Supporting Information for

Intersectional Inequality Index (Triple I)

Guilherme Lichand, Thiago da Costa, Rodrigo Megale, and Gustavo Moraes

Guilherme Lichand.
E-mail: glichand@stanford.edu

This PDF file includes:

Figs. S1 to S47
Table S1

A. Proofs

This section proves that Triple I satisfies all desirable properties outlined in the main text. For conciseness, we do not carry the subscript s throughout the proofs.

Proposition 1. ***Non-anonymity**: Triple I (III_s) satisfies non-anonymity.*

Proof. Let

$$\text{III} = \frac{1}{2}\sum_{k\in K}(p_k - P_k)^2$$

Without loss of generality, let $y_{i'h} = 1$ and $y_{i''j} = 0$ (A1).

Let III^0 be Triple I *before* swapping i' and i''.

We have that:

$$2\text{III}^0 = \sum_{k\in k}(p_k - P_k)^2 = \underbrace{(p_h - P_h)^2}_{(1)} + \underbrace{(p_j - P_j)^2}_{(2)} + \sum_{k\in K\setminus\{h,j\}}(p_k - P_k)^2$$

Let $C_s \coloneqq \sum_{k\in K}\sum_{i\in k} Y_{isk}$, the cardinality of the success set ($Y = 1$) in s.

As such:

$$\begin{aligned}
(1):\quad (p_h - P_h)^2 &= \left(\frac{\sum_{i\in h} Y_{i,h}}{\sum_{k\in K}\sum_{i\in k} Y_{i,k}} - P_h\right)^2 \\
&= \left(\frac{\sum_{i\in h} Y_{i,h}}{C_s} - P_h\right)^2 \\
&\overset{A1}{=} \left(\frac{\sum_{i\in h\setminus i'} Y_{i,h} + 1}{C_s} - P_h\right)^2 \\
&= \left(\frac{1}{C_s} + \underbrace{\left(\frac{\sum_{i\in h\setminus i'} Y_{i,h}}{C_s} - P_h\right)}_{:=\sigma}\right)^2 \\
&= \left(\frac{1}{C_s}\right)^2 + 2\sigma\frac{1}{C_s} + \sigma^2,
\end{aligned}$$

and

$$\begin{aligned}
(2):\quad (p_j - P_j)^2 &= \left(\frac{\sum_{i\in j} Y_{i,j}}{\sum_{k\in K}\sum_{i\in k} Y_{i,k}} - P_j\right)^2 \\
&= \left(\frac{\sum_{i\in j} Y_{i,j}}{C_s} - P_j\right)^2 \\
&\overset{A1}{=} \left(\frac{\sum_{i\in j\setminus i''} Y_{i,j} + 0}{C_s} - P_j\right)^2 \\
&= \left(\underbrace{\frac{\sum_{i\in j\setminus i''} Y_{i,j}}{C_s} - P_j}_{:=\theta}\right)^2 \\
&= \theta^2
\end{aligned}$$

Finally, we have:

$$2\text{III}^0 = \left(\frac{1}{C_s}\right)^2 + 2\sigma\frac{1}{C_s} + \sigma^2 + \theta^2 + \sum_{k\in K\setminus\{h,j\}}(p_k - P_k)^2$$

Now let III^1 be Triple I *after* swapping i' and i''. First, note that since we are swapping one individual in h for one individual in j, P_h and P_j remain constant. Also, since we are not changing the total number of individuals with success outcomes, C_s remains constant too. Hence,

$$2\text{III}^1 = \sum_{k\in k} (p_k - P_k)^2 = \underbrace{\left(p'_h - P_h\right)^2}_{(3)} + \underbrace{\left(p'_j - P_j\right)^2}_{(4)} + \sum_{k\in K\setminus\{h,j\}} (p_k - P_k)^2 \,,$$

where

$$\begin{aligned}(3):\quad (p'_h - P_h)^2 &= \left(\frac{\sum_{i\in h} Y_{i,h}}{\sum_{k\in K}\sum_{i\in k} Y_{i,k}} - P_h\right)^2 \\ &= \left(\frac{\sum_{i\in h} Y_{i,h}}{C_s} - P_h\right)^2 \\ &\overset{A1}{=} \left(\frac{\sum_{i\in h\setminus i''} Y_{i,h} + 0}{C_s} - P_h\right)^2\end{aligned}$$

Since $\sum_{i\in h\setminus i''} Y_{i,h} = \sum_{i\in h\setminus i'} Y_{i,h}$

,

$$\begin{aligned}&= \left(\underbrace{\frac{\sum_{i\in h\setminus i'} Y_{i,h}}{C_s} - P_h}_{=\sigma}\right)^2 \\ &= \sigma^2,\end{aligned}$$

and

$$\begin{aligned}(4):\quad (p'_j - P_j)^2 &= \left(\frac{\sum_{i\in j} Y_{i,j}}{\sum_{k\in K}\sum_{i\in k} Y_{i,k}} - P_j\right)^2 \\ &= \left(\frac{\sum_{i\in j} Y_{i,j}}{C_s} - P_j\right)^2 \\ &\overset{A1}{=} \left(\frac{\sum_{i\in j\setminus i'} Y_{i,j} + 1}{C_s} - P_j\right)^2\end{aligned}$$

Since $\sum_{i\in j\setminus i'} Y_{i,j} = \sum_{i\in j\setminus i''} Y_{i,j}$

,

$$\begin{aligned}&= \left(\frac{1}{C_s} + \underbrace{\left(\frac{\sum_{i\in j\setminus i''} Y_{i,j}}{C_s} - P_j\right)}_{=\theta}\right)^2 \\ &= \left(\frac{1}{C_s}\right)^2 + 2\theta\frac{1}{C_s} + \theta^2\end{aligned}$$

Thus,

$$2\text{III}^1 = \sigma^2 + \left(\frac{1}{C_s}\right)^2 + 2\theta\frac{1}{C_s} + \theta^2 + \sum_{k\in K\setminus\{h,j\}} (p_k - P_k)^2$$

Taking the difference between 2III^1 and 2III^0, we arrive at

$$2 \cdot \mathrm{III}^1 - 2 \cdot \mathrm{III}^0 = \left(\sigma^2 + \left(\frac{1}{C_s}\right)^2 + 2\theta\frac{1}{C_s} + \theta^2 + \sum_{k \in K \setminus \{h,j\}} (p_k - P_k)^2\right)$$

$$- \left(\left(\frac{1}{C_s}\right)^2 + 2\sigma\frac{1}{C_s} + \sigma^2 + \theta^2 + \sum_{k \in K \setminus \{h,j\}} (p_k - P_k)^2\right)$$

$$= 2\theta\frac{1}{C_s} - 2\sigma\frac{1}{C_s}$$

$$\mathrm{III}^1 - \mathrm{III}^0 = \frac{1}{C_s}(\theta - \sigma)$$

Substituting the expressions for σ and θ, we get:

$$\mathrm{III}^1 - \mathrm{III}^0 = \frac{1}{C_s}\left[\left(\frac{\sum_{i \in j \setminus i''} Y_{i,j}}{C_s} - P_j\right) - \left(\frac{\sum_{i \in h \setminus i'} Y_{i,h}}{C_s} - P_h\right)\right]$$

Since the non-anonymity property guarantees that

$$\left(\frac{\sum_{i \in h \setminus i'} Y_{i,h}}{C_s} - P_h\right) \neq \left(\frac{\sum_{i \in j \setminus i''} Y_{i,j}}{C_s} - P_j\right),$$

we have proved that

$$\mathrm{III}^1 \neq \mathrm{III}^0$$

□

Proposition 2. ***Exchanges**: Triple I (III_s) satisfies exchanges.*

Proof. Given the binary nature of the outcome, $Y_{i'hs} > Y_{i''js} \Leftrightarrow Y_{i'hs} = 1 \quad \text{and} \quad Y_{i''js} = 0$. This makes exchanges a special case of the non-Anonymity property.

In 1, we had that $\mathrm{III}^1 - \mathrm{III}^0$ (the difference in Triple I when swapping i' and i'') is:

$$\mathrm{III}^1 - \mathrm{III}^0 = \frac{1}{C_s}\left[\left(\frac{\sum_{i \in j \setminus i''} Y_{i,j}}{C_s} - P_j\right) - \left(\frac{\sum_{i \in h \setminus i'} Y_{i,h}}{C_s} - P_h\right)\right]$$

The exchanges property requires that $p_h - P_h > p_j - P_j + \frac{1}{C_s}$. Specializing it for Triple I yields:

$$p_h - P_h > p_j - P_j + \frac{1}{C_s} \Leftrightarrow \frac{\sum_{i \in h \setminus i'} Y_{i,h} + 1}{C_s} - P_h > \frac{\sum_{i \in j \setminus i''} Y_{i,j} + 0}{C_s} - P_j + \frac{1}{C_s}$$

$$\Leftrightarrow \frac{\sum_{i \in h \setminus i'} Y_{i,h}}{C_s} - P_h > \frac{\sum_{i \in j \setminus i''} Y_{i,j}}{C_s} - P_j + \frac{1}{C_s} - \frac{1}{C_s}$$

$$\Leftrightarrow \frac{\sum_{i \in h \setminus i'} Y_{i,h}}{C_s} - P_h > \frac{\sum_{i \in j \setminus i''} Y_{i,j}}{C_s} - P_j$$

So, we can conclude that

$$\mathrm{III}^1 - \mathrm{III}^0 < 0$$

$$\mathrm{III}^1 < \mathrm{III}^0$$

□

Proposition 3. ***Group symmetry**: Triple I (III_s) satisfies group symmetry.*

Proof. This is trivially true for the Triple I, since:

$$|p_{hs} - P_{hs}| = |p_{js} - P_{js}| \Leftrightarrow (p_{hs} - P_{hs})^2 = (p_{js} - P_{js})^2$$

□

Proposition 4. ***Robustness to mergers and splits**: Triple I (III_s) satisfies robustness to mergers and splits.*

Proof. The goal is to show that Triple I's computation does not mechanically increase or decrease with the number of social groups K. It suffices to prove that as K increases, III could either increase, decrease or stay constant, depending on the specifics of population and success partitions.

Let $K = 2$, with social groups h and j. Let $p_h = \alpha$ and $p_j = 1 - \alpha$, and $P_h = \beta$, $P_j = 1 - \beta$

We have:

$$\begin{aligned} \text{III}_0 &= \frac{1}{2}\left[(\alpha - \beta)^2 + ((1 - \alpha) - (1 - \beta))^2\right] \\ &= (\alpha - \beta)^2 \end{aligned}$$

Now, let us partition h further into h_1 ($p_{h_1} = \alpha_1$ and $P_{h_1} = \beta_1$) and h_2 ($p_{h_2} = \alpha_2$ and $P_{h_2} = \beta_2$), with j as before. Computed with $K = 3$, Triple I becomes:

$$\begin{aligned} \text{III}_1 &= (\alpha_1 - \beta_1)^2 + (\alpha_2 - \beta_2)^2 + ((1 - \alpha) - (1 - \beta))^2 \\ &= (\alpha_1 - \beta_1)^2 + ((\alpha - \alpha_1) - (\beta - \beta_1))^2 + (\beta - \alpha)^2 \\ &= 2(\alpha_1 - \beta_1)^2 + 2(\alpha - \beta)^2 - 2(\alpha - \beta)(\alpha_1 - \beta_1) \end{aligned}$$

As such,

$$\begin{aligned} \Delta\text{III} &= \text{III}_1 - \text{III}_0 \\ &= 2(\alpha_1 - \beta_1)^2 - 2(\alpha - \beta)(\alpha_1 - \beta_1) \\ &= 2(\alpha_1 - \beta_1)\left((\alpha_1 - \beta_1) - (\alpha - \beta)\right) \end{aligned}$$

We analyze the conditions for ΔIII's each possible sign below:

- Invariance; i.e., $\Delta\text{III} = 0$:

  $$(\alpha_1 - \beta_1)\left((\alpha_1 - \beta_1) - (\alpha - \beta)\right) = 0$$

  Triple I is invariant to the increase in K if at least one of the following conditions is satisfied:

  1. $\alpha_1 = \beta_1$

  2. $\alpha_1 - \beta_1 = \alpha - \beta$

- Increasing; i.e., $\Delta\text{III} > 0$:

  $$(\alpha_1 - \beta_1)\left((\alpha_1 - \beta_1) - (\alpha - \beta)\right) > 0$$

  Triple I increases with K if one of the following conditions is satisfied:

  1. $\alpha_1 - \beta_1 > 0$ and $\alpha_1 - \beta_1 > \alpha - \beta$

  2. $\alpha_1 - \beta_1 < 0$ and $\alpha_1 - \beta_1 < \alpha - \beta$

- Decreasing; i.e., $\Delta\text{III} < 0$:

  $$(\alpha_1 - \beta_1)\left((\alpha_1 - \beta_1) - (\alpha - \beta)\right) < 0$$

  Triple I decreases with K if one of the following conditions is satisfied:

1. $\alpha_1 - \beta_1 < 0$ and $\alpha_1 - \beta_1 > \alpha - \beta$

2. $\alpha_1 - \beta_1 > 0$ and $\alpha_1 - \beta_1 < \alpha - \beta$

Now we proceed to show that all conditions are achievable depending on the population and success partitions.

Let $\alpha_1 = x\alpha$, with $x \in (0,1)$ and $\beta_1 = y\beta$ with $y \in (0,1)$.

- $\Delta\text{III} = 0$:

  1. $\alpha_1 = \beta_1 \implies x\alpha = y\beta$
     So we have the following conditions

$$\begin{cases} \alpha = \beta \implies x = y \\ \alpha \neq \beta \implies x = y\dfrac{\beta}{\alpha} \quad \text{with} \quad y < \dfrac{\alpha}{\beta} \end{cases}$$

  2. $\alpha_1 - \beta_1 = \alpha - \beta \implies x\alpha - y\beta = \alpha - \beta$
     In this case, we need that

$$x = \alpha - \beta(1-y)$$

     Since $x, y \in (0,1)$, the expression above needs to satisfy the following:

$$\alpha - \beta(1-y) > 0 \quad \text{and} \quad \alpha - \beta(1-y) < 1$$
$$1 - \frac{\alpha}{\beta} < y < 1 - \frac{\alpha}{\beta} + \frac{1}{\beta}$$

- $\Delta\text{III} > 0$:

  1. $\alpha_1 - \beta_1 > 0$ and $\alpha_1 - \beta_1 > \alpha - \beta$

     We need that:

     $\underbrace{x\alpha - y\beta > 0}_{(A1)}$ and $\underbrace{x\alpha - y\beta > \alpha - \beta}_{(A2)}$

     (A1) : $x > y\frac{\beta}{\alpha}$ with $y < \frac{\alpha}{\beta}$

     (A2) : $x > 1 - (1-y)\frac{\beta}{\alpha}$

     Since (A2) is less restrictive, the final condition is:

     $1 - (1-y)\frac{\beta}{\alpha} < x < 1 \quad \text{and} \quad 0 < y < 1$

  2. $\alpha_1 - \beta_1 < 0$ and $\alpha_1 - \beta_1 < \alpha - \beta$

     We need that:

     $\underbrace{x\alpha - y\beta < 0}_{(A3)}$ and $\underbrace{x\alpha - y\beta < \alpha - \beta}_{(A4)}$

     (A3) : $x < y\frac{\beta}{\alpha}$

     (A4) : $x < 1 - (1-y)\frac{\beta}{\alpha}$ with $y > \frac{\alpha}{\beta} - 1$

     As such, the final restrictions are given by:

$$\begin{cases} 0 < x < y\dfrac{\beta}{\alpha} \quad \text{and} \quad 0 < y < 1 \quad \text{if} \quad \beta > \alpha \\ 0 < x < x < 1 - (1-y)\dfrac{\beta}{\alpha} \quad \text{and} \quad \dfrac{\alpha}{\beta} - 1 < y < 1 \quad \text{if} \quad \beta < \alpha \end{cases}$$

- $\Delta\text{III} < 0$:

  1. $\alpha_1 - \beta_1 < 0$ and $\alpha_1 - \beta_1 > \alpha - \beta$

     We need that:

     $$\underbrace{x\alpha - y\beta < 0}_{(A5)} \text{ and } \underbrace{x\alpha - y\beta > \alpha - \beta}_{(A6)}$$

     $$\text{(A5)}: \; x < y\frac{\beta}{\alpha}$$

     $$\text{(A6)}: \; x > 1 - (1-y)\frac{\beta}{\alpha}$$

     $$\text{(A5)} + \text{(A6)}: \; 1 - (1-y)\frac{\beta}{\alpha} < x < y\frac{\beta}{\alpha} \quad \text{and} 0 < y < 1 \quad \text{, achievable when} \quad \beta > \alpha$$

  2. $\alpha_1 - \beta_1 > 0$ and $\alpha_1 - \beta_1 < \alpha - \beta$

     We need that:

     $$\underbrace{x\alpha - y\beta > 0}_{(A7)} \text{ and } \underbrace{x\alpha - y\beta < \alpha - \beta}_{(A8)}$$

     $$\text{(A7)}: \; x > y\frac{\beta}{\alpha} \text{ with } y < \frac{\alpha}{\beta}$$

     $$\text{(A8)}: \; x < 1 - (1-y)\frac{\beta}{\alpha} \text{ with } y > 1 - \frac{\alpha}{\beta}$$

     $$\text{(A7)} + \text{(A8)}: \; y\frac{\beta}{\alpha} < x < 1 - (1-y)\frac{\beta}{\alpha} \quad \text{and} 0 < y < 1 \quad \text{, achievable when} \quad \beta < \alpha$$

□

B. Monte Carlo simulations of Triple I estimates by site size (N)

In this section, we investigate how sample sizes affect Triple I estimates relative to the population Triple I (the ‘ground truth’).

We ran simulations across three different sites, varying the ‘ground truth’ across them: low ($III = 0$), intermediate ($III = 8.125$), and high ($III = 37.87$). These figures are merely illustrative and stylized: we assume that all groups have identical population shares across all sites; ‘low’ reflects a setting in which each group’s success share is identical to its population share; ‘medium’, one in which one group’s success share is 30%, another is 25%, another is 20%, another 15%, yet another is 10%, and all others’ are 0%; and ‘high’, one in which one group’s success share is 90%, another is 10%, and all others’ are 0%. For each site, we generated samples of 20, 40, 80, 160, 240, 400, 600, and 1000 individuals. Results for each sample size are averaged across 500 repetitions.

Following the main text, each individual belongs to one out of 16 social groups. We closely replicate the decision tree used to compute Triple I in the main text: once the sample is drawn, we require at least five observations of each social group to compute Triple I ($M = 5$); if some group does not have at least five observations in that site, we aggregate sequentially until that condition is met (to K=8, then K=4 and, finally, K=2). As such, when we simulate sample sizes below 80, we always compute Triple I with $K < 16$.

Figure S1 plots the bias in Triple I’s estimates by site size and by inequality level. Based on the maximum number of groups, when sample sizes are below 80, bias is larger. For sites larger than 160 observations, the sample bias is already close to zero.

The simulations suggest a trade-off between the granularity of social groups and bias. A larger number of groups requires larger samples to better approximate population parameters. Nonetheless, past aggregation issues ($N > 80$, for $K = 16$ and $M = 5$), the sample Triple I slightly overestimates inequities as it converges to its population value. Such overestimation in small samples is akin to the small-sample properties of alternative inequality indicators widely used in the segregation literature.

C. Data and methodology

C1. US data. For the United States, the data are drawn from the US decennial censuses, which are conducted by the US Census Bureau. These censuses provide comprehensive demographic, social, and economic information on the U.S. population, and the IPUMS International database standardizes these variables to enable cross-national comparisons. We utilize a variety of harmonized variables in the analysis (reported as labelled by IPUMS). The **household variables** include `COUNTRY` (country), `YEAR` (year), `GEO1_US` (state-level geographic identifier), `GEO2_US` (consistent PUMA for 2000-2020), `GEO2ALT_US` (alternate consistent PUMA for 1980-2010), `OWNERSHIP` (ownership of dwelling), `FUELHEAT` (fuel for heating), `PHONE` (telephone availability), `AUTOS` (automobiles available), `ROOMS` (number of rooms), and `BEDROOMS` (number of bedrooms). The **person-level variables** include `PERWT` (person weight), `AGE` (age), `SEX` (sex), `RACE` (race or color), `EMPSTAT` (employment status).

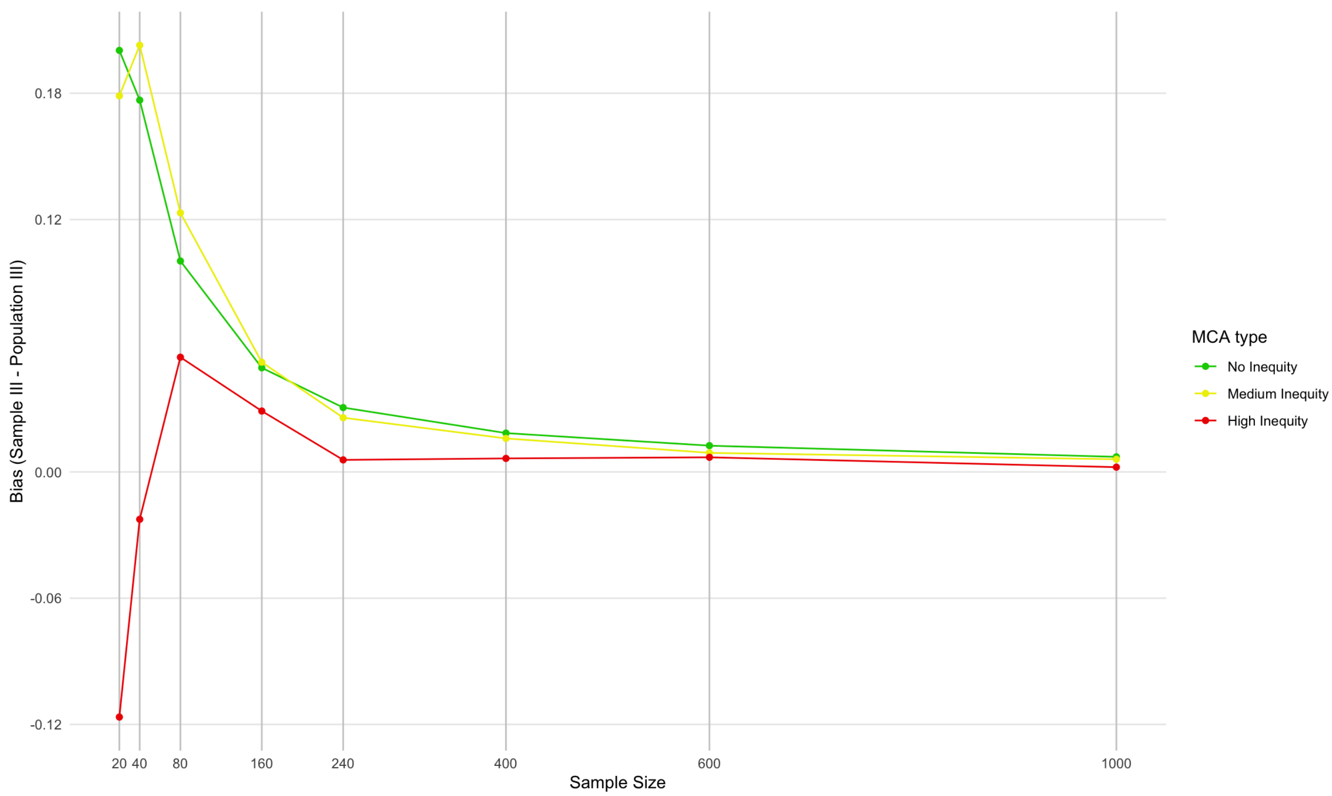

Fig. S1. Monte Carlo Simulations for Sample Bias in Triple I

Notes: The graph computes the average sample bias (Sample III - Population III) for different sample sizes and population parameters. We ran 500 repetitions for each sample size and utilized 3 different population parameters for Unemployment Triple I: $III_{high} = 37.87$, $III_{medium} = 8.125$, $III_{low} = 0$.

C2. Brazil data. For Brazil, the data are obtained from the censuses conducted by the Brazilian Institute of Geography and Statistics (IBGE), for the years of 1980, 1991, 2000 and 2010. These censuses provide detailed information on the Brazilian population, and, like the U.S. data, are harmonized by IPUMS International to ensure comparability over time and across countries. We utilize a variety of harmonized variables in the analysis (reported as labelled by IPUMS). The **household variables** include `COUNTRY` (country of residence), `YEAR` (year of data collection), `GEO1_BR` (state-level geographic identifier), `GEO2_BR` (MCA-level geographic identifier, consistent from 1980 through 2010), `OWNERSHIP` (ownership of dwelling), `ELECTRIC` (electricity availability), `WATSUP` (water supply type), `SEWAGE` (sewage availability), `PHONE` (telephone availability), `AUTOS` (automobiles available), `REFRIG` (refrigerator availability), `TV` (television set), `RADIO` (radio in household), `ROOMS` (number of rooms), `BEDROOMS` (number of bedrooms), `BATHROOMS` (number of bathrooms), and `WALL` (wall or building material). The **person-level variables** include `PERWT` (person weight), `AGE` (age), `SEX` (sex), `RACE` (race or color), `EMPSTAT` (employment status).

C3. Main variables and social groups. Unemployment status:
To construct our unemployment variable, we restrict attention to 25 to 55 year-olds and participants in the labor force. Unemployment status is determined by the `EMPSTAT` variable, which categorizes individuals according to their employment status.

Race:
We utilize two levels of aggregation for race. In the more refined one, we separate individuals as follows:

- White: if the individual is reported as white.
- Black or brown: if the individual is reported as black or brown.[*]
- Asian: if the individual is reported as Asian. [†]
- Under-represented minority (URM): any race not reported above is classified as URM.

For the second level of aggregation, we pool whites and Asians, and blacks, browns and URMs.

Socioeconomic status:
The socioeconomic status (SES) variable is constructed using Principal Component Analysis (PCA) based on a set of household attributes. For the USA, we utilize the following variables: household ownership, house has fuel heating, telephone ownership, number of automobiles owned, total rooms and bedrooms in the house. For Brazil, we utilize the following: household ownership, house has sewage treatment, water supply and masonry built, refrigerator, television, telephone and radio ownership, number of automobiles owned, total rooms, bedrooms and bathrooms in the house.[‡]

To account for changes in the importance of these variables over time, the PCA is applied separately for each year. First, the selected variables are standardized within each year. Then, we proceed with the PCA calculation, using the *prcomp* function in R. The first principal component, which captures the greatest variation among the standardized variables, is extracted and used as the SES index. Finally, for each year we calculate the median of the SES index to classify individuals according to its relative position, separating them in the Top 50% and Bottom 50% of the SES index.

Social groups:
In our analysis, we utilize 4 levels of aggregation to social groups.

- $K = 16$**:** gender (male / female), race/ethnicity (Asian / black or brown / white / URM), and SES (below / above median wealth).
- $K = 8$**:** gender (male / female) $\times$ race/ethnicity (black, brown or URM / Asian or white) $\times$ SES (below / above median wealth)
- $K = 4$**:** gender (male / female) $\times$ race/ethnicity (black, brown or URM / Asian or white)
- $K = 2$**:** gender (male / female)

A. C4. Shapley values. The unweighted shapley values are calculated according to equation (**??**). For each group, shapley values are obtained by excluding the social group of the MCA population and redistributing its total population and unemployed population homogeneously across all remaining groups. Then, the Triple I calculation is redone. Even in MCAs where we are not utilizing $k = 16$ social groups, we calculate shapley values utilizing those groups. The computation is the similar, we remove the group from the sample and redistribute its population and unemployed population equally among the other groups. The additional step is to follow the decision tree and aggregate to the level in which we have at least 5 observations for each group, then we proceed with the computation of the new Triple I. Finally to obtain the group weighted shapley values ($S^w(k)$ we divide $S(k)$ by P_k, that is, $S^w(k) = \frac{S(k)}{P_k}$. The calculation is performed at the MCA level and then aggregated for the US utilizing MCA population weights.

[*] Only Brazil separates blacks and browns, so to maintain comparability to the USA, we aggregate both races.
[†] Here we include Chinese, Japanese, Korean, Vietnamese, Filipino, Indian, Pakistani, Bangladeshi, and other Asians.
[‡] For 1980, we do not have data on the number of bathrooms and for 2000 we do not have data on masonry, so we did not use them in those years.

274 D. Descriptive statistics

275 In this section, we provide descriptive statistics to complement our analysis.

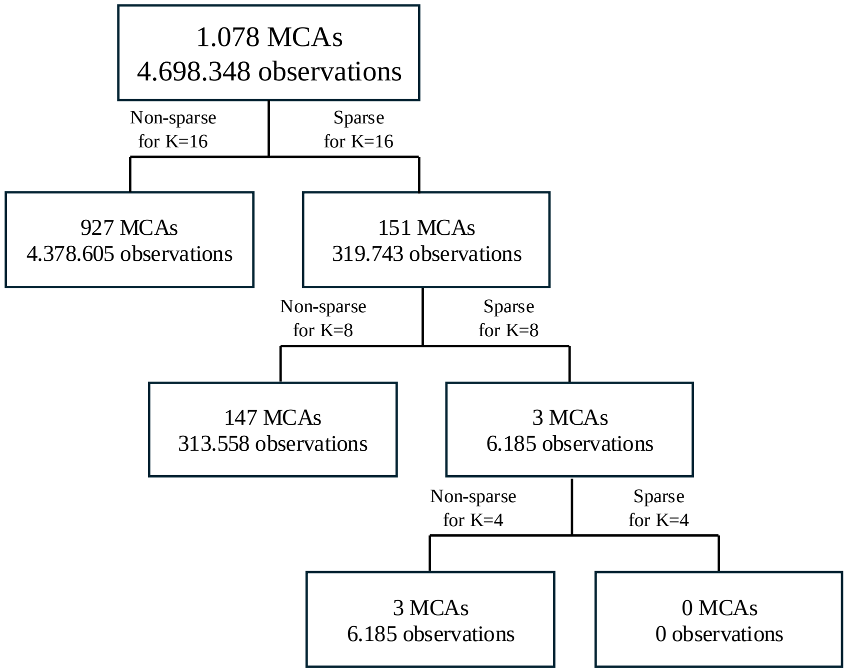

Fig. S2. Decision Tree for USA, 2020

This figure represents the decision tree for how we aggregate individuals for the calculation of the Unemployment Triple I for the USA, 2020. If a MCA does not satisfy that criterion using the more refined population partition (k=16), we use the this decision tree to compute the indicator in these cases, with a progressively coarser definition of social groups.

Table S1. Proportion (%) of MCAs by level of aggregation in Triple I's computation

	BRA				USA				
Level of Aggregation	1980	1991	2000	2010	1980	1990	2000	2010	2020
Race-Gender-Wealth	0.0	0.9	6.3	12.6	52.7	62.6	89.5	62.4	86.1
Race Aggregated - Gender - Wealth	89.8	96.3	93.3	87.2	42.9	34.4	10.3	35.0	13.6
Race Aggregated - Gender	8.8	2.7	0.4	0.2	3.1	2.6	0.2	2.4	0.3
Gender	1.4	0.1	0.0	0.0	1.3	0.4	0.0	0.2	0.0
Total	100	100	100	100	100	100	100	100	100

E. Additional results: US

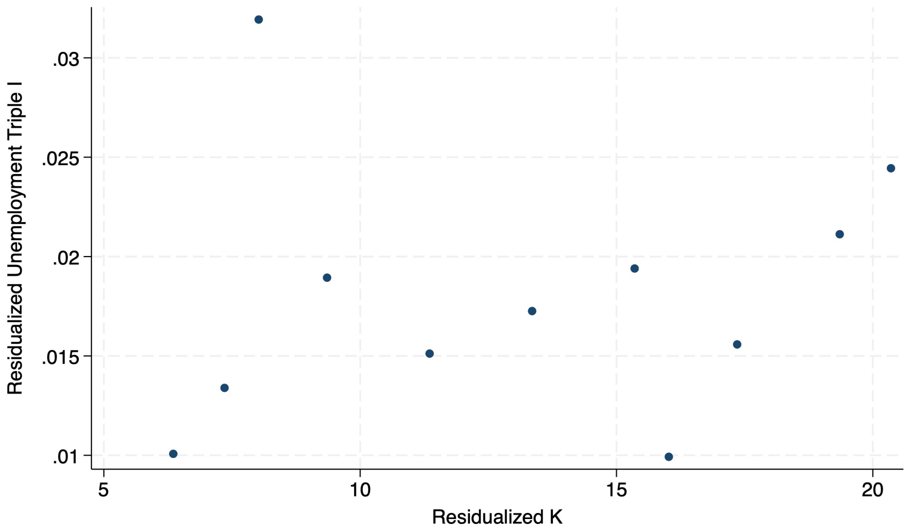

Fig. S3. Relationship between Unemployment Triple I and number of social groups utilized (k), residualized by MCA's total population, USA.
Notes: The graph shows the relationship between the residualized values of 'Triple I Unemployment' and 'K', after controlling for MCA's total population, using data pooled across 1980 to 2020 for the USA. k represents the number of social groups used in the calculation of the Unemployment Triple I and can take values of 16, 8, 4, or 2.

E1. Triple I's relationship with number of social groups utilized.

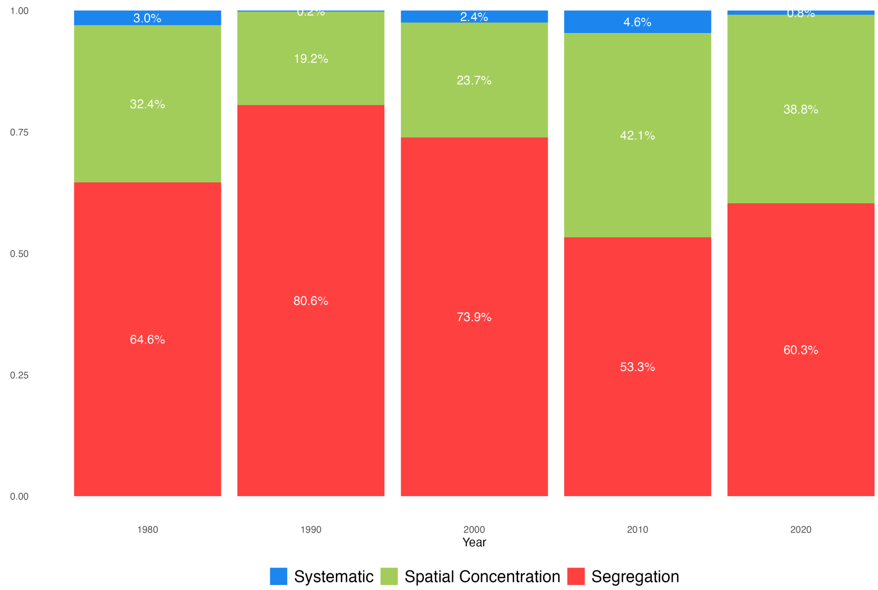

Fig. S4. Inequality sources, USA

Notes: the figure above display the decomposition of the Unemployment Triple I for the USA, across the years. The decomposition is at the country-level, when applicable, we aggregate utilizing MCA population weights. The Systematic portion corresponds to the calculation of the Triple I at the country-level, utilizing the most aggregated composition of the groups (k=2, males and females). The Spatial Concentration portion corresponds to the difference between the computation of the Triple I at the mca-level utilizing the most aggregated composition and the systematic portion. The Segregation portion corresponds to the difference between the original Triple I and the computation of the Triple I at the mca-level utilizing the most aggregated composition.

E2. Triple I decomposition.

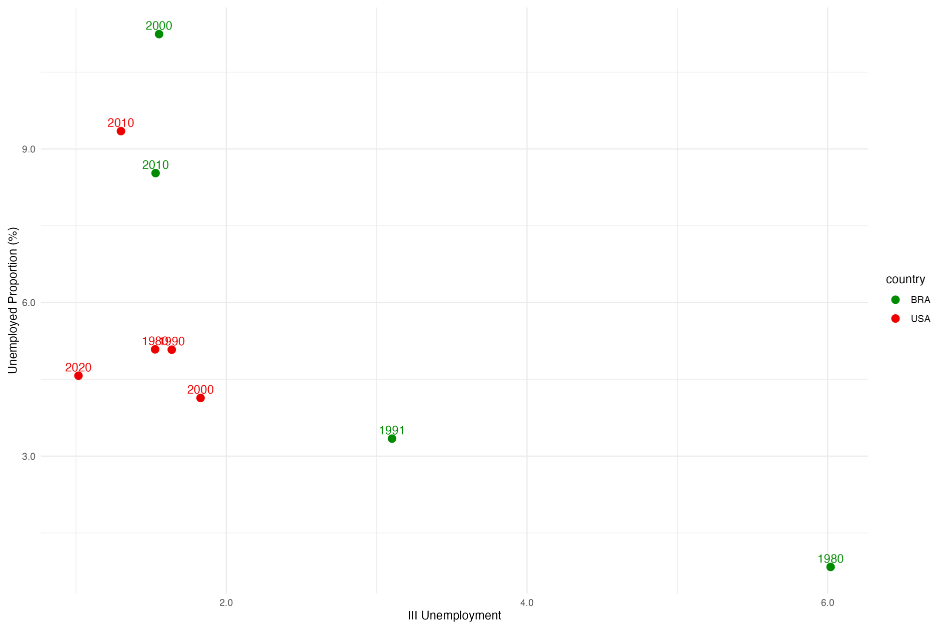

Fig. S5. Triple I vs Unemployment rate over the years

Notes: the figure above displays the relationship between the Rate of Unemployment (y-axis) and the Unemployment Triple I (x-axis) across the years, for Brazil (green dots) and the USA (red dots). Both the variables are computed at the MCA-level and aggregated to country-level using MCA population weights.

E3. Triple I and unemployment rates.

Fig. S6. Population and unemployment partitions into social groups, USA (2010)

Notes: These figures display the partition of (a) total population and of (b) total unemployed population, collected utilizing the US Census of 2010. The partition is divided among all social groups utilized in the analysis of the Triple I, and all proportions are calculated using individual sample weights.

E4. Population and unemployment partitions.

Fig. S7. Population and unemployment partitions into social groups, USA (2000)

Notes: These figures display the partition of (a) total population and of (b) total unemployed population, collected utilizing the US Census of 2000. The partition is divided among all social groups utilized in the analysis of the Triple I, and all proportions are calculated using individual sample weights.

Fig. S8. Population and unemployment partitions into social groups, USA (1990)

Notes: These figures display the partition of (a) total population and of (b) total unemployed population, collected utilizing the US Census of 1990. The partition is divided among all social groups utilized in the analysis of the Triple I, and all proportions are calculated using individual sample weights.

Fig. S9. Population and unemployment partitions into social groups, USA (1980)

Notes: These figures display the partition of (a) total population and of (b) total unemployed population, collected utilizing the US Census of 1980. The partition is divided among all social groups utilized in the analysis of the Triple I, and all proportions are calculated using individual sample weights.

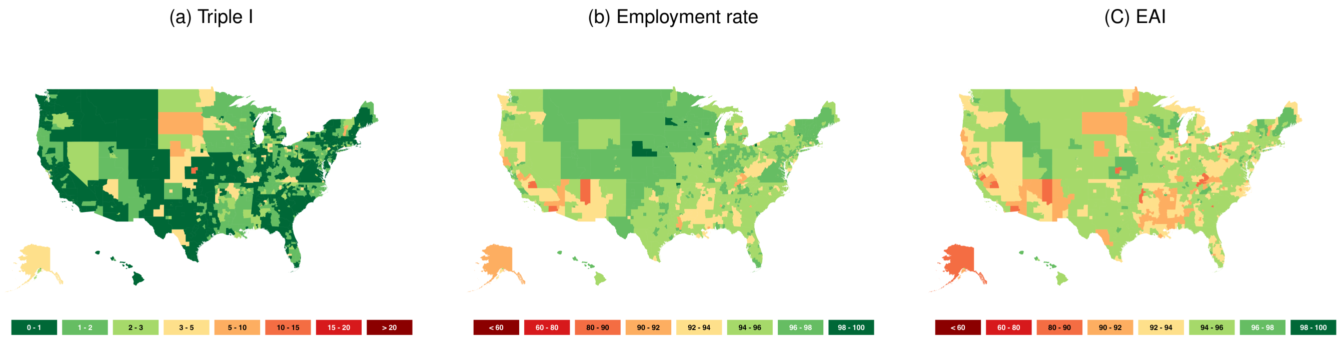

Fig. S10. Unemployment Triple I, employment rate and EAI by MCA, USA (2020)

Notes: These maps display the (a) Triple I for Unemployment (b) Employment Rate and (c) EAI of employment by mca. All calculations are performed at the MCA level and then aggregated to the MCA level using MCA population weights. Each map divides mcas among 8 intervals, and each mca is colored according to the interval it falls in. EAI's value is obtained by $P_s^{emp}(1 - III_s^{unemp})$, where P_s^{emp} is the mca's employment rate and III_s^{unemp} is the mca's unemployment Triple I.

E5. Triple I, unemployment rate and EAI.

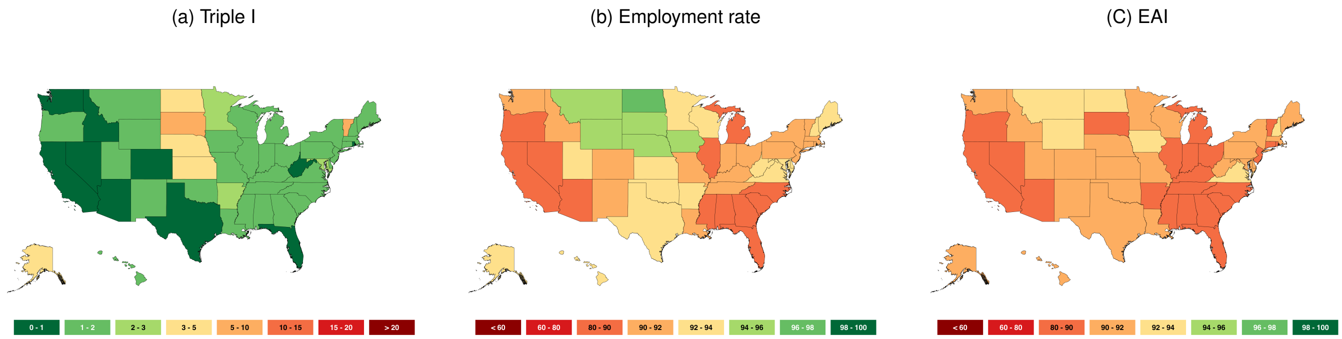

Fig. S11. Unemployment Triple I, employment rate and EAI by State, USA (2010)

Notes: These maps display the (a) Triple I for Unemployment (b) Employment Rate and (c) EAI of employment by state. All calculations are performed at the MCA level and then aggregated to the State level using MCA population weights. Each map divides states among 8 intervals, and each state is colored according to the interval it falls in. EAI's value is obtained by $P_s^{emp}(1 - III_s^{unemp})$, where P_s^{emp} is the state's employment rate and III_s^{unemp} is the state's unemployment Triple I.

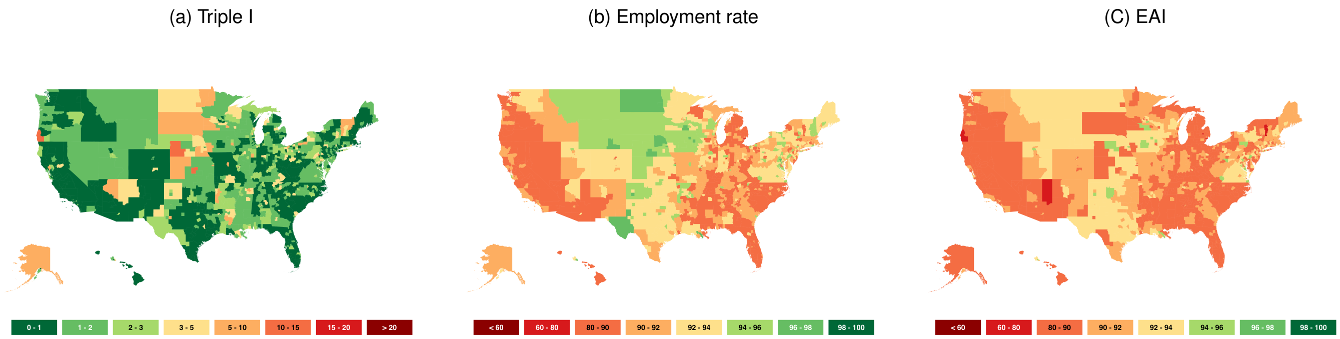

Fig. S12. Unemployment Triple I, employment rate and EAI by MCA, USA (2010)

Notes: These maps display the (a) Triple I for Unemployment (b) Employment Rate and (c) EAI of employment by mca. All calculations are performed at the MCA level and then aggregated to the MCA level using MCA population weights. Each map divides mcas among 8 intervals, and each mca is colored according to the interval it falls in. EAI's value is obtained by $P_s^{emp}(1 - III_s^{unemp})$, where P_s^{emp} is the mca's employment rate and III_s^{unemp} is the mca's unemployment Triple I.

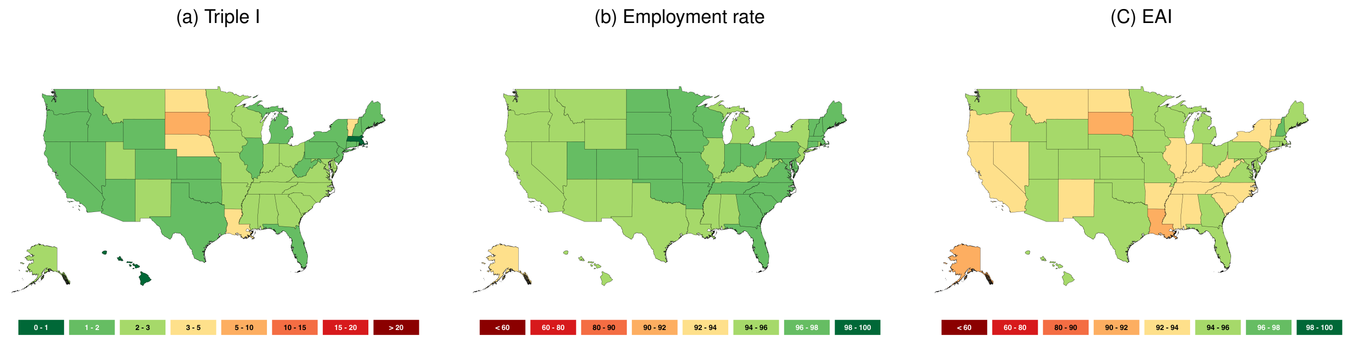

Fig. S13. Unemployment Triple I, employment rate and EAI by State, USA (2000)

Notes: These maps display the (a) Triple I for Unemployment (b) Employment Rate and (c) EAI of employment by state. All calculations are performed at the MCA level and then aggregated to the State level using MCA population weights. Each map divides states among 8 intervals, and each state is colored according to the interval it falls in. EAI's value is obtained by $P_s^{emp}(1 - III_s^{unemp})$, where P_s^{emp} is the state's employment rate and III_s^{unemp} is the state's unemployment Triple I.

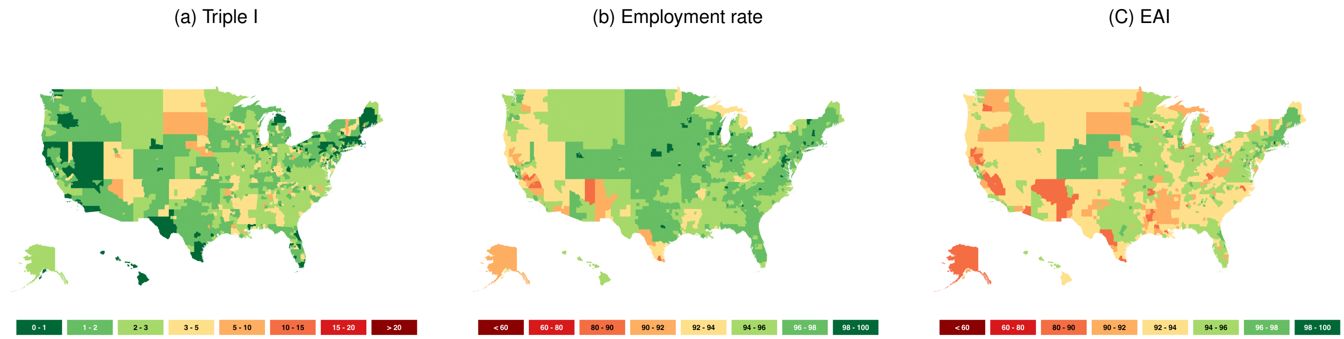

Fig. S14. Unemployment Triple I, employment rate and EAI by MCA, USA (2000)

Notes: These maps display the (a) Triple I for Unemployment (b) Employment Rate and (c) EAI of employment by mca. All calculations are performed at the MCA level and then aggregated to the MCA level using MCA population weights. Each map divides mcas among 8 intervals, and each mca is colored according to the interval it falls in. EAI's value is obtained by $P_s^{emp}(1 - III_s^{unemp})$, where P_s^{emp} is the mca's employment rate and III_s^{unemp} is the mca's unemployment Triple I.

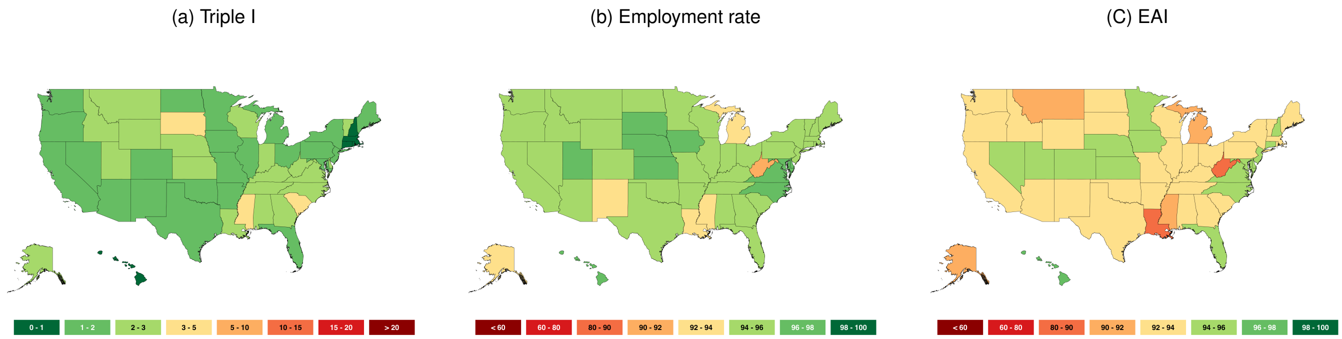

Fig. S15. Unemployment Triple I, employment rate and EAI by State, USA (1990)

Notes: These maps display the (a) Triple I for Unemployment (b) Employment Rate and (c) EAI of employment by state. All calculations are performed at the MCA level and then aggregated to the State level using MCA population weights. Each map divides states among 8 intervals, and each state is colored according to the interval it falls in. EAI's value is obtained by $P_s^{emp}(1 - III_s^{unemp})$, where P_s^{emp} is the state's employment rate and III_s^{unemp} is the state's unemployment Triple I.

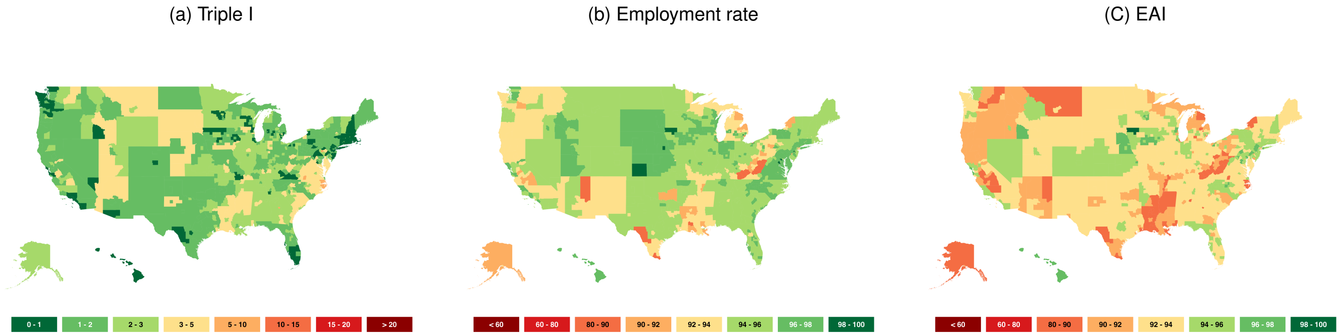

Fig. S16. Unemployment Triple I, employment rate and EAI by MCA, USA (1990)

Notes: These maps display the (a) Triple I for Unemployment (b) Employment Rate and (c) EAI of employment by mca. All calculations are performed at the MCA level and then aggregated to the MCA level using MCA population weights. Each map divides mcas among 8 intervals, and each mca is colored according to the interval it falls in. EAI's value is obtained by $P_s^{emp}(1 - III_s^{unemp})$, where P_s^{emp} is the mca's employment rate and III_s^{unemp} is the mca's unemployment Triple I.

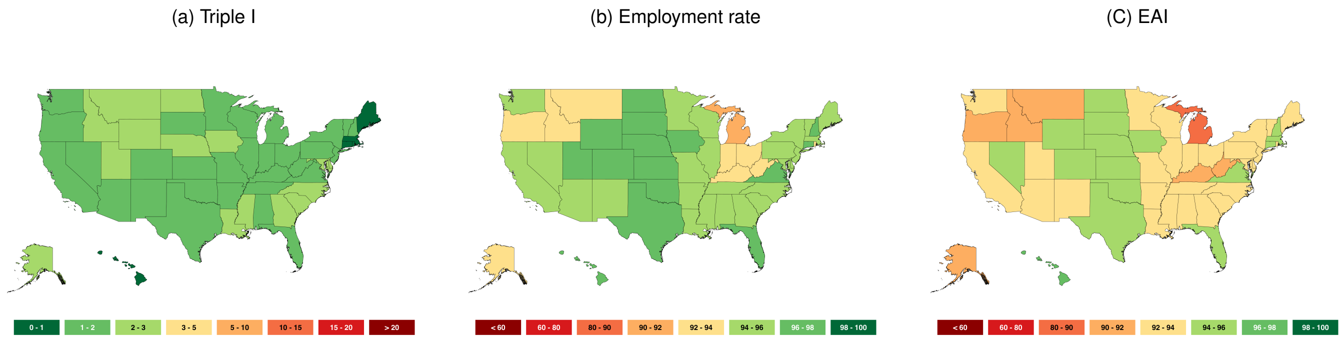

Fig. S17. Unemployment Triple I, employment rate and EAI by State, USA (1980)

Notes: These maps display the (a) Triple I for Unemployment (b) Employment Rate and (c) EAI of employment by state. All calculations are performed at the MCA level and then aggregated to the State level using MCA population weights. Each map divides states among 8 intervals, and each state is colored according to the interval it falls in. EAI's value is obtained by $P_s^{emp}(1 - III_s^{unemp})$, where P_s^{emp} is the state's employment rate and III_s^{unemp} is the state's unemployment Triple I.

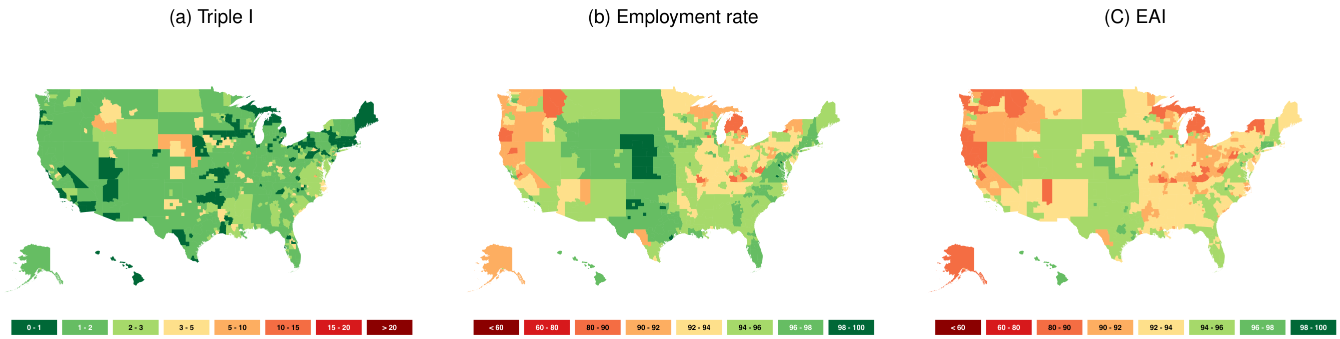

Fig. S18. Unemployment Triple I, employment rate and EAI by MCA, USA (1980)

Notes: These maps display the (a) Triple I for Unemployment (b) Employment Rate and (c) EAI of employment by mca. All calculations are performed at the MCA level and then aggregated to the MCA level using MCA population weights. Each map divides mcas among 8 intervals, and each mca is colored according to the interval it falls in. EAI's value is obtained by $P_s^{emp}(1 - III_s^{unemp})$, where P_s^{emp} is the mca's employment rate and III_s^{unemp} is the mca's unemployment Triple I.

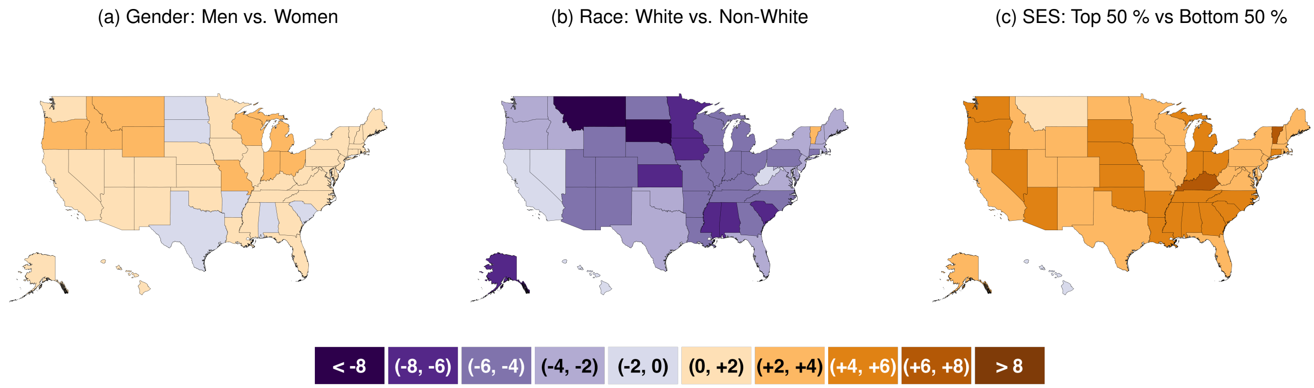

Fig. S19. Differences in unemployment rates across social groups by State, USA (2010)

Notes: These maps display the difference in unemployment rate for (a) Men - Women (b) White - Non-White and (c) SES top 50% - SES bottom 50%. All calculations are performed at the MCA level and then aggregated to the State level using MCA population weights. Each map divides states among 8 intervals, and each state is colored according to the interval it falls in.

B. E6. Differences in unemployment rates across social groups.

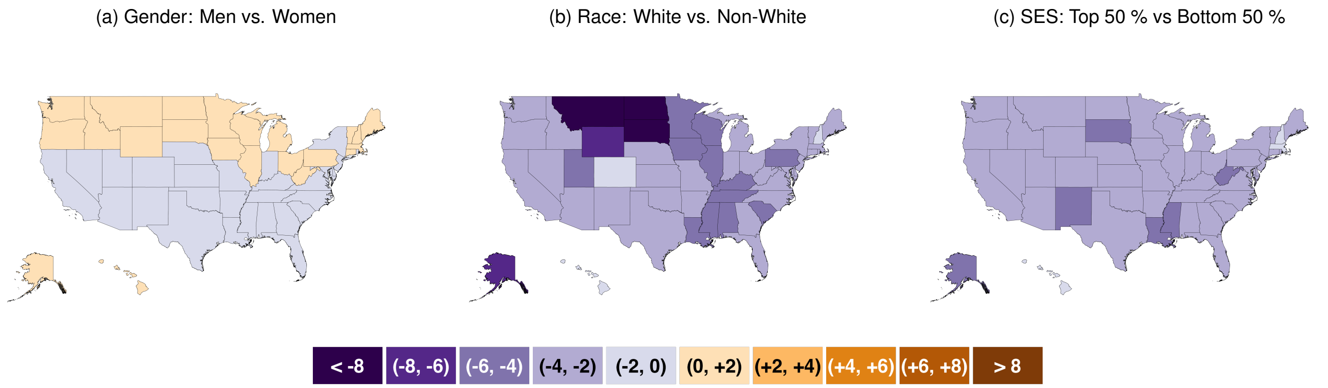

Fig. S20. Differences in unemployment rates across social groups by State, USA (2000)

Notes: These maps display the difference in unemployment rate for (a) Men - Women (b) White - Non-White and (c) SES top 50% - SES bottom 50%. All calculations are performed at the MCA level and then aggregated to the State level using MCA population weights. Each map divides states among 8 intervals, and each state is colored according to the interval it falls in.

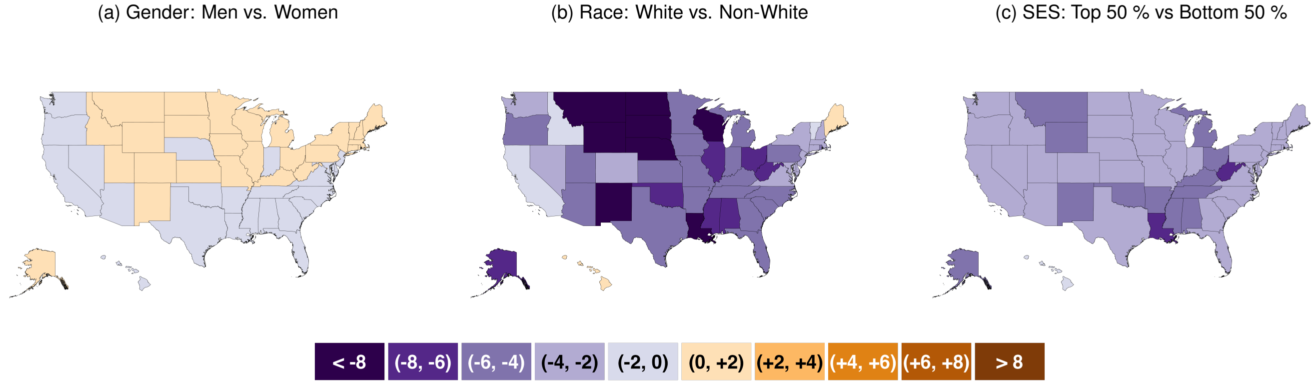

Fig. S21. Differences in unemployment rates across social groups by State, USA (1990)

Notes: These maps display the difference in unemployment rate for (a) Men - Women (b) White - Non-White and (c) SES top 50% - SES bottom 50%. All calculations are performed at the MCA level and then aggregated to the State level using MCA population weights. Each map divides states among 8 intervals, and each state is colored according to the interval it falls in.

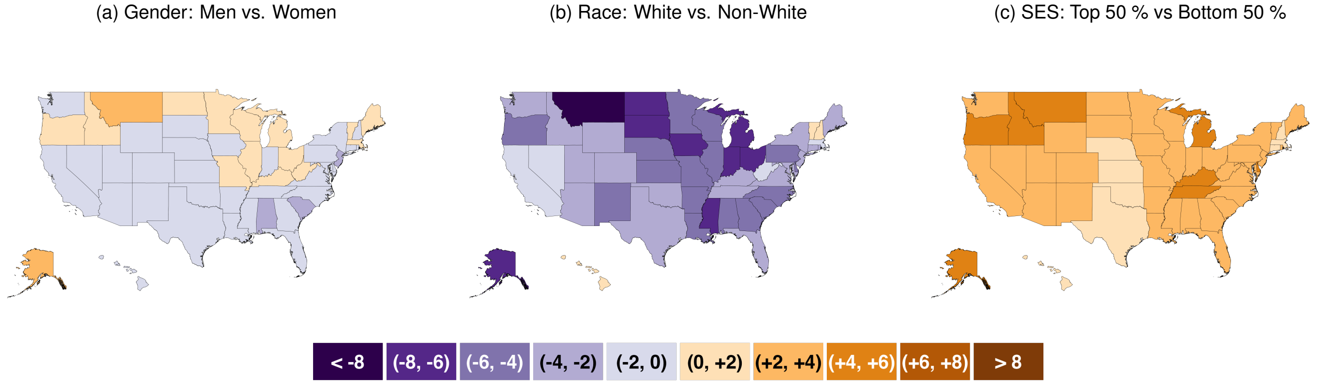

Fig. S22. Differences in unemployment rates across social groups by State, USA (1980)

Notes: These maps display the difference in unemployment rate for (a) Men - Women (b) White - Non-White and (c) SES top 50% - SES bottom 50%. All calculations are performed at the MCA level and then aggregated to the State level using MCA population weights. Each map divides states among 8 intervals, and each state is colored according to the interval it falls in.

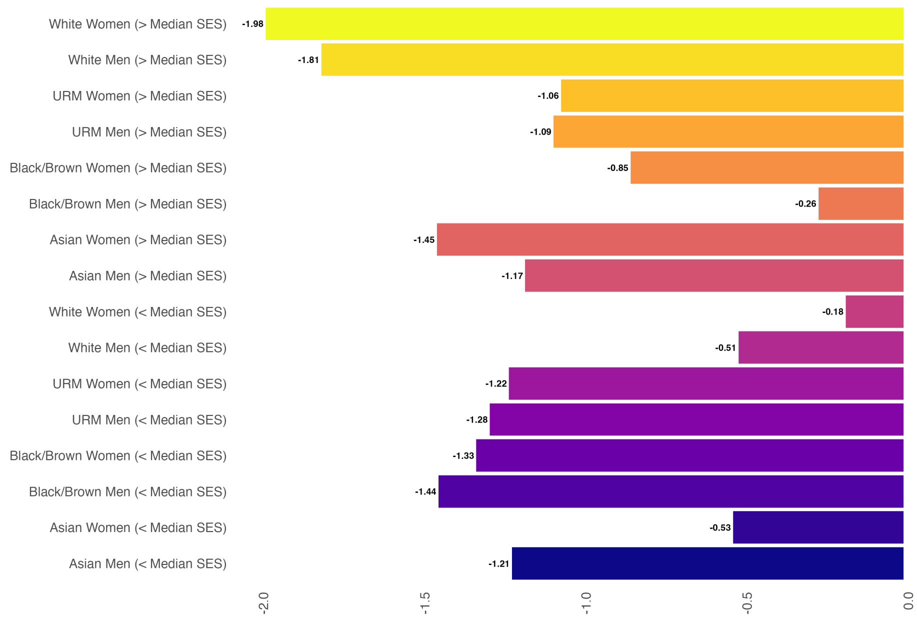

Fig. S23. Group-weighted Shapley values for Unemployment Triple I, USA (2010)

Notes: The figure above displays the **group-weighted shapley values** for Unemployment Triple I for each social group utilized in the analysis. The calculation is performed at the MCA level and then aggregated for the US utilizing MCA population weights. For each group, shapley values are obtained by excluding the social group of the MCA population and redistributing its total population and unemployed population homogeneously across all remaining groups. Then, the Triple I calculation is redone. The unweighted shapley values are obtained according to equation (**??**). Finally to obtain the group weighted shapley values ($S^w(k)$ we divide $S(k)$ by P_k, that is, $S^w(k) = \frac{S(k)}{P_k}$.

283 **E7. Shapley values.**

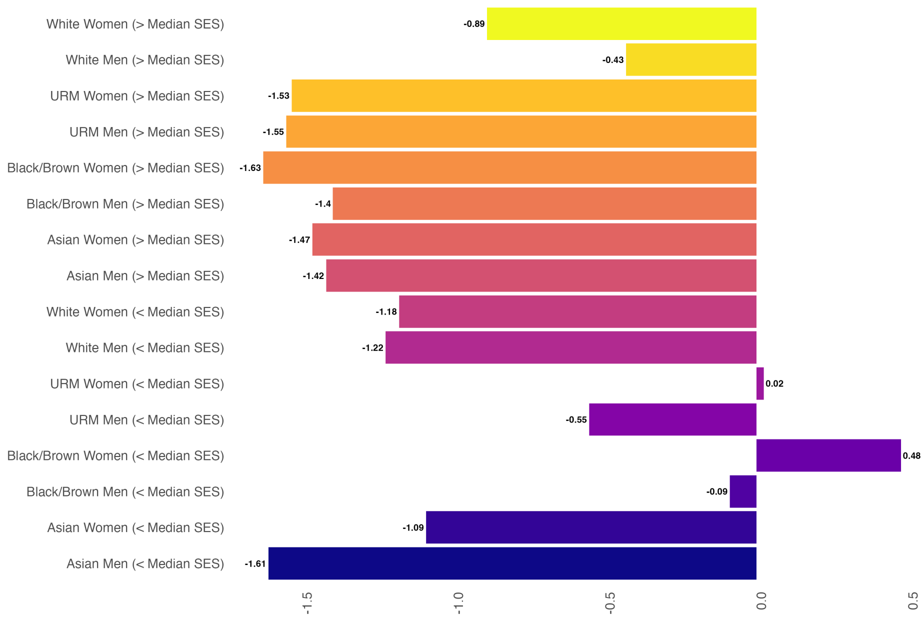

Fig. S24. Group-weighted Shapley values for Unemployment Triple I, USA (2000)

Notes: The figure above displays the **group-weighted shapley values** for Unemployment Triple I for each social group utilized in the analysis. The calculation is performed at the MCA level and then aggregated for the US utilizing MCA population weights. For each group, shapley values are obtained by excluding the social group of the MCA population and redistributing its total population and unemployed population homogeneously across all remaining groups. Then, the Triple I calculation is redone. The unweighted shapley values are obtained according to equation (**??**). Finally to obtain the group weighted shapley values ($S^w(k)$ we divide $S(k)$ by P_k, that is, $S^w(k) = \frac{S(k)}{P_k}$.

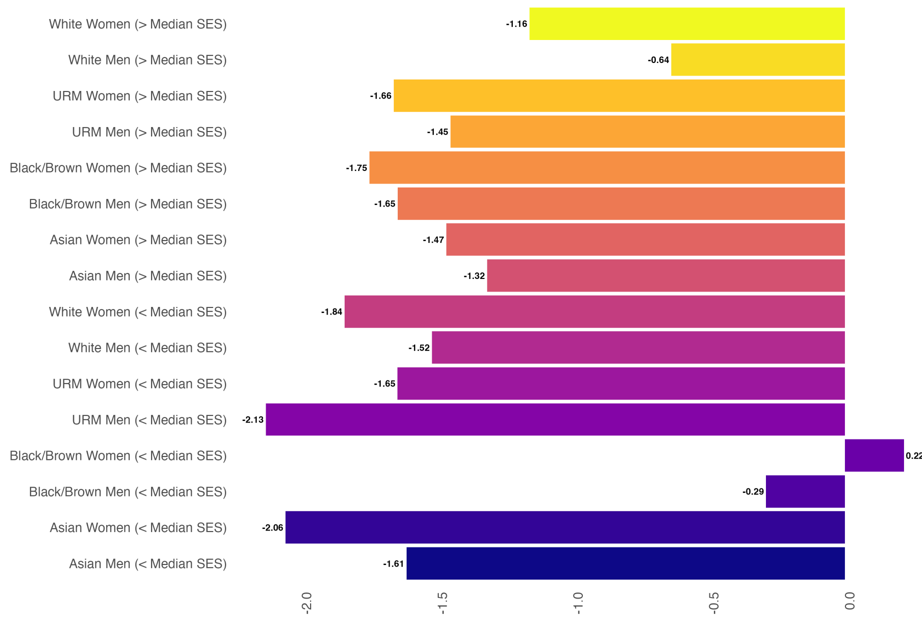

Fig. S25. Group-weighted Shapley values for Unemployment Triple I, USA (1990)

Notes: The figure above displays the **group-weighted shapley values** for Unemployment Triple I for each social group utilized in the analysis. The calculation is performed at the MCA level and then aggregated for the US utilizing MCA population weights. For each group, shapley values are obtained by excluding the social group of the MCA population and redistributing its total population and unemployed population homogeneously across all remaining groups. Then, the Triple I calculation is redone. The unweighted shapley values are obtained according to equation (**??**). Finally to obtain the group weighted shapley values ($S^w(k)$ we divide $S(k)$ by P_k, that is, $S^w(k) = \frac{S(k)}{P_k}$.

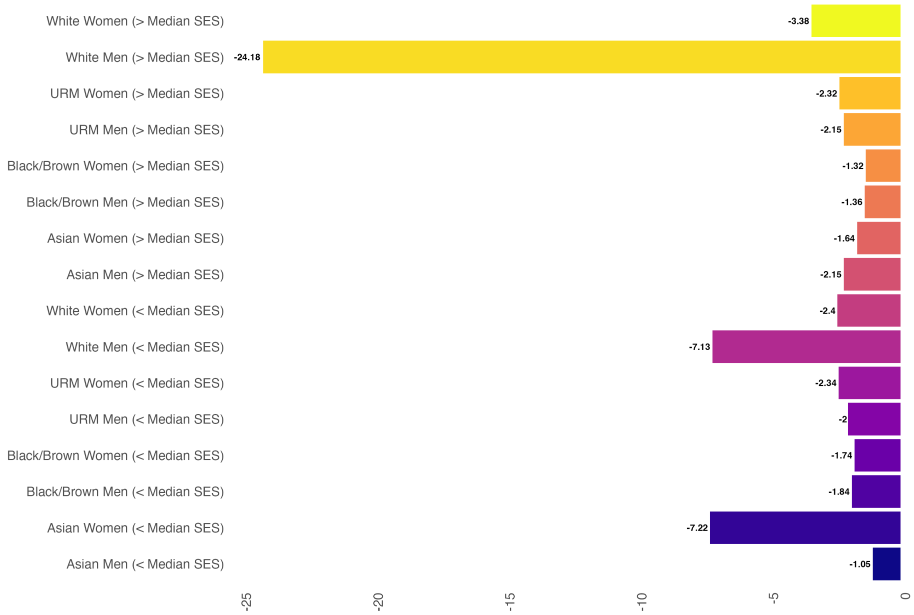

Fig. S26. Group-weighted Shapley values for Unemployment Triple I, USA (1980)

Notes: The figure above displays the **group-weighted shapley values** for Unemployment Triple I for each social group utilized in the analysis. The calculation is performed at the MCA level and then aggregated for the US utilizing MCA population weights. For each group, shapley values are obtained by excluding the social group of the MCA population and redistributing its total population and unemployed population homogeneously across all remaining groups. Then, the Triple I calculation is redone. The unweighted shapley values are obtained according to equation (**??**). Finally to obtain the group weighted shapley values ($S^w(k)$ we divide $S(k)$ by P_k, that is, $S^w(k) = \frac{S(k)}{P_k}$.

284 ## F. Additional results: Brazil

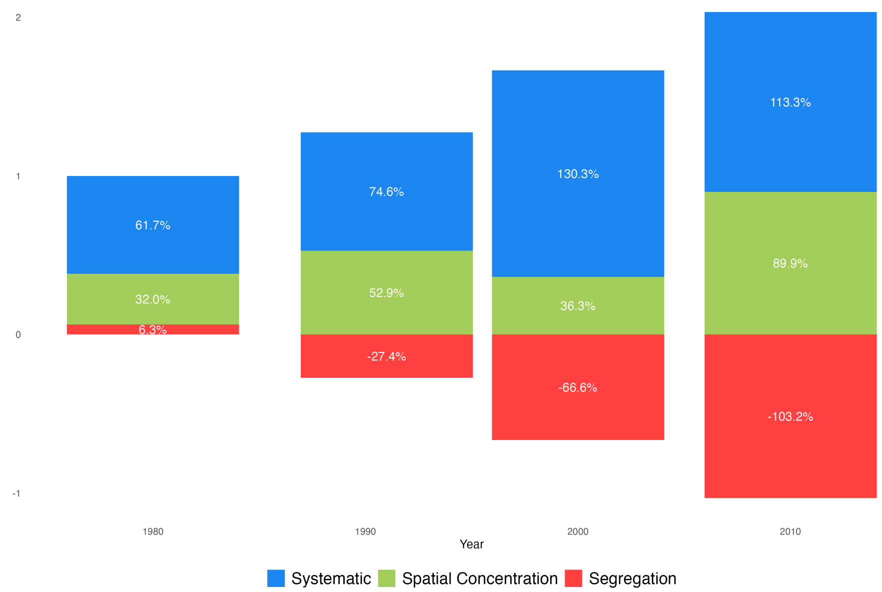

Fig. S27. Inequality sources, BRA

Notes: the figure above display the decomposition of the Unemployment Triple I for Brazil, across the years. The decomposition is at the country-level, when applicable, we aggregate utilizing MCA population weights. The Systematic portion corresponds to the calculation of the Triple I at the country-level, utilizing the most aggregated composition of the groups (k=2, males and females). The Spatial Concentration portion corresponds to the difference between the computation of the Triple I at the mca-level utilizing the most aggregated composition and the systematic portion. The Segregation portion corresponds to the difference between the original Triple I and the computation of the Triple I at the mca-level utilizing the most aggregated composition.

285 ### C. F1. Triple I decomposition.

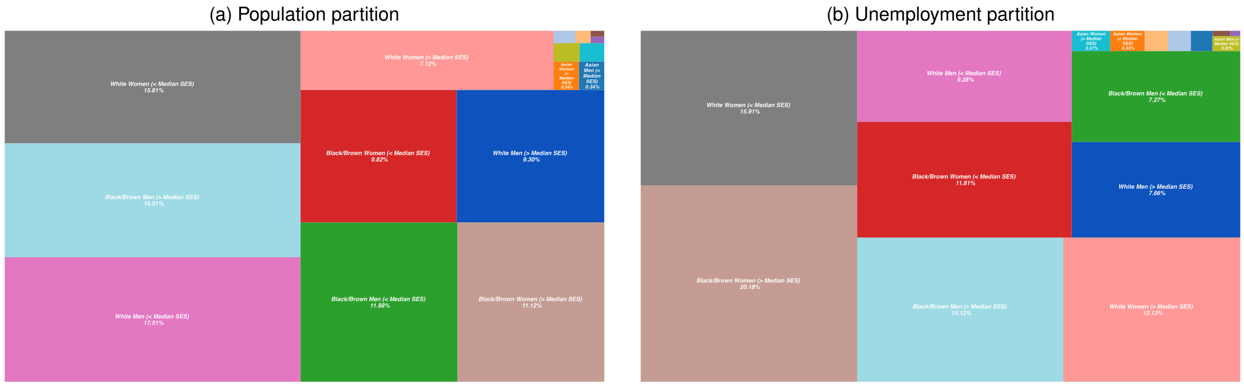

Fig. S28. Population and unemployment partitions into social groups, Brazil (2010)

Notes: These figures display the partition of (a) total population and of (b) total unemployed population, collected utilizing the Brazilian Census of 2010. The partition is divided among all social groups utilized in the analysis of the Triple I, and all proportions are calculated using individual sample weights.

286 **F2. Population and unemployment partitions.**

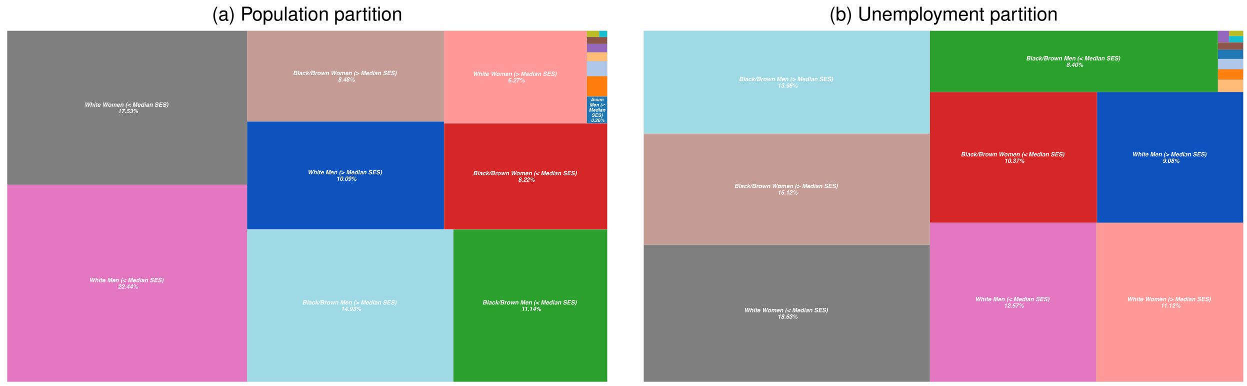

Fig. S29. Population and unemployment partitions into social groups, Brazil (2000)

Notes: These figures display the partition of (a) total population and of (b) total unemployed population, collected utilizing the Brazilian Census of 2000. The partition is divided among all social groups utilized in the analysis of the Triple I, and all proportions are calculated using individual sample weights.

Fig. S30. Population and unemployment partitions into social groups, Brazil (1991)

Notes: These figures display the partition of (a) total population and of (b) total unemployed population, collected utilizing the Brazilian Census of 1991. The partition is divided among all social groups utilized in the analysis of the Triple I, and all proportions are calculated using individual sample weights.

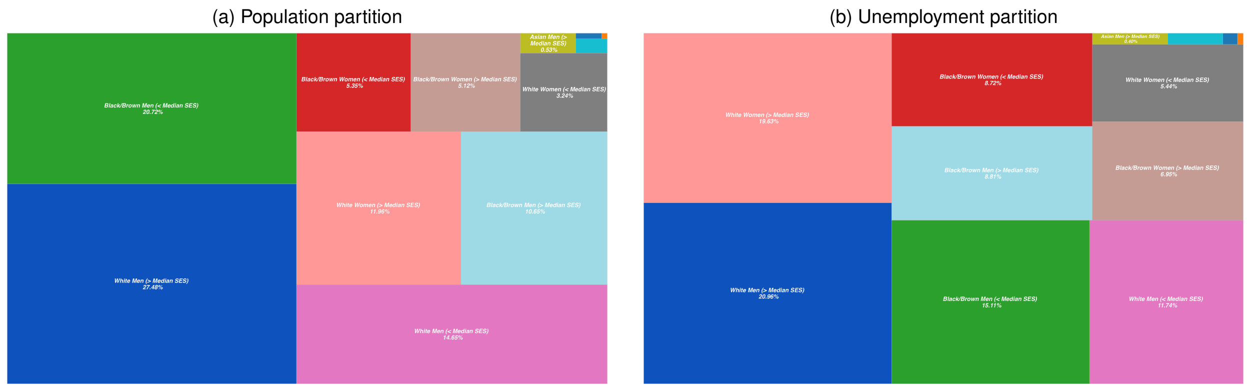

Fig. S31. Population and unemployment partitions into social groups, Brazil (1980)

Notes: These figures display the partition of (a) total population and of (b) total unemployed population, collected utilizing the Brazilian Census of 1980. The partition is divided among all social groups utilized in the analysis of the Triple I, and all proportions are calculated using individual sample weights.

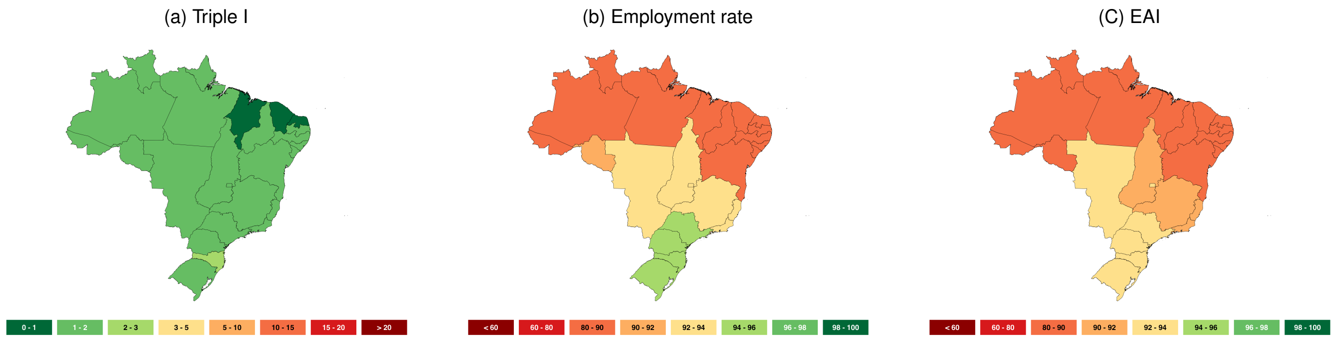

Fig. S32. Unemployment Triple I, employment rate and EAI by State, Brazil (2010)

Notes: These maps display the (a) Triple I for Unemployment (b) Employment Rate and (c) EAI of employment by state. All calculations are performed at the MCA level and then aggregated to the State level using MCA population weights. Each map divides states among 8 intervals, and each state is colored according to the interval it falls in. EAI's value is obtained by $P_s^{emp}(1 - III_s^{unemp})$, where P_s^{emp} is the state's employment rate and III_s^{unemp} is the state's unemployment Triple I.

F3. Triple I, Unemployment rate and EAI.

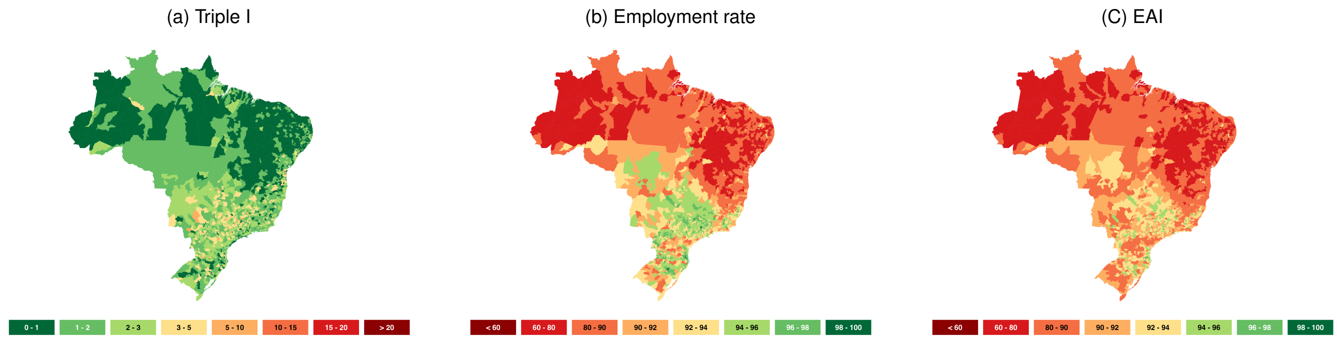

Fig. S33. Unemployment Triple I, employment rate and EAI by MCA, Brazil (2010)

Notes: These maps display the (a) Triple I for Unemployment (b) Employment Rate and (c) EAI of employment by mca. All calculations are performed at the MCA level and then aggregated to the MCA level using MCA population weights. Each map divides mcas among 8 intervals, and each mca is colored according to the interval it falls in. EAI's value is obtained by $P_s^{emp}(1 - III_s^{unemp})$, where P_s^{emp} is the mca's employment rate and III_s^{unemp} is the mca's unemployment Triple I.

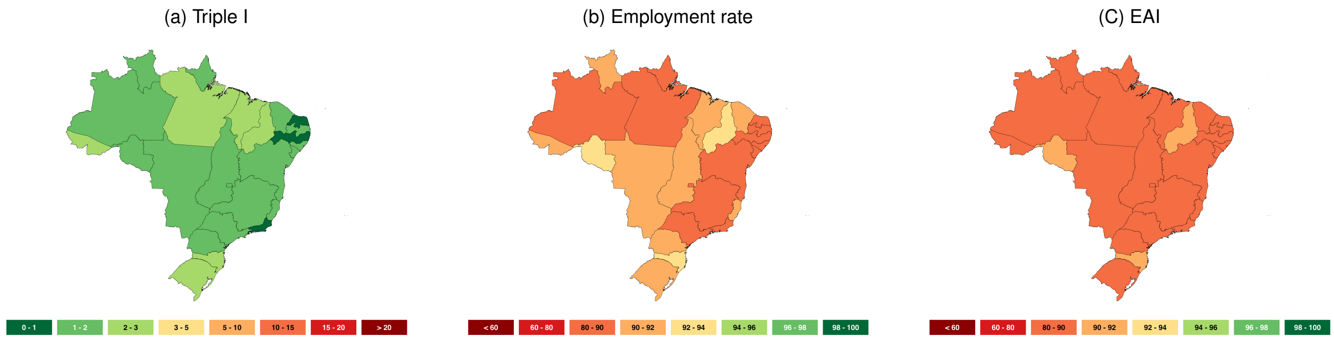

Fig. S34. Unemployment Triple I, employment rate and EAI by State, Brazil (2000)

Notes: These maps display the (a) Triple I for Unemployment (b) Employment Rate and (c) EAI of employment by state. All calculations are performed at the MCA level and then aggregated to the State level using MCA population weights. Each map divides states among 8 intervals, and each state is colored according to the interval it falls in. EAI's value is obtained by $P_s^{emp}(1 - III_s^{unemp})$, where P_s^{emp} is the state's employment rate and III_s^{unemp} is the state's unemployment Triple I.

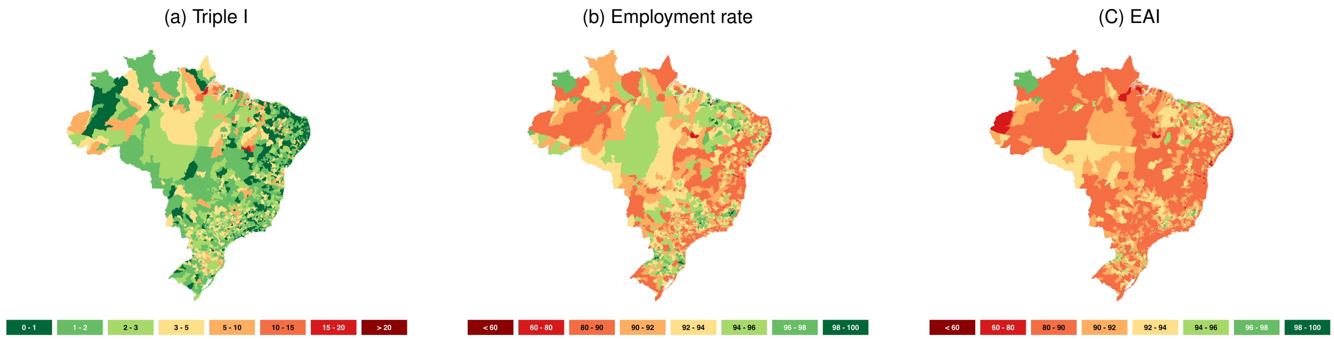

Fig. S35. Unemployment Triple I, employment rate and EAI by MCA, Brazil (2000)

Notes: These maps display the (a) Triple I for Unemployment (b) Employment Rate and (c) EAI of employment by mca. All calculations are performed at the MCA level and then aggregated to the MCA level using MCA population weights. Each map divides mcas among 8 intervals, and each mca is colored according to the interval it falls in. EAI's value is obtained by $P_s^{emp}(1 - III_s^{unemp})$, where P_s^{emp} is the mca's employment rate and III_s^{unemp} is the mca's unemployment Triple I.

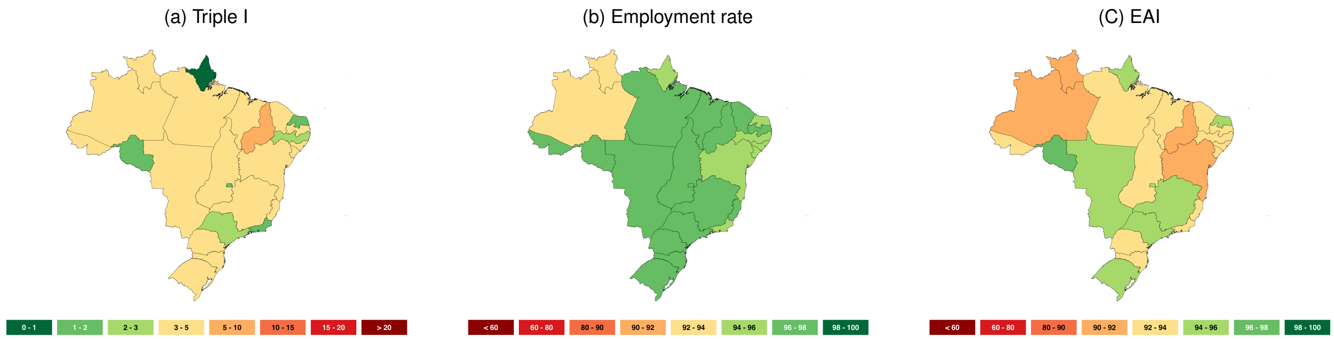

Fig. S36. Unemployment Triple I, employment rate and EAI by State, Brazil (1991)

Notes: These maps display the (a) Triple I for Unemployment (b) Employment Rate and (c) EAI of employment by state. All calculations are performed at the MCA level and then aggregated to the State level using MCA population weights. Each map divides states among 8 intervals, and each state is colored according to the interval it falls in. EAI's value is obtained by $P_s^{emp}(1 - III_s^{unemp})$, where P_s^{emp} is the state's employment rate and III_s^{unemp} is the state's unemployment Triple I.

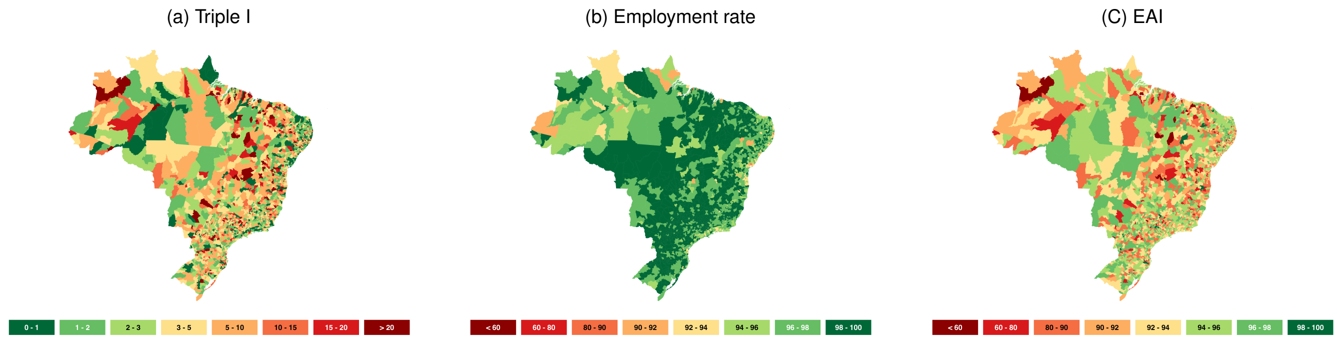

Fig. S37. Unemployment Triple I, employment rate and EAI by MCA, Brazil (1991)

Notes: These maps display the (a) Triple I for Unemployment (b) Employment Rate and (c) EAI of employment by mca. All calculations are performed at the MCA level and then aggregated to the MCA level using MCA population weights. Each map divides mcas among 8 intervals, and each mca is colored according to the interval it falls in. EAI's value is obtained by $P_s^{emp}(1 - III_s^{unemp})$, where P_s^{emp} is the mca's employment rate and III_s^{unemp} is the mca's unemployment Triple I.

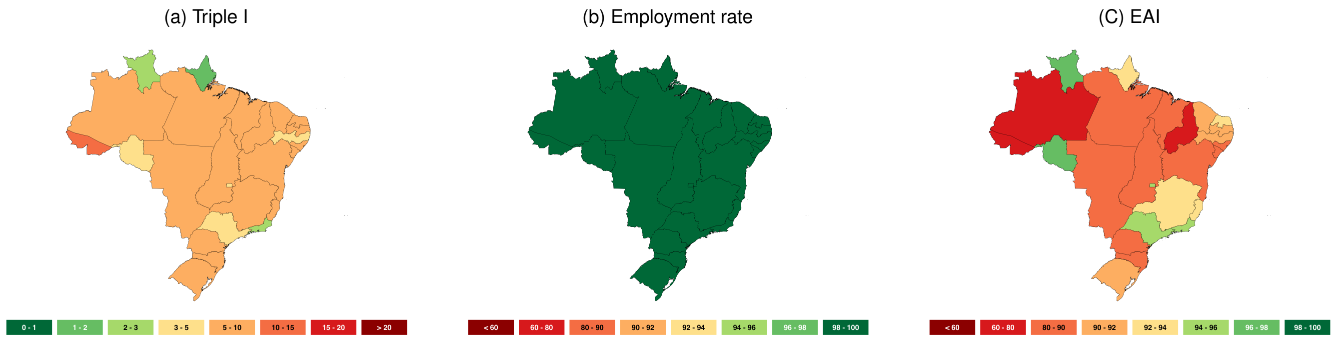

Fig. S38. Unemployment Triple I, employment rate and EAI by State, Brazil (1980)

Notes: These maps display the (a) Triple I for Unemployment (b) Employment Rate and (c) EAI of employment by state. All calculations are performed at the MCA level and then aggregated to the State level using MCA population weights. Each map divides states among 8 intervals, and each state is colored according to the interval it falls in. EAI's value is obtained by $P_s^{emp}(1 - III_s^{unemp})$, where P_s^{emp} is the state's employment rate and III_s^{unemp} is the state's unemployment Triple I.

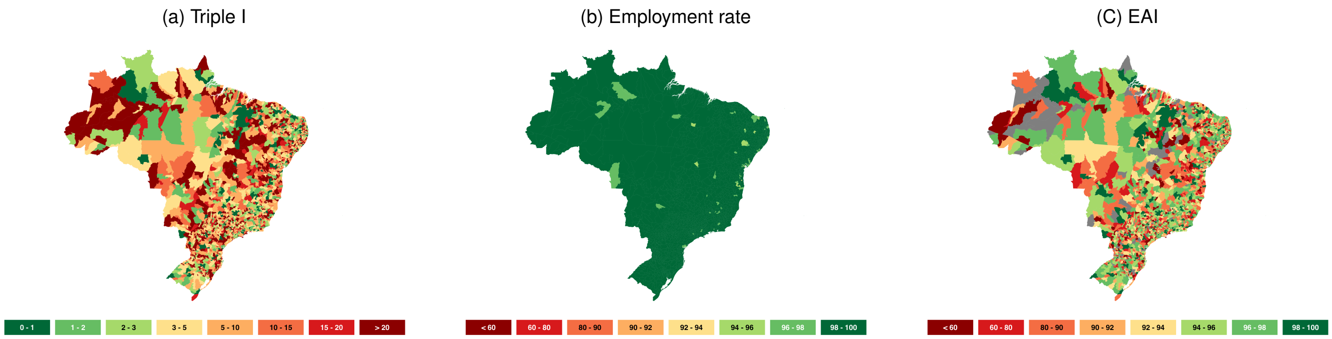

Fig. S39. Unemployment Triple I, employment rate and EAI by MCA, Brazil (1980)

Notes: These maps display the (a) Triple I for Unemployment (b) Employment Rate and (c) EAI of employment by mca. All calculations are performed at the MCA level and then aggregated to the MCA level using MCA population weights. Each map divides mcas among 8 intervals, and each mca is colored according to the interval it falls in. EAI's value is obtained by $P_s^{emp}(1 - III_s^{unemp})$, where P_s^{emp} is the mca's employment rate and III_s^{unemp} is the mca's unemployment Triple I.

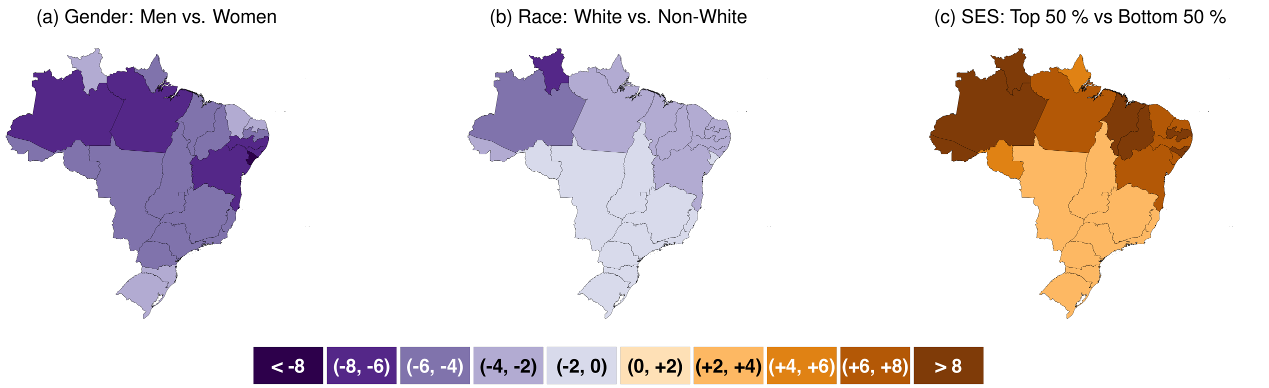

Fig. S40. Differences in unemployment rates across social groups by State, Brazil (2010)

Notes: These maps display the difference in unemployment rate for (a) Men - Women (b) White - Non-White and (c) SES top 50% - SES bottom 50%. All calculations are performed at the MCA level and then aggregated to the State level using MCA population weights. Each map divides states among 8 intervals, and each state is colored according to the interval it falls in.

D. F4. Differences in unemployment rates across social groups.

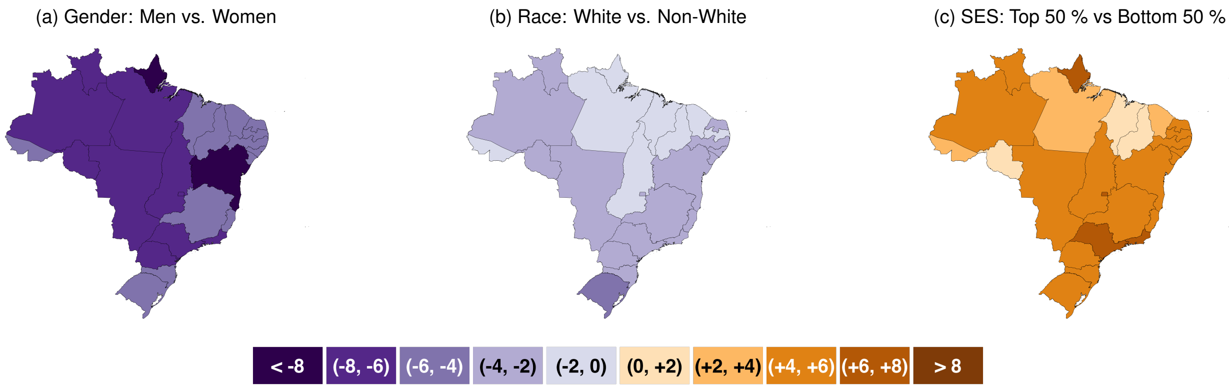

Fig. S41. Differences in unemployment rates across social groups by State, Brazil (2000)

Notes: These maps display the difference in unemployment rate for (a) Men - Women (b) White - Non-White and (c) SES top 50% - SES bottom 50%. All calculations are performed at the MCA level and then aggregated to the State level using MCA population weights. Each map divides states among 8 intervals, and each state is colored according to the interval it falls in.

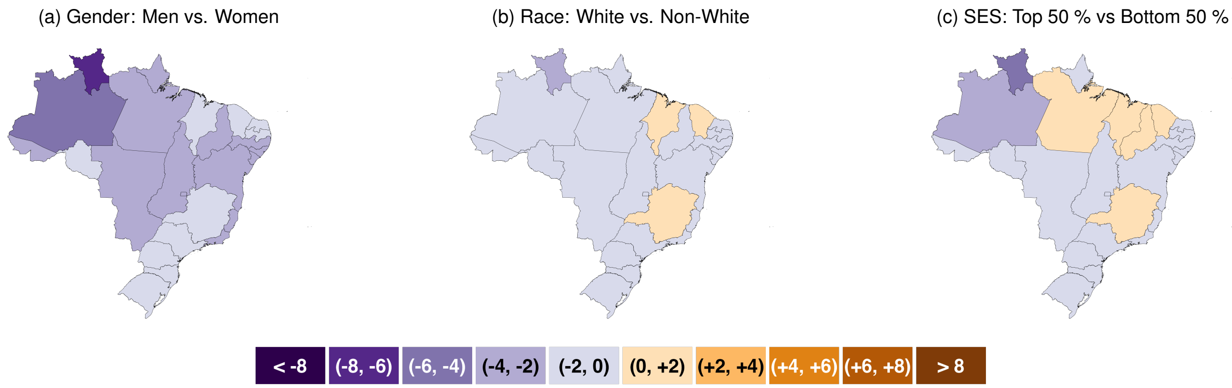

Fig. S42. Differences in unemployment rates across social groups by State, Brazil (1991)

Notes: These maps display the difference in unemployment rate for (a) Men - Women (b) White - Non-White and (c) SES top 50% - SES bottom 50%. All calculations are performed at the MCA level and then aggregated to the State level using MCA population weights. Each map divides states among 8 intervals, and each state is colored according to the interval it falls in.

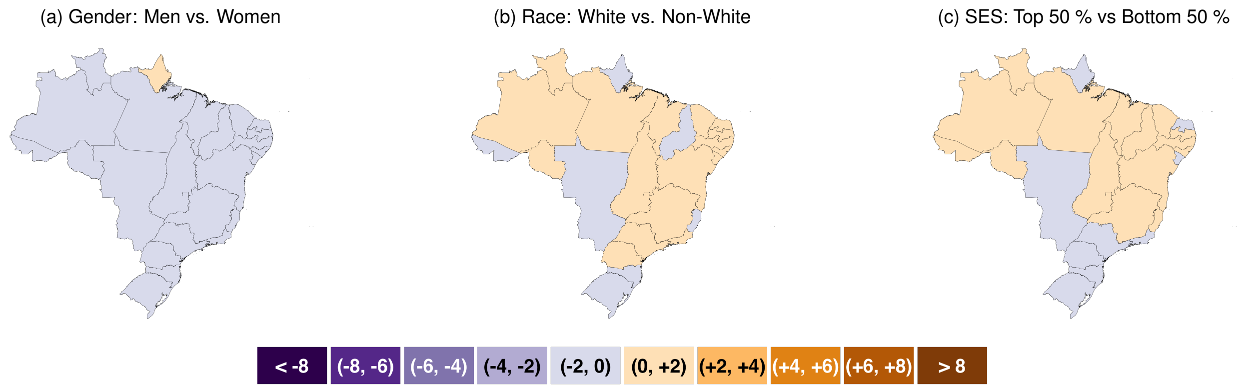

Fig. S43. Differences in unemployment rates across social groups by State, Brazil (1980)

Notes: These maps display the difference in unemployment rate for (a) Men - Women (b) White - Non-White and (c) SES top 50% - SES bottom 50%. All calculations are performed at the MCA level and then aggregated to the State level using MCA population weights. Each map divides states among 8 intervals, and each state is colored according to the interval it falls in.

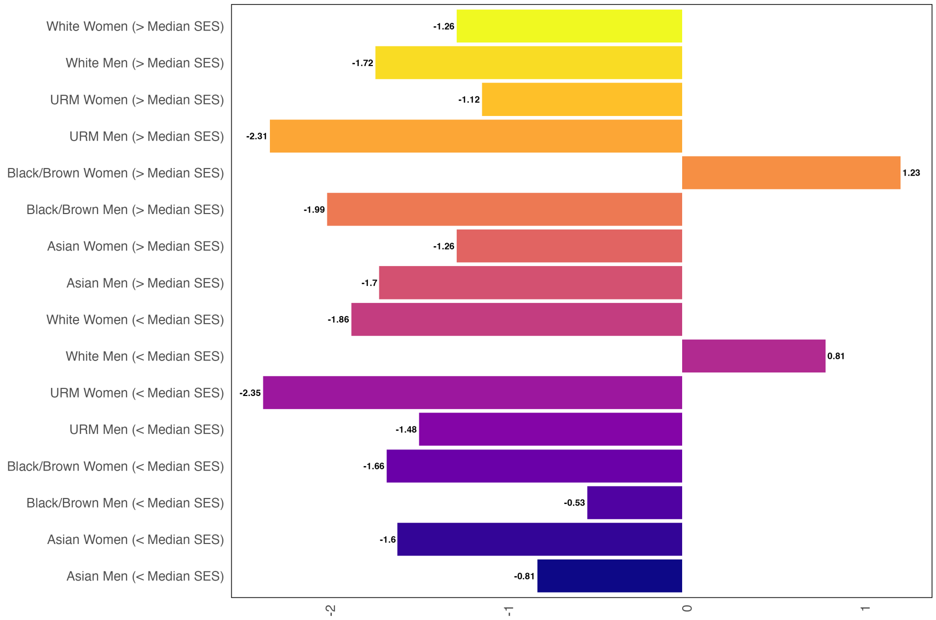

Fig. S44. Group-weighted Shapley values for Unemployment Triple I, Brazil (2010)

Notes: The figure above displays the **group-weighted shapley values** for Unemployment Triple I for each social group utilized in the analysis. The calculation is performed at the MCA level and then aggregated for the US utilizing MCA population weights. For each group, shapley values are obtained by excluding the social group of the MCA population and redistributing its total population and unemployed population homogeneously across all remaining groups. Then, the Triple I calculation is redone. The unweighted shapley values are obtained according to equation (**??**). Finally to obtain the group weighted shapley values ($S^w(k)$ we divide $S(k)$ by P_k, that is, $S^w(k) = \frac{S(k)}{P_k}$.

289 **E. F5. Shapley values.**

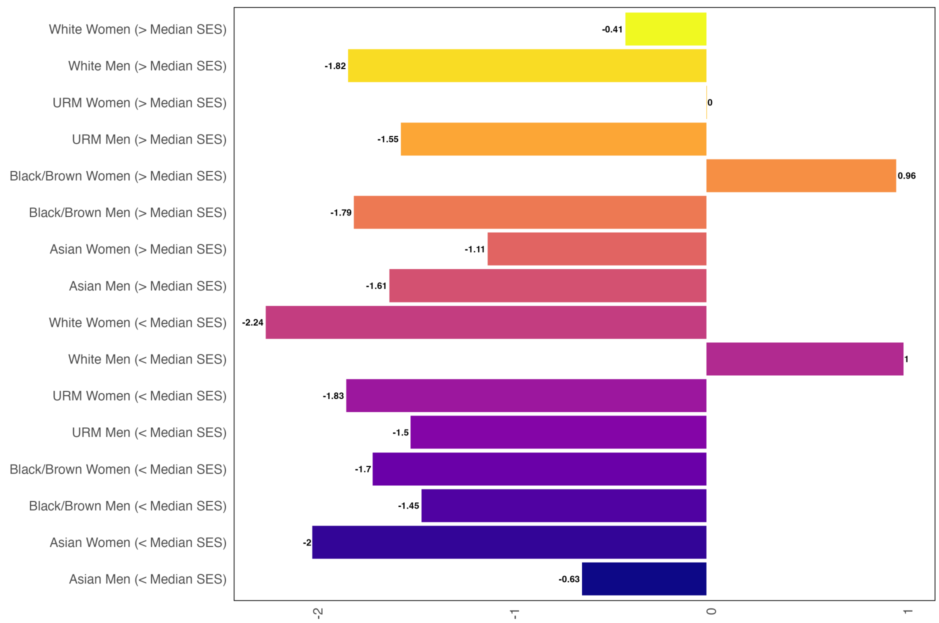

Fig. S45. Group-weighted Shapley values for Unemployment Triple I, Brazil (2000)

Notes: The figure above displays the **group-weighted shapley values** for Unemployment Triple I for each social group utilized in the analysis. The calculation is performed at the MCA level and then aggregated for the US utilizing MCA population weights. For each group, shapley values are obtained by excluding the social group of the MCA population and redistributing its total population and unemployed population homogeneously across all remaining groups. Then, the Triple I calculation is redone. The unweighted shapley values are obtained according to equation (**??**). Finally to obtain the group weighted shapley values ($S^w(k)$ we divide $S(k)$ by P_k, that is, $S^w(k) = \frac{S(k)}{P_k}$.

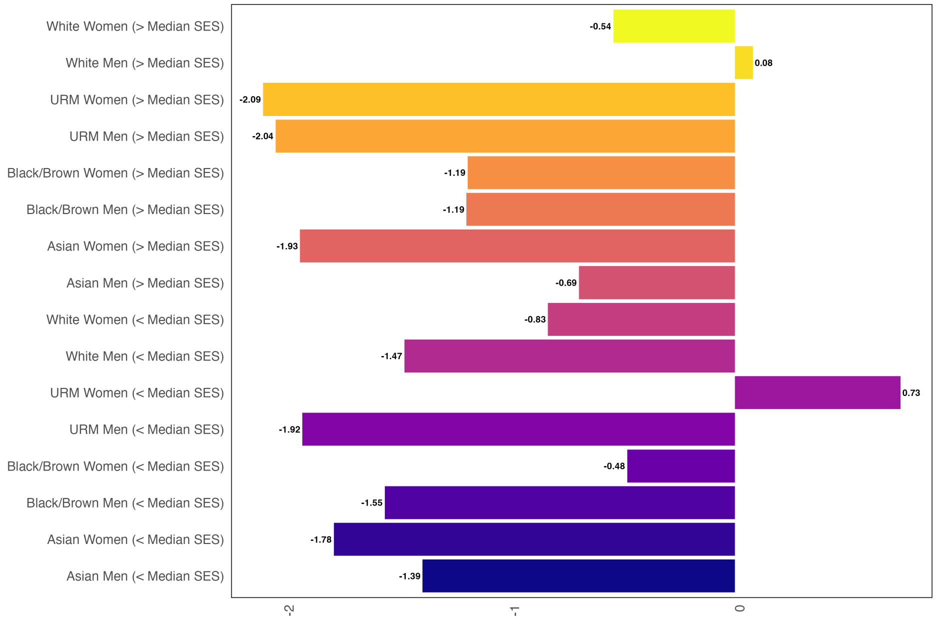

Fig. S46. Group-weighted Shapley values for Unemployment Triple I, Brazil (1991)

Notes: The figure above displays the **group-weighted shapley values** for Unemployment Triple I for each social group utilized in the analysis. The calculation is performed at the MCA level and then aggregated for the US utilizing MCA population weights. For each group, shapley values are obtained by excluding the social group of the MCA population and redistributing its total population and unemployed population homogeneously across all remaining groups. Then, the Triple I calculation is redone. The unweighted shapley values are obtained according to equation (**??**). Finally to obtain the group weighted shapley values ($S^w(k)$ we divide $S(k)$ by P_k, that is, $S^w(k) = \frac{S(k)}{P_k}$.

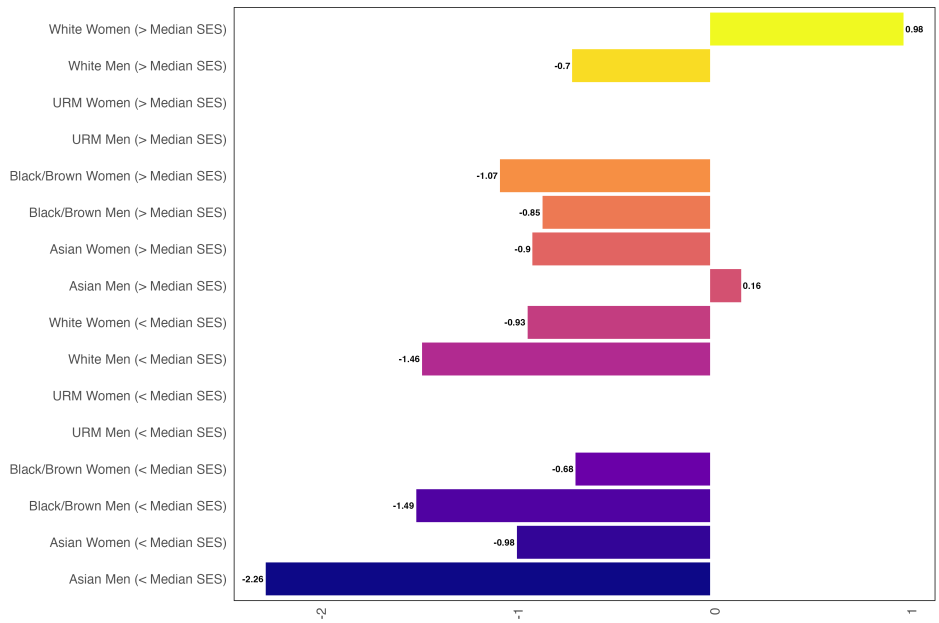

Fig. S47. Group-weighted Shapley values for Unemployment Triple I, Brazil (1980)

Notes: The figure above displays the **group-weighted shapley values** for Unemployment Triple I for each social group utilized in the analysis. The calculation is performed at the MCA level and then aggregated for the US utilizing MCA population weights. For each group, shapley values are obtained by excluding the social group of the MCA population and redistributing its total population and unemployed population homogeneously across all remaining groups. Then, the Triple I calculation is redone. The unweighted shapley values are obtained according to equation (**??**). Finally to obtain the group weighted shapley values ($S^w(k)$ we divide $S(k)$ by P_k, that is, $S^w(k) = \frac{S(k)}{P_k}$.