Collective Behavior and Memory States in Flow Networks with Tunable Bistability: Supplementary Information

I. PREISACH MODEL

The transition states of non-interacting hysterons are defined according to the Preisach model [1], which is generalized for any object with an input-output relationship that contains a multi-branch nonlinearity. Transitions between branches occur at extrema of this relationship, and for a network of hysterons all subject to the same global field H, one may write down switching field values for each hysteron in both the positive and negative direction, h_i^+ and h_i^-, such that when H is increasing, the next hysteron to transition to its upper branch will be the one with the minimum positive switching field among those hysterons that are currently in the lower branch, $\text{argmin}(h_i^+)$. Conversely, when H is decreasing, the next hysteron to transition to its lower branch will be the one with the maximum negative switching field among those hysterons that are currently in the upper branch, $\text{argmax}(h_i^-)$. Because hysterons do not interact with each other in this model, these switching fields are independent of network state and determined entirely by the properties of the hysteron itself.

In our system of NDR devices in series, such as the one depicted if Fig. S1(a), the IV curves $I(\Delta V_i)$ are the relevant input-output relationship, and the two positive differential resistance regions are the two branches that are generally accessible because the negative differential resistance region is generally unstable. The global field here is the current through this network, although we control it indirectly through the source voltage ΔV_T, and the switching fields of each NDR edge are the currents at the peak and valley of its respective IV curve, I^i_{max} and I^i_{min}. The IV curves depicted in Fig. S1(b) are such that $I^1_{max} > I^2_{max} > I^3_{max}$ and $I^1_{min} < I^2_{min} < I^3_{min}$. In this tuning configuration, the edges will switch up in sequential order and switch down in the same sequential order, giving rise to the transition graph depicted in Fig. S1(c). With this transition graph, every binary string state is accessible from the two saturated states, "000" and "111". It should be noted that, while in this demonstration we have placed the devices in geometric order along the series chain to correspond with their switching order, their physical placement in the circuit is irrelevant for switching order.

This ordering of positive and negative switching fields may be extended to any number of devices in series, although drawing the transition graphs becomes cumbersome. The demonstration in Fig. 4 obeys the same ordering, and therefore can access any binary string state by modifying the source voltage alone.

It was pointed out by [2] that a network of elements such as ours in series under voltage control would not obey the Priesach model explicitly, but would rather exhibit geometric interactions resulting from our indirect control of the true global switching field, I. In a model with interactions, the switching fields of each edge will depend on the network's state, leading to exotic effects like avalanches. While we do observe avalanches in our system for specific tuning configurations, we find that the non-interacting model is sufficient for the demonstration in Fig. 4, and interactions in this regime are negligible. For more information on avalanches in our system, refer to the Discussion and Supplemental Video 3.

II. THE ROLE OF CAPACITANCE

Section III describes the ideal behavior of a collection of bistable elements that respond to changes in the total voltage applied. There is a predicted discontinuous jump in ΔV as the element switches from the low-voltage state to the high-voltage state (and vice-versa). However, a truly discontinuous jump is not physical. In fact, a SPICE simulation of the system is unreliable unless a capacitor is added in parallel with each NDR element as shown in Fig. S2(a). The addition of this capacitor satisfies the solver regardless of how small it is, indicating that in the real system, the requirement is satisfied by parasitic capacitance. From the MOSFET data sheet, we estimate this value to be on the order of 1 pF.

To understand how the discontinuous jump and the capacitor are related, consider why we predicted a discontinuous jump: Kirchhoff's laws require that the current through each element in the circuit is always equal. What would it mean to relax this constraint? If there is a mismatch in the current through two consecutive elements, there must be an accumulation of charge on the wire between them. An ideal wire cannot store charge, but any real conductor has some (usually negligible) ability to store charge. Adding capacitors to the system and considering the effect as their capacitance approaches zero is a good model for ideal hysteron behavior.

Consider a system of NDR elements in series

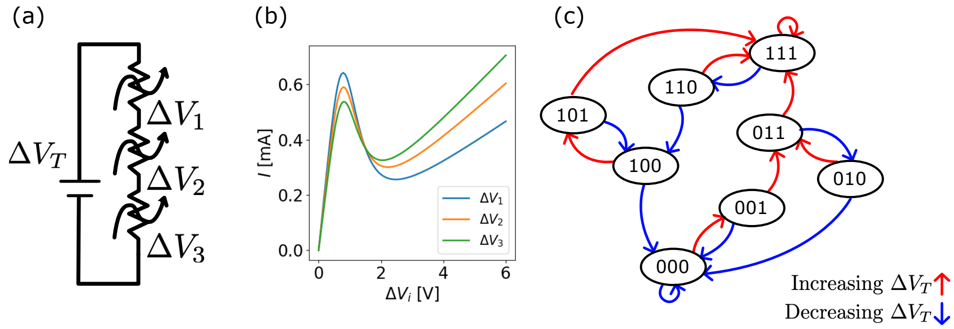

Figure S1. Preisach model for transition states of a three-edge network in series. (a) The network diagram shows three nonlinear edges in series, which each have a voltage drop ΔV_i but share the same current I. There is a total voltage drop across the network of ΔV_T. (b) IV characteristics of each of the three nonlinear elements. Edge 1's characteristic reaches the lowest peak I_{max} and the highest valley I_{min}, and each successive element has a higher peak and lower valley. (c) Resulting transition state graph from the peak/valley ordering. Because of the ordering described in (b), all binary string states are reachable from the two saturated states, "000" and "111".

where some are in the low-voltage state and some are in the high-voltage state, such as the one depicted in Fig. S2(a). There are capacitors in parallel across each NDR element, and all of the capacitors are the same size. An electrical node is defined as the point of connection between two or more circuit elements and all the conducting material (*e.g.*, wiring) that is directly connected and therefore held at the same voltage. The voltage drop ΔV_{ij} across the NDR element connecting nodes i and j is calculated as $V_j - V_i$. When some NDR elements are in one state and some are in the other, there is a voltage drop mismatch. This can be seen in Fig. S2(b) and (c) as the sharp kink that travels across the system over time. The charge on a capacitor is proportional to the voltage drop across it, so the capacitors corresponding to the elements in the high-voltage state must be more charged than those corresponding to the elements in the low-voltage state. Nodes 1 through 4 each include the negative plate of one capacitor and the positive plate of another, so the net charge on the node between an element in the high-voltage state and an element in the low-voltage state must be nonzero. Consider the governing equation for capacitance:

$$Q_{cap,ij} = C\Delta V_{ij} \tag{S.1}$$

We can use this to arrive at an equation that relates the voltage at each node to the total charge accumulated on it.

$$\begin{aligned} Q_i &= Q_{cap,i(i+1)} - Q_{cap,(i-1)i} \\ &= C\Delta V_{i(i+1)} - C\Delta V_{(i-1)i} \\ &= C(2V_i - V_{i-1} - V_{i+1}) \end{aligned}$$

$$\mathbb{Q} = C\hat{L}\mathbb{V}, \tag{S.2}$$

where $\mathbb{Q}$ is the vector containing the charges on each node, $\hat{L}$ is the graph laplacian, and $\mathbb{V}$ is the vector containing the voltages at each node. $C\hat{L} = \mathbb{C}$, where $\mathbb{C}$ is the Maxwell capacitance matrix for this system of interacting conductors.

As the total voltage provided to the system is ramped up and down, the charge accumulation travels between nodes. While a node is accumulating charge, the net current into it must be nonzero.

$$\frac{dQ_i}{dt} = I_{net,i} = I_{(i-1)i} - I_{i(i+1)} \tag{S.3}$$

Integrating (S.3) over time provides another way to calculate the charge on each node.

$$Q_i(t) = \int_0^t \left(I_{(i-1)i} - I_{i(i+1)}\right) dt \tag{S.4}$$

All of this hints at a switching timescale that depends directly on the capacitance. However, the exact form of this timescale is unclear due to the nonlinear nature of the elements. That said, parasitic capacitance is small enough that we can treat our experiments as quasistatic.

For small values of capacitance, the charge appears to move instantaneously from one node to the next. For slightly larger values of capacitance, we can observe a timescale. Fig. S2(d) shows the current mismatch between elements 2 and 3 as a function of time for two different values of capacitance. When the capacitance is doubled, the width of the peak increases, indicating that the amount of time it takes for charge to be transferred increases. Integrating the net current into a node with respect to time yields the amount of charge stored on that node, as shown in Fig. S2(e). Note that the slight deviation from zero between t = 20 s and t = 40 s is due to

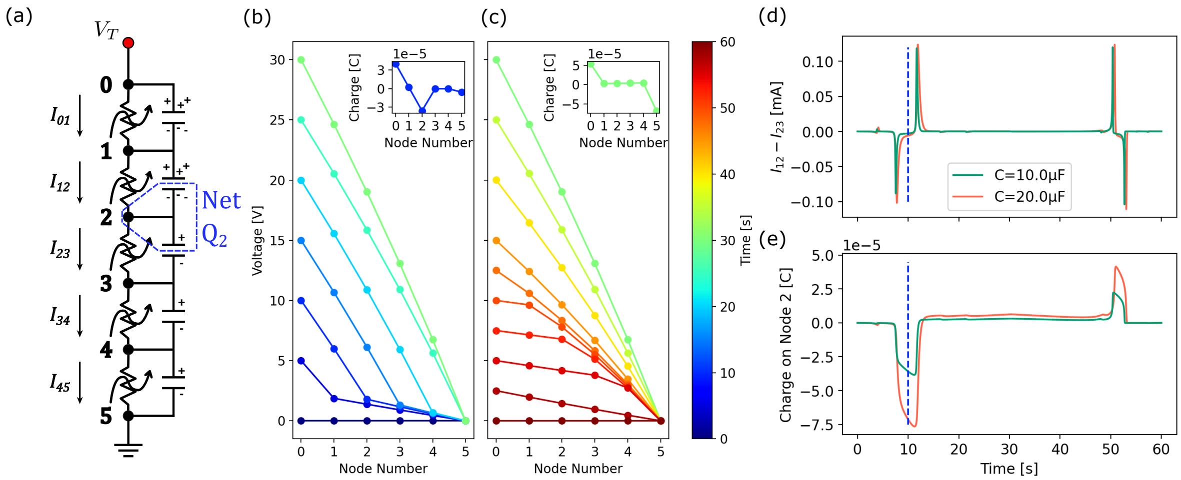

Figure S2. The effect of capacitance on network dynamics, as demonstrated with LTspice. (a) Circuit schematic. The charge indicated on each capacitor describes the system at time $t = 10$ s, when elements 01 and 12 are in the high voltage state and elements 23, 34, and 45 are in the low voltage state. The dotted blue line highlights the resulting net charge on node 2. (b)-(e) LTspice Circuit Simulator data. (b) Voltage at each node as the DC source voltage V_T is ramped up from 0 to 30 V. Inset: The net charge on each node at time $t = 10$ s. (c) Voltage at each node as V_T is ramped down from 30 V to 0. Inset: The net charge on each node when $t = 30$; all elements are in the high voltage state, so nodes 1-4 have zero net charge. (d) Net current into node 2 as a function of time for $C = 10\,\mu$F (green) and $C = 20\,\mu$F (orange). Blue dotted line indicated time $t = 10$ s. (e) Net charge on node 2 as a function of time as calculated by integrating (d) in time.

the slight differences in the tuning of the NDR elements. Unsurprisingly, the total amount of charge stored scales with the size of the capacitors.

III. SUPPLEMENTAL VIDEO 1: GEOMETRIC SOLUTION FOR THE CURRENT PASSING THROUGH A SMALL NETWORK

A NDR device is in series with a linear resistor, subject to a total voltage drop ΔV_T. As ΔV_T increases, the NDR moves from the low-voltage branch to a region of multistability, but only transitions to the high-voltage branch when that is the singly stable state, at sufficiently high ΔV_T. Conversely, when ΔV_T is decreasing, the NDR transitions down to the low-voltage branch once it has left the multistable region.

IV. SUPPLEMENTAL VIDEO 2: COLLECTIVE MEMORY IN A TUNED NETWORK OF NINE NONLINEAR ELEMENTS

The total source voltage V_0 is measured by a multimeter. First, V_0 is ramped from 0 V to 28 V then back down to 0 V. The LEDs of each element turn on in order from 1 to 9 while V_0 is increasing, and turn off in the same order while V_0 is decreasing, demonstrating the transition state ordering as dictated in Eq. (4).

The network undergoes the sequence in Fig. 4(e) to achieve the "+" state, shown in Fig. 4(c). The source voltage is then reset to 0 V and the network then undergoes the sequence in Fig. 4(f) to achieve the "×" state, shown in Fig. 4(d).

V. SUPPLEMENTAL VIDEO 3: ANOMALOUS AVALANCHES IN A DETUNED NETWORK OF NINE NONLINEAR ELEMENTS

When a network of nine NDR devices in series is detuned slightly, geometric interactions between devices can occur. Avalanches of the form *uu* (monotonic) and *uudddd* (dissonant) are observed, both of which occur while V¡sub¿0¡/sub¿ is increasing.

[1] Isaak D Mayergoyz and G Friedman. Generalized preisach model of hysteresis. *IEEE transactions on Magnetics*, 24(1):212–217, 1988.

[2] Dor Shohat and Martin van Hecke. Geometric control and memory in networks of bistable elements. *arXiv preprint arXiv:2409.07804*, 2024.