Supplementary Information for "Lorentz Skew Scattering and Giant Nonreciprocal Magneto-Transport"

CONTENTS

SUPPLEMENTARY NOTE 1: FORMULATION OF LSK

The Boltzmann equation can be solved by the method of successive approximation which collects terms at each order of fields and scattering strength. In the main text, substituting the distribution function (3) into the kinetic Boltzmann equation (1) and collecting terms at each order of fields and scattering strength, we obtain a set of coupled linear equations. For example, for terms linear in E ($i = 1$), we have

$$\hat{\mathcal{I}}_{\rm c} f^{(1,2)} = \hat{\mathcal{D}}_{\rm E} f^0, \qquad \hat{\mathcal{I}}_{\rm c} f_B^{(1,4)} = \hat{\mathcal{D}}_{\rm L} f^{(1,2)}, \tag{S1}$$

and the remaining equations share common forms of

$$\hat{\mathcal{I}}_{\rm c} f^{(1,j)} = -\hat{\mathcal{I}}_{\rm sk} f^{(1,j+1)} \qquad (j < 2), \tag{S2}$$

$$\hat{\mathcal{I}}_{\rm c} f_B^{(1,j)} = \hat{\mathcal{D}}_{\rm L} f^{(1,j-2)} - \hat{\mathcal{I}}_{\rm sk} f_B^{(1,j+1)} \qquad (j < 4). \tag{S3}$$

Similarly, one can write down the equations at E^2 ($i = 2$) order. These equations include

$$\hat{\mathcal{I}}_{\rm c} f^{(2,4)} = \hat{\mathcal{D}}_{\rm E} f^{(1,2)}, \qquad \hat{\mathcal{I}}_{\rm c} f_B^{(2,6)} = \hat{\mathcal{D}}_{\rm E} f_B^{(1,4)} + \hat{\mathcal{D}}_{\rm L} f^{(2,4)}, \tag{S4}$$

and the remaining equations share the common forms of

$$\hat{\mathcal{I}}_{\rm c} f^{(2,j)} = \hat{\mathcal{D}}_{\rm E} f^{(1,j-2)} - \hat{\mathcal{I}}_{\rm sk} f^{(2,j+1)} \qquad (j < 4), \tag{S5}$$

$$\hat{\mathcal{I}}_{\rm c} f_B^{(2,j)} = \hat{\mathcal{D}}_{\rm E} f_B^{(1,j-2)} + \hat{\mathcal{D}}_{\rm L} f^{(2,j-2)} - \hat{\mathcal{I}}_{\rm sk} f_B^{(2,j+1)} \qquad (j < 6). \tag{S6}$$

From these equations, we can successively solve each $f^{(2,j)}$ and $f_B^{(2,j)}$, and obtain the components needed for LSK contribution to non-reciprocal magneto-transport. Note that in our notation, one has $\hat{\mathcal{I}}_{\rm c} = \hat{\mathcal{I}}_{\rm c}^{(0,-2)}$ and $\hat{\mathcal{I}}_{\rm sk} = \hat{\mathcal{I}}_{\rm sk}^{(0,-3)}$. One can thus check that in each equation, the (E, V^{-1}, B) order is balanced on the two sides.

These equations allow us to sequentially solve $f^{(i,j)}$ and $f_B^{(i,j)}$ at each order. For instance, $f^{(1,2)} = \hat{\mathcal{I}}_{\rm c}^{-1}\hat{\mathcal{D}}_{\rm E} f^0 \sim \tau\hat{\mathcal{D}}_{\rm E} f^0$ is the familiar one responsible for Drude conductivity, $f_B^{(1,4)} = \hat{\mathcal{I}}_{\rm c}^{-1}\hat{\mathcal{D}}_{\rm L} f^{(1,2)} \sim \tau^2\hat{\mathcal{D}}_{\rm L}\hat{\mathcal{D}}_{\rm E} f_0$, and so on. Moreover, the structure of these equations enables a systematic diagrammatic approach, as explained in the main text.

SUPPLEMENTARY NOTE 2: RESULTS FOR 2D DIRAC MODEL

The [001] surface of topological crystalline insulators SnTe and $\mathrm{Pb_{1-x}Sn_xTe(Se)}$ hosts four massless Dirac points, which are protected by the presence of two mirror symmetries [1]. It was shown that at low temperature, the surface states undergoes a structural phase transition, which spontaneously breaks one of the mirrors (while the other remains intact). As a result, two of the surface Dirac cones become massive (gapped), while the other two remain massless [2]. These two massless Dirac points have zero Berry curvature and vanishing third-order skew scattering rate ω^a, thus do not contribute to the LSK transport. Therefore, one needs only consider the two gapped Dirac cones, which are described by the low-energy Hamiltonian

$$H = \tau w k_y + v_x k_x \sigma_y - \tau v_y k_y \sigma_x + \Delta\sigma_z. \tag{S7}$$

Here $\tau = \pm 1$ labels two Dirac cones connected by time-reversal operation $\mathcal{T}$, σ_i's are the Pauli matrices, and w, v and Δ are real model parameters. The mass term Δ opens a direct energy gap, rendering a finite Berry curvature. Such tilted Dirac cone dispersion has been observed in experiment [3].

To obtain analytic expressions for the LSK conductivity, we take the approximation $w/v \ll 1$, which is satisfied for the surface states of SnTe ($w/v = 0.1$ there [4]). For scattering, we consider short-range random impurities with disorder concentration n_i and average disorder strength V_0. Applying our theory, we find that the nonzero components of the $\chi^{\rm LSK}$ tensor are

$$\chi^{\rm LSK}_{yyyz} \approx -\frac{e^4\tau^4 n_i V_0^3 w\Delta(\mu^2-\Delta^2)(96\Delta^6 - 134\Delta^4\mu^2 + 5\Delta^2\mu^4 + 49\mu^6)}{128\pi\hbar^6\mu^8 v^2}, \tag{S8}$$

$$\chi^{\rm LSK}_{yxxz} \approx \frac{e^4\tau^4 n_i V_0^3 w\Delta(\mu^2-\Delta^2)(72\Delta^6 - 114\Delta^4\mu^2 + 21\Delta^2\mu^4 + 29\mu^6)}{128\pi\hbar^6\mu^8 v^2}, \tag{S9}$$

$$\chi^{\rm LSK}_{xyxz} \approx \frac{e^4\tau^4 n_i V_0^3 w\Delta(\mu^2-\Delta^2)(72\Delta^6 - 124\Delta^4\mu^2 + 61\Delta^2\mu^4 - 17\mu^6)}{128\pi\hbar^6\mu^8 v^2}, \tag{S10}$$

$$\chi^{\rm LSK}_{xxyz} \approx -\frac{e^4\tau^4 n_i V_0^3 w\Delta(\mu^2-\Delta^2)(96\Delta^6 - 152\Delta^4\mu^2 + 15\Delta^2\mu^4 + 57\mu^6)}{128\pi\hbar^6\mu^8 v^2}, \tag{S11}$$

where

$$\frac{1}{\tau} = \frac{n_i V_0^2}{4\hbar}\frac{\mu^2 + 3\Delta^2}{v^2\Delta} \tag{S12}$$

is the inverse transport relaxation time [5].

Given the geometric character of LSK mechanism, the LSK conductivity should be enhanced near the band edges, where the Berry curvature is concentrated. Thus, we give a closer look at the region where $\mu \sim \Delta$. By substituting Eq. (S12) into Eq. (S8)-(S11), the magnitudes of LSK conductivities for current response along y direction can be estimated as

$$\chi^{\rm LSK}_{yyyz} \sim -\frac{e^4 w}{2\pi^2\hbar^5 D}\frac{\tau^4}{\tau_{\rm sk}}, \tag{S13}$$

$$\chi^{\rm LSK}_{yxxz} \sim \frac{e^4 w}{4\pi^2\hbar^5 D}\frac{\tau^4}{\tau_{\rm sk}}, \tag{S14}$$

where $D = \Delta/2\pi v^2$ being the density of states at conduction band bottom, and

$$\frac{1}{\tau_{\rm sk}} = \int [\mathrm{d}\mathbf{k}]\omega^{(3a)}_{ll'}\sin\phi_{\mathbf{k}\mathbf{k}'} = \frac{\Delta(\mu^2-\Delta^2)n_i V_0^3}{8\hbar\mu v^4} \tag{S15}$$

is the characteristic time for skew scattering [6], with $\phi_{\mathbf{k}\mathbf{k}'}$ being the angle between wavevectors $\mathbf{k}$ and $\mathbf{k}'$.

SUPPLEMENTARY NOTE 3: COMPOSITION OF TWO SKEW SCATTERING

The composition of two skew scattering processes was first identified in Ref. [7] and the resultant current density can be expressed as

$$j_a = -\tau^4 \int [\mathrm{d}\mathbf{k}] v_a \Big[\hat{\mathcal{D}}_E \hat{\mathcal{I}}^{(0,-3)}_{\rm sk}\hat{\mathcal{I}}^{(0,-3)}_{\rm sk} + \hat{\mathcal{I}}^{(0,-3)}_{\rm sk}\hat{\mathcal{D}}_E\hat{\mathcal{I}}^{(0,-3)}_{\rm sk} + \hat{\mathcal{I}}^{(0,-3)}_{\rm sk}\hat{\mathcal{I}}^{(0,-3)}_{\rm sk}\hat{\mathcal{D}}_E\Big]\hat{\mathcal{D}}_E f_0. \tag{S16}$$

The mechanism can be also illustrated by our diagrammatic approach in the main text [see $f^{(2,2)}$ in Fig. 2]. This quadratic response may give rise to another $\sim E^2 B$ signal, where the magnetic field enters through Zeeman coupling, which modifies the wavefunction of Bloch electron. Such a contribution scales as $\tau^4/\tau_{\rm sk}^2$, thus is much smaller than the Lorentz skew scattering, which scales as $\tau^4/\tau_{\rm sk}$, in highly conductive materials where $\tau_{\rm sk}$ is large. For example, in the low temperature limit where only the static impurity scattering dominates, τ, $\tau_{\rm sk} \sim 1/n_i$, where n_i is the impurity concentration. In highly conductive materials n_i is very small hence the Lorentz skew scattering contribution (scales as n_i^{-3}) entirely dominates over the B-field corrected composition of two skew scattering (scales as n_i^{-2}). Our calculations on the surface-state transport of topological crystalline insulator SnTe also explicitly verify this conclusion, as pointed out in the main text.

[1] Y. Ando and L. Fu, Annu. Rev. Condens. Matter Phys. **6**, 361 (2015).

[2] Y. Okada, M. Serbyn, H. Lin, D. Walkup, W. Zhou, C. Dhital, M. Neupane, S. Xu, Y. J. Wang, R. Sankar, F. Chou, A. Bansil, M. Z. Hasan, S. D. Wilson, L. Fu, and V. Madhavan, Science **341**, 1496 (2013).

[3] Y. Tanaka, Z. Ren, T. Sato, K. Nakayama, S. Souma, T. Takahashi, K. Segawa, and Y. Ando, Nat. Phys. **8**, 800 (2012).

[4] I. Sodemann and L. Fu, Phys. Rev. Lett. **115**, 216806 (2015).

[5] Z. Z. Du, C. M. Wang, S. Li, H.-Z. Lu, and X. C. Xie, Nat. Commun. **10**, 3047 (2019).

[6] N. A. Sinitsyn, J. E. Hill, H. Min, J. Sinova, and A. H. MacDonald, Phys. Rev. Lett. **97**, 106804 (2006).

[7] Y.-X. Huang, C. Xiao, S. A. Yang, and X. Li, Phys. Rev. B **111**, 155127 (2025).