APPENDIX II

The detailed step by step derivation of an input state evolution according to the ports notations shown on Fig.2.:

The source is a single coherent input in port 1 representing a normalized vector state:

$$|\psi_{source}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}\widehat{D}_{1,v}(\alpha_{1,v})\\ 0\\ \widehat{D}_{1,h}(\alpha_{1,h})\\ 0\end{pmatrix}|0\rangle \quad (13)$$

with $|0\rangle$ being a vacuum state, $\widehat{D}_{1,v}(\alpha_{1,v})$ and $\widehat{D}_{1,h}(\alpha_{1,h})$ being displacement operators [10] in spatial mode 1 in vertical and horizontal polarization modes accordingly. The action of PBS is represented by the matrix (14) which is CNOT gate in quantum computing [5]:

$$B_p = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix} \quad (14)$$

hence, yielding:

$$|\psi_{PBS}\rangle \to B_p|\psi_{source}\rangle \to \frac{1}{\sqrt{2}}\begin{pmatrix}\widehat{D}_{2,v}(\alpha_{2,v})\\ 0\\ 0\\ \widehat{D}_{3,h}(\alpha_{3,h})\end{pmatrix}|0\rangle \quad (15)$$

where $\widehat{D}_{2,v}(\alpha_{2,v})$ and $\widehat{D}_{3,h}(\alpha_{3,h})$ displacement operators in vertical and horizontal polarization modes shifted to spatial modes 2 and 3 accordingly. The phase modulation induced by the AOM is occurring in mode 3 and it is convenient to continue treating the diffracted beam which we selected for the recombination as spatial mode 3 as well. We derived the mode shifting for the diffracted beam using the function of phase modulation operator provided in [9], which adapted to our terms takes the following form:

$$\hat{S}_{PM} = \exp\left(\sum_1^\infty \hat{a}^\dagger_{\omega+\Omega} J_1(\omega)\hat{a}_\omega\right) \quad (16)$$

where: $\hat{a}_\omega$ is annihilation operator in the initial frequency mode of the source, $\hat{a}^\dagger_{\omega+\Omega}$ is creation operator in the AOM upshifted frequency mode and $J_1(\omega)$ is the first order Bessel function coefficient. It's worth adding a remark that expression of $\hat{S}_{PM}$ in the original paper contained a single-photon scattering coefficient instead of $J_1(\omega)$, however for a coherent state the expectation value of scattering coefficients summed over the infinity would plausibly result in $J_1(\omega)$. To show an action of the operator $\hat{S}_{PM}$ we represent the displacement operator $\widehat{D}_{3,h}(\alpha_{3,h})$ explicitly in a single frequency mode ω: $\widehat{D}_{3,h,\omega}(\alpha_{3,h,\omega})$. Now we apply $\hat{S}_{PM}$ transformation suggested in [9] upon the displacement operator which shifts it to frequency mode $\omega+\Omega$:

$$\hat{S}_{PM}\widehat{D}_{3,h,\omega}(\alpha_{3,h,\omega})\hat{S}^\dagger_{PM} = J_1(\omega)\widehat{D}_{3,h,\omega+\Omega}(\alpha_{3,h,\omega+\Omega}) \quad (17)$$

Using the equation (17) we can represent the state evolution as follows:

$$|\psi_{AOM}\rangle \to \frac{1}{\sqrt{2}}\begin{pmatrix}\widehat{D}_{2,v,\omega}(\alpha_{2,v,\omega})\\ \hat{S}_{PM}\hat{a}_{3,v,\omega}\hat{S}^\dagger_{PM}\\ \hat{a}_{2,h,\omega}\\ \hat{S}_{PM}\widehat{D}_{3,h,\omega}(\alpha_{3,h,\omega})\hat{S}^\dagger_{PM}\end{pmatrix}|0\rangle \to$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix}\widehat{D}_{2,v,\omega}(\alpha_{2,v,\omega})\\ 0\\ 0\\ J_1(\omega)\widehat{D}_{3,h,\omega+\Omega}(\alpha_{3,h,\omega+\Omega})\end{pmatrix}|0\rangle \quad (18)$$

where: $\widehat{D}_{2,v,\omega}(\alpha_{2,v,\omega})$ is the displacement operator in vertical mode 2 and frequency ω; $\hat{a}_{3,v,\omega}$ and $\hat{a}_{2,h,\omega}$ are annihilation operators in vertical and horizontal modes 3 and 2 and frequency ω accordingly. Adhering to formalities we have indicated the action of $\hat{S}_{PM}$ on the operator $\hat{a}_{3,v,\omega}$ as phase modulation applies to all operators in spatial mode 3. The resulting state $|\psi_{AOM}\rangle$ acquires different frequency modes and it is not normalized any longer due to $J_1(\omega)$ coefficient. Now, as discussed in previous chapter in the pre-measurement sequence step 2 by adjusting the HWP1 we equalize amplitudes for modes 2 and 3 and normalize the state accordingly obtaining:

$$|\psi'_{AOM}\rangle \to \frac{1}{\sqrt{2}}\begin{pmatrix}\widehat{D}_{2,v,\omega}(\alpha_{2,v,\omega})\\ 0\\ 0\\ \widehat{D}_{3,h,\omega+\Omega}(\alpha_{3,h,\omega+\Omega})\end{pmatrix}|0\rangle \quad (19)$$

The normalization is hence accounting for keeping the Bessel function coefficient $J_1(\omega)$ implicit throughout the derivation. It will also become apparent now why formalities with mode shifting were given in details when we consider the state in Schrödinger's picture, i.e. state $|\psi'_{AOM}\rangle$ evolving in time τ [11]:

$$|\psi'_{AOM}(\tau)\rangle \to \frac{1}{\sqrt{2}}\begin{pmatrix}\widehat{D}_{2,v,\omega}\left(\alpha_{2,v,\omega}(\tau)\right)\\ 0\\ 0\\ \widehat{D}_{3,h,\omega+\Omega}\left(\alpha_{3,h,\omega+\Omega}(\tau)\right)\end{pmatrix}|0\rangle \to$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix}e^{-i\omega\tau}\widehat{D}_{2,v,\omega}\left(\alpha_{2,v,\omega}(0)\right)\\ 0\\ 0\\ e^{-i(\omega+\Omega)\tau}\widehat{D}_{3,h,\omega+\Omega}\left(\alpha_{3,h,\omega+\Omega}(0)\right)\end{pmatrix}|0\rangle \to$$

$$\frac{e^{-i\omega\tau}}{\sqrt{2}}\begin{pmatrix}\widehat{D}_{2,v,\omega}\left(\alpha_{2,v,\omega}(0)\right)\\ 0\\ 0\\ e^{-i\Omega\tau}\widehat{D}_{3,h,\omega+\Omega}\left(\alpha_{3,h,\omega+\Omega}(0)\right)\end{pmatrix}|0\rangle \quad (20)$$

It is apparent from the expression (20) that we deduced the phase factor $e^{-i\Omega\tau}$, which will affect the oscillation dynamics of an observable – light intensity. The state (20) can be simplified by omitting common phase factor $e^{-i\omega\tau}$ in future calculations. Additionally, we can treat the coherent states in modes ω and $\omega+\Omega$ as fully tangential, i.e. $\langle\alpha_\omega|\alpha_{\omega+\Omega}\rangle = 1$, since physically our detection equipment is equally sensitive to these modes within the actual shift of ≈107 MHz for Ω. So, in further notations we can as well omit explicit indication of the frequency modes. Thus, we obtain the following normalized state:

$$|\psi'_{AOM}\rangle \to \frac{1}{\sqrt{2}}\begin{pmatrix}\hat{D}_{2,v}(\alpha_{2,v})\\0\\0\\e^{-i\varphi}\hat{D}_{3,h}(\alpha_{3,h})\end{pmatrix}|0\rangle \tag{21}$$

with φ being modulation phase $\varphi = \Omega\tau$. Using expression (21) we can define input states for test configurations. The action of HWP2 and HWP3 with FA set at $\frac{\pi}{4}$ is expressed in a matrix form using adapted Jones calculus definition [12], namely:

$$WP_2(\beta) = \begin{pmatrix}\cos 2\beta & 0 & \sin 2\beta & 0\\0 & 1 & 0 & 0\\\sin 2\beta & 0 & \cos 2\beta & 0\\0 & 0 & 0 & 1\end{pmatrix} \tag{22}$$

for the HWP2 with β being an angle of the wave-plate's FA and:

$$WP_3(\gamma) = \begin{pmatrix}1 & 0 & 0 & 0\\0 & \cos 2\gamma & 0 & \sin 2\gamma\\0 & 0 & 1 & 0\\0 & \sin 2\gamma & 0 & \cos 2\gamma\end{pmatrix} \tag{23}$$

for the HWP3 with γ being an angle of the wave-plate's FA.

$$|\psi^I_{in}\rangle \to WP_2\left(\frac{\pi}{4}\right)|\psi'_{AOM}\rangle \to \frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\\hat{D}_{4,h}(\alpha_{4,h})\\e^{-i\varphi}\hat{D}_{5,h}(\alpha_{5,h})\end{pmatrix}|0\rangle \tag{24}$$

where $\hat{D}_{4,h}(\alpha_{4,h})$ and $\hat{D}_{5,h}(\alpha_{5,h})$ are displacement operators in horizontal polarization mode shifted to spatial modes 4 and 5 accordingly. Hence, in the first test configuration we have the following output state:

$$|\psi^I_{out}\rangle \to \frac{1}{2}\begin{pmatrix}0\\0\\(1+e^{-i\varphi})\hat{D}_{4,h}(\alpha_{4,h})\\(e^{-i\varphi}-1)\hat{D}_{5,h}(\alpha_{5,h})\end{pmatrix}|0\rangle \tag{25}$$

Likewise, the input state of the test configuration 2 yields:

$$|\psi^{II}_{in}\rangle \to WP_3\left(\frac{\pi}{4}\right)|\psi'_{AOM}\rangle \to \frac{1}{\sqrt{2}}\begin{pmatrix}\hat{D}_{4,v}(\alpha_{4,v})\\e^{-i\varphi}\hat{D}_{5,v}(\alpha_{5,v})\\0\\0\end{pmatrix}|0\rangle \tag{26}$$

So, taking into account the polarization selectivity of dispersive prisms P1 and P2 we obtain:

$$|\psi^{II}_{out}\rangle \to B|\psi^{II}_{in}\rangle \to \frac{1}{2}\begin{pmatrix}0.73(1-e^{-i\varphi})\hat{D}_{4,v}(\alpha_{4,v})\\0.73(1+e^{-i\varphi})\hat{D}_{5,v}(\alpha_{5,v})\\0\\0\end{pmatrix}|0\rangle \tag{27}$$

where the $|\psi^{II}_{out}\rangle$ output state of the test configuration 2 is correlated to factor 0.73 so we can work in the same normalization scale for voltage response as in previous case. While one may suggest omitting the common factor 0.73 and consider the model normalization independently it may, however, appear confusing given the actual oscilloscope response. Thus, to have more quantitative approach we want to contrast the scales of all predicted models under the same normalization amplitudes. The factor 0.73 comes from $\sqrt{0.73}$ transmission amplitude for TE (vertical) mode per surface of dispersive prisms as discussed in experimental setup chapter.

In the third and fourth test configurations we use no waveplate but LP's with different angle settings. The effect of LP1 and LP2 can be defined according to the Jones calculus [12] as the following matrix suited for our 4-dimensional state vector:

$$LP(\theta_1,\theta_2) = \begin{pmatrix}\cos^2\theta_1 & 0 & \cos\theta_1\sin\theta_1 & 0\\0 & \cos^2\theta_2 & 0 & \cos\theta_2\sin\theta_2\\\cos\theta_1\sin\theta_1 & 0 & \sin^2\theta_1 & 0\\0 & \cos\theta_2\sin\theta_2 & 0 & \sin^2\theta_2\end{pmatrix} \tag{28}$$

with θ_1 and θ_2 angles of transmission axes for LP1 and LP2 accordingly. Hence, in test configuration 3 we obtain:

$$|\psi^{III}_{in}\rangle \to |\psi'_{AOM}\rangle \tag{29}$$

$$|\psi^{III}_{out}\rangle \to LP\left(\frac{\pi}{4},\frac{\pi}{4}\right)B|\psi^{III}_{in}\rangle \to \frac{1}{4}\begin{pmatrix}0.73(1+e^{-i\varphi})\hat{D}_{4,v}(\alpha_{4,v})\\0.73(1+e^{-i\varphi})\hat{D}_{5,v}(\alpha_{5,v})\\(1+e^{-i\varphi})\hat{D}_{4,h}(\alpha_{4,h})\\(1+e^{-i\varphi})\hat{D}_{5,h}(\alpha_{5,h})\end{pmatrix}|0\rangle \tag{30}$$

where we can see the factor 0.73 cannot be omitted as it affects relative amplitudes. The expression (30) shows that all of four amplitudes are in phase, so by subtracting the intensities at port 4 and 5 which now have two components per port: $I^{III}_4(\varphi) = I^{III}_5(\varphi) = \left|\frac{0.73}{4}(1+e^{-i\varphi})\right|^2 + \left|\frac{1}{4}(1+e^{-i\varphi})\right|^2 = \frac{1+0.73^2}{4}\cos^2\frac{\varphi}{2}$ and as they are both in phase subtracting the signals provides no oscillation. In the same manner test configuration 4 output state becomes:

$$|\psi^{IIII}_{out}\rangle \to LP\left(\frac{\pi}{4},-\frac{\pi}{4}\right)B|\psi^{IIII}_{in}\rangle \to \frac{1}{4}\begin{pmatrix}0.73(1+e^{-i\varphi})\hat{D}_{4,v}(\alpha_{4,v})\\0.73(1-e^{-i\varphi})\hat{D}_{5,v}(\alpha_{5,v})\\(1+e^{-i\varphi})\hat{D}_{4,h}(\alpha_{4,h})\\(-1+e^{-i\varphi})\hat{D}_{5,h}(\alpha_{5,h})\end{pmatrix}|0\rangle \tag{31}$$

For the input vectors $|\psi^{V}_{in}\rangle$ and $|\psi^{VI}_{in}\rangle$, i.e. of last two test configurations, we perform a polarization state swap by applying both waveplate matrices upon vector $|\psi'_{AOM}\rangle$ (for which the order does not matter as these matrices commute):

$$\left|\psi_{in}^{V}\right\rangle = \left|\psi_{in}^{VI}\right\rangle \rightarrow WP_3\left(\frac{\pi}{4}\right)WP_2\left(\frac{\pi}{4}\right)\left|\psi'_{AOM}\right\rangle \rightarrow$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ e^{-i\varphi}\widehat{D}_{5,v}\left(\alpha_{5,v}\right) \\ \widehat{D}_{4,h}\left(\alpha_{4,h}\right) \\ 0 \end{pmatrix}|0\rangle \tag{32}$$

Thus, resulting output state for configuration 5 is:

$$\left|\psi_{out}^{V}\right\rangle \rightarrow LP\left(\frac{\pi}{4},\frac{\pi}{4}\right)B\left|\psi_{in}^{V}\right\rangle \rightarrow$$

$$\frac{1}{4}\begin{pmatrix} 0.73(-e^{-i\varphi}+1)\widehat{D}_{4,\mathrm{v}}\left(\alpha_{4,v}\right) \\ 0.73(e^{-i\varphi}-1)\widehat{D}_{5,v}\left(\alpha_{5,v}\right) \\ (-e^{-i\varphi}+1)\widehat{D}_{4,h}\left(\alpha_{4,h}\right) \\ (e^{-i\varphi}-1)\widehat{D}_{5,h}\left(\alpha_{5,h}\right) \end{pmatrix}|0\rangle \tag{33}$$

Finally, in last configuration of the output state we apply the anti-symmetric position of LP's:

$$\left|\psi_{out}^{VI}\right\rangle \rightarrow LP\left(\frac{\pi}{4},-\frac{\pi}{4}\right)B\left|\psi_{in}^{VI}\right\rangle \rightarrow$$

$$\frac{1}{4}\begin{pmatrix} 0.73(-e^{-i\varphi}+1)\widehat{D}_{4,\mathrm{v}}\left(\alpha_{4,v}\right) \\ 0.73(e^{-i\varphi}+1)\widehat{D}_{5,v}\left(\alpha_{5,v}\right) \\ (-e^{-i\varphi}+1)\widehat{D}_{4,h}\left(\alpha_{4,h}\right) \\ (-e^{-i\varphi}-1)\widehat{D}_{5,h}\left(\alpha_{5,h}\right) \end{pmatrix}|0\rangle \tag{34}$$