Equitability and explosive synchronisation in multiplex and higher-order networks Supplementary Information

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1 Theoretical results

1.1 Dynamical system and synchronisation

We consider the dynamical system introduced in the Main Text, that is, the system of $N \geq 1$ differential equations with N dynamical variables $\mathbf{x}_1, \ldots, \mathbf{x}_N : T \to \mathbb{R}^D$, for some $T \subseteq \mathbb{R}$ open interval (*time domain*) and $D \geq 1$ integer (*dimension*), given by

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) + \mathbf{h}_i(\mathbf{x}_1, \ldots, \mathbf{x}_N) \qquad i = 1, \ldots, N \tag{I.1}$$

with either $\mathbf{h}_i = \mathbf{h}_i^{multi}$ for all i (called the *multilayer case*), or $\mathbf{h}_i = \mathbf{h}_i^{hyper}$ for all i (called the *hypergraph case*), where

$$\mathbf{h}_i^{multi}(\mathbf{x}_1, \ldots, \mathbf{x}_N) = \sum_{m=1}^{M} \sigma_m \sum_{j=1}^{N} A_{i,j}^{(m)} \mathbf{g}^{(m)}(\mathbf{x}_i, \mathbf{x}_j), \text{ and} \tag{I.2}$$

$$\mathbf{h}_i^{hyper}(\mathbf{x}_1, \ldots, \mathbf{x}_N) = \sum_{m=1}^{M} \sigma_m \sum_{j_1,\ldots,j_m=1}^{N} A_{i,j_1,\ldots,j_m}^{(m)} \mathbf{g}^{(m)}(\mathbf{x}_i, \mathbf{x}_{j_1}, \ldots, \mathbf{x}_{j_m}). \tag{I.3}$$

Here, we assume the following:

- $\mathbf{f}\colon \mathbb{R}^D \to \mathbb{R}^D$ is a smooth function;
- each $\mathbf{g}^{(m)}$ is a smooth function of the form $\mathbf{g}^{(m)}\colon \mathbb{R}^{2\times D} \to \mathbb{R}^D$ (multilayer case) or $\mathbf{g}^{(m)}\colon \mathbb{R}^{(M+1)\times D} \to \mathbb{R}^D$ (hypergraph case);
- $\sigma_m > 0$ are non-negative real parameters (coupling strength for layer m or order m);
- $A^{(m)}$ is the adjacency matrix for layer m, or the hypergraph adjacency tensor of order $m+1$;
- the coupling functions are synchronisation non-invasive, that is, they satisfy $\mathbf{g}^{(m)}(\mathbf{x}, \ldots, \mathbf{x}) = \mathbf{0}$ for all m, and any $\mathbf{x} \in \mathbb{R}^D$.

The function $\mathbf{f}$ represents the internal dynamics at each node, the functions $\mathbf{g}^{(m)}$ the coupling dynamics on layer m (multilayer case) or the $(m+1)$-body dynamics (hypergraph case), the matrix or tensor $A^{(m)}$ model the pairwise (multilayer case) or many-body (hypergraph case) interactions between the dynamical variables, and the coupling parameter σ_m model the overall coupling strengths on layer m (multilayer case) or on m-body interactions (hypergraph case). One important example is the Laplacian operator, $\mathbf{g}^{(m)}(\mathbf{x}_i, \mathbf{x}_j) = \mathbf{x}_i - \mathbf{x}_j$ — this is indeed the only linear non-invasive operator in two variables up to a constant.

Remark 1.1. *Mathematically, the coupling parameter can be absorbed into the adjacency matrix, respectively tensor. In practice, we use them to parametrise families of dynamical systems. For instance, in simulations, we will typically increase each σ_m from zero, or very low values, to higher values to study the transition into (cluster and/or global) synchronisation of the system.*

Note that, for $M = 1$, both the multilayer and the hypergraph cases reduce to the network (monolayer) case,

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) + \sigma \sum_{j=1}^{N} a_{ij}\, \mathbf{g}(\mathbf{x}_j, \mathbf{x}_i). \tag{I.4}$$

We can represent the interactions between the variables $\mathbf{x}_1, \ldots, \mathbf{x}_N$ using a multilayer network (multilayer case) or a hypergraph (hypergraph case). In the multilayer case, we define the *interaction multilayer network* of this dynamical system as the weighted multilayer network with M layers of N vertices and inter-layer weighted adjacency matrices $A^{(m)}$ $(1 \leq m \leq M)$. In the hypergraph case, the *interaction hypergraph* is the weighted hypergraph with vertices $V = \{1, \ldots, N\}$ and weighted hyperedges $\{i_1, \ldots, i_m\}$ whenever $A^{(m)}_{i_1,\ldots,i_m} \neq 0$, which then becomes the weight of that hyperedge.

Remark 1.2 (Multiplex and multilayer dynamics)**.** *Strictly speaking, our multilayer model is a multiplex, as we have the same N nodes (dynamical units) on each layer, that is, we have M different 'types' of pairwise interactions between the same N nodes. In multilayer network, we have different nodes on each layer and both inter- and intra-layer pairwise interactions. The dynamics on a multilayer network with N nodes organised into M layers of N_α nodes each (so that $N = \sum_{\alpha=1}^{M} N_\alpha$) would have a coupling of the form*

$$\mathbf{h}_i^{\text{multilayer}} = \sum_{\beta=1}^{M} \sigma_{\alpha,\beta} \sum_{j \in V_\beta} A^{\alpha,\beta}_{i,j} \mathbf{g}^{\alpha,\beta}(\mathbf{x}_i, \mathbf{x}_j) \,, \tag{I.5}$$

where i is a node in the α layer and V_β is the set of nodes in the β layer. If all nodes (dynamical units) are identical (same internal dynamics given by the same function $\mathbf{f}$), this case can be seen as a multiplex network on N nodes and M^2 layers $m = (\alpha, \beta)$ such that $A^{(m)}_{ij}$ is zero unless $i \in V_\alpha$ and $i \in V_\beta$. The generic multilayer case (different internal dynamics, that is, functions $\mathbf{f}^{(\alpha)}$, on each layer α) is out of the scope of the present paper and part of our future work on equitablity and synchronisation.

A *solution* $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_N)$ of the dynamical system (I.1) consists of differentiable functions $\mathbf{x}_1, \ldots, \mathbf{x}_N \colon T \to \mathbb{R}^m$ that satisfy the system of equations. A *synchronised solution* is a solution which satisfies $\mathbf{x}_1 = \ldots = \mathbf{x}_N$ (identical synchronisation). By the non-invasive property above, $\mathbf{x}_s$ is a solution of the differential equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ if and only if $\mathbf{x}_1 = \ldots = \mathbf{x}_N = \mathbf{x}_s$ is a (synchronised) solution of (I.1).

A subset C of the node set $V = \{1, \ldots, N\}$ is called a *cluster*. A solution $\mathbf{x}_1, \ldots, \mathbf{x}_N$ is C-*synchronised*, or *synchronised on* C, if $\mathbf{x}_i = \mathbf{x}_j$ for all $i, j \in C$. Note that if $C = \{i\}$ a singleton (a set with one element), every solution is C-synchronised. If $\mathcal{P}$ is a partition of the node set V (a collection of pairwise disjoint clusters whose union is V), we call a solution $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_N)$ $\mathcal{P}$-*synchronised* if it is C-synchronised for each $C \in \mathcal{P}$.

1.2 Dynamical and structural equitability

In this section, we discuss dynamical and structural equitability, in more generality than in the Main Text. We focus on the multiplex case, and discuss the hypergraph case in Section 1.5.

Correspondence to terminology in the Main Text. Dynamical equitability in the Main Text corresponds to weak dynamically equitability (WDE) in this SI. Similarly, structural equitability corresponds to structural equitability per layer (SEL), and linear independence in cluster synchronisation to linear independence with respect to the families $\mathcal{F}(C)$.

Consider the multilayer dynamical system given by Equations (I.1) and (I.2). Let $\mathbf{x}$ be a solution, $C \subseteq V$ a cluster, $i \in V$ a node, and $m \in \{1, \ldots, M\}$ a layer. We extend the notation from the Main Text to

$$\mathbf{h}_i = \sum_{m=1}^{M} \sigma_m \sum_{j=1}^{N} A_{i,j}^{(m)} \mathbf{g}^{(m)}(\mathbf{x}_i, \mathbf{x}_j), \qquad h_i = \sum_{m=1}^{M} \sigma_m \sum_{j=1}^{N} A_{i,j}^{(m)}, \tag{I.6}$$

$$\mathbf{h}_i^C = \sum_{m=1}^{M} \sigma_m \sum_{j\in C} A_{i,j}^{(m)} \mathbf{g}^{(m)}(\mathbf{x}_i, \mathbf{x}_j), \qquad h_i^C = \sum_{m=1}^{M} \sigma_m \sum_{j\in C} A_{i,j}^{(m)}, \tag{I.7}$$

$$\mathbf{h}_i^{C,m} = \sum_{j\in C} A_{i,j}^{(m)} \mathbf{g}^{(m)}(\mathbf{x}_i, \mathbf{x}_j), \qquad h_i^{C',m} = \sum_{j\in C} A_{i,j}^{(m)}, \tag{I.8}$$

and note the equalities

$$\mathbf{h}_i = \sum_{C\in\mathcal{P}} \mathbf{h}_i^C, \qquad \mathbf{h}_i^C = \sum_{m=1}^{M} \mathbf{h}_i^{C,m}, \qquad h_i = \sum_{C\in\mathcal{P}} h_i^C, \qquad \text{and} \qquad h_i^C = \sum_{m=1}^{M} h_i^{C,m}. \tag{I.9}$$

The rows above (I.6) to (I.8) encode the dynamical (left) respectively structural (right) input to node i from all other nodes (top row), all nodes on cluster C from all layer (middle row), and all nodes on cluster C and layer m (bottom row). The quantities on the left column, $\mathbf{h}_i$, $\mathbf{h}_i^C$ and $\mathbf{h}_i^{C,m}$, depend on the solution $\mathbf{x}$ and thus are functions of $t \in T$, and note that $\mathbf{h}_i$ is simply $\mathbf{h}_i^{\text{multi}}$ in Eq. (I.2). The quantifies on the right column depend only on the underlying interaction multiplex (they can be seen as total or partial node degrees in the multiplex) and thus independent on any solution or exact form of the coupling functions — they only depend on the pattern and strength of interactions between dynamical units.

Next, we define several notions of external equitability. In all of them, equitability refers to 'equal input' for each node in a cluster $C \subseteq V$ from nodes outside the cluster ('external'). The different notions arise from whether the input refers to the overall contribution from all nodes outside the cluster ('weak') or from nodes on each cluster in a partition separately, or on each cluster and layer ('per layer'), and to either the dynamical input with respect to a solution ('dynamical') or to the structural input with respect to the pattern and strength of interactions between dynamical units ('structural'). In what follows, we drop the word 'external', for simplicity.

Definition 1.3. *We call a cluster $C \subseteq V$* dynamically equitable *with respect to a solution $\mathbf{x}$ if every pair of nodes $i, j \in C$ have the same dynamical input from the nodes outside C, that is,*

$$\mathbf{h}_i^{V\setminus C} = \mathbf{h}_i^{V\setminus C}. \tag{I.10}$$

We call a cluster $C \subseteq V$ structurally equitable *if every pair of nodes $i, j \in C$ have the same structural input from the nodes outside C, that is,*

$$h_i^{V\setminus C} = h_i^{V\setminus C}. \tag{I.11}$$

Let $\mathcal{P}$ be a partition of the node set and $\mathbf{x}$ *a solution of the dynamical system. We call the partition $\mathcal{P}$* weakly dynamically equitable (WDE) *(with respect to* $\mathbf{x}$*) if every cluster $C \in \mathcal{P}$ is dynamically equitable (with respect to* $\mathbf{x}$*). We call the partition $\mathcal{P}$* weakly structurally equitable (WSE) *if every cluster $C \in \mathcal{P}$ is structurally equitable. We call the partition $\mathcal{P}$* dynamically equitable (DE) *(with respect to* $\mathbf{x}$*) if for every $C, C' \in \mathcal{P}$, $C \neq C'$, and every $i, j \in C$,*

$$\mathbf{h}_i^{C'} = \mathbf{h}_j^{C'}. \tag{I.12}$$

We call the partition $\mathcal{P}$ structurally equitable (SE) *if for every $C, C' \in \mathcal{P}$, $C \neq C'$, and every $i, j \in C$,*

$$h_i^{C'} = h_j^{C'}. \tag{I.13}$$

We call the partition $\mathcal{P}$ dynamically equitable per layer (DEL) *(with respect to* $\mathbf{x}$*) if for every $C, C' \in \mathcal{P}$, $C \neq C'$, and every $i, j \in C$,*

$$\mathbf{h}_i^{C',m} = \mathbf{h}_j^{C',m}. \tag{I.14}$$

A partition $\mathcal{P}$ of the node set is structurally equitable per layer (SEL) *if for every $C, C' \in \mathcal{P}$, $C \neq C'$, and every $i, j \in C$,*

$$h_i^{C',m} = h_j^{C',m}. \tag{I.15}$$

Remark 1.4. *Note that these definitions (WDE, WSE, DE, SE, DEL and SEL) are 'external', in the sense that we do not need to hold in the case $C = C'$.*

Next, we show that strong equitability implies weak equitability, equitability per layer implies equitability, and structural equitability per layer implies dynamical equitability for synchronised solutions. This, proved in Theorem 1.7 below, can be represented schematically as

$$\begin{array}{ccccc} DEL & \Longrightarrow & DE & \Longrightarrow & WDE \\ {\scriptstyle CS}\Big\Uparrow & & & & \\ SEL & \Longrightarrow & SE & \Longrightarrow & WSE \end{array} \tag{I.16}$$

where CS that stands for 'cluster synchronisation' (that is, $\mathcal{P}$-synchronisation), and the notation $A \overset{B}{\Longrightarrow} C$ means 'A and B implies C'. Note that structural concepts (SEL, SE, WSE) are with respect to a fixed partition $\mathcal{P}$ of the node set, and the dynamical concepts (DEL, DE, WDE, CS) are with respect to the partition $\mathcal{P}$ and a solution $\mathbf{x}$ of the dynamical system.

Remark 1.5. *The concepts of SEL and SE, respectively DEL and DE, coincide if there is only one layer $m = 1$, that is, in the monolayer, or network, case, but otherwise they are distinct. Similarly, we can give examples, both in the dynamical and the structural sense, of weakly equitable partitions that are not equitable.*

Remark 1.6. *Neither SE nor WSE necessarily imply dynamical equitability in any of its forms even under cluster synchronisation (it is not difficult to give counterexamples — mathematically, it amounts to the condition $\sum_i a_i = \sum_i b_i$ not implying $\sum_i c_i a_i = \sum_i c_i b_i$*

for more than two summands and constants a_i, b_i, c_i) and there does not seem to be a natural condition that would make that occur. In any case, cluster synchronisation relates to the strongest form of structural equitability, SEL, as shown later (Section 1.4) so we consider SEL the appropriate notion of structural equitability in multiplex networks, as presented in the Main Text.

Theorem 1.7. *Consider the multilayer dynamical system given by Equations* (I.1) *and* (I.2). *Let $\mathcal{P}$ be a partition of the node set and $\mathbf{x}$ a solution of the dynamical system. Then,*

(1) if $\mathcal{P}$ is dynamically, respectively structurally, equitable, then each $C \in \mathcal{P}$ is weakly dynamically, respectively structurally, equitable;

(2) if $\mathcal{P}$ is dynamically, respectively structurally, equitable per layer, then it is dynamically, respectively structurally, equitable;

(3) if $\mathcal{P}$ is structurally equitable per layer and $\mathbf{x}$ is $\mathcal{P}$-synchronised, then it is dynamically equitable per layer.

Proof. (1) This follows directly from the definitions. If $i, j \in C \in \mathcal{P}$ then

$$\mathbf{h}_i^{C\setminus V} = \sum_{C \neq C' \in \mathcal{P}} \mathbf{h}_i^{C'} = \sum_{C \neq C' \in \mathcal{P}} \mathbf{h}_j^{C'} = \mathbf{h}_i^{C\setminus V}, \tag{I.17}$$

and, similarly, we can show that $h_i^{C\setminus V} = h_j^{C\setminus V}$.

(2) This follows directly from the definitions. If $i \in C \in \mathcal{P}$ then

$$\mathbf{h}_i^{C} = \sum_{m=1}^{M} \mathbf{h}_i^{C,m} = \sum_{m=1}^{M} \mathbf{h}_j^{C,m} = \mathbf{h}_j^{C} \tag{I.18}$$

and, similarly, we can show that $h_i^C = h_j^C$.

(3) If $\mathbf{x}$ is $\mathcal{P}$-synchronised, we can write $\mathbf{x}_C$ for the synchronised solution on a cluster $C \in \mathcal{P}$, that is, $\mathbf{x}_i = \mathbf{x}_C$ for any $i \in C$. Then, given $i, j \in C$, $C \neq C' \in \mathcal{P}$, and $m \in \{1, \ldots, M\}$,

$$\mathbf{h}_i^{C',m} = \sum_{k \in C'} A_{i,k}^{(m)} \mathbf{g}^{(m)}(\mathbf{x}_i, \mathbf{x}_k) = \sum_{k \in C'} A_{i,k}^{(m)} \mathbf{g}^{(m)}(\mathbf{x}_C, \mathbf{x}_{C'}) = \mathbf{g}^{(m)}(\mathbf{x}_C, \mathbf{x}_{C'})\, h_i^{C',m}, \tag{I.19}$$

and similarly for $\mathbf{h}_j^{C',m}$. Therefore, $h_i^{C',m} = h_j^{C',m}$ implies $\mathbf{h}_i^{C',m} = \mathbf{h}_j^{C',m}$ and we have dynamical equitability per layer. □

1.3 Linear independence over trajectories

Given functions $f_1(t), \ldots, f_K(t)$, $t \in T \subseteq \mathbb{R}$, the set $\mathcal{F} = \{f_1, \ldots, f_K\}$ is *linearly independent over T* if the condition $a_1 f_1(t) + \ldots + a_K f_K(t) = 0$ for all $t \in T$, where $a_i \in \mathbb{R}$ are constants, implies $a_1 = \ldots = a_K = 0$. Linear independence can be checked for differentiable functions by using the Wronskian $W(t)$, a $K \times K$ determinant of the functions and their higher derivatives: if $W(t)$ is not identically zero over T, the functions must be linearly independent [1]. A single function $f_1(t)$ is linearly independent if (and only if) it is not identically zero on T, and $f_1, \dots, f_K$, $K \geq 2$ are linearly independent if no function f_i is an exact linear combination of the rest, over T.

Definition 1.8. *Let $\mathcal{P}$ be a partition of the node set, $C, C' \in \mathcal{P}$ clusters with $C \neq C'$, and $\mathbf{x}$ a $\mathcal{P}$-synchronised solution. We define the following sets of functions on T:*

$$\mathcal{F}(C) = \{\mathbf{g}^{(m)}(\mathbf{x}_C, \mathbf{x}_{C'}) \mid C' \in \mathcal{P}, 1 \leq m \leq M\}, \tag{I.20}$$

$$\mathcal{F}(C, C') = \{\mathbf{g}^{(m)}(\mathbf{x}_C, \mathbf{x}_{C'}) \mid 1 \leq m \leq M\}, \tag{I.21}$$

$$\mathcal{F}(C, C', m) = \{\mathbf{g}^{(m)}(\mathbf{x}_C, \mathbf{x}_{C'})\}. \tag{I.22}$$

Note that $\mathcal{F}(C, C', m)$ only has one element, and the inclusions

$$\mathcal{F}(C, C', m) \subseteq \mathcal{F}(C, C') \subseteq \mathcal{F}(C) \tag{I.23}$$

for any $C, C' \in \mathcal{P}$, $1 \leq m \leq M$. Any subset of a linearly independent set is linearly independent, which means that

- if $\mathcal{F}(C)$ is linearly independent, so is $\mathcal{F}(C, C')$ for any $C' \in \mathcal{P}$, $C' \neq C$;
- if $\mathcal{F}(C, C')$ is linearly independent, so is $\mathcal{F}(C, C', m)$ for any m.

Schematically, we can represent this as

$$\mathcal{F}(C) \Longrightarrow \mathcal{F}(C, C') \Longrightarrow \mathcal{F}(C, C', m). \tag{I.24}$$

As in the Main Text, we define linear independence over the first family of sets.

Definition 1.9. *Let $\mathcal{P}$ be a partition of the node set and $\mathbf{x}$ a $\mathcal{P}$-synchronised solution. We call $\mathbf{x}$* linearly independent *(with respect to $\mathcal{P}$) if $\mathcal{F}(C)$ is linearly independent for each $C \in \mathcal{P}$.*

The following result clarifies the relation between linear independence and different notions of equitability.

Theorem 1.10. *Consider the multilayer dynamical system given by Equations* (I.1) *and* (I.2)*. Let $\mathcal{P}$ be a partition of the node set and $\mathbf{x}$ a $\mathcal{P}$-synchronised solution of the dynamical system.*

(1) Suppose that $\mathcal{F}(C, C', m)$ is linearly independent (equivalently, $\mathbf{g}^{(m)}(x_C, x_{C'})$ is not identically zero over T) for all $C, C' \in \mathcal{P}$, $C \neq C'$ and $1 \leq m \leq M$. Then $\mathcal{P}$ is DEL with respect to $\mathbf{x}$ if and only if $\mathcal{P}$ is SEL.

(2) Suppose that $\mathcal{F}(C, C')$ is linearly independent for all $C, C' \in \mathcal{P}$, $C \neq C'$. Then $\mathcal{P}$ is DE with respect to $\mathbf{x}$ if and only if $\mathcal{P}$ is SEL.

(3) Suppose that $\mathcal{F}(C)$ is linearly independent for all $C \in \mathcal{P}$. Then $\mathcal{P}$ is WDE with respect to $\mathbf{x}$ if and only if $\mathcal{P}$ is SEL.

The results of this theorem can be represented schematically as

$$\begin{tikzcd}
DEL \arrow[r, Leftrightarrow, "\mathcal{F}(C,C')"] \arrow[dr, Leftrightarrow, "\mathcal{F}(C,C',m)"'] & DE \arrow[r, Leftrightarrow, "\mathcal{F}(C)"] \arrow[d, Leftrightarrow, "\mathcal{F}(C,C')"] & WDE \arrow[dl, Leftrightarrow, "\mathcal{F}(C)"] \\
 & SEL &
\end{tikzcd} \qquad \text{(CS)} \tag{I.25}$$

where we assume cluster synchronisation (CS), we have used $DEL \implies DE \implies WDE$ from the previous theorem, and the right-to-left arrows are a consequence of the other arrows and (I.24). In particular, if $\mathbf{x}$ is $\mathcal{P}$-synchronised and linearly independent ($\mathcal{F}(C)$ is linearly independent for each $C \in \mathcal{P}$), then all four notions are SEL, DEL, DE, and WDE are equivalent.

Proof. We will write $\mathbf{x}_C$ for the synchronised solutions on a cluster $C \in \mathcal{P}$, that is, $\mathbf{x}_C = \mathbf{x}_i$ for any $i \in C$.

(1) Let $C, C' \in \mathcal{P}$, $C \neq C'$, $i, j \in C$, $m \in \{1, \dots, M\}$, and $t \in T$ time domain. We have

$$\begin{gathered}
\mathbf{h}_i^{C',m}(t) = \mathbf{h}_j^{C',m}(t) \iff \\
\sum_{k \in C'} A_{i,k}^{(m)} \mathbf{g}^{(m)}(\mathbf{x}_i(t), \mathbf{x}_k(t)) = \sum_{k \in C'} A_{j,k}^{(m)} \mathbf{g}^{(m)}(\mathbf{x}_j(t), \mathbf{x}_k(t)) \iff \\
\mathbf{g}^{(m)}(\mathbf{x}_C(t), \mathbf{x}_{C'}(t)) \sum_{k \in C'} \left(A_{i,k}^{(m)} - A_{j,k}^{(m)} \right) = 0 \iff \\
\mathbf{g}^{(m)}(\mathbf{x}_C(t), \mathbf{x}_{C'}(t)) \left(h_i^{C',m} - h_j^{C',m} \right) = 0
\end{gathered}$$

with each equation satisfied for all $t \in T$. By the linear independence hypothesis, this is equivalent to

$$h_i^{C,m} - h_j^{C,m} = 0 \text{ for all } m \iff h_i^{C,m} = h_j^{C,m} \text{ for all } m.$$

All in all, we have DEL holds if and only if SEL holds. We note that one implication (SEL implies DEL) does not need the linear independence hypothesis, just $\mathcal{P}$-synchronisation, which we knew from Theorem 1.7 or diagram I.16.

(2) The proof is very similar, and sketched below (we drop the time parameter t), with

the same notation as in (1):

$$\begin{aligned}
\mathbf{h}_i^{C'} = \mathbf{h}_j^{C'} &\iff \\
\sum_{m=1}^{M} \sigma_m \sum_{k \in C'} A_{i,k}^{(m)} \mathbf{g}^{(m)}(\mathbf{x}_i, \mathbf{x}_k) = \sum_{m=1}^{M} \sigma_m \sum_{k \in C'} A_{j,k}^{(m)} \mathbf{g}^{(m)}(\mathbf{x}_j, \mathbf{x}_k) &\iff \\
\sum_{m=1}^{M} \sigma_m \mathbf{g}^{(m)}(\mathbf{x}_C, \mathbf{x}_{C'}) \sum_{k \in C'} \left(A_{i,k}^{(m)} - A_{j,k}^{(m)} \right) = 0 &\iff \\
\sum_{m=1}^{M} \sigma_m \mathbf{g}^{(m)}(\mathbf{x}_C, \mathbf{x}_{C'}) \left(h_i^{C',m} - h_j^{C',m} \right) = 0 &\overset{\mathcal{F}(C,C')}{\iff} \\
\sigma_m \left(h_i^{C,m} - h_j^{C,m} \right) = 0 &\overset{\sigma_m \neq 0}{\iff} \\
h_i^{C',m} = h_j^{C',m}.
\end{aligned}$$

(3) Analogously to (1) and (2), and with the same notation, we have

$$\begin{aligned}
\mathbf{h}_i^{V \setminus C} = \mathbf{h}_j^{V \setminus C} &\iff \\
\sum_{m=1}^{M} \sigma_m \sum_{k \notin C} A_{i,k}^{(m)} \mathbf{g}^{(m)}(\mathbf{x}_i, \mathbf{x}_k) = \sum_{m=1}^{M} \sigma_m \sum_{k \notin C} A_{j,k}^{(m)} \mathbf{g}^{(m)}(\mathbf{x}_j, \mathbf{x}_k) &\iff \\
\sum_{m=1}^{M} \sigma_m \sum_{C' \in \mathcal{P}, C' \neq C} \sum_{k \in C'} \mathbf{g}^{(m)}(\mathbf{x}_C, \mathbf{x}_{C'}) \left(A_{i,k}^{(m)} - A_{j,k}^{(m)} \right) = 0 &\iff \\
\sum_{m=1}^{M} \sum_{C' \in \mathcal{P}, C' \neq C} \sigma_m \left(h_i^{C',m} - h_j^{C',m} \right) \mathbf{g}^{(m)}(\mathbf{x}_C, \mathbf{x}_{C'}) = 0 &\overset{\mathcal{F}(C)}{\iff} \\
\sigma_m \left(h_i^{C,m} - h_j^{C,m} \right) = 0 &\overset{\sigma_m \neq 0}{\iff} \\
h_i^{C',m} = h_j^{C',m}. & \qquad \square
\end{aligned}$$

Now that we have explored the different notions of equitability and their relationships, we move on to study cluster synchronisation in the context of equitability.

1.4 Synchronisation and equitability

We first show that equitability is a necessary condition for cluster synchronisation, in the following sense.

Theorem 1.11. *Consider the multilayer dynamical system given by Equations* (I.1) *and* (I.2). *Let $\mathcal{P}$ be a partition of the node set $\{1, \ldots, N\}$ and $\mathbf{x}$ a $\mathcal{P}$-synchronised solution. Then,*

(1) the partition $\mathcal{P}$ is WDE with respect to $\mathbf{x}$;

(2) if $\mathbf{x}$ is linearly independent ($\mathcal{F}(C)$ is linearly independent for all $C \in \mathcal{P}$), then $\mathcal{P}$ is SEL.

Proof. (1) Let $C \in \mathcal{P}$, $i, j \in C$. Since $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_N)$ is a solution the dynamical system,

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) + \mathbf{h}_i(\mathbf{x}_1, \ldots, \mathbf{x}_N) = \mathbf{f}(\mathbf{x}_i) + \mathbf{h}_i^C + \mathbf{h}_i^{V\setminus C}, \tag{I.26}$$

and similarly for j. Since $\mathbf{x}$ is $\mathcal{P}$-synchronised, $\mathbf{h}_i^C = \mathbf{h}_j^C = 0$ (as the coupling functions are non-invasive), and $\mathbf{x}_i = \mathbf{x}_j$. In particular,

$$\dot{\mathbf{x}}_i = \dot{\mathbf{x}}_j \implies \mathbf{h}_i^{V\setminus C} = \mathbf{h}_j^{V\setminus C} \tag{I.27}$$

and $\mathcal{P}$ is WDE with respect to $\mathbf{x}$ by definition.

(2) By (1), we have WDE, which, under the linear independence hypothesis, is equivalent to SEL (and DEL, and DE) by Theorem 1.22; see diagram I.25. □

Next, we show the reciprocal, that is, that equitability is also a sufficient condition, using the quotient dynamics.

As in the Main Text, we define the *quotient* of the dynamical system (I.1), (I.2) with respect to a partition $\mathcal{P}$ of the node set $\{1, \ldots, N\}$ into $K = |\mathcal{P}|$ clusters as the dynamical system with dynamical units $\mathbf{y}_1, \ldots, \mathbf{y}_K$ and K differential equations

$$\dot{\mathbf{y}}_i = \mathbf{f}(\mathbf{y}_i) + \mathbf{h}_i(\mathbf{y}_1, \ldots, \mathbf{y}_K), \qquad \text{where} \tag{I.28}$$

$$\mathbf{h}_i(\mathbf{y}_1, \ldots, \mathbf{y}_K) = \sum_{m=1}^{M} \sigma_m \sum_{j=1}^{K} B_{i,j}^{(m)}\, \mathbf{g}^{(m)}(\mathbf{y}_i, \mathbf{y}_j) \qquad \text{and} \tag{I.29}$$

$$B_{ij}^{(m)} = \frac{1}{|C_i|} \sum_{k\in C_i, l\in C_j} A_{kl}^{(m)}, \tag{I.30}$$

the quotient matrix of $A^{(m)}$ with respect to $\mathcal{P}$.

Theorem 1.12. *Consider the multilayer dynamical system given by Equations* (I.1) *and* (I.2)*, which we refer to as the parent dynamical system, a partition of the node set $\mathcal{P}$, and the associated quotient dynamical system given by Equations* (I.28) *to* (I.30)*. Suppose that $\mathcal{P}$ is SEL. Then, $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_N)$ is a $\mathcal{P}$-synchronised solution of the parent dynamical system if and only if $\mathbf{y} = (\mathbf{y}_1, \ldots, \mathbf{y}_K)$ is a solution of the quotient dynamical system, where $\mathbf{y}_k = \mathbf{x}_i$ whenever $i \in C_k$.*

Proof. Note that the condition $\mathbf{y}_k = \mathbf{x}_i$ whenever $i \in C_k$ uniquely defines $\mathbf{y}$ from $\mathbf{x}$ (since $\mathbf{x}$ is $\mathcal{P}$-synchronised), and $\mathbf{x}$ from $\mathbf{y}$ (since $\mathcal{P}$ is a partition of the node set). Using the equitability condition (SEL), we can write

$$B_{kl}^{(m)} = \frac{1}{|C_k|} \sum_{i\in C_k, j\in C_l} A_{ij}^{(m)} = \frac{1}{|C_k|} \sum_{i\in C_k} \overbrace{\sum_{j\in C_l} A_{ij}^{(m)}}^{=h_i^{C_l}} = \frac{1}{|C_k|} |C_k| h_i^{C_l} = h_i^{C_l} = \sum_{j\in C_l} A_{ij}^{(m)}, \tag{I.31}$$

for any choice $i \in C_k$. Suppose that $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_N)$ and $\mathbf{y} = (\mathbf{y}_1, \dots, \mathbf{y}_K)$ satisfy $\mathbf{y}_k = \mathbf{x}_i$ whenever $i \in C_k$. Let $k \in \{1, \dots, K\}$ and $i \in C_k$. Then

$$\mathbf{h}_k(\mathbf{y}_1, \dots, \mathbf{y}_K) = \sum_{m=1}^{N} \sigma_m \sum_{l=1}^{K} B_{kl}^{(m)} \mathbf{g}^{(m)}(\mathbf{y}_k, \mathbf{y}_l) \tag{I.32}$$

$$\stackrel{(I.58)}{=} \sum_{m=1}^{N} \sigma_m \sum_{l=1}^{K} \sum_{j \in C_l} A_{ij}^{(m)} \mathbf{g}^{(m)}(\mathbf{y}_k, \mathbf{y}_l) \tag{I.33}$$

$$= \sum_{m=1}^{N} \sigma_m \sum_{l=1}^{K} \sum_{j \in C_l} A_{ij}^{(m)} \mathbf{g}^{(m)}(\mathbf{x}_i, \mathbf{x}_j) \tag{I.34}$$

$$= \sum_{m=1}^{N} \sigma_m \sum_{j=1}^{N} A_{ij}^{(m)} \mathbf{g}^{(m)}(\mathbf{x}_i, \mathbf{x}_j) = \mathbf{h}_i(\mathbf{x}_1, \dots, \mathbf{x}_N). \tag{I.35}$$

All in all, if $i \in C_k$, the differential equation for $\mathbf{x}_i$ in the parent dynamical system is the same as the differential equation for $\mathbf{y}_k$ in the quotient dyanamical system. This completes the proof. □

1.5 Synchronisation and equitability for hypergraphs

We now discuss the hypergraph case. Consider the hypergraph dynamical system given by Equations (I.1) and (I.3). Let $\mathcal{P}$ be a partition of the node set $\{1, \dots, N\}$, $1 \le m \le M$, $C, C_1, \dots, C_m \in \mathcal{P}$, and $i \in C$. Consider the following notation, which extends the notation in the Main Text,

$$h_i^{C_1,\dots,C_m} = \sum_{j_1 \in C_1} \cdots \sum_{j_1 \in C_m} A_{i,j_1,\dots,j_m}^{(m)}, \tag{I.36}$$

$$h_i^{(m)} = \sum_{C_1,\dots,C_m \in \mathcal{P}} h_i^{C_1,\dots,C_m}, \tag{I.37}$$

$$h_i = \sum_{m=1}^{M} \sigma_m h_i^{(m)}. \tag{I.38}$$

and corresponding dynamical quantities $\mathbf{h}_i^{C_1,\dots,C_m}$, $\mathbf{h}_i^{(m)}$ and $\mathbf{h}_i$, given a solution $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_N)$, by adding $\mathbf{g}^{(m)}(\mathbf{x}_i, \mathbf{x}_{j_1}, \dots, \mathbf{x}_{j_m})$ to Eq. (I.36) above.

Definition 1.13. *We call the partition* $\mathcal{P}$ dynamically equitable per layer (DEL)*, respectively* dynamically equitable (DE)*, respectively* weakly dynamically equitable (WDE) *with respect to a solution* $\mathbf{x}$*, if, for each* $C \in \mathcal{P}$ *and* $i, j \in C$*, we have*

$$\mathbf{h}_i^{C_1,\dots,C_m} = \mathbf{h}_j^{C_1,\dots,C_m}, \ \textit{respectively} \tag{I.39}$$

$$\mathbf{h}_i^{(m)} = \mathbf{h}_j^{(m)}, \ \textit{respectively} \tag{I.40}$$

$$\mathbf{h}_i^{ext} = \mathbf{h}_j^{ext}, \tag{I.41}$$

where each equation must hold for all $C_1, \dots, C_m \in \mathcal{P}$ *not all equal to* C*, and all* $1 \le m \le M$*, and we write* $\mathbf{h}_i^{ext} = \mathbf{h}_i - \sum_{m=1}^{M} \sigma_m \mathbf{h}_i^{C,\overset{m}{\dots},C}$ *where* $C,\overset{m}{\dots},C$ *represents* C *repeated* m *times.*

Remark 1.14. *We keep a similar terminology to the multiplex case, even though it does not make much sense to talk about 'layers' in this case but rather about 'multi-body interactions'.*

Remark 1.15. *This is still an 'external' notion of equitability, as we exclude the case $C_1 = \ldots = C_m = C$, as argued in the Main Text.*

Definition 1.16. *We call a partition $\mathcal{P}$* structurally equitable per layer *(SEL) if, for each $C \in \mathcal{P}$ and $i, j \in C$, we have*

$$h_i^{C_1,\ldots,C_m} = h_j^{C_1,\ldots,C_m} \tag{I.42}$$

for all $C_1 \ldots, C_m \in \mathcal{P}$ not all equal to C, and all $1 \leq m \leq M$.

Remark 1.17. *As in Section 1.2, we could have defined structural equitability (SE) and weak structural equitability (WSE) and shown $SEL \implies SE \implies WSE$. However, as in the multilayer case, SEL is the appropriate notion of structural equitability, for analogous reasons (see Remark 1.6).*

As in the multilayer case, we can show that strong equitability implies weak equitability, equitability per layer implies equitability, and structural equitability per layer implies dynamical equitability for synchronised solutions.

Theorem 1.18. *Consider the hypergraph dynamical system given by Equations* (I.1) *and* (I.3)*. Let $\mathcal{P}$ be a partition of the node set and $\mathbf{x}$ a solution of the dynamical system. Then,*

(1) if $\mathcal{P}$ is DE, then it is WDE;

(2) if $\mathcal{P}$ is DEL, then it is DE;

(3) if $\mathcal{P}$ is SEL and $\mathbf{x}$ is $\mathcal{P}$-synchronised, then $\mathcal{P}$ is DEL.

Proof. (1) and (2) follow directly from the definitions, namely Eqs. (I.37) and (I.38).

(3) If $\mathbf{x}$ is $\mathcal{P}$-synchronised, we can write $\mathbf{x}_C$ for the synchronised solution on a cluster $C \in \mathcal{P}$, that is, $\mathbf{x}_i = \mathbf{x}_C$ for any $i \in C$. Let $C \in \mathcal{P}$, $i, j \in C$, $m \in \{1, \ldots, M\}$, and $C_1, \ldots, C_m \in \mathcal{P}$. Then

$$\mathbf{h}_i^{C_1,\ldots,C_m} = \sum_{j_1 \in C_1,\ldots,j_m \in C_m} A^{(m)}_{i,j_1,\ldots,j_m} \mathbf{g}^{(m)}(\mathbf{x}_i, \mathbf{x}_{j_1}, \ldots, \mathbf{x}_{j_m}) \tag{I.43}$$

$$= \sum_{j_1 \in C_1,\ldots,j_m \in C_m} A^{(m)}_{i,j_1,\ldots,j_m} \mathbf{g}^{(m)}(\mathbf{x}_C, \mathbf{x}_{C_1}, \ldots, \mathbf{x}_{C_m}) \tag{I.44}$$

$$= \mathbf{g}^{(m)}(\mathbf{x}_C, \mathbf{x}_{C_1}, \ldots, \mathbf{x}_{C_m}) \sum_{j_1 \in C_1,\ldots,j_m \in C_m} A^{(m)}_{i,j_1,\ldots,j_m} \tag{I.45}$$

$$= \mathbf{g}^{(m)}(\mathbf{x}_C, \mathbf{x}_{C_1}, \ldots, \mathbf{x}_{C_m})\, h_i^{C_1,\ldots,C_m}, \tag{I.46}$$

and similarly for $\mathbf{h}_j^{C_1,\ldots,C_m}$. By hypothesis, $h_i^{C_1,\ldots,C_m} = h_j^{C_1,\ldots,C_m}$ and hence, by the above, $\mathbf{h}_i^{C_1,\ldots,C_m} = \mathbf{h}_j^{C_1,\ldots,C_m}$. □

The results of the theorem can be represented schematically as follows,

$$\begin{array}{ccccc} DEL & \Longrightarrow & DE & \Longrightarrow & WDE \\ {\scriptstyle CS}\Big\Uparrow & & & & \\ SEL & & & & \end{array} \tag{I.47}$$

with the same notational conventions as in (I.16).

Remark 1.19. *The concepts of DEL and DE coincide in the monolayer, or network, case ($m = 1$), but otherwise they are distinct. Similarly, we can give examples of WDE partitions that are not DE.*

Next, we define the corresponding notions of linear independence in the hypergraph case.

Definition 1.20. *Let $\mathcal{P}$ be a partition of the node set, $C, C_1 \ldots, C_m \in \mathcal{P}$, and $\mathbf{x}$ a $\mathcal{P}$-synchronised solution. We define the following sets of functions on T:*

$$\begin{aligned}
&\mathcal{F}(C) = \left\{\mathbf{g}^{(m)}(\mathbf{x}_C, \mathbf{x}_{C_1}, \ldots, \mathbf{x}_{C_m}) \mid 1 \le m \le M, C_1, \ldots, C_m \in \mathcal{P} \text{ not all equal to } C\right\}, \\
&\mathcal{F}(C, m) = \left\{\mathbf{g}^{(m)}(\mathbf{x}_C, \mathbf{x}_{C_1}, \ldots, \mathbf{x}_{C_m}) \mid C_1, \ldots, C_m \in \mathcal{P} \text{ not all equal to } C\right\}, \\
&\mathcal{F}(C, C_1, \ldots, C_m) = \left\{\mathbf{g}^{(m)}(\mathbf{x}_C, \mathbf{x}_{C_1}, \ldots, \mathbf{x}_{C_m})\right\}.
\end{aligned}$$

Note that $\mathcal{F}(C, C_1, \ldots, C_m)$ has only one element, and the inclusions

$$\mathcal{F}(C, C_1, \ldots, C_m) \subseteq \mathcal{F}(C, m) \subseteq \mathcal{F}(C) \tag{I.48}$$

for any $C, C_1, \ldots, C_m \in \mathcal{P}$, $1 \le m \le M$. Any subset of a linearly independent set is linearly independent, which means that

- if $\mathcal{F}(C)$ is linearly independent, so is $\mathcal{F}(C, m)$ for any $1 \le m \le M$;
- if $\mathcal{F}(C, m)$ is linearly independent, so is $\mathcal{F}(C, C_1, \ldots, C_m)$ for any $C_1, \ldots, C_m \in \mathcal{P}$.

Schematically, we can represent this as

$$\mathcal{F}(C) \Longrightarrow \mathcal{F}(C, m) \Longrightarrow \mathcal{F}(C, C_1, \ldots, C_m). \tag{I.49}$$

As in the Main Text, we define linear independence over the first family of sets.

Definition 1.21. *Let $\mathcal{P}$ be a partition of the node set and $\mathbf{x}$ a $\mathcal{P}$-synchronised solution. We call $\mathbf{x}$* linearly independent *(with respect to $\mathcal{P}$) if $\mathcal{F}(C)$ is linearly independent for each $C \in \mathcal{P}$.*

The following result clarifies the relation between linear independence and different notions of equitability in the hypergraph case. It is analogous to Theorem 1.22 and the proof (omitted) is very similar.

Theorem 1.22. *Consider the hypergraph dynamical system given by Equations* (I.1) *and* (I.3). *Let $\mathcal{P}$ be a partition of the node set and $\mathbf{x}$ a $\mathcal{P}$-synchronised solution of the dynamical system.*

(1) Suppose that $\mathcal{F}(C, C_1, \ldots, C_m)$ is linearly independent (equivalently, the coupling function $\mathbf{g}^{(m)}(\mathbf{x}_C, \mathbf{x}_{C_1}, \ldots, \mathbf{x}_{C_m})$ is not identically zero over T) for all $C \in \mathcal{P}$, $C_1, \ldots, C_m \in \mathcal{P}$ not all equal to C, and $1 \leq m \leq M$. Then $\mathcal{P}$ is DEL with respect to $\mathbf{x}$ if and only if $\mathcal{P}$ is SEL.

(2) Suppose that $\mathcal{F}(C, m)$ is linearly independent for all $C \in \mathcal{P}$, and $1 \leq m \leq M$. Then $\mathcal{P}$ is DE with respect to $\mathbf{x}$ if and only if $\mathcal{P}$ is SEL.

(3) Suppose that $\mathcal{F}(C)$ is linearly independent for all $C \in \mathcal{P}$. Then $\mathcal{P}$ is WDE with respect to $\mathbf{x}$ if and only if $\mathcal{P}$ is SEL.

The results in this theorem can be represented schematically as

$$\begin{tikzcd}
DEL \arrow[r, Rightarrow, shift right] \arrow[dr, Rightarrow, shift left] & DE \arrow[l, Rightarrow, shift right, "\mathcal{F}(C,m)"'] \arrow[r, Rightarrow, shift right] \arrow[d, Rightarrow, shift left] & WDE \arrow[l, Rightarrow, shift right, "\mathcal{F}(C)"'] \arrow[dl, Rightarrow, shift left] & \\
& SEL \arrow[ul, Rightarrow, shift left, "\mathcal{F}(C,C_1,\ldots,C_m)"] \arrow[u, Rightarrow, shift left, "\mathcal{F}(C,m)"] \arrow[ur, Rightarrow, shift left, "\mathcal{F}(C)"'] & & (\mathrm{CS})
\end{tikzcd} \tag{I.50}$$

where we assume cluster synchronisation (CS), we have used $DEL \implies DE \implies WDE$ from the previous theorem, Theorem 1.18, and the right-to-left arrows are a consequence of the other arrows and (I.49). In particular, if $\mathbf{x}$ is $\mathcal{P}$-synchronised and linearly independent ($\mathcal{F}(C)$ is linearly independent for each $C \in \mathcal{P}$), then all four notions are SEL, DEL, DE, and WDE are equivalent.

Finally, we related cluster synchronisation and stability in the next two results, which adapt Theorems 1.11 and 1.12 to the hypergraph case.

Theorem 1.23. *Consider the dynamical system* (I.1) *with* (I.3)*. Let $\mathcal{P}$ be a partition of the node set $\{1, \ldots, N\}$ and $\mathbf{x}$ a $\mathcal{P}$-synchronised solution. Then,*

(1) the partition $\mathcal{P}$ is WDE with respect to $\mathbf{x}$;

(2) if $\mathbf{x}$ is linearly independent ($\mathcal{F}(C)$ is linearly independent for all $C \in \mathcal{P}$), then $\mathcal{P}$ is SEL.

Proof. (1) Let $i, j \in C \in \mathcal{P}$. Since the solution is $\mathcal{P}$-synchronised, $\mathbf{x}_i = \mathbf{x}_j$ and so $\dot{\mathbf{x}}_i = \dot{\mathbf{x}}_j$, which implies

$$\mathbf{h}_i(\mathbf{x}_1, \ldots, \mathbf{x}_N) = \mathbf{h}_j(\mathbf{x}_1, \ldots, \mathbf{x}_N) \implies \mathbf{h}_i^{\text{ext}}(\mathbf{x}_1, \ldots, \mathbf{x}_N) = \mathbf{h}_j^{\text{ext}}(\mathbf{x}_1, \ldots, \mathbf{x}_N), \tag{I.51}$$

since $\mathbf{h}_i^{C,\ldots,C} = \mathbf{h}_i^{C,\ldots,C} = 0$ as the coupling functions are non-invasive $\mathbf{g}^{(m)}(\mathbf{x}, \ldots, \mathbf{x}) = 0$.
(2) Write $\mathbf{x}_C = \mathbf{x}_i$ whenever $i \in C$. Let $i, j \in C$, then $\mathbf{x}_i = \mathbf{x}_j$ so $\dot{\mathbf{x}}_i = \dot{\mathbf{x}}_j$ which implies

(substitute differential equation, equate, rearrange, write $\mathbf{x}_l$ as $\mathbf{x}_{C'}$ whenever $l \in C'$)

$$\sum_{m=1}^{M} \sigma_m \sum_{j_1,\ldots,j_m=1}^{N} \left(A^{(m)}_{i,j_1,\ldots,j_m} - A^{(m)}_{j,j_1,\ldots,j_m} \right) \mathbf{g}^{(m)}(\mathbf{x}_C, \mathbf{x}_{C_1}, \ldots, \mathbf{x}_{C_m}) = 0 \iff \tag{I.52}$$

$$\sum_{m=1}^{M} \sigma_m \left(h_i^{C_1,\ldots,C_m} - h_j^{C_1,\ldots,C_m} \right) \mathbf{g}^{(m)}(\mathbf{x}_C, \mathbf{x}_{C_1}, \ldots, \mathbf{x}_{C_m}) = 0 \overset{\mathcal{F}(C)}{\iff} \tag{I.53}$$

$$\sigma_m \left(h_i^{C_1,\ldots,C_m} - h_j^{C_1,\ldots,C_m} \right) = 0 \overset{\sigma_m \neq 0}{\iff} h_i^{C_1,\ldots,C_m} = h_j^{C_1,\ldots,C_m} \tag{I.54}$$

for all m, $C_1, \ldots, C_m$, that is, the partition is SEL. $\square$

As in the Main Text, we define the *quotient* of the hypergraph dynamical system (I.1) and (I.3) with respect to a partition $\mathcal{P}$ of the node set $\{1, \ldots, N\}$ into $K = |\mathcal{P}|$ clusters as the dynamical system with dynamical units $\mathbf{y}_1, \ldots, \mathbf{y}_K$ and K differential equations

$$\dot{\mathbf{y}}_i = \mathbf{f}(\mathbf{y}_i) + \mathbf{h}_i(\mathbf{y}_1, \ldots, \mathbf{y}_K), \quad \text{where} \tag{I.55}$$

$$\mathbf{h}_i(\mathbf{y}_1, \ldots, \mathbf{y}_K) = \sum_{m=1}^{M} \sigma_m \sum_{j_1,\ldots,j_m=1}^{K} B^{(m)}_{i,j_1,\ldots,j_m} \mathbf{g}^{(m)}(\mathbf{y}_i, \mathbf{y}_{j_1}, \ldots, \mathbf{y}_{j_m}) \quad \text{and} \tag{I.56}$$

$$B^{(m)}_{ij_1\ldots j_m} = \frac{1}{|C_i|} \sum_{k\in C_i, k_1 \in C_{j_1}, \ldots, k_m \in C_{j_m}} A^{(m)}_{k,k_1,\ldots,k_m}, \tag{I.57}$$

the quotient tensor of $A^{(m)}$ with respect to $\mathcal{P}$.

Theorem 1.24. *Consider the hypergraph dynamical system given by Equations* (I.1) *and* (I.3)*, which we refer to as the parent dynamical system, a partition of the node set $\mathcal{P}$, and the associated quotient dynamical system given by Equations* (I.55) *to* (I.57)*. Suppose that $\mathcal{P}$ is SEL. Then, $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_N)$ is a $\mathcal{P}$-synchronised solution of the parent dynamical system if and only if $\mathbf{y} = (\mathbf{y}_1, \ldots, \mathbf{y}_K)$ is a solution of the quotient dynamical system, where $\mathbf{y}_k = \mathbf{x}_i$ whenever $i \in C_k$.*

Proof. Note that the condition $\mathbf{y}_k = \mathbf{x}_i$, $i \in C_k$, uniquely defines $\mathbf{y}$ from $\mathbf{x}$ (since $\mathbf{x}$ is $\mathcal{P}$-synchronised), and $\mathbf{x}$ from $\mathbf{y}$ (since $\mathcal{P}$ is a partition of the node set). Using the equitability condition (SEL), we can write

$$\begin{aligned} B^{(m)}_{kl_1\ldots l_m} &= \frac{1}{|C_k|} \sum_{i\in C_k, j_1 \in C_{l_1}, \ldots, j_m \in C_{l_m}} A^{(m)}_{ij_1\ldots j_m} = \frac{1}{|C_k|} \sum_{i \in C_k} \overbrace{\sum_{j \in C_l} A^{(m)}_{ij}}^{=h_i^{C_l}} \\ &= \frac{1}{|C_k|} |C_k| h_i^{C_l} = h_i^{C_l} = \sum_{j \in C_l} A^{(m)}_{ij}, \end{aligned} \tag{I.58}$$

for any choice $i \in C_k$. Suppose that $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_N)$ and $\mathbf{y} = (\mathbf{y}_1, \ldots, \mathbf{y}_K)$ satisfy

$\mathbf{y}_k = \mathbf{x}_i$ whenever $i \in C_k$. Let $k \in \{1, \ldots, K\}$ and $i \in C_k$. Then

$$\mathbf{h}_k(\mathbf{y}_1, \ldots, \mathbf{y}_K) = \sum_{m=1}^{N} \sigma_m \sum_{l=1}^{K} B_{kl}^{(m)} \mathbf{g}^{(m)}(\mathbf{y}_k, \mathbf{y}_l) \tag{I.59}$$

$$\overset{(I.58)}{=} \sum_{m=1}^{N} \sigma_m \sum_{l=1}^{K} \sum_{j \in C_l} A_{ij}^{(m)} \mathbf{g}^{(m)}(\mathbf{y}_k, \mathbf{y}_l) \tag{I.60}$$

$$= \sum_{m=1}^{N} \sigma_m \sum_{l=1}^{K} \sum_{j \in C_l} A_{ij}^{(m)} \mathbf{g}^{(m)}(\mathbf{x}_i, \mathbf{x}_j) \tag{I.61}$$

$$= \sum_{m=1}^{N} \sigma_m \sum_{j=1}^{N} A_{ij}^{(m)} \mathbf{g}^{(m)}(\mathbf{x}_i, \mathbf{x}_j) = \mathbf{h}_i(\mathbf{x}_1, \ldots, \mathbf{x}_N). \tag{I.62}$$

All in all, if $i \in C_k$, the differential equation for $\mathbf{x}_i$ in the parent dynamical system is the same as the differential equation for $\mathbf{y}_k$ in the quotient dynamical system. This completes the proof. □

2 Numerical Simulations

In this section, we comment in more detail on the different simulations displayed in the Main Text, both for the multilayer as well as for the hypergraph case.

2.1 Multilayer case

We simulated Lorenz [2] oscillators $\mathbf{f}(\mathbf{x}) = (\sigma(y-z), x(\rho-z)-y, xy-\beta z)$ with parameters $\sigma = 10$, $\rho = 28$, $\beta = 2$ and Laplacian couplings in both layers 1 and 2, in the form $\mathbf{g}^{(1)}(\mathbf{x}_j, \mathbf{x}_i) = (x_j - x_i, 0, 0)$, $\mathbf{g}^{(2)}(\mathbf{x}_j, \mathbf{x}_i) = (0, y_j - y_i, 0)$, where $\mathbf{x}_i = (x_i, y_i, z_i)$ and $\mathbf{x}_j = (x_j, y_j, z_j)$.

The dynamics were numerically computed in Julia with an RK45 algorithm, up to final time $t_f = 1000$ with a timestep of $\Delta t = 10^{-3}$. The synchronization error S_k of the kth cluster is computed at late times ($t \in [900, 1000]$ at intervals of 0.25) as the time-averaged root-mean-square deviation defined by

$$S_k = \left\langle \sqrt{\frac{1}{N_k} \sum_{i \in C_k} |\mathbf{f}(x_i) - \overline{\mathbf{x}}_k|^2} \right\rangle_T , \tag{II.63}$$

where C_k is the set of nodes contained in cluster k, $\overline{\mathbf{x}}_k$ is the ensemble average of $\mathbf{x} = (x_i)_{i \in V}$ within C_k, and $\langle . \rangle_T$ denotes the temporal average over the late time window T.

The multiplex network $G_A = \{G_A^{(1)}, G_A^{(2)}\}$, for which both layers simultaneously satisfy the structural equitability condition, is shown in Fig. 1: nodes $\{7, 8, 9, 10\}$ form a cluster which is structurally equitable with all other nodes in both layers. As shown in the Main Text, this cluster synchronises for smaller values of coupling strengths (σ_1, σ_2) than the whole network.

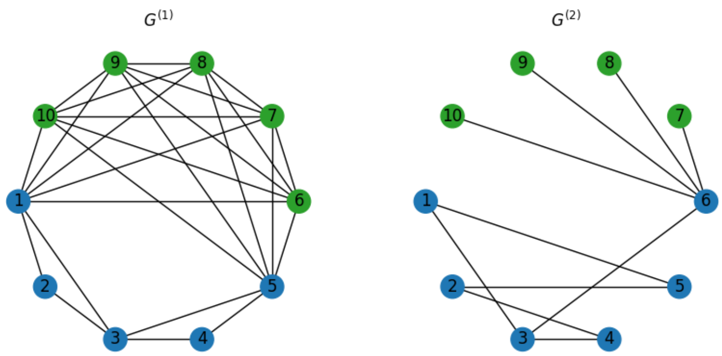

Figure 1: Multilayer network G_A with the same non-trivial external equitable (or structurally equitable) partition in both layers.

The multilayer network $G_B = \{G_B^{(1)}, G_B^{(2)}\}$, which does not satisfy the structural equitability condition, is shown in Fig.2. Although there are non-trivial structurally equitable partitions for each layer separately, with almost identical nodes ($\{6, 7, 8, 9, 10, 11\}$ for the first layer and $\{7, 8, 9, 10, 11\}$ for the second), there are no common structurally equitable partition for both layers. As it was shown in the Main Text, in this case there is no cluster synchronization before total synchronisation when both layers are active.

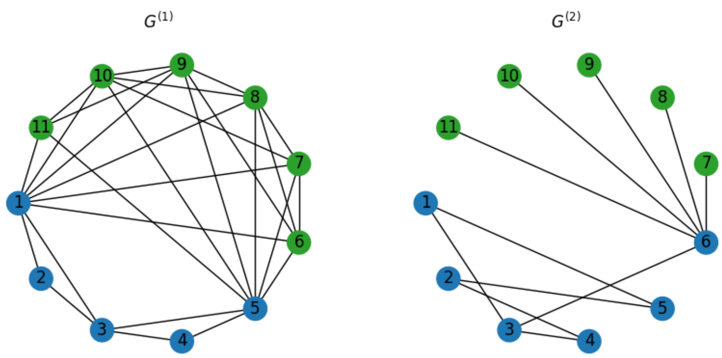

Figure 2: Multilayer network G_B which does not have any common non-trivial external equitable (or structurally equitable) partition in both layers.

2.2 Hypergraph case

We now consider hypergraph dynamics. We simulated Lorenz [2] oscillators as in the multilayer case, with parameters $\sigma = 10$, $\rho = 28$, $\beta = 8/3$. For simplicity, we assume the couplings to be diffusive, i.e. $\mathbf{g}^{(1)}(\mathbf{x}_i, \mathbf{x}_j) = \mathbf{h}^{(1)}(\mathbf{x}_j) - \mathbf{h}^{(1)}(\mathbf{x}_i)$ and $\mathbf{g}^{(2)}(\mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_k) = \mathbf{h}^{(2)}(\mathbf{x}_j, \mathbf{x}_k) - \mathbf{h}^{(2)}(\mathbf{x}_i, \mathbf{x}_i)$, and so-called *natural coupling* $\mathbf{h}^{(2)}(\mathbf{x}, \mathbf{x}) = \mathbf{h}^{(1)}(\mathbf{x})$. This means that the coupling from pairwise and triadic interactions are essentially the same: a three-body interaction with two nodes on the same state is equivalent to a two-body interaction.

Under these assumptions, we chose the two-body and three-body couplings as $\mathbf{h}^{(1)}(\mathbf{x}_j) = (x_j^3, 0, 0)^T$ and $\mathbf{h}^{(2)}(\mathbf{x}_j, \mathbf{x}_k) = (x_j^2 x_k, 0, 0)^T$ (cf. [3] where the Master Stability Function for this system is calculated, see Figure 5a in loc. cit.)

The computation was performed in C with a RK45 algorithm, up to final time $t_f = 1000$ with a timestep of $\Delta t = 10^{-3}$. The synchronization error S_k of the kth cluster is computed over the last 500 timesteps. The synchronization error is again defined by equation (II.63).

As for the hypergraph itself, it is defined as follows (we give a text description as a visualisation is probably not very insightful). The hypergraph has 20 nodes $V = \{1, ..., 20\}$ and 4 non-trivial clusters:

- Cluster $C_1 = \{1, 2, 3\}$ with connectivity
  - Pairwise: internal edges (1, 2) and (2, 3), external edges (i, 15), (i, 17), (i, 18), for each $i \in C_1$;
  - Triadic: no internal hyperedges, external hyperedges (1, 17, 20) and (2, 15, 19).

  (In this cluster, the pairwise structural equitability condition is satisfied, but not the triadic one.)

- Cluster $C_2 = \{4, 5, 6, 7\}$ with connectivity
  - Pairwise: internal edges (4, 5), (4, 7) and (7, 5), external edges (4, 15) and (7, 20);
  - Triadic: internal hyperedge (5, 6, 7), external hyperedges (i, 18, 20), (i, 17, 15) and (i, 15, 16) for each $i \in C_2$;

  (In this cluster, the pairwise structural equitability condition is not satisfied, but the triadic one is.)

- Cluster $C_3 = \{8, 9, 10\}$ with connectivity
  - Pairwise: no internal edges, external edges (8, 15) and (8, 16);
  - Triadic: internal hyperedge (8, 9, 10), external hyperedges (9, 15, 16) and (10, 15, 16);

  (In this cluster, neither the pairwise nor the triadic structural equitability conditions are satisfied.)

- Cluster $C_4 = \{11, 12, 13, 14\}$ with connectivity
  - Pairwise: internal edges (11, 12), (11, 13) (12, 13) and (13, 14), external edges (i, 18) and (i, 20) for each $i \in C_4$;
  - Triadic: internal hyperedge (12, 13, 14), external hyperedges (i, 16, 17), (i, 19, 20) for each $i \in C_4$;

- The remaining nodes (15 to 20) are not clustered, that is, they are trivial or one-node clusters and there is no additional connectivity among them.