# SUPPLEMENTARY INFORMATION: Collective Behavior in Active Swarms: Dynamics at the Edge of Disorder


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## I. POLARIZATION ORDER PARAMETER

The ordering in the system is measured by the polarization order parameter

$$P = \frac{1}{N} \left| \sum_j \exp(i\theta_j(t)) \right|_t \tag{1}$$

where $\theta_j(t)$ is the polar angle of the orientation vector $\boldsymbol{e}_j(t)$ of particle $j$ and the sum is over all particles in the system. One can also define a non-equilibrium susceptibility, which is a measure of the fluctuations in the system as

$$\chi_p = N(\langle P^2 \rangle - \langle P \rangle^2) \tag{2}$$

## II. DYNAMIC STRUCTURE FACTOR

To measure the density fluctuations in the flock, we calculate the dynamic structure factor, starting from the microscopic density field $\rho(\mathbf{r}, t)$. The microscopic density field $\rho(\mathbf{r}, t)$ is defined as

$$\rho(\mathbf{r}, t) = \sum_{j=1}^{N} \delta\left(\mathbf{r} - \mathbf{r}_j(t)\right), \tag{3}$$

where $\mathbf{r}_j(t)$ is the position of particle $j$ at time $t$, and $N$ is the total number of particles.

The spatial Fourier transform of the density field is then given by

$$\rho(\mathbf{q}, t) = \int d\mathbf{r}\, e^{-i\mathbf{q}\cdot\mathbf{r}} \rho(\mathbf{r}, t) = \sum_{j=1}^{N} e^{-i\mathbf{q}\cdot\mathbf{r}_j(t)}. \tag{4}$$

The intermediate scattering function $F(\mathbf{q}, t)$ is defined as the autocorrelation of the Fourier-transformed density:

$$F(\mathbf{q}, t) = \frac{1}{N} \left\langle \rho(\mathbf{q}, t_0 + t) \rho(-\mathbf{q}, t_0) \right\rangle_{t_0}, \tag{5}$$

and the dynamic structure factor $S(\mathbf{q}, \omega)$ is then obtained via the temporal Fourier transform

$$S(\mathbf{q}, \omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t} F(\mathbf{q}, t). \tag{6}$$

In isotropic systems, we can use radially averaged dynamic structure factor and intermediate scattering function by averaging over all wavevectors $\mathbf{q}$ of the same magnitude $q = |\mathbf{q}|$.

The radially averaged intermediate scattering function is defined as

$$F(q, t) = \frac{1}{N_q} \sum_{|\mathbf{q}|=q} F(\mathbf{q}, t), \tag{7}$$

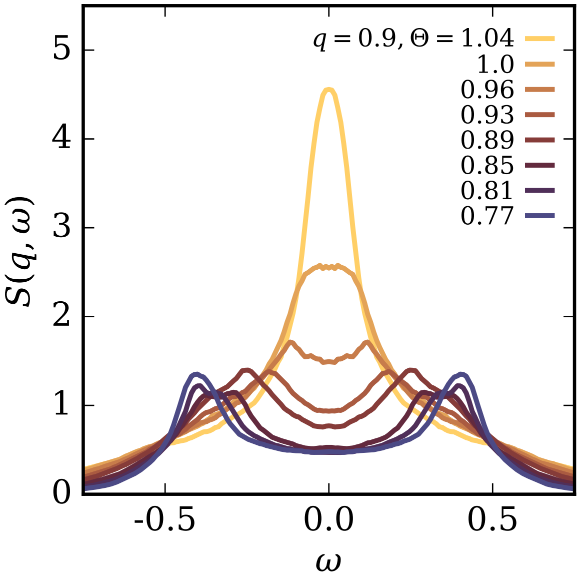


FIG. 1: The dynamic structure factor at a fixed $q = 0.9$ for different $\Theta$. For large $\Theta$, the structure is Lorentzian and therefore the density fluctuations are diffusive. As $\Theta$ is increased, a peak at small $v_\rho = \omega/q$ develops. When $\Theta$ is further increased, a 'two-peak' structure emerges, which signals the existence of two propagating excitations with different velocities. Here $N = 6000$, $K = 80$, and $eta = 1$.

where $N_q$ is the number of wavevectors with magnitude $q$. As before, the radially averaged dynamic structure factor is then computed as

$$S(q,\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t} F(q,t). \tag{8}$$

## III. BOUNDARY LENGTH ESTIMATION

To estimate the boundary length, we first determine the number of edge particles, $N_b$, by counting those with fewer than $n_c$ neighbours, where $n_c$ is a cutoff parameter. Different choices of $n_c$ yield varying estimates of $N_b$, as shown in the Fig. 2. Notably, when $\Omega_f$ is small, the average inter-particle spacing decreases, making a smaller $n_c$ more suitable for estimating the boundary length. Conversely, for larger $\Omega_f$, a higher $n_c$ may be more appropriate. Since our focus is on the trend with $\Omega_f$ rather than an exact estimate of $N_b$, we average over different cutoff choices, $n_c \in [12, 15]$. The boundary length is then given as $l_b \simeq R_a \langle N_b \rangle$. The boundary length of a circular, disk-like arrangement of $N$ particles can be estimated by assuming they occupy a disk perimeter $l_0$ and radius $R_0$. Given the inter-agent separation $R_a$, the radius is approximated as

$$R_0 = R_a \sqrt{N} \tag{9}$$

From this, the flock boundary length is obtained as

$$l_0 = 2\pi R_0 = 2\pi R_a \sqrt{N} \tag{10}$$

To determine the relation between $\Omega_f$ and line tension, we consider the Monge parmaterization of the flock boundary at a point $x$ as a function $h(\boldsymbol{x})$, representing the height of the flock boundary above a reference line. Any fluctuations of this boundary incur an energy cost given by

$$E = \int dx\, \sigma_{\text{eff}} \left(\nabla h(x)\right)^2 \tag{11}$$

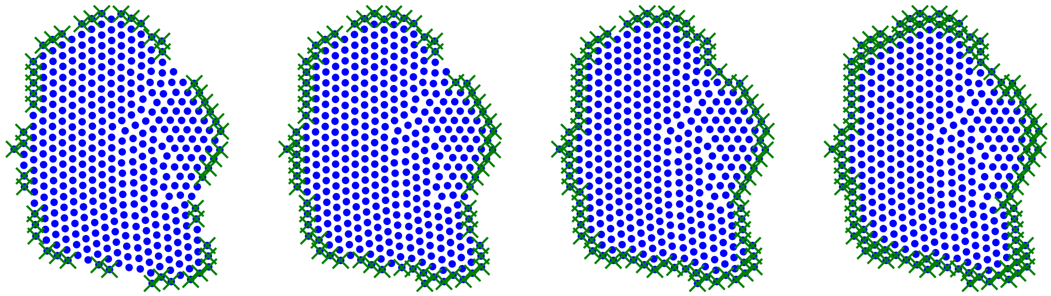

FIG. 2: Flock snapshots showing the boundary particles (marked in green crosses) for different threshold choices, $n_c =$ (a) 12 (b) 13 (c) 14 and (d) 15. For $n_c = 12$, the boundary length is under estimated while for $n_c = 15$, there is an overestimation. The final length is determined by averaging over these choices.

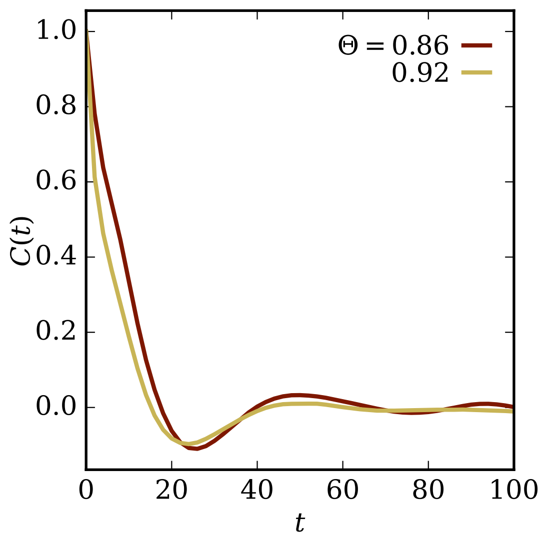


FIG. 3: Velocity auto-correlation $C(t)$ of the flocking showing negative correlations. Here $N = 400$, $\eta = 0.5$, and $K = 40$.

where $\sigma_{\rm eff}$ is an effective tension. Employing the Fourier transform $h(q) = (2\pi)^{-1} \int dx\, h(x) \exp{(iqx)}$, the mode spectrum for the height fluctuations is obtained as

$$\langle h(q)h(q')\rangle = \frac{\pi T_{\rm eff}}{\sigma_{\rm eff} q^2}\delta(q-q') \tag{12}$$

where $T_{\rm eff}$ is some measure of temperature in the system (due to noise, collision avoidance etc). In the small-gradient approximation, the line element of the flock boundary is $dl = d\boldsymbol{r}\sqrt{1+(\nabla h)^2} \simeq (1+(\nabla h)^2/2)d\boldsymbol{r}$.

With the assumption that the flock boundary deviates locally from a circle, the average flock boundary is then obtained as

$$\langle l_b\rangle = l_0 + \frac{T_{\rm eff}\pi}{2\sigma_{\rm eff} R_a}. \tag{13}$$

where $l_0 = 2\pi R_a\sqrt{N}$ is the length of the flock boundary for a perfectly circular flock made of $N$ particles maintaining an average separation $R_a$.

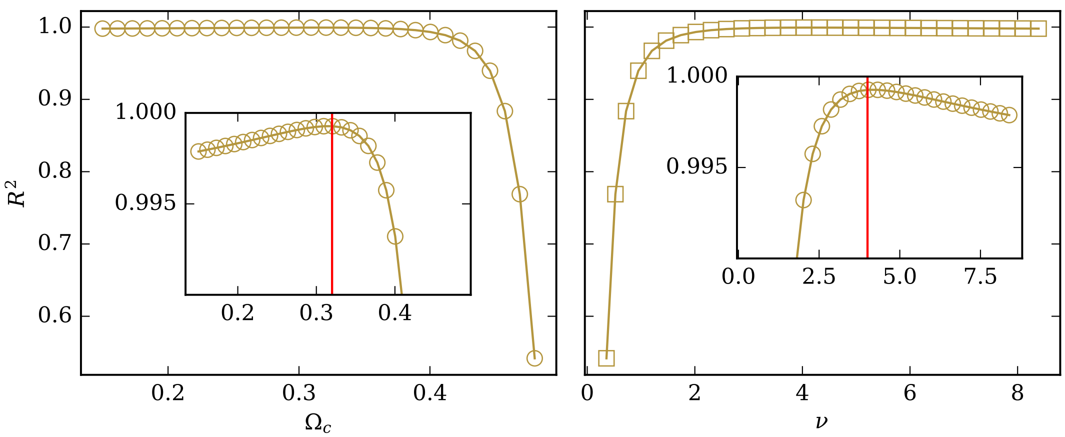


FIG. 4: The goodness of fit, measured via the coefficient of determination $R^2$ for different $\nu$ and $\Omega_c/K$ pairs. Here for a given $\Omega_c/K$, we fit the equation $\Omega_c(L)/K = \Omega_c/K + AL^{-\nu}$. It can be seen that for $\nu > 3$ and $\Omega_c/K < 0.35$, the fit is degenerate $R^2 \simeq 0.99$

## IV. FIT DEGENERACY

For the change of the finite-size critical point $\Omega_a(L)$, we fit the generalized from

$$\Omega_c(L) = \Omega_c^\infty + AL^{-1/\nu} \tag{14}$$

where $A$ is a constant obtained from the fitting procedure. For different choices of $\Omega_c^\infty$, we find that any value of $\nu \gtrsim 3.0$ and $\Omega_c^\infty \lesssim 0.35$ provides a good fit to the data, given the associated error, i.e., a coefficient of determination $R^2 > 0.99$ (see Fig. 4). Notably, the scaling of the mean polarization $\langle P \rangle$ and susceptibility $\chi_p$ for the fitted parameter sets is also degenerate, suggesting that the standard finite-size scaling analysis of second-order transitions may not apply. This indicates the possibility that the transition is not of second order.

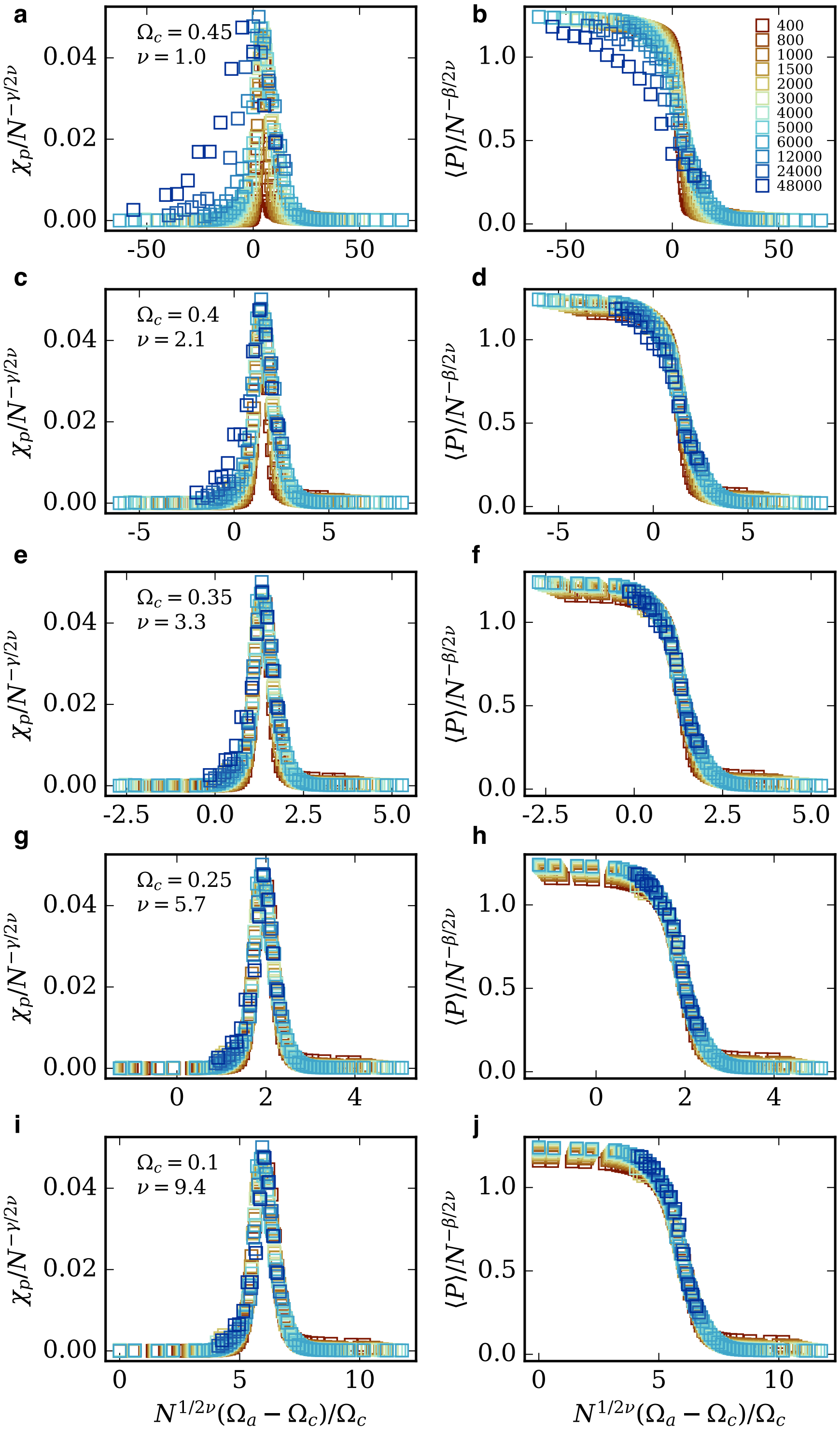


FIG. 5: (a,c,e,g,i) Scaled polarization and (b,d,f,h,j) susceptibility curves for different $\Omega_c$ and $\nu$ values. As in Fig. 4, the scaling is degenerate for $\nu > 3$ and $\Omega_c/K < 0.35$

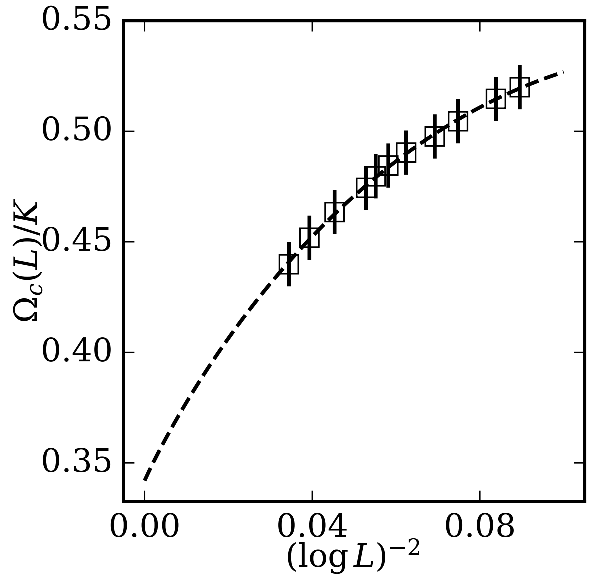


FIG. 6: Finite-size scaling of $\Omega_c$ with the assumption of a logarithmic growth for the correlation function. Here, the fit function has the form $\Omega_c(L)/K = \Omega_c/K + A/(\log L)^2 + B/(\log L)^3$. The best fit parameters are $\Omega_c/K = 0.34$, $A = 4.3$, and $B = 7.8$.

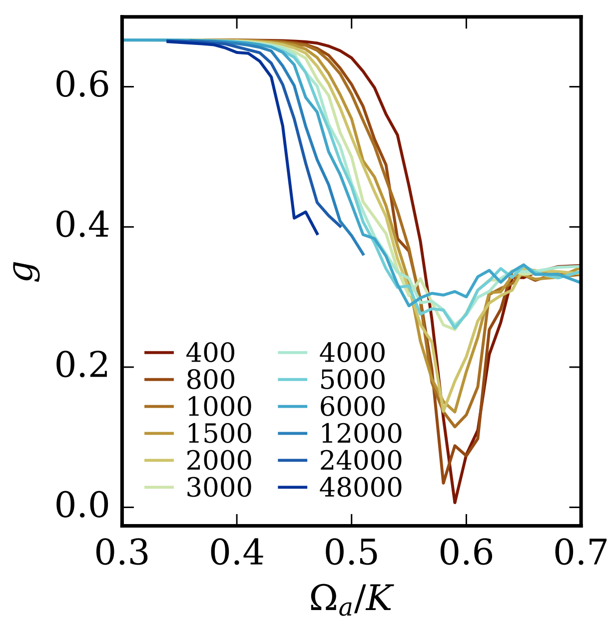


FIG. 7: Binder cumulant $g = 1 - \langle P^4 \rangle / 3\langle P^2 \rangle^2$ for different system sizes. No clear crossing develops, therefore $\Omega_c$ cannot be determined via this analysis.

## V. MOVIE CAPTIONS

*Movie M1–* Flock motion for $N = 400$ and $\Theta = 0.12$ ($K = 40$, $\eta = 0.5$). The flock has a large persistence length and shows ballistic motion. Circles show the position of the center of mass with time.

*Movie M2–* Flock motion for $N = 400$ and $\Theta = 0.74$ ($K = 40$, $\eta = 0.5$). The flock has a small persistence length and shows diffusive motion. Circles show the position of the center of mass with time.

*Movie M3–* Time evolution of the density map $\rho(\mathbf{r}, t)$ for $N = 6000$ and $\Theta = 0.94$ ($K = 80$, $\eta = 1$) exhibiting large shape and density fluctuations of the flock.

*Movie M4–* Time evolution of the density map $\rho(\mathbf{r}, t)$ for $N = 6000$ and $\Theta = 0.81$ ($K = 80$, $\eta = 1$) exhibiting a 'breathing mode', i.e. expansion and contraction of the flock.

*Movie M5–* Internal flock dynamics of a large system with $N = 48000$, and $\Theta = 0.89$ ($K = 80$, $\eta = 1$). Here, several vortices can be seen forming and breaking apart, signalling a possible connection to the BKT transition.