A Study of Heavy Flavored Mesons with Killingbeck Potential Approach*

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In non-perturbative Quantum Chromodynamics (QCD), the formalism of the potential model has proven effective in predicting various static and dynamic properties of heavy-flavored mesons. Within the Schrödinger's framework, the interaction potential plays a crucial role in determining the meson spectrum. In this work, we incorporate the Killingbeck potential in the Hamiltonian and apply the quantum mechanical variational method to estimate the masses, decay constants, and oscillation frequencies of selected heavy-flavored mesons, using a Coulombic wave function as the trial function. Our results are systematically compared with experimental data and other theoretical findings available in the literature.

I. INTRODUCTION

Quantum chromodynamics (QCD) is the fundamental theory of strong interaction that describes the dynamics of quarks and gluons. These form hadrons such as protons, neutrons, and mesons. Based on the $SU(3)_C$ gauge symmetry, QCD introduces the concept of "colour charge", namely red, green, and blue, as the source of strong interactions [1]. Unlike quantum electrodynamics (QED), where photons do not interact with each other, QCD's gluons, the mediators of the strong force, carry colour charge and thus engage in self-interactions, adding complexity to the theory. The two fundamental aspects of QCD are -(1)Perturbative,(2)Non-perturbative [2]. While perturbative approach deals with processes involving high momentum transfer[3] like deep inelastic scattering(DIS)[4] and hard scatterings at LHC[5], the non-perturbative approach(NPQCD) deals with low momentum transfer processes[3] like hadron masses, decay constants[6] and form factor of hadrons. The most widely popular approach in NPQCD is lattice QCD. A less ambitious approach is the analytical study of the NPQCD with potential models where the known property of hadron-like confinement is imposed through a suitable confinement parameter, b.

The most suitable confinement parameter, b, was introduced in 1975 when De Rujula, Georgi, and Glashow [7] applied an effective potential of Quantum Electrodynamics (QED) to quark physics. It is worth noting that the potential is a concept from non-relativistic quantum mechanics, not from relativistic quantum field theory, where interactions are visualized as arising from the exchange of quanta. However, for slowly moving particles, it corresponds to the three-dimensional Fourier transform of the lowest-order covariant matrix element. On the other hand, heavy-flavored mesons contain heavy quarks such as charm (c) or bottom (b) quarks, and their structure is analogous to that of the hydrogen atom, although the strong interaction governs the interaction force. The mass of the heavy quark in heavy-flavored mesons effectively dominates its momentum and behaves almost like a static source of the strong force, interacting with the light quarks and gluons within the bound state. Hence, non-relativistic frameworks, such as potential models[8], are applicable for studying the static and dynamic properties of heavy-flavored mesons.

Despite some limitations[9][10], the potential model[11][12] in Quantum Chromodynamics (QCD) has been quite successful in explaining various static and dynamic properties of heavy-flavored mesons. The most widely used potentials in QCD are:

1. **Cornell Potential[13][14]**

$$V(r) = -\frac{4\alpha_s}{r} + br$$

A combination of a Coulomb-like term and a linear confining term, commonly used in quarkonium models.

2. **Power Law Potential[15][16]:**

$$V(r) = \frac{-a}{r} + br^n$$

A general potential form where the interaction strength varies with distance raised to power n.

3. **Martin Potential[17]:**

$$V(r) = br^{0.1} + c$$

A phenomenological potential with a very weak power dependence used in hadronic spectroscopy.

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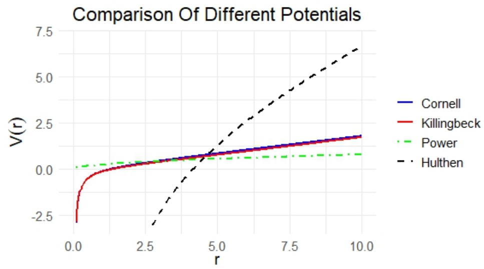

FIG. 1. Comparison of different potentials.

4. **Killingbeck Potential[18]:**

$$V(r) = ar^2 + br + \frac{c}{r}$$

A hybrid potential combining harmonic, linear, and Coulomb-like terms used in atomic and nuclear models.

5. **Logarithmic Potential[9]:**

$$V(r) = a\ln(r)$$

The potential is used to model the confinement property[19], where the force increases slowly with distance, and the Coulomb-like short-range interaction between the quarks.

6. **Hulthén Potential[9]:**

$$V(r) = -V_0\frac{e^{-\alpha r}}{1-e^{-\alpha r}}$$

A short-range potential resembling the Coulomb potential at small r, but decaying exponentially at large distances. It is used in nuclear and particle physics to describe screening effects and bound states in mesons.

While the Cornell potential[13] and power law potential[15] have been widely explored in Quantum Chromodynamics, the other ones have not. In this work, specifically, we consider the Killingbeck potential(2) to study several static and dynamic properties of heavy-flavor mesons. The potential contains harmonic (ar^2), linear (br), and Coulomb terms[18][20], which defines linear confinement at larger distance [19][21] (specifically the terms $ar^2 + br$) and Coulomb-like short ranged interaction between quarks ($\frac{c}{r}$). This potential is effective in studying quarkonium system[22], calculating properties like Isgur-wise function, which describes the semileptonic heavy meson decays[20].

The Schrodinger equation with the Killingbeck potential is not exactly solvable. Hence, we have to look for available approximation methods in Quantum mechanics. The most widely used approximation methods in quantum mechanics are:

- Perturbation Theory[23][24][25]
- The Variational method[26][27]
- WKB approximation method[28]

The application of the variational method in the approach of the QCD potential model is not new, dating back to 1996 [29], when Hwang et al. [29] used it with a linear plus Coulomb Cornell potential to estimate the masses and decay constants of heavy-flavor mesons. Later, in [16], Rai et al. used it with a power law potential, and in [30], Vega and Flores used it with a supersymmetric potential. Inspired by this and keeping in mind the success of the Killingbeck potential in determining the static and dynamic properties of heavy-flavor mesons[20][31], we applied the variational method, considering the Coulomb wave function (1) as the trial wave function. The Coulomb wave function was chosen because of the static nature of heavy quarks in heavy mesons, which makes the system analogous to that of the hydrogen atom.

In section 2 we provide the formalism where we used the variational method with a Coulombic trial wave-function. In section 3 we provide the results obtained with this formalism on masses, decay constants, and oscillation frequencies of neutral mesons. Detailed comparison is done with available experimental data as well as with other models such as QCD Sum rules[32], Lattice QCD[33], and Relatively Harmonic Confinement Model(RHCM)[34]. In section 4, we summarize the conclusion.

II. FORMALISM

A. Variational Method

We consider a Coulombic trial wave-function as,

$$\psi(r) = \frac{(\mu\alpha)^{3/2}}{\sqrt{\pi}}e^{-\mu\alpha r} \tag{1}$$

where μ is the reduced mass of the quark-antiquark pairs, and α is the variational parameter. The interaction potential used is the Killingbeck potential[18], which is given by:

$$V(r) = ar^2 + br + \frac{c}{r}, \tag{2}$$

where a, b, and c are potential parameters. In this work, we adopt the values $a = 0.00193235$, $b = 0.15743235$, and $c = -0.288421$ [20].

Now, the Hamiltonian of the $Q\bar{q}$ bound state of a heavy-flavor meson is given by:

$$\hat{H} = -\frac{1}{2\mu}\left[\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right] + V(r) \tag{3}$$

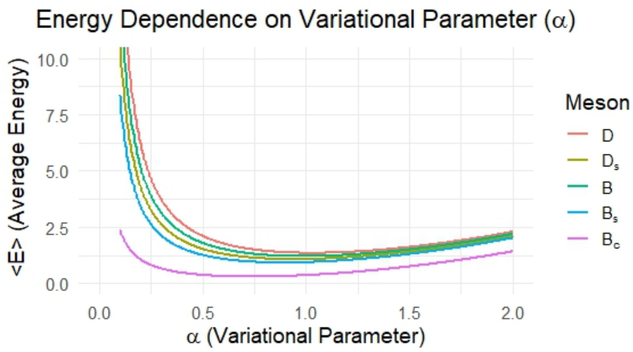

FIG. 2. Plot of average energy vs variational parameter (α)

Following the variational scheme, the ground state energy is:

$$\langle E\rangle = \langle\psi|\hat{H}|\psi\rangle \tag{4}$$

Now from Equation (2), we have:

$$\frac{\partial\psi}{\partial r} = -\alpha\mu\psi, \quad \frac{\partial^2\psi}{\partial r^2} = \alpha^2\mu^2\psi \tag{5}$$

Hence,

$$\langle\psi|\hat{H}|\psi\rangle = \langle E\rangle = \frac{1}{2}\mu\alpha^2 + c\mu\alpha + \frac{3b}{2\mu\alpha} + \frac{3a}{\mu^2\alpha^2} \tag{6}$$

Now, minimizing the energy, we can find the variational parameter for different heavy flavored mesons,

$$\frac{\partial\langle E\rangle}{\partial\alpha} = \mu\alpha - \frac{6a}{\mu^2\alpha^3} - \frac{3b}{2\mu\alpha^2} + c\mu = 0 \tag{7}$$

And we get,

$$\mu\alpha - \frac{6a}{\mu^2\alpha^3} - \frac{3b}{2\mu\alpha^2} + c\mu = 0 \tag{8}$$

Solving this equation in Mathematica 9, we get the variational parameter for different mesons, which is shown in Table(1) in the result section.

B. Masses of Heavy Flavored Mesons:

The mass formula for pseudo-scalar mesons is given by[31][35][20]:

$$M_p = M_Q + M_{\bar{Q}} + \langle E\rangle + \langle H_{\mathrm{SS}}\rangle \tag{9}$$

where M_p is the mass of the pseudo-scalar meson, M_Q and $M_{\bar{Q}}$ are the masses of the constituent quark and anti-quark respectively, $\langle E\rangle$ is the ground-state energy of the system, and $\langle H_{\mathrm{SS}}\rangle$is the energy shift due to mass splitting from spin interaction in perturbation theory, which is given by[36][31][10]:

$$\langle H_{\mathrm{SS}}\rangle = \frac{32\pi\alpha_s}{9M_QM_{\bar{Q}}}\left(\vec{S}_Q\cdot\vec{S}_{\bar{Q}}\right)|\psi(0)|^2 \tag{10}$$

Now, for pseudo-scalar mesons, the spin-spin interaction term becomes[31]:

$$\vec{S}Q\cdot\vec{S}\bar{Q} = -\frac{3}{4}$$

and so the mass formula becomes[37],[31]

$$M_p = M_Q + M_{\bar{Q}} + \langle E\rangle - \frac{8\pi\alpha_s}{3M_QM_{\bar{Q}}}|\psi(0)|^2 \tag{11}$$

C. Decay Constant

Charged mesons can decay to a lepton-neutrino pair through annihilation via a virtual $W^{\pm}$ boson. The weak decays of mesons can be classified into three categories as: Leptonic decay[38], semi-leptonic decay[39], and non-leptonic decay[40] according to the final products of the decay. Among these, leptonic decays are rarer but have a clear experimental signature because of the presence of a highly energetic lepton in the final state [39], where a charged lepton ($l = e, \mu, \tau$) and its corresponding neutrino (v) are created. leptonic decays are characterized by a parameter called the decay constant, which physically describes the wave-function overlap of the quark and anti-quark. This is an important parameter to estimate the Cabibbo-Kobayashi-Maskawa (CKM) matrix elements[41], where the rate of decay depends on. According to the Van-Royen-Weisskopf formula[42][37][31], the decay constant of a pseudo-scalar meson is related to the wave-function at the origin as[35][31][36][37]

$$f_P\left(|\psi(0)|^2, M_p\right) = \sqrt{\frac{12\,|\psi(0)|^2}{M_p}} \tag{12}$$

Now introducing the QCD correction term [31][36] it becomes,

$$f_P = \cdot\sqrt{\frac{12\,|\psi(0)|^2}{M_p}\bar{C}^2} \tag{13}$$

where,

$$\bar{C}^2 = \sqrt{1 - \frac{\alpha_s}{\pi}\left[2 - \frac{M_Q - M_q}{M_Q + M_q}\ln\left(\frac{M_Q}{M_{\bar{Q}}}\right)\right]} \tag{14}$$

D. Oscillation Frequency

Neutral mesons, B_D and B_S exhibit the property of particle-antiparticle mixing[37][43][44] where particles make transitions into their antiparticle states and vice versa. The transitions $B_q^0 \leftrightarrow \bar{B}_q^0$ are due to the weak interaction. [37][31] and caused due to the difference between their weak and mass eigenstates. This causes them to oscillate between these states[44][37], and it is characterised by a mixing mass parameter, ΔM_B, known as their frequency of oscillation[37][44][31].

$$\Delta m_B = \frac{G_F^2 m_t^2 M_{B_q} f_{B_q}^2}{8\pi} g(x_t) \eta_t |V_{tq} V_{tb}|^2, \tag{15}$$

and,

$$q = d, s \tag{16}$$

Where,

$$g(x_t) = \frac{1}{4} + \frac{9}{4(1-x_t)} - \frac{3}{2(1-x_t)^2} - \frac{3x_t^2}{2(1-x_t)^3}$$

Here, G_F is the Fermi constant, m_t is the top quark mass, M_{B_q} and f_{B_q} are the mass and decay constant of the B_q meson,$g(x_t)$ is a loop function with $x_t = \frac{m_t^2}{M_W^2}$, η_t is a QCD correction factor, and V_{tq}, V_{tb} are CKM matrix elements.

III. RESULTS

A. Mass

As given in the formalism section, meson masses are calculated using the mass relation equation(11), where, M_Q, $M_{\bar{Q}}$, $\langle E_{\rm GS}\rangle$ and $\langle H_{\rm SS}\rangle$ are taken as input parameters, and listed in Table II. The values of the variational parameter and reduced mass are listed in Table I. The quark masses are taken from [35], where the masses of the down (M_d), charm (M_c), strange (M_s), and bottom (M_b) quarks are assumed to be 0.336, 1.550, 0.483, and 4.950 GeV, respectively. The coupling constant α_s in the expression for $\langle H_{SS}\rangle$ is taken to be 0.39 and 0.23 for the charmonium and bottomonium ranges, respectively[45].

The calculated meson masses are compared with the experimental values [35], as well as with theoretical predictions from other models such as QCD sum rules, lattice QCD, and RHCM models [35]. Figure (3) is the scattered plot of our calculated values and theoretically predicted values from the mentioned model..

B. Decay Constant

The corresponding values of the decay constant of mesons are listed in Table IV. It is calculated using (13), where our calculated meson masses, as input parameters, are taken from the table II. The correction factor $\bar{C}$ is considered as 0.9[31]. Our calculated values of decay constants are listed in table IV. Also, the calculated values are compared with the experimental values as well as theoretical predicted values from QCD in table V.

The calculated meson masses are compared with the experimental values [35], as well as with theoretical predictions from other models such as QCD sum rules, lattice QCD, and RHCM models [35]. Figure (3) is the scattered plot of our calculated values and theoretically predicted values from the mentioned model.

C. Oscillation Frequency

As explained in Section 2.4, the oscillation frequencies for B_D and B_S mesons are calculated using Equations (16) and (17). The other parameters involved in these expressions include $m_t = 174$ GeV, $m_W = 80.403$ GeV, and the elements of the CKM matrix: $|V_{td}| = 0.0074$, $|V_{ts}| = 0.004$, and $|V_{tb}| = 1$. The QCD correction factor η_t is taken as 0.55[46]. Our calculated values and the comparison of these values with the experimental values and theoretically predicted values of the model mentioned in table VI.

TABLE I. Reduced masses and variational parameters for different mesons.

Meson	Reduced Mass (GeV)	α
$D(cd)$	0.2761	1.6119
$D_S(c\bar{s})$	0.3682	1.3495
$B(b\bar{d})$	0.3180	1.4754
$B_S(b\bar{s})$	0.4400	1.2055
$B_C(b\bar{c})$	1.1803	0.6787

TABLE II. Meson properties with mesons as rows and parameters as columns.

	M_Q (in GeV)	$M_{\bar{Q}}$ (in GeV)	$\langle E\rangle_{\mathrm{GS}}$ (in GeV)	$\|\psi(0)\|^2$ (in GeV^3)	$\langle H_{SS}\rangle$ (in GeV)	Calculated Mass (in GeV)	Experimental Mass (in GeV)
$D(c\bar{d})$	1.550	0.336	0.790	0.0280	0.3152	2.360	1.869 ± 0.0016
$D_S(c\bar{s})$	1.550	0.483	0.693	0.0390	0.3055	2.420	1.968 ± 0.0033
$B(b\bar{d})$	4.950	0.336	0.742	0.0328	0.1168	5.861	5.279 ± 0.0017
$B_S(b\bar{s})$	4.950	0.483	0.622	0.0457	0.1177	5.867	5.366 ± 0.0024
$B_C(b\bar{c})$	4.950	1.550	0.344	0.1634	0.1263	6.667	6.277 ± 0.0060

TABLE III. Comparison of calculated meson masses with QCD sum rules, lattice QCD, and RHCM model (in GeV).

	Calculated Mass (in GeV)	QCD Sum Rules (in GeV)	Lattice QCD (in GeV)	RHCM Model (in GeV)
$D(c\bar{d})$	2.360	1.870	1.885	2.653
$D_S(c\bar{s})$	2.420	1.970	1.969	2.778
$B(b\bar{d})$	5.861	5.280	5.283	6.192
$B_S(b\bar{s})$	5.867	5.370	5.366	6.264
$B_C(b\bar{c})$	6.667	–	6.278	6.929

TABLE IV. Calculated meson masses and decay constants with and without correction factors, compared to experimental values (all in GeV).

	Calculated Mass M_P (in GeV)	f_P (Without Correction) (in GeV)	f_P (With Correction) (in GeV)	Experimental f_P (in GeV)
D	2.360	0.380	0.342	$0.205 \pm 0.085 \pm 0.025$
D_S	2.420	0.487	0.490	0.254 ± 0.059
B_U	5.861	0.259	0.233	0.198 ± 0.014
B_S	5.867	0.311	0.280	0.237 ± 0.017
B_C	6.667	0.546	0.492	0.562

TABLE V. Comparison of calculated decay constants (f_P) with QCD sum rule, lattice QCD, and RHCM model (all in GeV).

	Calculated f_P (in GeV)	Lattice QCD (in GeV)	QCD Sum Rule (in GeV)	RHCM Model (in GeV)
D	0.342	0.220 ± 0.003	0.206 ± 0.002	0.336
D_S	0.439	0.258 ± 0.001	0.245 ± 0.015	0.387
B_U	0.233	0.218 ± 0.005	0.193 ± 0.012	0.581
B_S	0.280	0.228 ± 0.010	0.232 ± 0.018	0.600
B_C	0.492	–	–	0.607

TABLE VI. Oscillation frequencies Δm_B for B_D and B_S mesons from different approaches (all values in ps^{-1}).

	B_D (ps^{-1})	B_S (ps^{-1})
Calculated Δm_B	0.39	17.1
QCD Sum Rules	0.48	14.6
Lattice QCD	0.63	19.6
Experimental	0.50	17.76

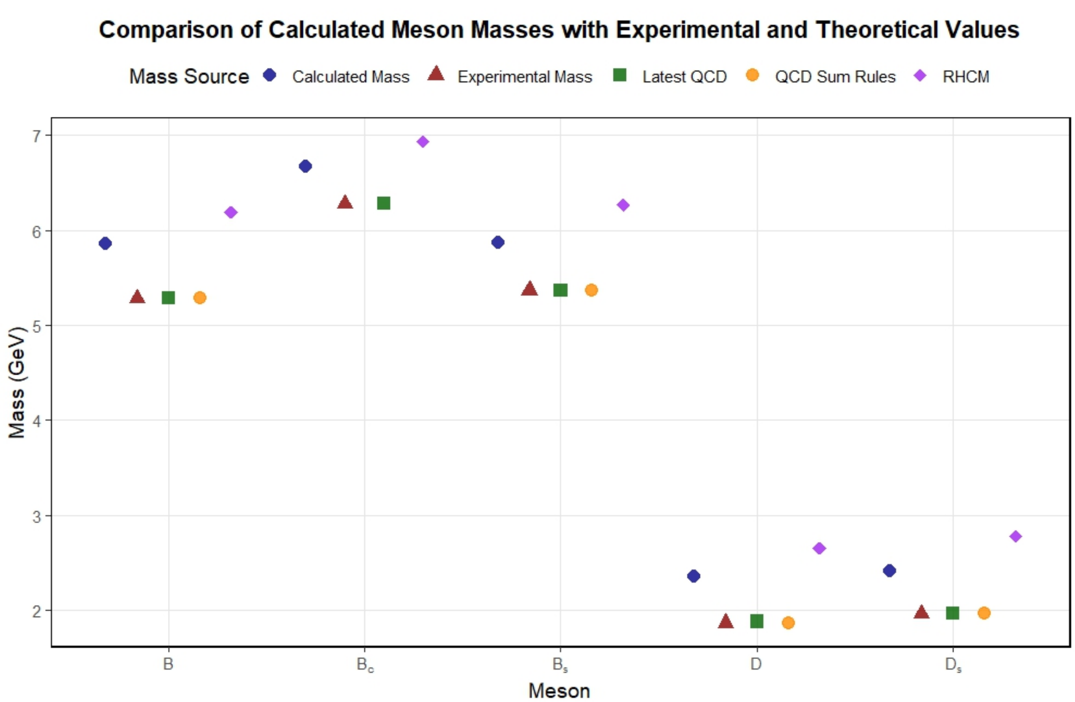

FIG. 3. Comparison of meson masses across different models.

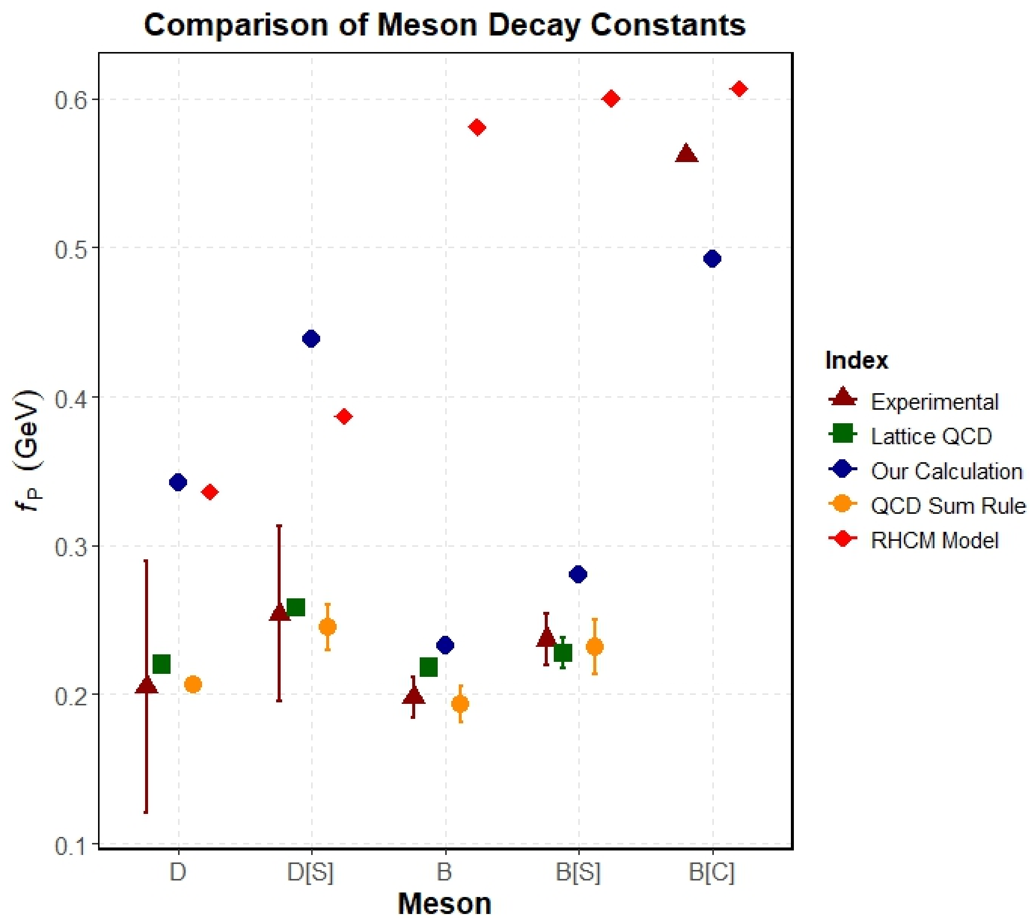

FIG. 4. Comparison of calculated decay constant (f_P) with QCD sum rule, lattice QCD, and RHCM model

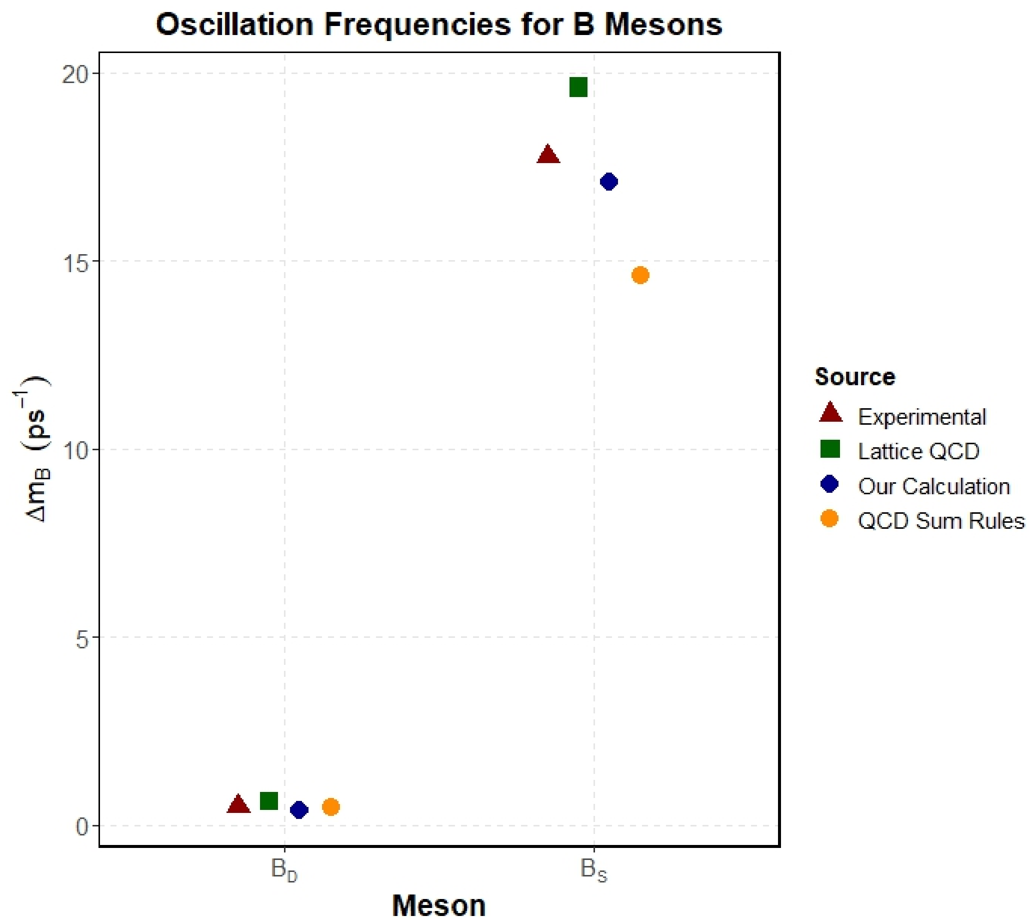

FIG. 5. Oscillation frequencies for different meson states.

IV. CONCLUSION

In this paper, we have employed the quantum mechanical variational method with the Killingbeck potentialto study several properties of heavy-flavored mesons. Dealing with a variational scheme is quite a challenge, because there is no rule for the choice of a trial wave-function with the unknown parameters, which is later optimized to estimate different properties. But following the recent works with the power law potential[16], Cornell potential[13] where it has been established that a Gaussian trial wave function is a good choice to study heavy-flavoured mesons. Although the linear plus Coulomb Cornell potential is explored quite elaborately to study static and dynamic properties of these mesons, there is no such development on the Killingbeck potential[18]. However, with the Killingbeck potential, the Coulombic trial wave function shows improvement in results, and the Coulombic wave function is physically more appealing too. Therefore, in this work, specifically, we adopted the Coulomb trial wave function and estimated masses, decay constants, and oscillation frequencies of a few heavy-flavoured mesons. The output results of the formalism substantially agree with available experimental data, as well as with the results of the RHCM model[], lattice QCD[], and QCD sum rules[]. It is an improvement over the previous perturbative approach, where an arbitrary choice between the linear and Coulomb part of the potential to consider as parent/perturbed term is necessary. The results for B mesons obtained are in excellent agreement with experimental data and with those of lattice QCD and QCD sum rule. Further, our results are in good harmony with some of the previous works. The success of the present formalism motivates its application to study other properties of heavy-flavoured mesons, such as Isgur-Wise function, leptonic decay width,Branching ratio, CKM matrix elements etc.

However, one limitation is that the results obtained are sensitive to the chosen trial wave function, so it is very crucial to choose a proper trial wave function for the $Q\bar{Q}$ system. Also, small inconsistencies in our results in the case of D, D_S mesons, where one quark/antiquark is light, reasonably indicate some relativistic correction is needed. Although in spite of such limitations, this model provides a simple way to investigate static and dynamic properties of heavy-flavoured mesons, at least as far as phenomenology is concerned.

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