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Reaching a group-agreement in finite time under acyclic interaction topology

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Abstract This paper discusses the finite time agreement problem of networks with acyclic partition topology. In view of the structural characteristics of such network topology, mathematical induction is particularly suitable to prove the main conclusions in the paper. In addition, for the consideration of the finite time consensus problem, in addition to using basic matrix theory to verify the solution of the problem, this brief also has a more detailed analysis of the time required to reach consensus. Based on these two points, it is observed that the solution of this problem is due to the features of acyclic partition interactions and the continuity of the related finite time protocol and contributes to the research on the grouping consensus of multiagent system. Furthermore, simulation examples are presented to verify the theoretical results.

Keywords Finite time consensus · acyclic partition · group consensus

1 Introduction

Since the concept of distributed mobile robots was proposed in the late 1980s, research on multiagent systems has developed rapidly. And the increasing development of this field in many scientific communities, allow research on multiagent systems

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have become a multidisciplinary field with contributions from mathematics, computer science, communication technology, and sociology. These lead to great application prospects of multiagent systems in the military, aerospace, industrial manufacturing, natural exploration, disaster prevention and treatment, and service industries, which has inspired a large number of researchers to study multiagent systems [1–5]. As is shown from itself proceeding development, most of the aforementioned works are concerned with the dynamics of the systems [6, 7] and the design of the update laws [8, 9]. Such as researches both on consensus problem of multiagent systems with first-order or second-order dynamics [10, 11] and on the convergence analysis for consensus protocol of multiagent systems [12, 13] are still in topic . However, the internal relationship of the multiagent system is complicated , which means that a suitable construction of a topology structure is also meaningful to those studies [14–16] . For this consideration, group consensus has attracted the attention of many researchers because of its importance and suitability for practical problems.

It is well known that development of incomplete consensus promotes the increase in the group consensus in the networks. Therefore, a large number of theoretical and practical results have been obtained. Group consensus (or cluster consensus) focuses the network of agents evolving into several groups, where agents in the same group reach complete consensus even if the motion of each different group is not the same [17] . Indeed, group consensus is an extended consensus problem containing complete consensus [14]. From this point of view, group consensus may be more practical than usual complete consensus for social production and life. In fact, various approaches for group consensus [18, 19] have been under development . Significantly, the topic of acyclic partition topology, as one of the most practical group topology networks, has increasingly wide appeal. It is commonly assumed that the mutual communication between robots is a sufficient and necessary condition for multirobot system cooperation. In other words, by using tools from graph theory, the study of consensus of multibody systems in a network with dynamic systems can be reached if and only if the topology contains a spanning tree [20, 21] . Then such network of acyclic partition topology has desirable features of forward-to-back information transmission between the groups and of requiring trivial interaction of information within the group [14]. Moreover, recent researches of such topology networks demonstrate its practical characteristics, such as the discussions about the consensus problems under cooperative-competitive [22, 23] .

What is more, the elementary requirement for information transfer within the group of such topology structure requires that each group meet the requirement of the existence of a spanning tree . That is closely associated with the result of finite time agreement problem [24]. Naturally, an interesting question emerges of whether multiple agents under noncyclical division can eliminate the influence between groups and hence achieve the finite time consensus. Additionally, the solution of this problem means the solution of the agreement problem of a large number of agents or multiple target tasks in (or within) a finite time, because finite time protocols has the advantage of a high convergence rate [25]. In conclusion, both for its theoretical importance and for practical applications, it is desirable to explore the problem of the finite time consensus of multiple agents with the acyclic partition network. Motivated by the above discussion, this paper will prove that multiple agents with an acyclic

partition topology network will make a finite time agreement. In particular, the paper focuses on the combination of the acyclic partition topology networks and the finite time protocols and finally focuses on achieving group consensus in finite consensus. The corresponding noteworthy innovations are listed as follows:

1). The group consensus problem under the acyclic partition topology networks is more suitable for practical applications, and is more suitable for most of the coordination work in practice. Therefore, this article has explored innovative points in the field of the finite time group consensus problem.

2). Since this article deals with the problem of group consensus rather than complete consensus, whether the proposed limited time protocol is not affected by the group is a problem that have to be overcome. This paper successfully solves this problem with the geometric characteristics of the acyclic partition topology.

As shown in the text below, the main theorem of this paper uses the property of zero-sum influence between groups to extend the general complete agreement to the group agreement. In addition, the proof of the theorem only uses the basic knowledge of advanced mathematics, and this work will help readers to analyze the finite time t^*. Furthermore, the logic of the whole proof process reflects the cooperation-competition of the acyclic partition systems. The rest of this paper is organized as follows. The problem is proposed in **Section 2**, then the self- contained proof process for the main Theorem is given in **Section 3**, **Section 4** provides the simulations.

2 Preliminaries

Before formally narrating the main conclusions, some basic concepts and theoretical knowledge need to be understood in advance. The following content will give a brief introduction to the required basic knowledge.

2.1 Basics of Algebra

We take $\mathbb{R}$ and $\mathbb{R}_{\times}$ as the notation for the sets of the real numbers and the non-positive real numbers, respectively, and $\mathbb{R}^n$ denotes the $n-$dimensional real space. It is necessary to mention that $m \times n$ matrices $\mathscr{A} = [a_{ij}] \in \mathbb{R}^{m\times n}$ where each component a_{ij} belongs to the real number field $\mathbb{R}$ for $i, j = 1, 2, \ldots, n$. The vectors $\mu \in \mathbb{R}^{n\times 1}$ are given usually in the form of $(\mu_1, \mu_2, \ldots, \mu_n)^T$. Here, we point out that $1_n = (1, 1, \ldots, 1)^T$, the column vector in $\mathbb{R}^{n\times 1}$ whose entries are 1. To facilitate the subsequent discussion, in this article $\mathscr{D}_{\mu} = \mathscr{D}_{(\mu_1,\mu_2,\ldots,\mu_m)^T} = diag(\mu_1, \mu_2, \ldots, \mu_m)$ is an $n-$th order square matrix diagonalized by the sequence $(\mu_1, \mu_2, \ldots, \mu_m)$, and $\lambda_{\mathscr{A}}$ is a vector consisting of the eigenvalues of matrix $\mathscr{A}$, respectively. $\widetilde{\max}\{S\}$ and $\widetilde{\min}\{S\}$ are the maximum and the minimum nonzero numbers of the set S.

2.2 Graph Theory

For multiagent systems, there is a mathematical model described by a weighted graph $\mathscr{G} \equiv (\mathscr{V}, \mathscr{E})$ of order n, with the nodes set $\mathscr{V} = \{v_1, v_2, \ldots, v_n\}$ and the edges set $\mathscr{E}$

in $\mathscr{V}\times\mathscr{V}$. In this paper, the discussed object is the directed graph, so that e_{ij} in $\mathscr{E}$ if and only if there exists a directed link from j to i. For such a weighted graph $\mathscr{G}$, $\mathscr{A}=[a_{ij}]_{n\times n}$ is the adjacency matrix of $\mathscr{G}$ if and only if the cost of the edge from j to i is a_{ij}. Hence, notation $\mathscr{G}\equiv(\mathscr{V},\mathscr{E},\mathscr{A})$ is used for a weighted graph $\mathscr{G}$. Based on practical considerations, it is always assumed that the given graph does not include any self-loops, implying that $a_{ii}=0$. P_{ij} are the (directed) paths from v_j to v_i that can be given by a series of edges $e_{j_1j},e_{j_2j_1},\ldots,e_{ij_s}$ in $\mathscr{E}$ for some indexes $j_1,j_2,\ldots,j_s$. As the most important concept in the discussion of the networks topology, $\mathscr{G}$ has a spanning tree if and only if there exists at least one path P_{ij} for each j to i. Furthermore, the Laplacian matrix $\mathscr{L}(\mathscr{A})=[l_{ij}]_{n\times n}$ related to a matrix $\mathscr{A}$ (or to graph $\mathscr{G}$) is defined by $l_{ij}=\begin{cases}-a_{ij}, & i\neq j\\ \sum\limits_{s\neq i}a_{is}, & i=j\end{cases}$. Considering all of the above, there exists an natural mapping from the graph category to the algebra category such that the discussion of graph $\mathscr{G}\equiv(\mathscr{V},\mathscr{E},\mathscr{A})$ is transformed into a discussion of the Laplacian matrix $\mathscr{L}(\mathscr{A})$. It is important to point out here that $\mathscr{L}(\mathscr{A})\neq\begin{pmatrix}\mathscr{L}_{11} & \mathscr{L}_{12}\\ \mathscr{L}_{21} & \mathscr{L}_{22}\end{pmatrix}$ if the matrix $\mathscr{A}$ has a block form as $\begin{pmatrix}\mathscr{A}_{11} & \mathscr{A}_{12}\\ \mathscr{A}_{21} & \mathscr{A}_{22}\end{pmatrix}$, where $\mathscr{L}_{ij}=\mathscr{L}(\mathscr{A}_{ij})$.

2.3 Dynamics of System

Consider a multiagent system composed of n identical networked agents for the state of $x\in\mathbb{R}^{n\times 1}$ agents. The agent's dynamics are described by the following equation,

$$\dot{x}_i(t)=u_i(t), i=1,2\ldots,n, \tag{1}$$

where each $u_i(t)\in\mathbb{R}$ is the control input. The protocol is designed by

$$u_i(t)=f_i(x(t)), i=1,2,\ldots,n. \tag{2}$$

Definition 1 With the system dynamics (1), if $x(t)=0$ for any $t\geq t^*$ under the protocols (2), then the protocols (2) are said to be solved consensus problem in finite time t^* for the multiagent system (1).

As usual, let $y=(y_1,y_2,\ldots,y_n)^T$ be the state information (continuous) communication, $y_i\in\mathbb{R}^{n\times 1}$ for $i=1,2,\ldots,n$ is given by using a differential equation as

$$y_i=\sum_{j=1}^{n}a_{ij}(x_j-x_i). \tag{3}$$

It is well known for differential equations that the direction of the gradient is more useful than its size [25]. Therefore, even if only keep the sign of the gradient to update the information, convergence can be obtained. In some cases, the use of function *sign* can reduce the noise of the gradient and obtain higher convergence rates. Thus, the articles [26, 27] aimed to introduce the finite (discrete) time protocols by applying

function *sign* on the model of information exchange. Oscillations will be obtained when the sign function replaces a part of the state interactions to update the whole system state. In other words, the advantage of this approach is that regardless of whether the initial value is the most appropriate, the signum function *sign* can help the control protocol jump out of the local state to achieve a better convergent state. Prior to specifying the protocols, it is necessary to give some notation, namely that the real function $\alpha \in \mathbb{R}$,sig^{α} is defined by $sig^{\alpha}(y_i) = \text{sign}(y_i)|y_i|^{\alpha}$ and $sig^{\alpha}(y)$ means that the function sig^{α} is applied on each component of y if y is a vector. Now, the protocols are given by modifying the function $y(t)$ of status updates,

$$u_i(t) = \beta sig^{\alpha}(y_i) + \gamma y_i, \tag{4}$$

where $0 < \alpha < 1$,$0 < \beta$,$0 < \gamma$.

In fact, β can reach 0, but then the case becomes generally linear. Additionally, the cases of $\alpha = 0$ and $\alpha > 1$ when $\gamma = 0$ that are in the scope of finite time problem in discontinuous case [28] and fixed-time consensus problem [29] should be mentioned.

Remark 1 The nonlinear consensus protocols (4) proposed in [24] to allow the related multiagent systems reach finite time consensus. Such protocols that have a continuous state feedback bridged the asymptotical consensus protocols and discontinuous finite-time consensus protocols. The fact that protocols (4) are continuous allows the studies for the finite consensus problems to be carried out using the continuity theory of function. These theoretical works provided a new method for exploring the finite time agreement problems of the multiagent system and had guided the work in this article to some extent.

3 Main Results

After preparing the above basic work, this section needs to make the following three assumptions for the establishment of the main theorem firstly.

Assumption 1 *Graph $\mathcal{G}$ has a acyclic partition $\{\mathcal{V}_1, \mathcal{V}_1, \ldots, \mathcal{V}_k\}$.*

For a further description, let the $i-$th node set $\mathcal{V}_i = \{v_{p_{i-1}+1}, v_{p_{i-1}+2}, \ldots, v_{p_{i-1}+p_i}\}$, where p_i is the cardinality of $\mathcal{V}_i$ and p_0 is given the value 0 to prevent ambiguity. Then, the relative Laplacian matrix will be in the form of

$$\mathcal{L} = \begin{pmatrix} \mathcal{L}_{11} & 0 & \cdots & 0 \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{L}_{k1} & \mathcal{L}_{k2} & \cdots & \mathcal{L}_{kk} \end{pmatrix}_{n \times n}, \tag{5}$$

where $\mathcal{L}_{ij} \in \mathbb{R}^{p_i \times p_j}$ is the Laplacian matrix associated with the information exchange between group $\mathcal{G}_i$ and group $\mathcal{G}_j$. Clearly, $\mathcal{L}_{ii} \in \mathbb{R}^{p_i \times p_i}, i = 1, 2, \ldots, k$ are Laplacian matrices corresponding to $\mathcal{G}_i$ as an independent group.

Assumption 2 *The sum of each row of $\mathcal{L}_{ij}$ is zero.*

This assumption is often used in the most of the reports in many literatures [22, 23], which means that the effects or compacts from $\mathscr{G}_j$ on $\mathscr{G}_i$ are intuitive, where $j < i$.

Assumption 3 *Each $\mathscr{G}_i$ has a spanning tree.*

Lemma 1 *Under the Assumptions (1) and (2), Assumption (3) holds if and only if the Laplacian matrix $\mathscr{L}$ has zero as its eigenvalue whose algebraic and geometric multiplicity are both k (i.e., 0 is a k−multiply simple eigenvalue of $\mathscr{L}$) and the other eigenvalues have positive real parts [14] . Moreover, $\mathscr{L}$ has $(\rho_1^{(i)}, \ldots, \rho_{i-1}^{(i)}, \mu_i, \mathbf{0}_i)$ as its left eigenvectors corresponding to 0, where*

$$\begin{cases} \mathbf{0}_i \text{ is the vector of dimension } (n - \sum_{j=1}^{i} p_j), i = 1,2,\ldots,k; \\ \mu_i = (\mu_{i1}, \ldots, \mu_{ik_i})^T \in \mathbb{R}_{\times}^{p_i \times 1} \text{ satisfies } \sum_{l=1}^{p_i} \mu_{il} = 1, i = 1,2,\ldots,k; \\ \rho_j^{(i)} = (\rho_{j1}^{(i)}, \ldots, \rho_{jp_j}^{(i)})^T \in \mathbb{R}^{p_j \times 1}, j = 1,2,\ldots,i-1 \\ \qquad\qquad \text{satisfies } \sum_{l=1}^{p_j} \rho_{jl}^{(i)} = 0, i = 2,3,\ldots,k. \end{cases} \tag{6}$$

Lemma 2 *[30] For the real numbers $r_i \geq 0, i = 1,2,\ldots,n$,if $0 < \rho \leq 1$, one has*

$$(\sum_{i=1}^{n} r_i)^{\rho} \leq \sum_{i=1}^{n} r_i^{\rho} \leq n^{1-\rho} (\sum_{i=1}^{n} r_i)^{\rho}. \tag{7}$$

.

Theorem 1 *With the protocol (4), the multiagent system (1) with the acyclic-partition topology $\mathscr{G}$ reaches the finite-time agreement under the* ASSUMPTIONS *(2) and (3) . More precisely, this multiagent system reaches the finite-time agreement within the time $t^* = \frac{1+\alpha}{1-\alpha} \sum_{i=1}^{k} \frac{V_i^{\frac{1-\alpha}{1+\alpha}}(0)}{\kappa_i}$, where each $\kappa_i > 0$ is time-invariant and is related to influence within and between groups.*

Proof Note that $\mathscr{G}$ is a graph with acyclic partition $\{\mathscr{V}_1, \mathscr{V}_1, \ldots, \mathscr{V}_k\}$, the fact that it is "acyclic" means that information can only flow in the $\mathscr{V}_i$-to-$\mathscr{V}_j$ $(j > i)$ direction and indicates that the theorem will be verified by induction on k, the number of clusters of $\mathscr{G}$.

Now for simplicity, denote that

$$\begin{cases} x_{(\bar{i})} = (x_{n_{i-1}+1}, x_{n_{i-1}+2}, \ldots, x_{n_{i-1}+n_i})^T \\ y_{(\bar{i})} = (y_{n_{i-1}+1}, y_{n_{i-1}+2}, \ldots, y_{n_{i-1}+n_i})^T \\ u_{(\bar{i})} = (u_{n_{i-1}+1}, u_{n_{i-1}+3}, \ldots, u_{n_{i-1}+n_i})^T \end{cases}, \tag{8}$$

for the acyclic interaction topology for $i = 1,2,\ldots,k$,there gives $n_0 = 0$.

I. $\mathscr{G}_1$ has a directed spanning tree from ASSUMPTION 3 means that the first group system solves the finite time agreement problem in finite time $t_1 = \frac{(1+\alpha)V_1^{\frac{1-\alpha}{1+\alpha}}(0)}{(1-\alpha)\kappa_1}$,

where κ_1 is a positive constant and the Lyapunov function is taken as $V_1(x(t)) = y_{(\bar{1})}^T \mathscr{D}_{\mu_1}\left(\frac{\beta}{1+\alpha} sig^{\alpha}(y_{(\bar{1})}) + \frac{\gamma}{1} y_{(\bar{1})}\right)$, μ_1 the left eigenvectors of $\mathscr{L}_{11}$ of eigenvalue 0 .

II. The next step will show the correctness of the illustration that the states of the agents in $\mathscr{V}_2$ reach consensus in finite time under the protocol (4), which is the key to the whole proof process.

Under the consideration that the corresponding agents only receive state information from the agents in $\mathscr{G}_1$(*the "front" group*), it is sufficient to explore the block matrix

$$\begin{pmatrix} \mathscr{L}_{11} & 0 \\ \mathscr{L}_{21} & \mathscr{L}_{22} \end{pmatrix}, \tag{9}$$

and the the corresponding neighborhood matrix is $\begin{pmatrix} \mathscr{A}_{11} & 0 \\ \mathscr{A}_{21} & \mathscr{A}_{22} \end{pmatrix}$.

Then, their state transformations have a form of the sum of two parts

$$\begin{aligned} y_{n1+i} = &\sum_{j=1}^{n_1} a_{n_1+i,j}(x_j - x_{n_1+i}) \\ &+ \sum_{j=1}^{n_2} a_{n_1+i,n_1+j}(x_{n_1+j} - x_{n_1+i}), \end{aligned} \tag{10}$$

where the first part describes the independent state communication of its own group, and the second part describes the influence of the previous group on the agent.

Its "partition" also can be illustrated in compact form as

$$y_{(\bar{2})} = -\Big(\mathscr{L}(\mathscr{A}_{21}) \mid \mathscr{L}(\mathscr{A}_{22})\Big)\, x. \tag{11}$$

Differentiating with respect to time t, we obtain that

$$\dot{y}_{(\bar{2})} = -(\mathscr{L}(\mathscr{A}_{22}) + \mathscr{D}_{21}) u_{(\bar{2})} - \mathscr{A}_{21} u_{(\bar{1})}. \tag{12}$$

By the analysis in the part **I**, the states of agents in the first group have already reached agreement and then $\dot{x}_{(\bar{1})} = 0$ as time evolves.

Accompanied by the results $\mathscr{D}_{21} = 0_{n_2 \times n_1}$ of the condition in ASSUMPTION (2), LEMMA *1* implies that matrix (9) has $\mathbf{1}_n$ and $(\rho_{(\bar{2})}, \mu_2)$ as the right and left eigenvectors, respectively, corresponding to eigenvalue 0.

Its final form is obtained as

$$\dot{y}_{(\bar{2})} = -\mathscr{L}(\mathscr{A}_{22})\, u_{(\bar{2})} = -\mathscr{L}_{22}\, u_{(\bar{2})}. \tag{13}$$

Now consider the Lyapunov candidate

$$\begin{aligned} V_2(x(t)) =& y_{(\bar{2})}^T \mathscr{D}_{\mu_2}\left(\frac{\beta}{1+\alpha} sig^{\alpha}(y_{(\bar{2})}) + \frac{\gamma}{2} y_{(\bar{2})}\right) \\ =& \sum_{j=1}^{n_2} \mu_{2j}\left(\frac{\beta}{1+\alpha} |y_{n_1+j}|^{1+\alpha} + \frac{\gamma}{2} y_{n_1+j}^2\right) \\ \geq& 0, \end{aligned} \tag{14}$$

and

$$\dot{V}_2(t) = \sum_{j=1}^{n_2} \mu_{2j} \Big(\beta sig^{\alpha}(y_{n_1+j}) + \gamma y_{n_1+j} \Big) \dot{y}_{n_1+j}. \tag{15}$$

By substituting equation (13) into (15), we obtain the compact form as

$$\dot{V}_2(t) = -{u_{(\bar{2})}}^T \mathscr{D}_{\mu_2} \mathscr{L}_{22} u_{(\bar{2})} \leq 0. \tag{16}$$

Moreover, because the influence of $\mathscr{V}_1$ on the each agent in $\mathscr{V}_2$ can be eliminated, we obtain

$$\max \lambda_{\mathscr{D}_{\mu_2}\mathscr{L}_{22}} {u_{(\bar{2})}}^T u_{(\bar{2})} \geq -\dot{V}_2(t) \geq \widetilde{\min} \lambda_{\mathscr{D}_{\mu_2}\mathscr{L}_{22}} {u_{(\bar{2})}}^T u_{(\bar{2})}. \tag{17}$$

And

$${u_{(\bar{2})}}^T u_{(\bar{2})} = \sum_{j=1}^{n_2} \Big(\beta |y_{n_1+j}|^{\alpha} + \gamma |y_{n_1+j}| \Big)^2 \tag{18}$$

With the initial state x_0, $V_2(t) \neq 0$ should be assumed, and then the following inequalities follow from LEMMMA (2)

$$\begin{aligned} V_2^{\frac{2\alpha}{\alpha+1}}(t) &= \Big(\sum_{j=1}^{n_2} \mu_{2j} \Big(\frac{\beta}{1+\alpha} |y_{n_1+j}|^{1+\alpha} + \frac{\gamma}{2} y_{n_1+j}^2 \Big) \Big)^{\frac{2\alpha}{\alpha+1}} \\ &\leq \sum_{j=1}^{n_2} \mu_{2j}^{\frac{2\alpha}{\alpha+1}} \Big(\frac{\beta}{1+\alpha} |y_{n_1+j}|^{1+\alpha} + \frac{\gamma}{2} y_{n_1+j}^2 \Big)^{\frac{2\alpha}{\alpha+1}} \\ &\leq \sum_{j=1}^{n_2} \mu_{2j}^{\frac{2\alpha}{\alpha+1}} \Big((\frac{\beta}{1+\alpha})^{\frac{2\alpha}{\alpha+1}} |y_{n_1+j}|^{2\alpha} + (\frac{\gamma}{2})^{\frac{2\alpha}{\alpha+1}} |y_{n_1+j}|^{\frac{4\alpha}{\alpha+1}} \Big). \end{aligned} \tag{19}$$

Note that

$$\sum_{j=1}^{n_2} \mu_{2j}^{\frac{2\alpha}{\alpha+1}} \Big((\frac{\beta}{1+\alpha})^{\frac{2\alpha}{\alpha+1}} |y_{n_1+j}|^{2\alpha} + (\frac{\gamma}{2})^{\frac{2\alpha}{\alpha+1}} |y_{n_1+j}|^{\frac{4\alpha}{\alpha+1}} \Big) > 0. \tag{20}$$

If not, $y_{(\bar{2})} = 0$ means that $x \in (0_{n_1}, 1_{n_2})^T \cdot \mathbb{R}$, leading to the conclusion that $u = 0$. All of the above derivations present a trivial case.

We further observe that

$$\begin{aligned} 0 < & y_{(\bar{2})}^T y_{(\bar{2})} = x^T \Big(\mathscr{L}(\mathscr{A}_{21}) \dashv \mathscr{L}_{22} \Big)^T \Big(\mathscr{L}(\mathscr{A}_{21}) \dashv \mathscr{L}_{22} \Big) x \\ & \leq \max_{z \in \mathscr{S}^n} (z^T (\mathscr{L}(\mathscr{A}_{21}) \dashv \mathscr{L}_{22})^T (\mathscr{L}(\mathscr{A}_{21}) \dashv \mathscr{L}_{22}) z) x_0^T x_0 \triangleq \delta, \end{aligned} \tag{21}$$

where $\mathscr{S}^n$ is the unit sphere.

These inequalities (17)-(21) imply that there exists a positive constant κ_2 such that

$$\frac{\dot{V}_2(t)}{V_2^{\frac{2\alpha}{\alpha+1}}(t)} \leq -\kappa_2. \tag{22}$$

This argument will be verified by the following two parts **i** and **ii**.

i. Consider the case that $1 \leq y_{(\bar{2})}^T y_{(\bar{2})} \leq \delta$. Since $\{z : 1 \leq z^T z \leq \delta\}$ is compact, then $\overline{\kappa_2} \triangleq \min\limits_{1\leq z^T z\leq \delta} \left\{ \frac{\left(\beta sig^{\alpha}(z)+\gamma z\right)^T \mathscr{D}_{\mu_2} \mathscr{L}_{22} \left(\beta sig^{\alpha}(z)+\gamma z\right)}{\left(z^T \mathscr{D}_{\mu_2}\left(\frac{\beta}{1+\alpha} sig^{\alpha}(z)+\frac{\gamma}{2} z\right)\right)^{\frac{2\alpha}{\alpha+1}}} \right\}$ exists and is larger than zero. Therefore,

$$\frac{\dot{V}_2(t)}{V_2^{\frac{2\alpha}{\alpha+1}}(t)} \leq -\overline{\kappa_2}. \tag{23}$$

ii. For the other case, that is, $0 < y_{(\bar{2})}^T y_{(\bar{2})} < 1$,the following inequality follows,

$$V_2^{\frac{2\alpha}{\alpha+1}}(t) \leq (\max\{\mu_2\})^{\frac{2\alpha}{\alpha+1}} \left[\left(\frac{\beta}{1+\alpha}\right)^{\frac{2\alpha}{\alpha+1}} + \left(\frac{\gamma}{2}\right)^{\frac{2\alpha}{\alpha+1}} \right] \sum_{j=1}^{n_2} |y_{n_1+j}|^{2\alpha}, \tag{24}$$

And then consider combining formula (18)-(21) with (23) to obtain the next formula

$$-\frac{\dot{V}_2(t)}{V_2^{\frac{2\alpha}{\alpha+1}}(t)} \geq \frac{\widetilde{\min}\lambda_{\mathscr{D}_{\mu_2}\cdot\mathscr{L}_{22}} \cdot (\beta+\gamma)}{(\max \mu_2)^{\frac{2\alpha}{\alpha+1}} \cdot \left(\left(\frac{\beta}{1+\alpha}\right)^{\frac{2\alpha}{\alpha+1}} + \left(\frac{\gamma}{2}\right)^{\frac{2\alpha}{\alpha+1}} \right)} \triangleq \underline{\kappa_2} \geq 0. \tag{25}$$

From formulas (24) and (25), there exists $\kappa_2 = \max\{\overline{\kappa_2}, \underline{\kappa_2}\}$ to let inequality (22) hold.

Based on the *Differential Comparison Principle*, $V_2(t)$ will reach 0 in finite time $t_2 = \frac{(1+\alpha)V_2^{\frac{1-\alpha}{1+\alpha}}(0)}{(1-\alpha)\kappa_2}$. Hence the system consisting of $\mathscr{V}_1$ and $\mathscr{V}_2$ reaches consensus in finite time $t^* = \frac{1+\alpha}{1-\alpha} \frac{\kappa_2 V_1^{\frac{1-\alpha}{1+\alpha}}(0) + \kappa_1 V_2^{\frac{1-\alpha}{1+\alpha}}(0)}{\kappa_1 \kappa_2}$.

III. In the last step, suppose that the state of agents corresponding to the vertex $\mathscr{V}_1, ..., \mathscr{V}_2$ have already reached an agreement in the time $t^* = \frac{1+\alpha}{1-\alpha} \sum\limits_{i=1}^{k-1} \frac{V_i^{\frac{1-\alpha}{1+\alpha}}(0)}{\kappa_i}$, where each constant $\kappa_i > 0$ is described in the second step.

Under the induction assumption, for the group $\mathscr{V}_k$, these previous groups $\mathscr{V}_1, ..., \mathscr{V}_2$ can be viewed as a group as a whole. The corresponding block Lagrange matrix will be $\mathscr{L} = \begin{pmatrix} \widetilde{\mathscr{L}_{11}} & 0 \\ \widetilde{\mathscr{L}_{21}} & \mathscr{L}_{kk} \end{pmatrix}$. Consequently, the finite time consensus problem of the last n_k agents can be solved by applying the results of part **II**, and it will take time $t_k = \frac{(1+\alpha)V_k^{\frac{1-\alpha}{1+\alpha}}(0)}{(1-\alpha)\kappa_k}$ after the others are in state consensus.

Finally, all of the above illustrations imply that the system (1) reaches a finite time agreement in $t^* = \frac{1+\alpha}{1-\alpha} \sum\limits_{i=1}^{k} \frac{V_i^{\frac{1-\alpha}{1+\alpha}}(0)}{\kappa_i}$.

By mathematical induction, under the protocol (4), the typical acyclic interaction topology $\mathscr{G}$ studied in this paper endows multiagent system (1) with a finite time convergence.

Remark 2 As is given in the proof process, the formula (12) $\dot{y}_{(\bar{2})} = -(\mathscr{L}(\mathscr{A}_{22}) + \mathscr{D}_{21})u_{(\bar{2})} - \mathscr{A}_{21}u_{(\bar{1})}$ indeed describes the dynamics of the state information of the

multiagent system. For each agent-i, in the detail, its information interactions depends on three parts, information exchange in its own group $-\mathscr{L}(\mathscr{A}_{22})u_{(\bar{2})}$, information exchange with the previous group $-\mathscr{D}_{21}u_{(\bar{2})}$ and the dynamics of the previous group $-\mathscr{A}_{21}u_{(\bar{1})}$. The features of such typical acyclic interaction topology give the sufficiency conditions to eliminate the impact from the previous groups on the later groups, which is definitely advantageous.

Remark 3 Different from the condition $\mu_i \perp y_{(\bar{i})}$ in [10], it is only satisfied that $\mu_i \perp \dot{y}_{(\bar{i})}$, so that the upper and lower bounds of the Lyapunov function $\dot{V}_i\big(x(t)\big)$ will not be available within the orthogonal space of $y_{(\bar{i})}$. However, this topology solves the finite time consensus time problem while giving rise to the above problem. Based on this problem, this paper obtains the upper and lower bounds of $\dot{V}_i\big(x(t)\big)$ in the fundamental algebraic approach, due to the feature of the non-negativity of the left eigenvector. Furthermore, using this approach the finite time t^* can be obtained more accurately. In other words, it is not necessary to provided an interaction matrix strictly for group consensus.

Remark 4 In geometric sense, the steps from (17) to (21) in this proof process reflect the difference between this article and the previous article [10]. Specifically, the geometric properties of acyclic partition to ensure the establishment of inequality (17) and to make the finite time control protocol effective.

Remark 5 From the discussion in step **II** , the convergent time t^* of each sub-system is given exactly. Moreover, in addition to the parameter and the initial value of the state, the value of finite time t^* is also related to the design of the network topology.

$$\max\Bigg\{\min_{1\le z^T z\le\delta}\frac{\big(\beta sig^{\alpha}(z)+\gamma z\big)^T\mathscr{D}_{\mu_2}\mathscr{L}_{22}\big(\beta sig^{\alpha}(z)+\gamma z\big)}{\big(z^T\mathscr{D}_{\mu_2}\big(\frac{\beta}{1+\alpha}sig^{\alpha}(z)+\frac{\gamma}{2}z\big)\big)^{\frac{2\alpha}{\alpha+1}}},\ \frac{(\beta+\gamma)}{\big((\frac{\beta}{1+\alpha})^{\frac{2\alpha}{\alpha+1}}+(\frac{\gamma}{2})^{\frac{2\alpha}{\alpha+1}}\big)}\frac{\widetilde{\min}\lambda_{\mathscr{D}_{\mu_2}\cdot\mathscr{L}_{ii}}}{(\max\mu_2)^{\frac{2\alpha}{\alpha+1}}}\Bigg\}. \tag{26}$$

Based on this result, the multirobot information interaction design can be more targeted to meet practical needs. The form of t^* shows that smaller α can lead to a higher convergence rate when the agents' states differ slightly from each other, and larger α can lead to a higher convergence rate when agents states differ strongly from each other. Furthermore, by some straightforward simulations, it will be derived that the system under signum protocols (4) converges faster than linearly.

4 Simulations

This section first consider solving finite time consensus problem for graphs $\mathscr{G}$ shown in Fig. 1, and then presents three simulations under the three network topologies, Fig. 2 (acyclic partition), Fig. 3 (balanced couple) and Fig. 4 (balanced couple) respectively.

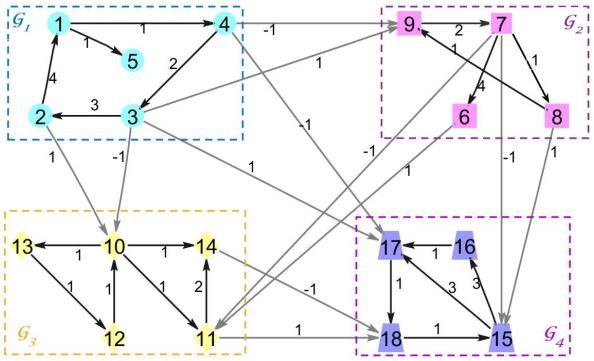


Fig. 1 The topology network with an acyclic partition of four groups

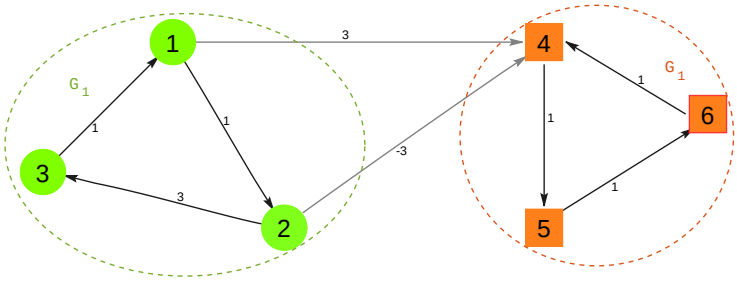


Fig. 2 The topology network with an acyclic partition of two groups

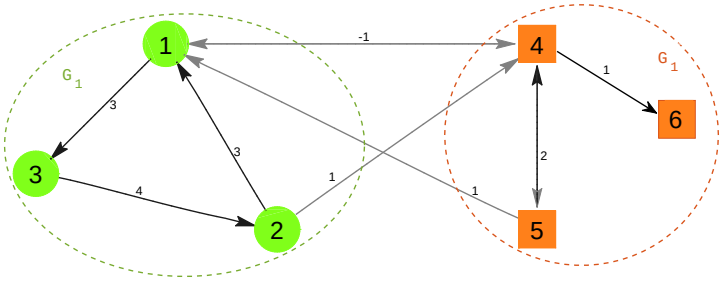


Fig. 3 The first balanced couple topology network of two groups

Figure 1 presents a certain acyclic partition of 18 agents with 4 groups $\mathcal{G}_1$ (green agents $\{v_1, v_2, v_3, v_4, v_5\}$), $\mathcal{G}_2$ (pink agents $\{v_6, v_7, v_8, v_9\}$), $\mathcal{G}_3$ (yellow agents $\{v_{10}, v_{11}, v_{12}, v_{13}, v_{14}\}$) and $\mathcal{G}_4$ (purple agents$\{v_{15}, v_{16}, v_{17}, v_{18}\}$). That clearly describes the interaction communication among the groups, that is, the first group $\mathcal{G}_1$ only sends state information to groups $\mathcal{G}_2$, $\mathcal{G}_3$ and $\mathcal{G}_4$, and the group $\mathcal{G}_2$ just receives information from the group $\mathcal{G}_1$ and sends information to other groups $\mathcal{G}_3$ and $\mathcal{G}_4$, etc. Figure 2 presents another certain acyclic partition of 6 agents with 2 groups $\mathcal{G}_1, \mathcal{G}_2$. In addition, this paper also gives attention to another group topology networks–balanced couple group topology, where Fig. 3 and Fig. 4 represent two cases.

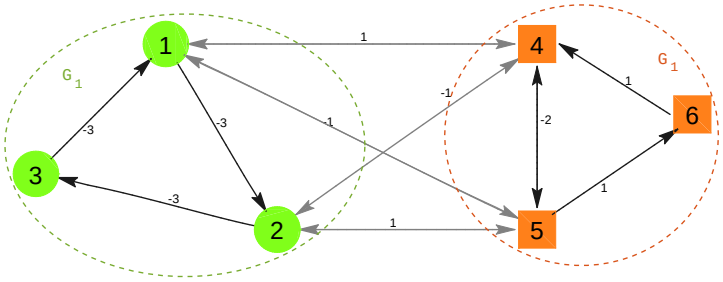


Fig. 4 The second balanced couple topology network of two groups

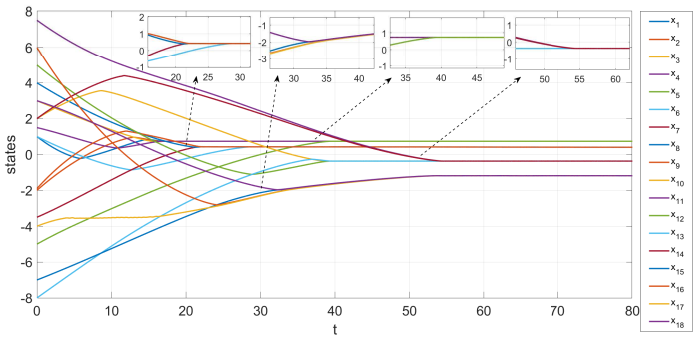


Fig. 5 The simulation result when the value of parameter α is 0.25 under the topology Fig. 1

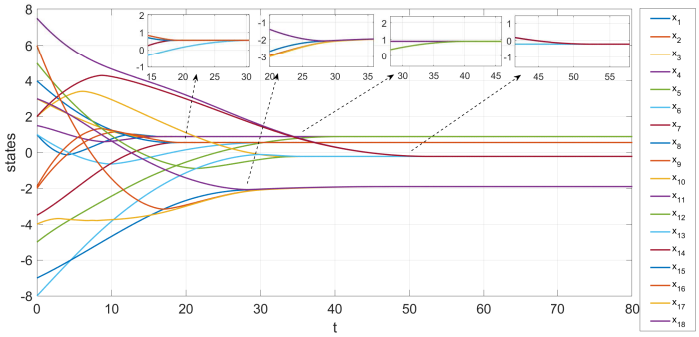


Fig. 6 The simulation result when the value of parameter α is 0.5 under the topology Fig. 1

As a consequence of the main result in this paper, Fig. 5, Fig. 6 and Fig. 7 show the simulation results under the consensus protocol (4) for a network with fixed topology on graph $\mathscr{G}$, where the initial states is $x(0) = (11,-2,3,1.5,-15,11,-3.5,4,-10.9,-21,17.5,6,-18,22,-27,36,-45,35)$ and α takes three different parameters 0.25, 0.50 and 0.75. It can be also easily seen that as the value of the parameter a becomes smaller, the convergence speed of this system becomes faster.

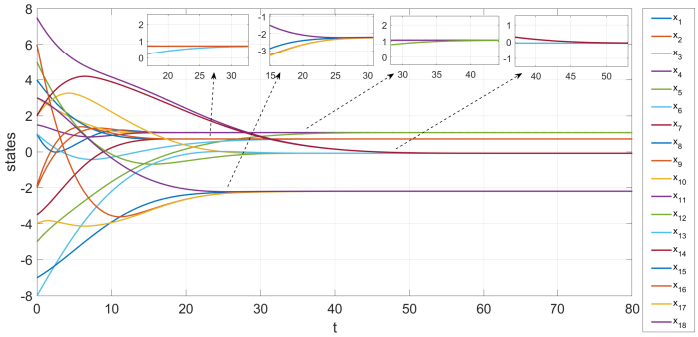


Fig. 7 The simulation result when the value of parameter α is 0.75 under the topology Fig. 1

The above simulation results also show that the finite time protocol can be well applied in the multi-agent system with acyclic partition. So can this type of finite time protocol also be used in other typical problems of group consensus? Based on this, this article immediately gives the simulation results as shown in Fig. 8, Fig. 9, and Fig. 10.

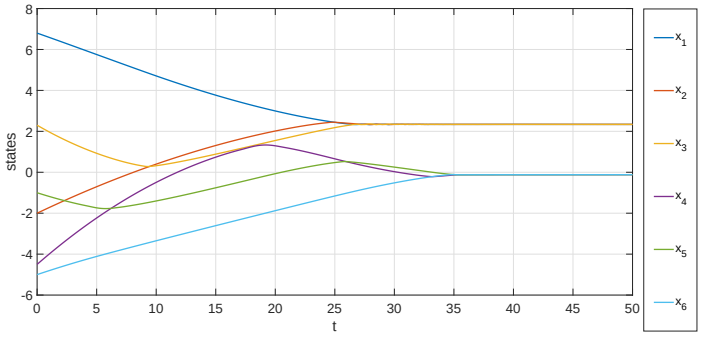


Fig. 8 The simulation result corresponding to topology Fig. 2

The topology corresponding to these three results are shown in Fig. 2, Fig. 3, and Fig. 4 with the initial state values are all $x(0) = (6.8, -2, 2.3, -4.5, -1, -5)$, where the value of α is 0.25. This result not only answers the question just raised, but also shows that the networks with acyclic partition topology is a sufficient and unnecessary condition for the agreement in finite time .

5 Conclusions

The early work guaranteed that the agents of each group reach the agreement in finite time. However for this paper, the fact that acyclic partition topology networks eliminate the integral influence among the groups under the finite time protocols was the

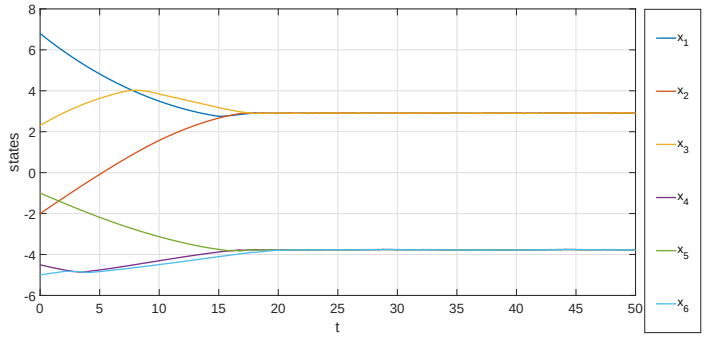


Fig. 9 The simulation result corresponding to topology Fig. 3

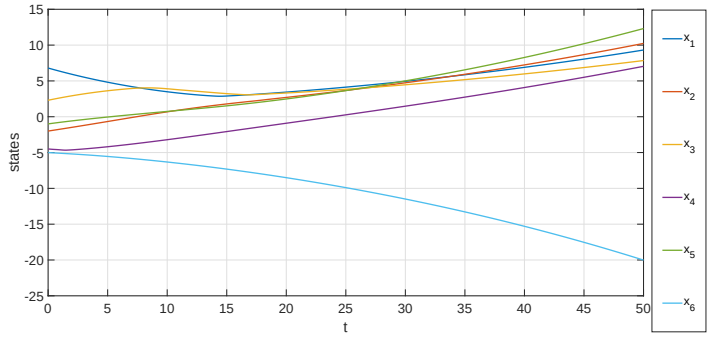


Fig. 10 The simulation result corresponding to topology Fig. 4

key for allowing the agent-systems to achieve consensus in finite time. As an additional conclusion, the finite time t^* was further analyzed and considered. Therefore, future research work will continue to consider the problem of cluster consensus in fixed time.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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