Supplementary Information for

The source of hardware-tailored codes and coding phases

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1. NUMERICAL OPTIMIZATION OF THE CHANNEL CAPACITY AND THE DISCOVERY OF THE $\mathbb{Z}_2$ SOURCE

The result of the optimization procedure described in the Methods is a density matrix ρ_{opt} which is a local maximum of the coherent information. For the point $q_z = 0.1$, $q_U = 0.05$ in the ORUM phase diagram, an optimization trace is shown in Fig. S1a showing smooth monotonic trajectory with a few small bumps along the way that presumably correspond to other nearby local maxima or saddlepoints. We then plot the density matrix ρ_{opt} in Fig. S1b as an image, finding that each matrix element has either zero magnitude or a magnitude of $1/2^N$, where N is the number of qubits in the system. Each non-zero matrix element has a phase which is 0 along the diagonal and generally non-zero in the off-diagonal elements. A quick computation of the purity reveals $\mathrm{Tr}\,\rho_{opt}^2 = 1/2$. By converting the two and column indices of the non-zero elements to binary, we find that they either correspond to two numbers with an even number of ones in the binary strings or an odd number of ones. We go into further detail in Section 2. In this way, optimization has revealed optimal input density matrices that represent encoded information with a surprising simplicity.

We numerically optimize the coherent information over all possible input density matrices to the ORUM channels. By using PyTorch [72] to implement the density matrix and channel superoperators, we optimize using autograd techniques familiar in machine learning together with a careful attention to preserving the norm of the density matrix. In the Methods section, we present details on how this optimization was carried out and on how we analyzed the resulting optimal density matrices to discover the $\mathbb{Z}_2$ source. Similarly, in Fig. S1, we present the optimization trace as well as the optimal density matrix that corresponds to the $\mathbb{Z}_2$ source.

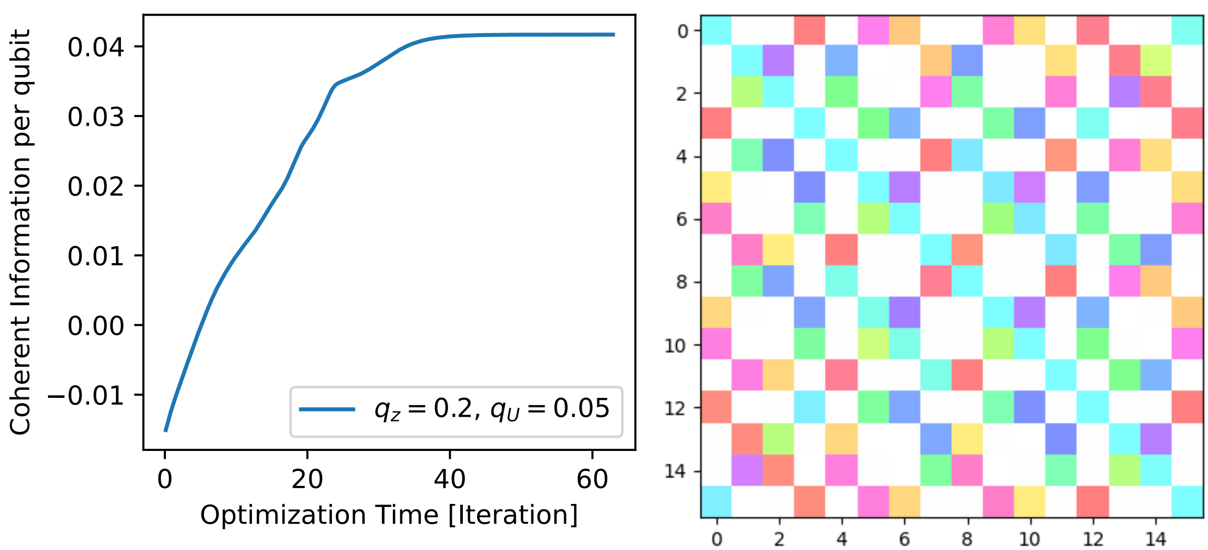

FIG. S1. **Discovery of the $\mathbb{Z}_2$ source for $N = 2$. a** optimization trace showing the rise of the coherent information I_c as a function of the gradient ascent optimization step. **b** An image plot of the (16×16) density matrix, corresponding to 2 system and 2 reference qubits, elements with white corresponding to zero matrix elements and color corresponding to matrix elements with a magnitude of $1/16$ and phase given by the color. The light blue matrix elements have zero phase, the light red matrix elements have a phase of π while all other colors are some other values of the phase between 0 and 2π.

2. $\mathbb{Z}_2$ SOURCE AS A SPIN LIQUID

The $\mathbb{Z}_2$ code optimal density matrix, found in section 1, is defined as follows. Let $|\text{even}\rangle_S$ be the state

$$|\text{even}\rangle_S = \frac{1}{2^{(n-1)/2}}\left(|00\ldots000\rangle_S + e^{i\phi_1}|00\ldots011\rangle_S + e^{i\phi_2}|00\ldots101\rangle_S + \ldots\right) \tag{S1}$$

where the sum extends over all basis states of the system $|x\rangle_S$ with an even number of 1's in the bit string x, each state having an arbitrary phase ϕ_i. Then define the state $|\text{odd}\rangle_S$ to be the state

$$|\text{odd}\rangle_S = \frac{1}{2^{(n-1)/2}}\left(|00\ldots001\rangle_S + e^{i\phi_3}|00\ldots010\rangle_S + e^{i\phi_4}|00\ldots100\rangle_S + \ldots\right), \tag{S2}$$

where the sum extends over all basis states of the system $|x\rangle_S$ with an odd number of bit strings, again each state having an arbitrary phase. Then, we can define a purification of this state observed in the quantum capacity calculations to be

$$|\psi_{\mathbb{Z}_2}\rangle_{RS} = \frac{1}{\sqrt{2}}\left(|0\rangle_R \otimes |\text{even}\rangle_S + |1\rangle_R \otimes |\text{odd}\rangle_S\right). \tag{S3}$$

and tracing over R shows us this state is

$$\rho_{\mathbb{Z}_2,S} = \frac{1}{2}(|\text{even}\rangle_{SS}\langle\text{even}| + |\text{odd}\rangle_{SS}\langle\text{odd}|) \tag{S4}$$

The arbitrary phases that enter the purification suggest the existence of a gauge symmetry. Applying the unitary transformation $U(\{\theta_i\}) = e^{i\sum_j \theta_j n_j}$ with $n_j = (1+Z_j)/2$ just alters the phases ϕ_1, ϕ_2, etc. in $|\text{even}\rangle_S$. Applying a similar transformation but with $n_1 = (1-Z_1)/2$, $n_j = (1+Z_j)/2$, $j \geq 2$ to $|\text{odd}\rangle_S$ similarly alters its phases. But if we set $\theta_i = \pi$, then we find $|\text{even}\rangle_S \to |\text{even}\rangle_S$ and $|\text{odd}\rangle_S \to -|\text{odd}\rangle_S$ which leaves $\rho_{\mathbb{Z}_2,S}$ invariant.

To show ORUM has a gauge symmetry, let us first define it more carefully. We can write the two-qubit depolarizing channel as

$$\mathcal{U}_{i,j}(\rho) = (1-q)\rho + q\sum_{a,b}\tau_{ia}\tau_{jb}\rho\tau_{ia}\tau_{jb} \tag{S5}$$

where τ_{ia} are the identity and three Pauli operators I_i, X_i, Y_i, Z_i on site i indexed by a. We can also write the one-qubit dephasing channel as

$$\mathcal{Z}_i(\rho) = (1-q)\rho + qZ_i\rho Z_i \tag{S6}$$

Then, if we pass a gauge-transformed state $U(\{\theta_j\})\rho U^\dagger(\{\theta_j\})$ through the ORUM quantum channel, we find

$$\mathcal{Z}_i\left(U(\{\theta_j\})\rho U^\dagger(\{\theta_j\}\right) = U(\{\theta_j\})\mathcal{Z}_i(\rho)U^\dagger(\{\theta_j\}) \tag{S7}$$

and

$$\mathcal{U}_{i,i+1}\left(U(\{\theta_j\})\rho U^\dagger(\{\theta_j\}\right) = U(\{\theta_j\})\mathcal{U}_{i,i+1}(\rho)U^\dagger(\{\theta_j\} \tag{S8}$$

for $U(\{\theta_j\})\tau_{ia}\tau_{i+1,b}U^\dagger(\{\theta_j\}) = \tau'_{ia}\tau'_{i+1,b}$ amounts to rotating the Pauli operators to a new local basis which still preserves the one-design property of these unitaries. Namely, after the gauge transformation, our expression for $\mathcal{U}_{i,i+1}$ has a different set of Kraus operators which define the same channel (Krauss operators are not unique). Hence, the phases we find numerically in $|\text{even}\rangle_S$ and $|\text{odd}\rangle_S$ are a consequence of a $U(1)^{\otimes n}$ gauge symmetry.

As a result of the gauge symmetry, the $\mathbb{Z}_2$ source is like a $\mathbb{Z}_2$ spin liquid, it breaks the $U(1)$ gauge symmetry of the dynamics down to the global $\mathbb{Z}_2$ symmetry of $\rho_{\mathbb{Z}_2,S}$.

3. SIMILARITIES BETWEEN QUANTUM CAPACITY AND THE MAXIMUM ENTROPY PRINCIPLE

In 1957, Jaynes, inspired by Shannon's invention of information theory [73], rederived statistical mechanics from an information perspective via the maximum entropy principle [74, 75]. This principle states that the entropy is maximized in equilibrium, subject to constraints of what is known about a system. Here we revisit this principle for quantum statistical mechanics to draw attention of the similarity between it and the quantum capacity theorem discussed in the main text.

The extension from classical statistical mechanics to quantum statistical mechanics was likely recognized immediately after Jaynes's papers, and a good early discussion on the topic is present in Douglas J. Scalopino's thesis [76]. One modern accessible reference is John Preskill's notes on quantum Shannon theory [77], where the Gibbs state is derived from a free energy minimization principle, but the focus of this treatment is on information theory, not statistical mechanics.

The maximum entropy principle states that the entropy should be maximized subject to known constraints. For a closed statistical mechanics system, the von Neumann entropy, viewed as a function of the density matrix, is constrained by the known energy $E = \mathrm{Tr}\,\rho H$ of the system fixed by the initial conditions. We want to maximize the entropy with respect to varying ρ while maintaining the energy constraint. But, since ρ itself is not a simple matrix but one defined by additional constraints, we need to view it as a general matrix subject to the additional unit trace $\mathrm{Tr}\,\rho = 1$, hermiticity $\rho^\dagger = \rho$, and non-negativity $\rho \geq 0$ constraints. Since hermiticity can be handled by imposing it directly on ρ and non-negativity can be checked after an optimum is found, this just amounts to the requirement of imposing a unit trace. Hence, we can proceed by imposing just two constraints using the method of Lagrange multipliers.

We begin with the functional:

$$S = -k_B \,\mathrm{Tr}\,\rho \log \rho - \alpha(\mathrm{Tr}\,\rho - 1) - k_B \beta(\mathrm{Tr}\,\rho H - E) \tag{S9}$$

When ρ obeys both constraints, this functional is the entropy. A natural solution is Boltzmann's equal a priori probabilities, the mixed state ρ of all states with $\mathrm{Tr}\,\rho H = E$, i.e.

$$\rho(E) = \frac{1}{\Omega}\delta_\Delta(H - E) \tag{S10}$$

where $\Omega = \mathrm{Tr}\,\delta_\Delta(H - E)$ and δ_Δ is a regularized *delta*-function of the Hamiltonian matrix H minus E with an energy width or cutoff Δ.

An alternative solution, which avoids a singular distribution, is to change variables from energy E to temperature T via a Legendre transformation

$$F = E - TS = \mathrm{Tr}\,\rho H + k_B T\,\mathrm{Tr}\,\rho \log \rho + T\alpha(\mathrm{Tr}\,\rho - 1), \tag{S11}$$

where we set the Lagrange multiplier $\beta = 1/k_B T$. Taking the derivative with respect to complex variable ρ_{ij}, with $\rho_{ji} = \rho^*_{ij}$ treated as a separate variable, we obtain

$$\frac{\partial F}{\partial \rho_{ij}} = \mathrm{Tr}\,E_{ij} H + k_B T\,\mathrm{Tr}\,E_{ij}\log\rho + k_B T\,\mathrm{Tr}\,\rho \frac{\partial}{\partial \rho_{ij}} \log \rho + T\alpha\,\mathrm{Tr}\,E_{ij} \tag{S12}$$

where $E_{ij} = \partial\rho/\partial\rho_{ij}$ is the matrix with matrix element ij equal to one, i.e. $(E_{ij})_{ij} = 1$, and all others zero. To make sense of this expression, we need to work out the derivative of $\log \rho$.

Defining the log of ρ using a Taylor series

$$\log \rho = \log(I + (\rho - I)) = \rho - I - \frac{1}{2}(\rho - I)^2 + \frac{1}{3}(\rho - I)^3 + \ldots, \tag{S13}$$

the derivative is

$$\frac{\partial}{\partial \rho_{ij}} \log \rho \;=\; E_{ij} \;-\; \frac{1}{2}\left(E_{ij}(\rho - I) + (\rho - I)E_{ij}\right) + \frac{1}{3}\left(E_{ij}(\rho - I)^2 + (\rho - I)E_{ij}(\rho - I) + (\rho - I)^2 E_{ij}\right) + \ldots. \tag{S14}$$

Multiplying this expression by ρ, taking the trace, and exploiting the cyclic property of the trace, we arrive at

$$\mathrm{Tr}\,\rho \frac{\partial}{\partial \rho_{ij}} \log \rho = \mathrm{Tr}\,\rho E_{ij}\left(I - (\rho - I) + (\rho - I)^2 - \ldots\right). \tag{S15}$$

Recognizing this as a geometric series we see it simplifies to

$$\mathrm{Tr}\,\rho \frac{\partial}{\partial \rho_{ij}} \log \rho = \mathrm{Tr}\,\rho E_{ij} \frac{I}{I + (\rho - I)} = \mathrm{Tr}\,\rho E_{ij} \rho^{-1} = \mathrm{Tr}\,E_{ij} \tag{S16}$$

Hence, we obtain

$$\frac{\partial F}{\partial \rho_{ij}} = \operatorname{Tr} E_{ij} H + k_B T \operatorname{Tr} E_{ij} \log \rho + (k_B + \alpha) T \operatorname{Tr} E_{ij} \tag{S17}$$

Now performing the trace leaves us with the matrix equation

$$\frac{\partial F}{\partial \rho_{ij}} = H_{ji} + k_B T (\log \rho)_{ji} + (k_B + \alpha) T \delta_{ij} = 0 \tag{S18}$$

Or

$$H + k_B T \log \rho + (k_B + \alpha) T I = 0 \tag{S19}$$

with solution $\rho = \frac{1}{Z} e^{-\beta H}$ with $Z = e^{-(k_B+\alpha)T}$ chosen to satisfy $\operatorname{Tr} \rho = 1$. In this way, we see that the maximum entropy principle is a derivation of statistical mechanics from a quantum information theory principle.

4. MEASUREMENT-INDUCED PHASE TRANSITION

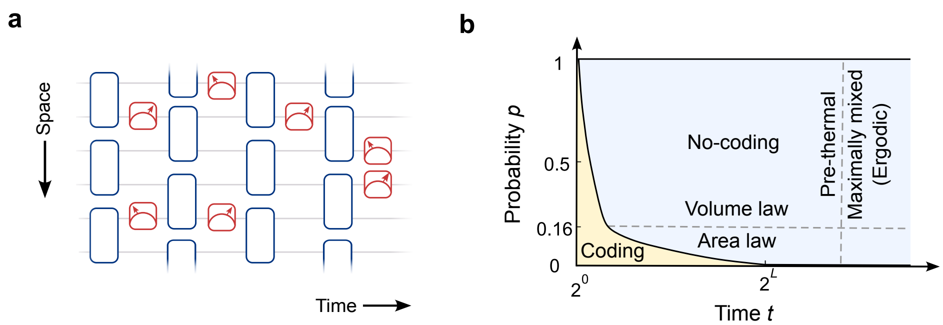

FIG. S2. **Purification in monitored circuits a,** Four layers of a monitored circuit (top), commonly used to study measurement-induced phase transition (MIPT), consisting of random 2-qubit unitaries (blue rectangles) and interspersed measurement gates (red squares) inserted with some measurement probability p. **b,** The purification phase diagram of a random Clifford model with the maximally mixed initial state. Below the critical point ($p < p_c$), the purification time diverges with the system size, whereas, for $p > p_c$, the maximally mixed state purifies at constant time.

The viewpoint of coding transition has been valuable to understanding measurement-induced phase transitions (MIPT) [43, 44]. MIPT arises due to a dynamical interplay between unitary gates vs. projective measurements, resulting in an entanglement phase transition from a volume law to an area law scaling of the entanglement entropy [39–42]. Furthermore, these transitions are also reflected in the divergence of purification time for a mixed initial state. This is a coding transition, as the purification signifies that the system no longer carries information about the initially entangled reference [43, 44]. While all states ultimately purify under monitored dynamics, they can reliably transmit quantum information for exponentially long timescales relative to system size. This phenomenon is captured in the system's coherent information, a key measure of its capacity to carry quantum information [50]. While the coding transition provides an alternative perspective to MIPT, observing it experimentally is still challenging due to the exponential sampling complexity of post-selection [78–80]. Nevertheless, identifying coding transitions by viewing the measurement-driven circuit as a noisy quantum channel generically doesn't require post-selection.

A typical circuit used to observe MIPT is shown in Fig. S2a (top), which consists of a "brickwork" circuit of two-qubit unitaries from, for instance, random Clifford ensemble interspersed with single-qubit measurements in the Z-basis. Starting from a pure state, the probability of measurement p, continuously drives the steady-state entanglement entropy from a volume-law to an area-law scaling [39–41]. However, one can also start from a mixed state, for instance, the maximally mixed state, resulting from an initial entanglement of the system S with a reference R. Although for any $p > 0$, the system purifies at exponentially long times, the purification time exhibits two distinct behaviors. Above a critical measurement probability p_c, the purification time is constant, whereas it sharply diverges with respect to the system size for $p > p_c$ (bottom). The p_c for the purification phase transition has been found to be the same as p_c for MIPT [43]. The purification transition is a "coding transition" because the system transitions from transmitting

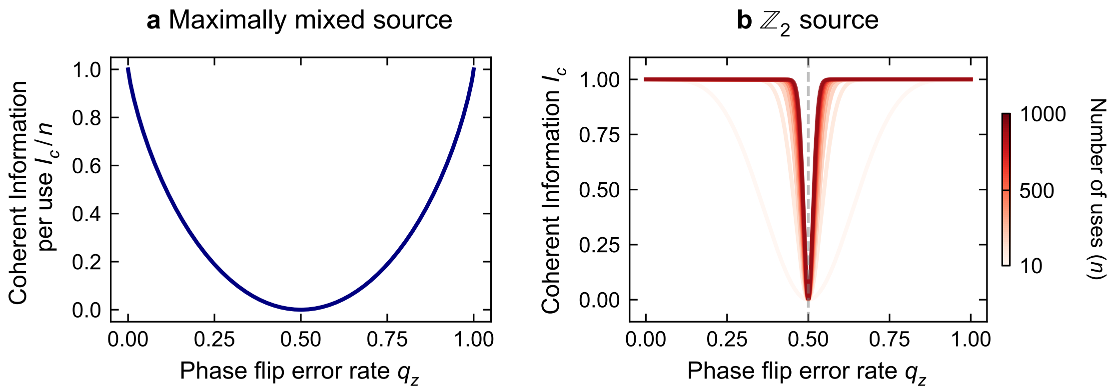

FIG. S3. **Coherent information along $q_U = 0$.** Coherent information of **a**, the maximally mixed code and **b**, the $\mathbb{Z}_2$ source along the line $q_U = 0$. Fort both the codes I_c vanishes at the multicritical point $(q_Z, q_U) = (0.5, 0)$.

finite quantum information via the maximally mixed state to transmitting zero quantum information. The coherent information $I_c = S(\rho'_S) - S(\rho_{RS})$ reflects the ability of an initial state ρ_{RS} to transmit quantum information through a channel $\mathcal{E}$.

Our work examines the coherent information by passing various states through the channel $\mathcal{E}$. The maximum coherent information defines the channel capacity, see Eq. (1). The trajectories generated by the quantum channel, which is often expressed in an operator-sum representation with Kraus operators, are quantitatively different from those produced by the original channel. A given channel can have many Kraus representations, so the procedure for generating trajectories is not unique if we take the channel as the fundamental dynamical law. For instance, a N-qubit depolarization channel can be defined as a sum over either Pauli, Clifford or the Haar-random unitaries as long as they form a 1-design. At a trajectory level, the entanglement phase transition for the Haar-random case has a higher p_c compared to the Clifford case. However, there is no transition in the entanglement phase in the Pauli case.

5. ANALYTICAL STUDY OF THE TRICRITICAL POINT

In this section, we analytically study the coherent information for the two types of sources in the coding regions near the tricritical point located at $(q_Z, q_U) = (1/2, 0)$. By restricting our analysis to the axis $q_U = 0$, the Open Random Unitary Model (ORUM) simplifies to a pure dephasing channel, rendering the spectrum of the output density matrices analytically tractable. We explicitly derive the coherent information $I_c(q_Z)$ for both the symmetry-respecting maximally-mixed source and the symmetry-breaking $\mathbb{Z}_2$ source. Explicit derivation of coherent information via statistical mechanics mapping for more sophisticated QEC codes subject to various other noise models can be found in Refs [81–83]. Our derivation demonstrates that both quantities vanish at the critical threshold $q_Z = 1/2$ and, crucially, that the channel capacity remains infinitely differentiable along this trajectory, showing the continuous nature of the zero-capacity critical point observed in the numerical optimization.

5.A. Coherent Information of the maximally mixed code along $q_U = 0$

Consider a single qubit in the maximally mixed state $\rho_1 = \mathbb{I}/2$. We purify this state with a reference qubit R into the Bell state in the X-basis:

$$\left|\Phi^+\right\rangle_{RS} = \frac{1}{\sqrt{2}} \left(\left|+\right\rangle_R \left|+\right\rangle_S + \left|-\right\rangle_R \left|-\right\rangle_S\right). \tag{S20}$$

The single-qubit dephasing channel $\mathcal{M}$ applies the Pauli-Z operator with probability $p = q/2$. In the X-basis, the Z operator acts as a bit-flip: $Z\left|+\right\rangle = \left|-\right\rangle$ and $Z\left|-\right\rangle = \left|+\right\rangle$. The action of the channel on the system qubit maps the initial state $\left|\Phi^+\right\rangle_{RS}$ to the orthogonal state $\left|\Psi^+\right\rangle_{RS}$:

$$(I \otimes Z)\left|\Phi^+\right\rangle_{RS} = \frac{1}{\sqrt{2}} \left(\left|+\right\rangle_R \left|-\right\rangle_S + \left|-\right\rangle_R \left|+\right\rangle_S\right) = \left|\Psi^+\right\rangle_{RS}. \tag{S21}$$

Thus, the output joint state is a classical mixture of these two orthogonal Bell states:

$$\rho'_{RS} = (1 - q_Z) \left|\Phi^+\right\rangle \left\langle\Phi^+\right| + q_Z \left|\Psi^+\right\rangle \left\langle\Psi^+\right|. \tag{S22}$$

Since $|\Phi^+\rangle$ and $|\Psi^+\rangle$ are orthogonal, the von Neumann entropy is simply the Shannon entropy of the mixing probabilities:

$$S(\rho'_{RS}) = H(q_Z) = -q_Z \log_2 q_Z - (1 - q_Z) \log_2 (1 - q_Z). \tag{S23}$$

The reduced state of the system remains maximally mixed, $\rho'_S = \mathbb{I}/2$, with entropy $S(\rho'_S) = 1$. The single-qubit coherent information is therefore $1 - H(q_Z)$. For the full system of n qubits, the result is:

$$I_c(q_Z) = n\left[1 - H(q_Z)\right]. \tag{S24}$$

We show the plot for I_c/n in Fig. S3a

5.B. Coherent information of the $\mathbb{Z}_2$ source along $q_U = 0$

The coherent information of the ORUM channel for the $\mathbb{Z}_2$ source at $q_U = 0$ can be computed analytically by determining the exact spectrum of the output density matrix. We consider the $\mathbb{Z}_2$ source state:

$$|\psi\rangle_{\mathbb{Z}_2,RS} = \frac{1}{\sqrt{2}} \left(|+ + \cdots +\rangle_{RS} + |- - \cdots -\rangle_{RS}\right). \tag{S25}$$

Tracing out the reference system R yields the mixed system state:

$$\rho_{\mathbb{Z}_2,S} = \frac{1}{2} \left(|+ + \cdots +\rangle_S \langle + + \cdots +|_S + |- - \cdots -\rangle_S \langle - - \cdots -|_S\right). \tag{S26}$$

Here, we consider the $q_U = 0$ caase, so a code ρ passes through n-uses of the single-qubit channel

$$\rho' = \mathcal{M}(\rho) = (1 - q_Z)\,\rho + q_Z Z \rho Z. \tag{S27}$$

After n-uses, we obtain

$$\rho' = \mathcal{M}^{(n)}(\rho) = \sum_{k=0}^{n} (1 - q_Z)^{n-k} (q_Z)^k \sum_{E_k} Z_{E_k} \rho Z_{E_k}, \tag{S28}$$

where the inner sum runs over all $\binom{n}{k}$ error configurations E_k consisting of Pauli-Z operators acting on k qubits. We define the probability weight for a weight-k error as:

$$\beta(k, q) = (1 - q_Z)^{n-k} (q_Z)^k. \tag{S29}$$

To find the spectrum of ρ', we analyze the action of the errors on the code basis states. Recall that the Pauli-Z operator flips the X-basis states: $Z|+\rangle = |-\rangle$ and $Z|-\rangle = |+\rangle$. Let $\mathbf{x} \in \{+, -\}^n$ denote a basis string in the X-basis.

- When a weight-k error Z_{E_k} acts on the term $|+ \cdots +\rangle \langle + \cdots +|$, it generates a state $|\mathbf{x}_k\rangle \langle \mathbf{x}_k|$ where the string $\mathbf{x}_k$ has exactly k minus signs.
- When a weight-j error Z_{E_j} acts on the term $|- \cdots -\rangle \langle - \cdots -|$, it generates a state $\left|\mathbf{x}'_j\right\rangle \left\langle \mathbf{x}'_j\right|$ where the string $\mathbf{x}'_j$ has j plus signs (and thus $n - j$ minus signs).

The output density matrix ρ' is diagonal in this $\{|\mathbf{x}\rangle\}$ basis. Consider a specific basis state $|\mathbf{x}\rangle$ containing k minus signs. This state is populated by two sources:

1. From the $|+ \cdots +\rangle$ branch: via an error of weight k. This occurs with probability $\frac{1}{2}\beta(k, q)$.
2. From the $|- \cdots -\rangle$ branch: via an error of weight $n - k$. This occurs with probability $\frac{1}{2}\beta(n - k, q_Z)$.

Thus, the eigenvalues of $\rho'_{\mathbb{Z}_2,S}$ depend only on the Hamming weight k of the basis state:

$$\lambda_k = \frac{1}{2}\left(\beta(k, q_Z) + \beta(n-k, q_Z)\right). \tag{S30}$$

Since there are $\binom{n}{k}$ such states for each weight k, the von Neumann entropy is:

$$\begin{aligned} S(\rho'_{\mathbb{Z}_2,S}) &= -\sum_{k=0}^{n} \binom{n}{k} \lambda_k \log_2(\lambda_k) \\ &= -\sum_{k=0}^{n} \binom{n}{k} \frac{\beta(k, q_Z) + \beta(n-k, q_Z)}{2} \log_2\left(\frac{\beta(k, q_Z) + \beta(n-k, q_Z)}{2}\right). \end{aligned} \tag{S31}$$

Next, we calculate the joint entropy $S(\rho'_{\mathbb{Z}_2,RS})$. Since the reference + system states $|+\cdots+\rangle$ and $|-\cdots-\rangle$ are orthogonal, they effectively label the two branches. The channel action does not mix these branches in the joint basis. The joint density matrix is block diagonal, equivalent to a classical mixture of the two error distributions. The eigenvalues are simply $\frac{1}{2}\beta(k, q)$ (occurring twice for each k, once for each branch). However, since we sum over the full distribution which is normalized, this simplifies to the Shannon entropy of the error distribution:

$$S(\rho'_{\mathbb{Z}_2,RS}) = -\sum_{k=0}^{n} \binom{n}{k} \beta(k, q_Z) \log_2(\beta(k, q_Z)) \equiv H(\beta). \tag{S32}$$

Finally, the coherent information $I_c = S(\rho'_{\mathbb{Z}_2,S}) - S(\rho'_{\mathbb{Z}_2,RS})$ is given explicitly by:

$$I_c(q_Z) = \sum_{k=0}^{n} \binom{n}{k} \left[\beta(k, q_Z) \log_2 \beta(k, q_Z) - \lambda_k \log_2 \lambda_k\right]. \tag{S33}$$

We plot this expression in Fig. S3b, where we can see that I_c for the $\mathbb{Z}_2$ source vanishes sharply at the critical point. In Fig. S3a, we saw that the coherent information of the maximally mixed state vanishes smoothly at the critical point. Indeed, all three phases, namely, the $\mathbb{Z}_2$ source coding phase, the maximally-mixed coding phase, and the no-coding phase, all meet together at this critical point; hence, it is a tri-critical point. Furthermore, in the limit of infinitely many uses, the maximally-mixed code maximizes the coherent information infinitesimally close to the critical point, so the channel capacity is also given by Fig. S3, an infinitely differentiable function.