Supplemental Information for "Macroscopic particle transport in dissipative long-range bosonic systems"

Hongchao Li,[1,*] Cheng Shang,[2,†] Tomotaka Kuwahara,[2,3,4,‡] and Tan Van Vu[5,§]

[1]*Department of Physics, University of Tokyo, 7-3-1 Hongo, Tokyo 113-0033, Japan*
[2]*Analytical quantum complexity RIKEN Hakubi Research Team, RIKEN Center for Quantum Computing (RQC), Wako, Saitama 351-0198, Japan*
[3]*PRESTO, Japan Science and Technology (JST), Kawaguchi, Saitama 332-0012, Japan*
[4]*RIKEN Cluster for Pioneering Research (CPR), Wako, Saitama 351-0198, Japan*
[5]*Center for Gravitational Physics and Quantum Information, Yukawa Institute for Theoretical Physics, Kyoto University, Kitashirakawa Oiwakecho, Sakyo-ku, Kyoto 606-8502, Japan*

CONTENTS

I. MACROSCOPIC PARTICLE TRANSPORT THEORY IN CLOSED QUANTUM SYSTEMS

In this section, we review the macroscopic particle transport theory in long-range closed quantum systems. Firstly, we show the proof of $\tau \geq \kappa_1^\varepsilon d_{XY}^{\alpha_\varepsilon}$. Since the density matrix follows the von Neumann equation, the dynamics of the boson number density $x_i(t) \coloneqq \mathrm{tr}(n_i \rho_t)/N$ is given by

$$\dot{x}_i = \frac{1}{N} \sum_{j, j\neq i} 2J_{ij}(t) \mathrm{Im}[\mathrm{tr}(b_j^\dagger b_i \rho_t)] =: \sum_{j(\neq i)} \phi_{ij}(t), \tag{S1}$$

where $\phi_{ij}(t)$ is the current flowing from the site j to the site i satisfying $\phi_{ij} = -\phi_{ji}$. On the one hand, since a fraction μ of bosons should be transported from X to Y in the time period τ, we have the relation

$$x_Y(\tau) - x_{X^c}(0) \geq \mu. \tag{S2}$$

Here, $X^c \coloneqq \Lambda \setminus X$ is the complement of X. Since we consider the Wasserstein distance between the final distribution $\boldsymbol{x}(\tau)$ and the initial distribution $\boldsymbol{x}(0)$, the coupling π_{mn} should satisfy $\sum_m \pi_{mn} = x_n(0)$ and $\sum_n \pi_{mn} = x_m(\tau)$. Accordingly, we obtain

$$x_Y(\tau) - x_{X^c}(0) = \sum_{i\in Y, j\in\Lambda} \pi_{ij} - \sum_{i\in\Lambda, j\in X^c} \pi_{ij} = \sum_{i\in Y, j\in X} \pi_{ij} - \sum_{i\in Y^c, j\in X^c} \pi_{ij} \leq \sum_{i\in Y, j\in X} \pi_{ij}. \tag{S3}$$

Hence, the Wasserstein distance can be lower bounded by

$$W(\boldsymbol{x}_0, \boldsymbol{x}_\tau) \geq \min_{i\in Y, j\in X} c_{ij} \sum_{i\in Y, j\in X} \pi_{ij} \geq \mu d_{XY}^{\alpha_\varepsilon}. \tag{S4}$$

On the other hand, since the current can be upper bounded by

$$|\phi_{ij}| \leq |J_{ij}(t)|(x_i + x_j). \tag{S5}$$

* lhc@cat.phys.s.u-tokyo.ac.jp
† cheng.shang@riken.jp
‡ tomotaka.kuwahara@riken.jp
§ tan.vu@yukawa.kyoto-u.ac.jp

With the Kantorovich-Rubinstein duality, we obtain

$$
\begin{aligned}
W(\boldsymbol{x}_0,\boldsymbol{x}_\tau) &\le \max_{\|h\|_L\le 1} h^T(\boldsymbol{x}(\tau)-\boldsymbol{x}(0)) = \max_{\|h\|_L\le 1}\sum_i h_i \int_0^\tau dt \sum_{j(\neq i)}\phi_{ij}(t)\\
&\le \frac{1}{2}\max_{\|h\|_L\le 1}\int_0^\tau dt\sum_{j\neq i}|\phi_{ij}||h_i-h_j| \le \frac{1}{2}\int_0^\tau dt\sum_{j\neq i}c_{ij}|\phi_{ij}(t)|.
\end{aligned}
\tag{S6}
$$

Here, we use $\|h\|_L\le 1$ to represent $|h_i-h_j|\le c_{ij}$. Therefore, the upper bound of the Wasserstein distance can be shown as

$$
\begin{aligned}
W(\boldsymbol{x}_0,\boldsymbol{x}_\tau) &\le \frac{1}{2}\int_0^\tau dt\sum_{j\neq i}c_{ij}|\phi_{ij}(t)| \le \sum_{i\neq j}J_{ij}c_{ij}\int_0^\tau x_i(t)dt \;= \int_0^\tau dt\sum_i x_i(t)\sum_{l=1}^{\infty}\sum_{j\in(i[l+1]\setminus i[l])}\frac{J}{\|i-j\|^{\alpha-\alpha_\varepsilon}}\\
&\le \int_0^\tau dt\sum_i x_i(t)\sum_{l=1}^{\infty}\frac{J}{l^{\alpha-\alpha_\varepsilon-D+1}} = J\varphi\zeta(\alpha-\alpha_\varepsilon-D+1)\tau.
\end{aligned}
\tag{S7}
$$

Hence,

$$
\tau \ge \frac{\mu}{J\varphi\zeta(\alpha-\alpha_\varepsilon-D+1)}d_{XY}^{\alpha_\varepsilon} =: \kappa_1^\varepsilon d_{XY}^{\alpha_\varepsilon}, \tag{S8}
$$

which is nothing but Eq. (5). Then, we move to prove the inequality $\langle P_{n_Y\ge N_0+\Delta N_0}\rangle_{\rho_\tau}\le\wp$, where

$$
\wp = (\Delta N_0 d_{XY}^{\alpha_\varepsilon})^{-1}NJ\varphi\zeta(\alpha-\alpha_\varepsilon-D+1)\tau. \tag{S9}
$$

By defining the probability distribution $p_{\vec N}(t) := \mathrm{tr}(\Pi_{\vec N}\rho_t)$, where $\Pi_{\vec N} = |\vec N\rangle\langle\vec N|$ be the projection onto the state $|\vec N\rangle$, the time evolution for $p_{\vec N}(t)$ can be derived from the von Neumann equation as

$$
\begin{aligned}
\dot p_{\vec N}(t) &= -i\mathrm{tr}\left(\Pi_{\vec N}\left[H_t,\rho_t\right]\right) = i\sum_{i\neq j}\langle\vec N|[\rho_t, J_{ij}(t)b_i^\dagger b_j]|\vec N\rangle = i\sum_{i\neq j}J_{ij}(t)\sqrt{n_i n_{j'}}(\langle\vec N|\rho_t|\vec N'\rangle-\langle\vec N'|\rho_t|\vec N\rangle)\\
&=: \sum_{i\neq j}\varphi_{\vec N\vec N'}(t),
\end{aligned}
\tag{S10}
$$

where $\varphi_{\vec N\vec N'}(t)$ represents all possible flows from state $|\vec N\rangle$ to state $|\vec N'\rangle$, which satisfies $n_i'=n_i-1, n_j'=n_j+1$ and $n_k'=n_k$ for all $k\neq i,j$. For the neighboring states $|\vec N\rangle$ and $|\vec N'\rangle$, the transport cost is defined as $c_{\vec N\vec N'} = \|i-j\|^{\alpha_\varepsilon}$. Following that, the cost between arbitrary two states $|\vec N\rangle$ and $|\vec M\rangle$ can be defined as the shortest-path cost over all possible paths connecting these states, $c_{\vec M\vec N} = \min\sum_{k=1}^K c_{\vec N_k,\vec N_{k-1}}$, where $c_{\vec N_0}=c_{\vec N}, c_{\vec N_K}=c_{\vec M}$, and $|\vec N_{k-1}\rangle$ and $|\vec N_k\rangle$ are neighboring states for all $1\le k\le K$. The Wasserstein distance reads:

$$
W(\boldsymbol{p}_0,\boldsymbol{p}_\tau) = \min_{\pi\in\mathcal{C}(\boldsymbol{p}_0,\boldsymbol{p}_\tau)}\sum_{\vec M,\vec N}c_{\vec M\vec N}\pi_{\vec M\vec N}, \tag{S11}
$$

where $\boldsymbol{p}_\tau$ and $\boldsymbol{p}_0$ represent the final and initial probability distributions, respectively. Similar to the previous analysis, we obtain

$$
W(\boldsymbol{p}_0,\boldsymbol{p}_\tau) \le \frac{1}{2}\int_0^\tau dt\sum_{\vec N}\sum_{i\neq j}c_{\vec N\vec N'}\left|\varphi_{\vec N\vec N'}(t)\right|. \tag{S12}
$$

By combining Eq. (S12) with the definition for $\varphi_{\vec N\vec N'}$ in Eq. (S10), the right-hand side of Eq. (S12) can be upper bounded as

$$
\frac{1}{2}\int_0^\tau\sum_{\vec N}\sum_{i\neq j}c_{\vec N\vec N'}|\varphi_{\vec N\vec N'}|dt \le \sum_{\vec N}\int_0^\tau dt p_{\vec N}(t)\sum_i n_i\sum_{j(\neq i)}\frac{J}{\|i-j\|^{\alpha-\alpha_\varepsilon}} \le N\tau J\varphi\zeta(\alpha-\alpha_\varepsilon-D+1), \tag{S13}
$$

where we apply the Cauchy-Schwarz inequality in the first inequality as

$$|\varphi_{\vec{N}\vec{N}'}(t)| \leq |J_{ij}|[n_i p_{\vec{N}}(t) + n_j' p_{\vec{N}'}(t)]. \tag{S14}$$

Next, we make the following definitions:

$$S_0 \coloneqq \{\vec{N}| \sum_{i\in X^c} n_i \leq N_0\}, \quad S_\tau \coloneqq \{\vec{N}| \sum_{i\in Y} n_i \geq N_0 + \Delta N_0\}. \tag{S15}$$

The process $\vec{N} \in S_0 \to \vec{M} \in S_\tau$ signifies that at least ΔN_0 particles are transported from region X to the region Y. To determine the lower bound of the Wasserstein distance, we first give the lower bound of $c_{\vec{M}\vec{N}}$ as

$$c_{\vec{M}\vec{N}} \geqslant \Delta N_0 \sum_{l=1}^{L} c_{\vec{N}_l,\vec{N}_{l-1}} \geqslant \Delta N_0 \sum_{l=1}^{L} \|i_l - i_{l-1}\|^{\alpha_\varepsilon} \geqslant \Delta N_0 d_{XY}^{\alpha_\varepsilon} \tag{S16}$$

with $\vec{N}_0 \to \cdots \to \vec{N}_L$ be a sequence of states that transfers one particle from X to Y. Following this, we obtain

$$W(\boldsymbol{p}_0, \boldsymbol{p}_\tau) = \min_\pi \sum_{\vec{M},\vec{N}} c_{\vec{M}\vec{N}} \pi_{\vec{M}\vec{N}} \geqslant \Delta N_0 d_{XY}^{\alpha_\varepsilon} \min_\pi \sum_{\vec{M}\in S_\tau, \vec{N}\in S_0} \pi_{\vec{M}\vec{N}}, \tag{S17}$$

and $\sum_{\vec{N}\in S_0} \pi_{\vec{M}\vec{N}} = p_{\vec{M}}(\tau)$ since $p_{\vec{N}}(0) = 0$ for $\vec{N} \notin S_0$. Using these facts, we obtain a lower bound for the Wasserstein distance:

$$W(\boldsymbol{p}_0, \boldsymbol{p}_\tau) \geqslant \Delta N_0 d_{XY}^{\alpha_\varepsilon} \sum_{\vec{M}\in S_\tau} p_{\vec{M}}(\tau) = \Delta N_0 d_{XY}^{\alpha_\varepsilon} \langle P_{n_Y \geqslant N_0+\Delta N_0}\rangle_{\rho_\tau}, \tag{S18}$$

where $P_{n_Y \geq N_0+\Delta N_0, N}$ is a projection operator given by

$$P_{n_Y \geq N_0+\Delta N_0} \coloneqq \sum_{\vec{N}:\langle\vec{N}|n_Y|\vec{N}\rangle \geq N_0+\Delta N_0} |\vec{N}\rangle\langle\vec{N}|, \tag{S19}$$

and $\langle P_{n_Y \geq N_0+\Delta N_0}\rangle_{\rho_\tau} \coloneqq \mathrm{Tr}[P_{n_Y \geq N_0+\Delta N_0}\rho_\tau]$ is the expectation value of the projection operator, which is nothing but the probability of finding $N_0 + \Delta N_0$ bosons in the region Y at time τ. Combining with Eq. (S13) and Eq. (S18) yields

$$\langle P_{n_Y \geqslant N_0+\Delta N_0}\rangle_{\rho_\tau} \leq \frac{NJ\varphi\zeta(\alpha - \alpha_\varepsilon - D + 1)\tau}{\Delta N_0 d_{XY}^{\alpha_\varepsilon}}, \tag{S20}$$

which is the same as Eq. (17).

II. TRIANGLE INEQUALITY OF THE GENERALIZED WASSERSTEIN DISTANCE

In this section, we aim to prove that the generalized Wasserstein distance defined in Eq. (20) satisfies the triangle inequality:

$$\tilde{W}(\boldsymbol{x}, \boldsymbol{y}) + \tilde{W}(\boldsymbol{y}, \boldsymbol{z}) \geq \tilde{W}(\boldsymbol{x}, \boldsymbol{z}), \tag{S21}$$

where $\|\boldsymbol{x}\|_1 \geq \|\boldsymbol{y}\|_1 \geq \|\boldsymbol{z}\|_1$.

Proof. We define $\boldsymbol{x}^{(1)}$ and $\boldsymbol{y}^{(1)}$ be the optimal vectors satisfying $\|\boldsymbol{x}^{(1)}\|_1 = \|\boldsymbol{y}^{(1)}\|_1$ to attain the generalized Wasserstein distance $\tilde{W}(\boldsymbol{x}, \boldsymbol{y})$, i.e., $W(\boldsymbol{x}^{(1)}, \boldsymbol{y}^{(1)}) = \tilde{W}(\boldsymbol{x}, \boldsymbol{y})$ where $W(\boldsymbol{x}^{(1)}, \boldsymbol{y}^{(1)})$ is defined as

$$W(\boldsymbol{x}^{(1)}, \boldsymbol{y}^{(1)}) \coloneqq \min_\pi \sum_{m,n} \pi_{mn}^{(1)} c_{mn}. \tag{S22}$$

Similarly, we can also define the vectors $\boldsymbol{y}^{(2)}$ and $\boldsymbol{z}^{(2)}$ to attain the generalized Wasserstein distance $\tilde{W}(\boldsymbol{y}, \boldsymbol{z})$ with

$\|\boldsymbol{y}^{(2)}\|_1 = \|\boldsymbol{z}^{(2)}\|_1$. Similarly, we also have

$$W(\boldsymbol{y}^{(2)}, \boldsymbol{z}^{(2)}) \coloneqq \min_{\pi} \sum_{m,n} \pi^{(2)}_{mn} c_{mn}. \tag{S23}$$

By defining $\pi_{mkn} \coloneqq \pi^{(1)}_{kn}\pi^{(2)}_{mk}/y^{(1)}_k$, we can verify that

$$\sum_n \pi_{mkn} = \sum_n \frac{\pi^{(1)}_{kn}\pi^{(2)}_{mk}}{y^{(1)}_k} = \pi^{(2)}_{mk}, \quad \sum_m \pi_{mkn} = \pi^{(1)}_{kn}\frac{y^{(2)}_k}{y^{(1)}_k} \le \pi^{(1)}_{kn}. \tag{S24}$$

By applying these two relations, we obtain

$$\begin{aligned} \tilde{W}(\boldsymbol{x}, \boldsymbol{y}) + \tilde{W}(\boldsymbol{y}, \boldsymbol{z}) &= W(\boldsymbol{x}^{(1)}, \boldsymbol{y}^{(1)}) + W(\boldsymbol{y}^{(2)}, \boldsymbol{z}^{(2)}) \\ &= \sum_{k,n} \pi^{(1)}_{kn} c_{kn} + \sum_{m,k} \pi^{(2)}_{mk} c_{mk} \ge \sum_{m,n,k} \pi_{mkn} c_{mn} = \sum_{m,n} \tilde{\pi}_{mn} c_{mn}, \end{aligned} \tag{S25}$$

where we apply the triangle inequality $c_{mk} + c_{kn} \ge c_{mn}$ and define a new coupling $\tilde{\pi}_{mn} \coloneqq \sum_k \pi_{mkn}$. The edge distributions of $\tilde{\pi}_{mn}$ are given by

$$\sum_n \tilde{\pi}_{mn} = \sum_{n,k} \pi_{mkn} = \sum_k \pi^{(2)}_{mk} = z^{(2)}_m, \quad \sum_m \tilde{\pi}_{mn} = \sum_{m,k} \pi_{mkn} = \sum_k \pi^{(1)}_{kn} \frac{y^{(2)}_k}{y^{(1)}_k} \eqqcolon \tilde{x}_n, \tag{S26}$$

where we have $\tilde{x}_n \le x^{(1)}_n \le x_n$, indicating $\boldsymbol{x} \succeq \tilde{\boldsymbol{x}}$. Consequently, we derive the following inequality:

$$\tilde{W}(\boldsymbol{x}, \boldsymbol{y}) + \tilde{W}(\boldsymbol{y}, \boldsymbol{z}) \ge \sum_{mn} \tilde{\pi}_{mn} c_{mn} \ge W(\tilde{\boldsymbol{x}}, \boldsymbol{z}^{(2)}) \ge \tilde{W}(\boldsymbol{x}, \boldsymbol{z}), \tag{S27}$$

which establishes the triangle inequality for the generalized Wasserstein distance and completes the proof. □