free variable importance testing with machine learning methods*"

This Supplemental file contains real data analysis and all theoretical proofs of the paper.

1 Real data analysis: Carworth Farms White (CFW) mice data

We consider a real dataset consisting of 1,200 male Carworth Farms White (CFW) mice, referred to as CFW mice data, to illustrate our proposed tests. This dataset is made publicly accessible by [1] and has been analyzed by [2] and [3]. It includes information on 92,734 SNPs of 1,200 male CFW mice. The CFW mice in this dataset have been subjected to various phenotyping assessments, such as conditioned fear, anxiety behavior, methamphetamine sensitivity, prepulse inhibition, fasting glucose levels, body weight, tail length, and testis weight. These phenotypes can be categorized into three main groups: behavioral, physiological, and expression quantitative traits. We concentrate on a phenotype: the muscle weights of extensor digitorum longus (EDL). The EDL muscle has been extensively utilized in the investigation of muscle loss in the elderly population, where the loss of muscle mass is already high. Further loss of muscle can lead to a decline in strength and may increase the risk of falls and muscular atrophy; see [4], [5], [6], and [7] for instance.

We apply the distance correlation based feature screening approach [8] to the data in order to reduce dimension of the predictors and select the first $N/\log(N)$ SNPs, which leaves a total of $N/\log(N)$ SNPs. In the study by [1], rs45690064 and rs27338905 were identified as significant SNPs for the weights of the EDL muscle, while rs46634317 on chromosome 2 was not. Now we apply our procedures to validate their results. By applying the Bonferroni correction, the results in Table 1 indicate that our tests identify both rs45690064 and rs27338905 as significant SNPs at the significance level of $\alpha = 0.05/3$. In contrast, the other procedures do not reject the null hypothesis at the significance level of $\alpha = 0.05/3$. For rs46634317, it is also observed that all the procedures do not reject the null hypothesis at the significance level of $\alpha = 0.05/3$. This implies that, given the other SNPs, rs46634317 does not provide significant additional information regarding the weights of the EDL muscle. Clearly, our results are consistent with the findings in [1], affirming the validity of identifying significant aspects.

Table 1 P-values for the CFW mice data

	rs45690064	rs27338905	rs46634317
V_n	2.258×10^{-3}	3.276×10^{-3}	0.428
V_{L_n}	1.225×10^{-6}	3.984×10^{-3}	0.291
U_n	4.284×10^{-3}	3.652×10^{-3}	0.171
pcm	0.100	0.084	0.574
pcm_M	0.516	0.492	0.647
vim	0.976	0.995	0.967
DSP	0.841	0.363	0.388
DSP_M	0.899	0.796	0.979
GCM	0.167	0.134	0.295
WGCM	0.458	0.474	0.380
WGCMe	0.753	0.579	0.696

2 Theoretical Proofs

Proof of Theorem 2.1. Note that

$$\begin{aligned}
T_n^{(l)} &= \frac{1}{\sqrt{n}} \sum_{j\in\mathcal{D}_2} [Y_j - m(X_j) + m(X_j) - \hat{m}_1(X_j)][h^{(l)}(W_j) - \phi^{(l)}(X_j) + \phi^{(l)}(X_j) - \hat{\phi}_1^{(l)}(X_j)] \\
&= \frac{1}{\sqrt{n}} \sum_{j\in\mathcal{D}_2} [Y_j - m(X_j)][h^{(l)}(W_j) - \phi^{(l)}(X_j)] + \frac{1}{\sqrt{n}} \sum_{j\in\mathcal{D}_2} [Y_j - m(X_j)][\phi^{(l)}(X_j) - \hat{\phi}_1^{(l)}(X_j)] \\
&\quad + \frac{1}{\sqrt{n}} \sum_{j\in\mathcal{D}_2} [m(X_j) - \hat{m}_1(X_j)][h^{(l)}(W_j) - \phi^{(l)}(X_j)] \\
&\quad + \frac{1}{\sqrt{n}} \sum_{j\in\mathcal{D}_2} [m(X_j) - \hat{m}_1(X_j)][\phi^{(l)}(X_j) - \hat{\phi}_1^{(l)}(X_j)] \\
&=: A_1 + A_2 + A_3 + A_4.
\end{aligned}$$

In the following we show that $A_i, i = 2, 3, 4$ are all $o_p(1)$.

Firstly for the term A_2, note that $\mathrm{E}[Y - m(X)|X] = 0$ and given $\mathcal{D}_1$, $[Y_j - m(X_j)][\phi^{(l)}(X_j) - \hat{\phi}_1^{(l)}(X_j)]$'s are i.i.d r. v.'s. Thus we have

$$\mathrm{E}(A_2) = \frac{1}{\sqrt{n}} \sum_{j\in\mathcal{D}_2} \mathrm{E}\left(\mathrm{E}\left[\{Y_j - m(X_j)\}\{\phi^{(l)}(X_j) - \hat{\phi}_1^{(l)}(X_j)\}|\mathcal{D}_1, X_j\right]\right) = 0.$$

Further note that

$$\mathrm{E}(A_2^2|\mathcal{D}_1) = \frac{1}{n} \sum_{j\in\mathcal{D}_2} \mathrm{E}\left[\{Y_j - m(X_j)\}^2\{\phi^{(l)}(X_j) - \hat{\phi}_1^{(l)}(X_j)\}^2|\mathcal{D}_1\right].$$

Hence

$$\mathrm{E}(A_2^2) = \frac{1}{n}\sum_{j\in\mathcal{D}_2} \mathrm{E}\left[\sigma^2(X_j)\{\phi^{(l)}(X_j) - \hat{\phi}_1^{(l)}(X_j)\}^2\right] = o(1).$$

Here $\sigma^2(X) = \mathrm{E}[\{Y - m(X)\}^2|X]$. Then $A_2 = o_p(1)$. Similarly, under condition (C2), we analogously show that $A_3 = o_p(1)$.

Next we turn to consider the term A_4. By Cauchy-Schwarz inequality, we have

$$|A_4| \le \frac{1}{\sqrt{n}}\sqrt{\sum_{j\in\mathcal{D}_2}[m(X_j) - \hat{m}_1(X_j)]^2}\sqrt{\sum_{j\in\mathcal{D}_2}[\phi^{(l)}(X_j) - \hat{\phi}_1^{(l)}(X_j)]^2} = o_p(1).$$

The last equation follows under the condition (C1).

It then follows that

$$T_n^{(l)} = \frac{1}{\sqrt{n}}\sum_{j\in\mathcal{D}_2}[Y_j - m(X_j)][h^{(l)}(W_j) - \phi^{(l)}(X_j)] + o_p(1).$$

We then have $\mathbf{T} \to N(0, \mathbf{\Omega})$.

In the following we aim to prove that $\hat{\Omega}_{kl}$ is a consistent estimator of Ω_{kl}. Recall that

$$\hat{\Omega}_{kl} = \frac{1}{n}\sum_{j\in\mathcal{D}_2}(\epsilon_j + \Delta m_j)^2(\eta_j^{(k)} + \Delta\phi_j^{(k)})(\eta_j^{(l)} + \Delta\phi_j^{(l)}).$$

Here $\epsilon_j = Y_j - m(X_j), \Delta m_j = m(X_j) - \hat{m}_1(X_j)$. We then have

$$\begin{aligned}
\hat{\Omega}_{kl} &= \frac{1}{n}\sum_{j\in\mathcal{D}_2}\epsilon_j^2\eta_j^{(k)}\eta_j^{(l)} + \epsilon_j^2\eta_j^{(k)}\Delta\phi_j^{(l)} + \epsilon_j^2\eta_j^{(l)}\Delta\phi_j^{(k)} + \epsilon_j^2\Delta\phi_j^{(k)}\Delta\phi_j^{(l)} \\
&+\frac{1}{n}\sum_{j\in\mathcal{D}_2}2\epsilon_j\Delta m_j\eta_j^{(k)}\eta_j^{(l)} + 2\epsilon_j\Delta m_j\eta_j^{(k)}\Delta\phi_j^{(l)} + 2\epsilon_j\Delta m_j\eta_j^{(l)}\Delta\phi_j^{(k)} + 2\epsilon_j\Delta m_j\Delta\phi_j^{(k)}\Delta\phi_j^{(l)} \\
&+\frac{1}{n}\sum_{j\in\mathcal{D}_2}\Delta m_j^2\eta_j^{(k)}\eta_j^{(l)} + \Delta m_j^2\eta_j^{(k)}\Delta\phi_j^{(l)} + \Delta m_j^2\eta_j^{(l)}\Delta\phi_j^{(k)} + \Delta m_j^2\Delta\phi_j^{(k)}\Delta\phi_j^{(l)} \\
&= \sum_{i=1}^{12} B_i.
\end{aligned}$$

Firstly note that

$$|B_2| \le \sqrt{\frac{1}{n}\sum_{j\in\mathcal{D}_2}\epsilon_j^2\eta_j^{(k)2}}\sqrt{\frac{1}{n}\sum_{j\in\mathcal{D}_2}\epsilon_j^2\Delta\phi_j^{(l)2}}.$$

Since $\mathrm{E}(\epsilon_j^2\Delta\phi_j^{(l)2}) = \mathrm{E}[\sigma^2(X_j)\Delta\phi_j^{(l)2}]$ and $\mathrm{E}(\epsilon^2\eta^{(k)2}) < \infty$, then from condition (C2), we have $B_2 = o_p(1)$. Similarly $B_3 = o_p(1)$ and $B_4 = o_p(1)$.

Next note that

$$|B_5| \le C\sqrt{\frac{1}{n}\sum_{j\in\mathcal{D}_2}\epsilon_j^2\eta_j^{(k)2}}\sqrt{\frac{1}{n}\sum_{j\in\mathcal{D}_2}\Delta m_j^2\eta_j^{(l)2}}.$$

Then similar to B_2, we have $B_5 = o_p(1)$. While for the term B_6, we have

$$\begin{aligned}|B_6| &\le C\sqrt{\frac{1}{n}\sum_{j\in\mathcal{D}_2}\epsilon_j^2\eta_j^{(k)2}}\sqrt{\frac{1}{n}\sum_{j\in\mathcal{D}_2}\Delta m_j^2\Delta\phi_j^{(l)2}}\\ &\le C\sqrt{\frac{1}{n}\sum_{j\in\mathcal{D}_2}\epsilon_j^2\eta_j^{(k)2}}\sqrt{n\cdot\frac{1}{n}\sum_{j\in\mathcal{D}_2}\Delta m_j^2\cdot\frac{1}{n}\sum_{j\in\mathcal{D}_2}\Delta\phi_j^{(l)2}}.\end{aligned}$$

Then under condition (C1), we have $B_6 = o_p(1)$. Similarly we conclude that $B_7 = o_p(1)$ and $B_8 = o_p(1)$. Further note that

$$|B_9| \le C\sqrt{\frac{1}{n}\sum_{j\in\mathcal{D}_2}\Delta m_j^2\eta_j^{(k)2}}\sqrt{\frac{1}{n}\sum_{j\in\mathcal{D}_2}\Delta m_j^2\eta_j^{(l)2}}.$$

Then from condition (C2), we have $B_9 = o_p(1)$. While for the term B_{10}, we have

$$|B_{10}| \le C\sqrt{\frac{1}{n}\sum_{j\in\mathcal{D}_2}\Delta m_j^2\eta_j^{(k)2}}\sqrt{\frac{1}{n}\sum_{j\in\mathcal{D}_2}\Delta m_j^2\Delta\phi_j^{(l)2}}.$$

Similar to D_6 and from conditions (C1) and (C2), we have $D_{10} = o_p(1)$. Other terms can be similarly handled. We then conclude that

$$\hat\Omega_{kl} = \frac{1}{n}\sum_{j\in\mathcal{D}_2}\epsilon_j^2\eta_j^{(k)}\eta_j^{(l)} + o_p(1) \to \Omega_{kl}.$$

Then by the Slusky theorem, we finish the proof of this theorem.

Proof of Theorem 2.2. Under the alternative hypotheses with $C_n = n^{-1/2}$, we have

$$\begin{aligned}T_n^{(l)} &= \frac{1}{\sqrt{n}}\sum_{j\in\mathcal{D}_2}[e_j + g(W_j) + m(X_j) - \hat m_1(X_j)][h^{(l)}(W_j) - \phi^{(l)}(X_j) + \phi^{(l)}(X_j) - \hat\phi_1^{(l)}(X_j)]\\ &= \frac{1}{\sqrt{n}}\sum_{j\in\mathcal{D}_2}e_j[h^{(l)}(W_j) - \phi^{(l)}(X_j)] + \frac{1}{\sqrt{n}}\sum_{j\in\mathcal{D}_2}e_j[\phi^{(l)}(X_j) - \hat\phi_1^{(l)}(X_j)]\\ &\quad + \frac{1}{n}\sum_{j\in\mathcal{D}_2}a(W_j)[h^{(l)}(W_j) - \phi^{(l)}(X_j)] + \frac{1}{n}\sum_{j\in\mathcal{D}_2}a(W_j)[\phi^{(l)}(X_j) - \hat\phi_1^{(l)}(X_j)]\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\sqrt{n}}\sum_{j\in\mathcal{D}_2}[m(X_j)-\hat{m}_1(X_j)][h^{(l)}(W_j)-\phi^{(l)}(X_j)]+\frac{1}{\sqrt{n}}\sum_{j\in\mathcal{D}_2}[m(X_j)-\hat{m}_1(X_j)][\phi^{(l)}(X_j)-\hat{\phi}_1^{(l)}(X_j)] \\
&=:\sum_{i=1}^{6}C_i.
\end{aligned}
$$

From the proof of Theorem 2.1, we know that $C_2=o_p(1), C_5=o_p(1), C_6=o_p(1)$. By Cauchy-Schwarz inequality, we get

$$
|C_4|\le\frac{1}{n}\sqrt{\sum_{j\in\mathcal{D}_2}a^2(W_j)}\sqrt{\sum_{j\in\mathcal{D}_2}[\phi^{(l)}(X_j)-\hat{\phi}_1^{(l)}(X_j)]^2}=o_p(1).
$$

The last equation follows the condition (C1). We then conclude that

$$
T_n^{(l)} = \frac{1}{\sqrt{n}}\sum_{j\in\mathcal{D}_2}e_j[h^{(l)}(W_j)-\phi^{(l)}(X_j)]+\mathrm{E}[a(W)\{h^{(l)}(W)-\phi^{(l)}(X)\}]+o_p(1).
$$

Then the asymptotic result for V_n follows.

Next we consider the fixed alternative hypotheses. In this situation, we similarly have

$$
\begin{aligned}
T_n^{(l)} &= \frac{1}{\sqrt{n}}\sum_{j\in\mathcal{D}_2}e_j[h^{(l)}(W_j)-\phi^{(l)}(X_j)]+\frac{1}{\sqrt{n}}\sum_{j\in\mathcal{D}_2}g(W_j)[h^{(l)}(W_j)-\phi^{(l)}(X_j)] \\
&\quad+\frac{1}{\sqrt{n}}\sum_{j\in\mathcal{D}_2}g(W_j)[\phi^{(l)}(X_j)-\hat{\phi}_1^{(l)}(X_j)]+o_p(1).
\end{aligned}
$$

For the last term, by Cauchy-Schwarz inequality, we get

$$
\left|\frac{1}{\sqrt{n}}\sum_{j\in\mathcal{D}_2}g(W_j)[\phi^{(l)}(X_j)-\hat{\phi}_1^{(l)}(X_j)]\right|\le\sqrt{n}\sqrt{\frac{\sum_{j\in\mathcal{D}_2}g^2(W_j)}{n}}\sqrt{\frac{\sum_{j\in\mathcal{D}_2}[\phi^{(l)}(X_j)-\hat{\phi}_1^{(l)}(X_j)]^2}{n}}=o_p(\sqrt{n}).
$$

The first term still converges to normal distribution. While the second term is equal to $\sqrt{n}\mathrm{E}[g(W)\{h^{(l)}(W)-\phi^{(l)}(X)\}]+o_p(\sqrt{n})$. Thus the theorem is proven.

Proof of Theorem 3.1. Note that

$$
\begin{aligned}
S_N^{(k,l)} &= \frac{1}{\sqrt{N}}\sum_{j=1}^{N}(\epsilon_j^{(k)}+\Delta m_j^{(k)})(\eta_j^{(l)}+\Delta\phi_j^{(l)}) \\
&= \frac{1}{\sqrt{N}}\sum_{j=1}^{N}\epsilon_j^{(k)}\eta_j^{(l)}+\frac{1}{\sqrt{N}}\sum_{j=1}^{N}\epsilon_j^{(k)}\Delta\phi_j^{(l)}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\sqrt{N}}\sum_{j=1}^{N}\Delta m_j^{(k)}\eta_j^{(l)}+\frac{1}{\sqrt{N}}\sum_{j=1}^{N}\Delta m_j^{(k)}\Delta\phi_j^{(l)} \\
=&:D_1+D_2+D_3+D_4.
\end{aligned}
$$

In the following, we aim to show that D_2, D_3, and D_4 are all $o_p(1)$.

Firstly for the term D_4, by Cauchy-Schwarz inequality, we have

$$
|D_4| \le \frac{1}{\sqrt{N}}\sqrt{\sum_{j=1}^{N}\Delta m_j^{(k)2}}\sqrt{\sum_{j=1}^{N}\Delta\phi_j^{(l)2}} = o_p(1).
$$

The last equation follows under the condition (C1').

Next we turn to consider the second term D_2. Recall that the null hypothesis is $H_0 : Y \perp\!\!\!\perp Z|X$. Then we have

$$
\mathrm{E}[\epsilon_j^{(k)}\Delta\phi_j^{(l)}|\{X_i, Z_i\}_{i=1}^N] = \Delta\phi_j^{(l)}\mathrm{E}[\epsilon_j^{(k)}|\{X_i, Z_i\}_{i=1}^N] = 0.
$$

Further we have

$$
\mathrm{E}[\epsilon_j^{(k)2}\Delta\phi_j^{(l)2}|\{X_i, Z_i\}_{i=1}^N] = \Delta\phi_j^{(l)2}\sigma^{(k)2}(X).
$$

It then follows that

$$
\begin{aligned}
\mathrm{E}[D_2|\{X_i, Z_i\}_{i=1}^N] &= \frac{1}{\sqrt{N}}\sum_{j=1}^{N}\mathrm{E}[\epsilon_j^{(k)}\Delta\phi_j^{(l)}|\{X_i, Z_i\}_{i=1}^N] = 0. \\
\mathrm{E}[D_2^2|\{X_i, Z_i\}_{i=1}^N] &= \frac{1}{N}\sum_{j=1}^{N}\mathrm{E}[\epsilon_j^{(k)2}\Delta\phi_j^{(l)2}|\{X_i, Z_i\}_{i=1}^N] = \frac{1}{N}\sum_{j=1}^{N}\Delta\phi_j^{(l)2}\sigma^{(k)2}(X) = o_p(1).
\end{aligned}
$$

We then conclude that $D_2 = o_p(1)$. Similarly we can show that $D_3 = o_p(1)$. We then conclude that

$$
S_N^{(k,l)} = \frac{1}{\sqrt{N}}\sum_{j=1}^{N}\epsilon_j^{(k)}\eta_j^{(l)} + o_p(1).
$$

The covariance of $S_N^{(k_1,l_1)}$ and $S_N^{(k_2,l_2)}$ is $\Sigma_{k_1l_1,k_2l_2} = \mathrm{E}(\epsilon^{(k_1)}\epsilon^{(k_2)}\eta^{(l_1)}\eta^{(k_2)})$, which can be estimated by:

$$
\hat{\Sigma}_{k_1l_1,k_2l_2} = \frac{1}{N}\sum_{j=1}^{N}\hat{\epsilon}_j^{(k_1)}\hat{\epsilon}_j^{(k_2)}\hat{\eta}_j^{(l_1)}\hat{\eta}_j^{(l_2)}.
$$

Note that

$$\hat{\Sigma}_{k_1l_1,k_2l_2} = \frac{1}{N}\sum_{j=1}^{N}(\epsilon_j^{(k_1)} + \Delta m_j^{(k_1)})(\epsilon_j^{(k_2)} + \Delta m_j^{(k_2)})(\eta_j^{(l_1)} + \Delta\phi_j^{(l_1)})(\eta_j^{(l_2)} + \Delta\phi_j^{(l_2)})$$

$$= \frac{1}{N}\sum_{j=1}^{N}\epsilon_j^{(k_1)}\epsilon_j^{(k_2)}\eta_j^{(l_1)}\eta_j^{(l_2)} + \epsilon_j^{(k_1)}\epsilon_j^{(k_2)}\eta_j^{(l_1)}\Delta\phi_j^{(l_2)} + \epsilon_j^{(k_1)}\epsilon_j^{(k_2)}\Delta\phi_j^{(l_1)}\eta_j^{(l_2)} + \epsilon_j^{(k_1)}\epsilon_j^{(k_2)}\Delta\phi_j^{(l_1)}\Delta\phi_j^{(l_2)}$$

$$+\frac{1}{N}\sum_{j=1}^{N}\epsilon_j^{(k_1)}\Delta m_j^{(k_2)}\eta_j^{(l_1)}\eta_j^{(l_2)} + \epsilon_j^{(k_1)}\Delta m_j^{(k_2)}\eta_j^{(l_1)}\Delta\phi_j^{(l_2)} + \epsilon_j^{(k_1)}\Delta m_j^{(k_2)}\Delta\phi_j^{(l_1)}\eta_j^{(l_2)} + \epsilon_j^{(k_1)}\Delta m_j^{(k_2)}\Delta\phi_j^{(l_1)}\Delta\phi_j^{(l_2)}$$

$$+\frac{1}{N}\sum_{j=1}^{N}\epsilon_j^{(k_2)}\Delta m_j^{(k_1)}\eta_j^{(l_1)}\eta_j^{(l_2)} + \epsilon_j^{(k_2)}\Delta m_j^{(k_1)}\eta_j^{(l_1)}\Delta\phi_j^{(l_2)} + \epsilon_j^{(k_2)}\Delta m_j^{(k_1)}\Delta\phi_j^{(l_1)}\eta_j^{(l_2)} + \epsilon_j^{(k_2)}\Delta m_j^{(k_1)}\Delta\phi_j^{(l_1)}\Delta\phi_j^{(l_2)}$$

$$+\frac{1}{N}\sum_{j=1}^{N}\Delta m_j^{(k_1)}\Delta m_j^{(k_2)}\eta_j^{(l_1)}\eta_j^{(l_2)} + \Delta m_j^{(k_1)}\Delta m_j^{(k_2)}\eta_j^{(l_1)}\Delta\phi_j^{(l_2)}$$

$$+\Delta m_j^{(k_1)}\Delta m_j^{(k_2)}\Delta\phi_j^{(l_1)}\eta_j^{(l_2)} + \Delta m_j^{(k_1)}\Delta m_j^{(k_2)}\Delta\phi_j^{(l_1)}\Delta\phi_j^{(l_2)}$$

$$= \sum_{i=1}^{16} E_i.$$

The consistency of $\hat{\Sigma}_{k_1l_1,k_2l_2}$ to $\Sigma_{k_1l_1,k_2l_2}$ can be similarly proven as the argument for the consistency of $\hat{\Omega}_{k,l}$ to $\Omega_{k,l}$ in the proof of Theorem 2.1. In the following, we only give a brief proof. Clearly the terms E_9, E_{10}, E_{11} and E_{12} are similar to the terms E_5, E_6, E_7 and E_8. For the term E_5, we have

$$|E_5| \leq \sqrt{\frac{1}{N}\sum_{j=1}^{N}\epsilon^{(k_1)2}\eta^{(l_1)2}}\sqrt{\frac{1}{N}\sum_{j=1}^{N}\Delta m_j^{(k_2)2}\eta^{(l_2)2}} = o_p(1).$$

The last result follows from the condition (C2'). Other terms can be similarly handled and the details are omitted here to save space. Thus we conclude that

$$\hat{\Sigma}_{k_1l_1,k_2l_2} = \frac{1}{N}\sum_{j=1}^{N}\epsilon_j^{(k_1)}\epsilon_j^{(k_2)}\eta_j^{(l_1)}\eta_j^{(l_2)} \to \Sigma_{k_1l_1,k_2l_2}.$$

Then by the Slutsky theorem, we prove this theorem.

Now we turn to consider fixed alternative hypotheses. Now denote $e_j^{(k)} = \varphi^{(k)}(Y_j) - \mathrm{E}[\varphi^{(k)}(Y_j)|W_j]$ and $g_j^{(k)} = \mathrm{E}[\varphi^{(k)}(Y_j)|W_j] - \mathrm{E}[\varphi^{(k)}(Y_j)|X_j]$. We then have

$$S_N^{(k,l)} = \frac{1}{\sqrt{N}}\sum_{j=1}^{N}(e_j^{(k)} + g_j^{(k)} + \Delta m_j^{(k)})(\eta_j^{(l)} + \Delta\phi_j^{(l)})$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{N}}\sum_{j=1}^{N} e_j^{(k)}\eta_j^{(l)} + \frac{1}{\sqrt{N}}\sum_{j=1}^{N} e_j^{(k)}\Delta\phi_j^{(l)} \\
&\quad + \frac{1}{\sqrt{N}}\sum_{j=1}^{N} g_j^{(k)}\eta_j^{(l)} + \frac{1}{\sqrt{N}}\sum_{j=1}^{N} g_j^{(k)}\Delta\phi_j^{(l)} \\
&\quad + \frac{1}{\sqrt{N}}\sum_{j=1}^{N} \Delta m_j^{(k)}\eta_j^{(l)} + \frac{1}{\sqrt{N}}\sum_{j=1}^{N} \Delta m_j^{(k)}\Delta\phi_j^{(l)} \\
&=: \sum_{i=1}^{6} F_i.
\end{aligned}
$$

By Cauchy-Schwarz inequality, we get

$$
|F_2| \le \sqrt{N}\sqrt{\frac{\sum_{j\in\mathcal{D}_2} e_j^{(k)2}}{N}}\sqrt{\frac{\sum_{j\in\mathcal{D}_2} \Delta\phi_j^{(l)2}}{N}} = o_p(N^{(1-\kappa_2)/2}).
$$

Similarly we have

$$
F_4 = o_p(N^{(1-\kappa_2)/2}),\ F_5 = o(N^{(1-\kappa_1)/2}), F_6 = o_p(1).
$$

We then have

$$
\frac{1}{\sqrt{N}} S_N^{(k,l)} = \frac{1}{N}\sum_{j=1}^{N} g_j^{(k)}\eta_j^{(l)} + o_p(1) \to \mathrm{E}[\{\varphi^{(k)}(Y) - m^{(k)}(X)\}h^{(l)}(W)].
$$

Given at least one of $\mathrm{E}[\{\varphi^{(k)}(Y) - m^{(k)}(X)\}h^{(l)}(W)]$'s is not zero, the asymptotic result follows.

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